8. The functions f and g are defined by

$$f : x \mapsto 1 - 2x^3, \ x \in \mathbb{R}$$

$$g : x \mapsto \frac{3}{x} - 4, \ x > 0, \ x \in \mathbb{R}$$

(a) Find the inverse function f^{-1}.

(2)

(b) Show that the composite function gf is

$$gf : x \mapsto \frac{8x^3 - 1}{1 - 2x^3}.$$

(4)

(c) Solve $gf(x) = 0$.

(2)

(d) Use calculus to find the coordinates of the stationary point on the graph of $y = gf(x)$.

(5)

a) $y = 1 - 2x^3$

$2x^3 = 1 - y$

$x^3 = \frac{1-y}{2}$

$x = \sqrt[3]{\frac{1-y}{2}}$

$f^{-1}(x) = \sqrt[3]{\frac{1-x}{2}}$

b) $gf : x \mapsto \frac{3}{1-2x^3} - 4$

$= \frac{3 - 4(1-2x^3)}{1-2x^3}$

$= \frac{8x^3 - 1}{1 - 2x^3}$

c) $$\frac{8x^3 - 1}{1 - 2x^3} = 0$$

$$8x^3 - 1 = 0$$

$$x^3 = \frac{1}{8}$$

$$x = 1/2$$

d) Stationary point when $\frac{dy}{dx} = 0$.

$$y = \frac{8x^3 - 1}{1 - 2x^3}$$

Quotient rule. $u = 8x^3 - 1$ $\quad v = 1 - 2x^3$

$\frac{du}{dx} = 24x^2$ $\quad \frac{dv}{dx} = -6x^2$

$$\frac{dy}{dx} = \frac{(1-2x^3)24x^2 - (8x^3-1)(-6x^2)}{(1-2x)^3}$$

$$= \frac{24x^2 - 48x^5 + 48x^5 - 6x^2}{(1-2x)^3}$$

When $\frac{dy}{dx} = 0$

$$0 = 18x^2$$

$x = 0$ substitute into: $y = -1$ $\quad (0, -1)$

Leave blank

5. The functions f and g are defined by

$$f : x \mapsto 3x + \ln x, \quad x > 0, \quad x \in \mathbb{R}$$
$$g : x \mapsto e^{x^2}, \quad x \in \mathbb{R}$$

(a) Write down the range of g.

(1)

(b) Show that the composite function fg is defined by

$$fg : x \mapsto x^2 + 3e^{x^2}, \quad x \in \mathbb{R}.$$

(2)

(c) Write down the range of fg.

(1)

(d) Solve the equation $\frac{d}{dx}\left[fg(x)\right] = x(xe^{x^2} + 2)$.

(6)

a) $g : x \mapsto e^{x^2}$ the x^2 prevents e being to a negative number
$e^0 = 1$

$g(x) \geq 1$

b) substitute e^{x^2} into $f(x)$

$fg : 3e^{x^2} + \ln e^{x^2}$

$: 3e^{x^2} + x^2$

(c) $fg(x) \geq 3$ smallest value is when $x^2 = 0$ which is $(3 \times 1 + 0) = 3$.

(d) $\frac{d}{dx}\left(x^2 + 3e^{x^2}\right)$ is $2x + 6xe^{x^2}$

$\Rightarrow 2x + 6xe^{x^2} = x(xe^{x^2} + 2)$

factorise with e^{x^2}

$\cancel{2x} + 6xe^{x^2} = x^2e^{x^2} + \cancel{2x}$

$e^{x^2}(6x - x^2) = 0$ $e^{x^2} \neq 0$ so $x = 0$ or 6.

6. The function f is defined by

$$\text{f}: x \mapsto \frac{3-2x}{x-5}, \quad x \in \mathbb{R}, \quad x \neq 5.$$

(*a*) Find $\text{f}^{-1}(x)$.

(3)

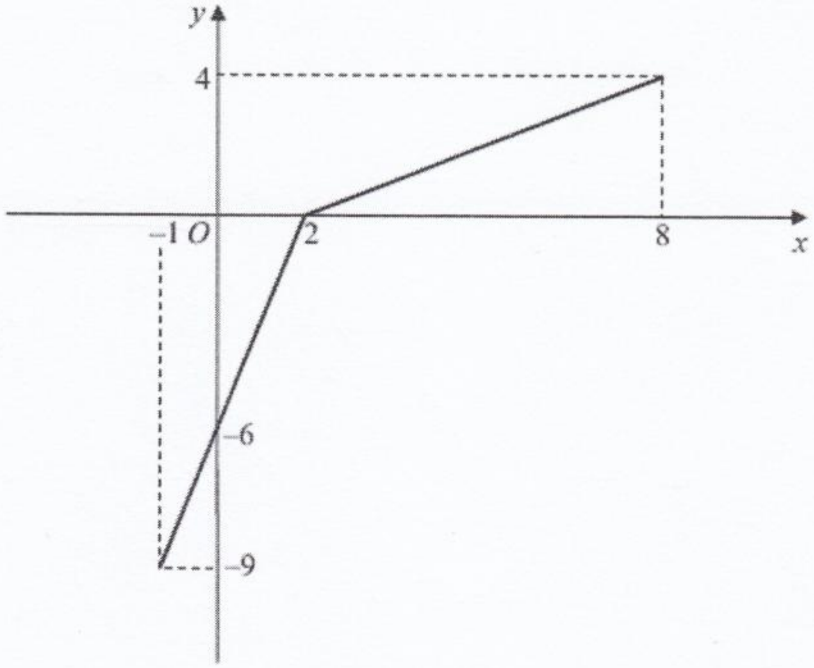


Figure 2

The function g has domain $-1 \leq x \leq 8$, and is linear from $(-1, -9)$ to $(2, 0)$ and from $(2, 0)$ to $(8, 4)$. Figure 2 shows a sketch of the graph of $y = \text{g}(x)$.

(*b*) Write down the range of g.

(1)

(*c*) Find gg(2).

(2)

(*d*) Find fg(8).

(2)

(*e*) On separate diagrams, sketch the graph with equation

(i) $y = |\text{g}(x)|$,

(ii) $y = \text{g}^{-1}(x)$.

Show on each sketch the coordinates of each point at which the graph meets or cuts the axes.

(4)

(*f*) State the domain of the inverse function g^{-1}.

(1)

a) <u>Inverse</u>

$$y = \frac{3-2x}{x-5}$$

$$y(x-5) = 3-2x$$

$$yx - 5y = 3 - 2x$$

$$yx + 2x = 3 + 5y$$

$$x(y+2) = 3 + 5y$$

$$x = \frac{3+5y}{y+2}$$

$$f^{-1}(x) = \frac{3+5x}{x+2}$$

(b) Range are y-axis values

$$-9 \leq g(x) \leq 4$$

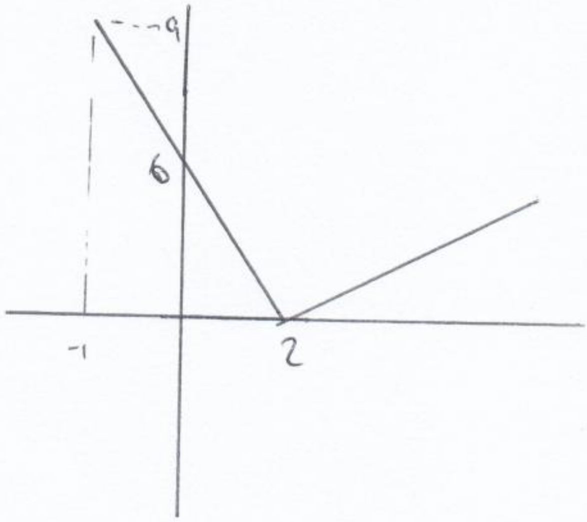
(c) $g(2) = 0$ from graph

$g(0) = -6$ $\Rightarrow$ $gg(2) = -6$

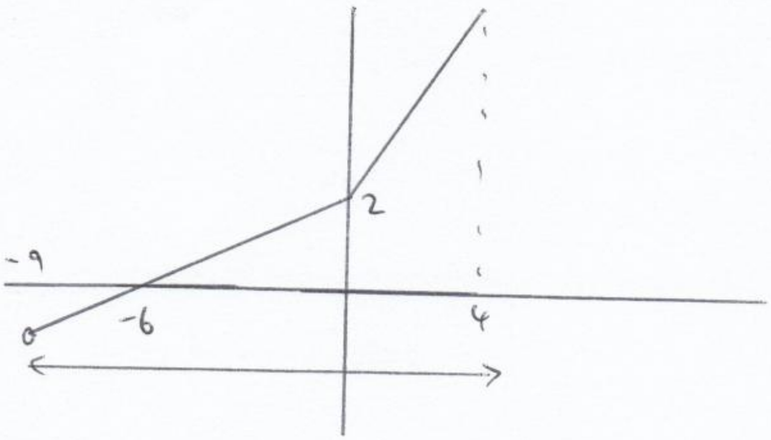
(d) $g(8) = 4$ from graph

$f(4) = \frac{3-2\times 4}{4-5} = 5$ $\quad \underline{fg(8) = 5.}$

(e) $y = |g(x)|$ reflect graph below x-axis in x-axis



(ii) $y = g^{-1}(x)$ reflect in line $y = x$



(f) domain x-values $-9 \le x < 4$

C3 Jan 2012

Leave blank

7. The function f is defined by

$$\mathrm{f}: x \mapsto \frac{3(x+1)}{2x^2+7x-4} - \frac{1}{x+4}, \qquad x \in \mathbb{R}, x > \frac{1}{2}$$

(a) Show that $\mathrm{f}(x) = \dfrac{1}{2x-1}$ **(4)**

(b) Find $\mathrm{f}^{-1}(x)$ **(3)**

(c) Find the domain of f^{-1} **(1)**

$$\mathrm{g}(x) = \ln(x+1)$$

(d) Find the solution of $\mathrm{fg}(x) = \dfrac{1}{7}$, giving your answer in terms of e. **(4)**

a) $f(x) = \dfrac{3(x+1)}{(2x-1)(x+4)} - \dfrac{1}{x+4}$

$= \dfrac{3(x+1) - (2x-1)}{(2x-1)(x+4)}$

$= \dfrac{3x+3-2x+1}{(2x-1)(x+4)}$

$= \dfrac{\cancel{(x+4)}}{(2x-1)\cancel{(x+4)}}$

$= \dfrac{1}{2x-1}$

C3 Jan 2012

Leave blank

Question 7 continued

(b) $f^{-1}(x)$ inverse

$$y = \frac{1}{2x-1}$$

$$y(2x-1) = 1$$

$$2xy - y = 1$$

$$2xy = 1+y$$

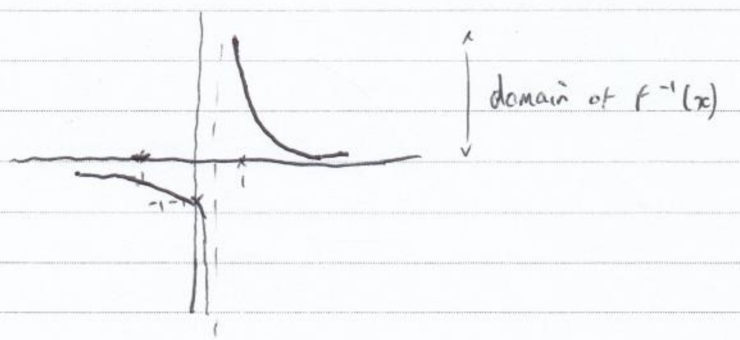
$$x = \frac{1+y}{2y}$$

$$f^{-1}(x) = \frac{1+x}{2x}$$

c) $x > 0$

Domain of $f^{-1}(x)$ is range of $f(x)$

$f(x) = \frac{1}{2x-1} \quad x > 1/2$



$x > 0$

P40084A01724

C3 Jan 2012

Leave blank

Question 7 continued

(d) $fg(x) = \frac{1}{7}$

$$\frac{1}{2\ln(x+1) - 1} = \frac{1}{7}$$

$$2\ln(x+1) - 1 = 7$$

$$2\ln(x+1) = 8$$

$$\ln(x+1) = 4$$

$$x+1 = e^4$$

$$x = e^4 - 1$$

Leave blank

5. The functions f and g are defined by

$$\mathrm{f}: x \mapsto \ln(2x-1), \qquad x \in \mathbb{R},\ x > \frac{1}{2},$$

$$\mathrm{g}: x \mapsto \frac{2}{x-3}, \qquad x \in \mathbb{R},\ x \neq 3.$$

(a) Find the exact value of fg(4). **(2)**

(b) Find the inverse function $\mathrm{f}^{-1}(x)$, stating its domain. **(4)**

(c) Sketch the graph of $y = |\mathrm{g}(x)|$. Indicate clearly the equation of the vertical asymptote and the coordinates of the point at which the graph crosses the y-axis. **(3)**

(d) Find the exact values of x for which $\left|\frac{2}{x-3}\right| = 3$. **(3)**

5a) $fg(4)$ (into g then f) $fg(4) = \ln 3$

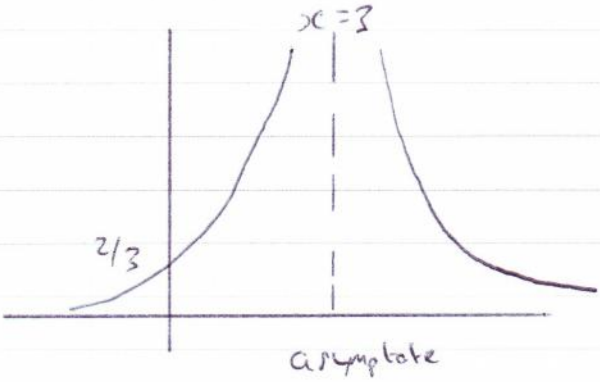
b) $y = \ln(2x-1)$

$e^y = 2x - 1$

$\frac{e^y + 1}{2} = x$

$f^{-1}(x) = \frac{e^x + 1}{2}$ Domain $x \in \mathbb{R}$

c)



N 2 6 1 0 9 A 0 1 0 2 4

Leave blank

Question 5 continued

d) $\frac{2}{x-3} = 3$ or $\frac{2}{x-3} = -3$

$2 = 3(x-3)$ $\quad$ $2 = -3(x-3)$

$2 = 3x - 9$ $\quad$ $2 = -3x + 9$

$3x = 11$ $\quad$ $-3x = -7$

$x = \frac{11}{3}$ $\quad$ $x = \frac{7}{3}$

N26109A01124

4. The function f is defined by

$$f: x \mapsto \frac{2(x-1)}{x^2-2x-3} - \frac{1}{x-3}, \quad x > 3.$$

(*a*) Show that $f(x) = \dfrac{1}{x+1}, \quad x > 3.$

(4)

(*b*) Find the range of f.

(2)

(*c*) Find $f^{-1}(x)$. State the domain of this inverse function.

(3)

The function g is defined by

$$g: x \mapsto 2x^2 - 3, \quad x \in \mathbb{R}.$$

(*d*) Solve $fg(x) = \dfrac{1}{8}$.

(3)

---

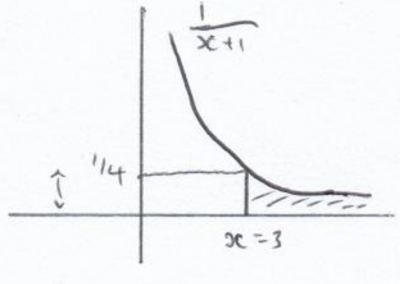
a) $f(x) = \dfrac{2(x-1)}{(x-3)(x+1)} - \dfrac{1}{x-3}$

$= \dfrac{2(x-1) - (x+1)}{(x-3)(x+1)}$

$= \dfrac{2x-2-x-1}{(x-3)(x+1)}$

$= \dfrac{\cancel{x-3}}{(\cancel{x-3})(x+1)} = \dfrac{1}{x+1}$

(b)



$x > 3$

range is $0 < f(x) < \frac{1}{4}$

range is y-axis

(c)

$$y = \frac{1}{x+1}$$

$$y(x+1) = 1$$

$$yx + y = 1$$

$$yx = 1 - y$$

$$x = \frac{1-y}{y}$$

$$f^{-1}(x) = \frac{1-x}{x}$$

Domain of Inverse = Range of function

so the same $(0, 1/4)$

(d) g into f

$$\frac{1}{2x^2 - 3 + 1} = \frac{1}{8}$$

$$\frac{1}{2x^2 - 2} = \frac{1}{8}$$

$$2x^2 - 2 = 8$$

$$2x^2 = 10$$

$$x^2 = 5$$

$$x = \pm\sqrt{5}$$

Leave blank

7. The function f is defined by

$$f(x) = 1 - \frac{2}{(x+4)} + \frac{x-8}{(x-2)(x+4)}, \quad x \in \mathbb{R}, x \neq -4, x \neq 2$$

(a) Show that $f(x) = \dfrac{x-3}{x-2}$ **(5)**

The function g is defined by

$$g(x) = \frac{e^x - 3}{e^x - 2}, \quad x \in \mathbb{R}, x \neq \ln 2$$

(b) Differentiate $g(x)$ to show that $g'(x) = \dfrac{e^x}{(e^x - 2)^2}$ **(3)**

(c) Find the exact values of x for which $g'(x) = 1$ **(4)**

a)

$$f(x) = \frac{(x-2)(x+4) - 2(x-2) + (x-8)}{(x-2)(x+4)}$$ common denominator

$$= \frac{x^2 + 2x - 8 - 2x + 4 + x - 8}{(x-2)(x+4)}$$

$$= \frac{x^2 + x - 12}{(x-2)(x+4)}$$

$$= \frac{(x-3)\cancel{(x+4)}}{(x-2)\cancel{(x+4)}}$$

$$f(x) = \frac{x-3}{x-2}$$

H 3 4 2 6 4 A 0 2 2 2 8

Question 7 continued

Leave blank

b) $u = e^x - 3$ $\qquad v = e^x - 2$ $\qquad$ quotient rule.

$\frac{du}{dx} = e^x \qquad \frac{dv}{dx} = e^x$

$\frac{dy}{dx} = \frac{(e^x - 2)e^x - (e^x - 3)e^x}{(e^x - 2)^2}$

$= e^{2x} - 2e^x - e^{2x} + 3e^x$

$\frac{dy}{dx} = \frac{e^x}{(e^x - 2)^2}$

(c) $g'(x) = 1 \qquad \frac{e^x}{(e^x - 2)^2} = 1$

$\Rightarrow \quad e^x = (e^x - 2)^2$

$e^x = e^{2x} - 4e^x + 4$

$0 = e^{2x} - 5e^x + 4$

$0 = (e^x - 1)(e^x - 4)$

so $\quad e^x = 1 \qquad e^x = 4$

$x = \ln 1 \qquad x = \ln 4$

$x = 0$

H 3 4 2 6 4 A 0 2 3 2 8

Turn over

4. The function f is defined by

$$f : x \mapsto |2x - 5|, \quad x \in \mathbb{R}.$$

(*a*) Sketch the graph with equation $y = f(x)$, showing the coordinates of the points where the graph cuts or meets the axes.

(2)

(*b*) Solve $f(x) = 15 + x$.

(3)

The function g is defined by

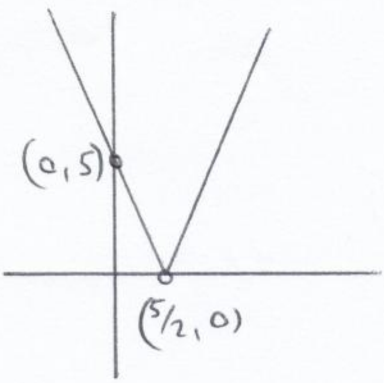
$$g : x \mapsto x^2 - 4x + 1, \quad x \in \mathbb{R}, \quad 0 \le x \le 5.$$

(*c*) Find fg(2).

(2)

(*d*) Find the range of g.

(3)



· reflect $y = 2x - 5$ below x-axis in x-axis

when $x = 0$ $y = 5$

when $y = 0$ $x = \frac{5}{2}$

(b) $|2x - 5| = 15 + x$

(15+x will cut through in the negative part of the graph)

$-2x + 5 = 15 + x$

$-10 = 3x$

$x = \frac{-10}{3}$

(c) $g \to f$ $g(2) = 2^2 - 4 \times 2 + 1$

$= -3$

$f(-3) = |2 \times -3 - 5|$

$= 11$

(d)

$$g(x) = x^2 - 4x + 1$$
$$= (x-2)^2 - 4 + 1$$
$$= (x-2)^2 - 3$$

Minimum of g is therefore -3 $\quad (x-2)^2$ always +ve.

$0 \leq x \leq 5$

$$g(5) = 25 - 4 \times 5 + 1$$
$$= 6$$

$$-3 \leq g(x) \leq 6$$

Leave blank

4. The function f is defined by

$$f: x \mapsto 4 - \ln(x+2), \quad x \in \mathbb{R},\ x \geqslant -1$$

(a) Find $f^{-1}(x)$. **(3)**

(b) Find the domain of f^{-1}. **(1)**

The function g is defined by

$$g: x \mapsto e^{x^2} - 2, \quad x \in \mathbb{R}$$

(c) Find $fg(x)$, giving your answer in its simplest form. **(3)**

(d) Find the range of fg. **(1)**

a) $f(x) = 4 - \ln(x+2)$

$y = 4 - \ln(x+2)$

$\ln(x+2) = 4 - y$

$x + 2 = e^{4-y}$

$x = e^{4-y} - 2$

$f^{-1}(x) = e^{4-x} - 2$

b) To find domain for $f^{-1}(x)$ it is the same as the range for $f(x)$

$f(x) = 4 - \ln(x+2) \quad x \geqslant -1$

$f(-1) = 4 - \ln 1 = 4$

$f(0) = 4 - 0.693 = 3.306$

$f(1) = 4 - \ln 3 = 2.901$ etc

So max value of $f(x) = 4$

$y \leq 4$ for $x \geqslant -1$

$\therefore$ domain $x \leq 4$ for $f^{-1}(x)$

P 3 8 1 5 9 A 0 8 2 4

Leave blank

Question 4 continued

c) $g(x) = e^{x^2} - 2 \quad x \in \mathbb{R}$

$fg(x) = 4 - \ln(e^{x^2} - 2 + 2)$

$= 4 - \ln e^{x^2}$

$= 4 - x^2$

d) Range of $fg(x)$ for $x \in \mathbb{R}$ is ≤ 4

Q4

(Total 8 marks)

P38159A0924

Turn over

Leave blank

6. The functions f and g are defined by

$$\mathrm{f} : x \mapsto \mathrm{e}^x + 2, \qquad x \in \mathbb{R}$$

$$\mathrm{g} : x \mapsto \ln x, \qquad x > 0$$

(a) State the range of f.

(1)

(b) Find $\mathrm{fg}(x)$, giving your answer in its simplest form.

(2)

(c) Find the exact value of x for which $\mathrm{f}(2x+3) = 6$

(4)

(d) Find f^{-1}, the inverse function of f, stating its domain.

(3)

(e) On the same axes sketch the curves with equation $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, giving the coordinates of all the points where the curves cross the axes.

(4)

6 a) $y = e^x + 2$

Range $y > 2$

(b) $fg(x) = e^{\ln x} + 2$

$= x + 2$

(c) $f(2x+3) \Rightarrow e^{2x+3} + 2 = 6$

$e^{2x+3} = 4$

$2x + 3 = \ln 4$

$x = \dfrac{\ln 4 - 3}{2}$

C3 June 2012

Leave blank

Question 6 continued

(d) $y = e^x + 2$

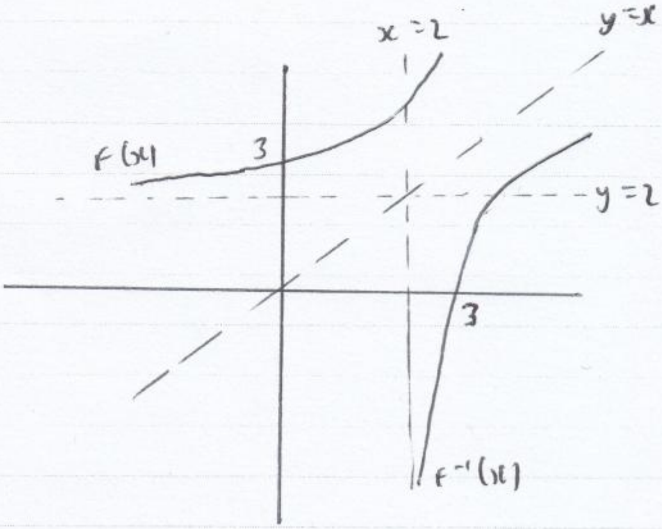
$y - 2 = e^x$

$\ln|y-2| = x$

$\Rightarrow \quad f^{-1}(x) = \ln(x-2)$

Domain $x > 2$ (same as range of $f(x)$)

(e)



P40686RA02132

Turn over

C3 JAN 2013

Leave blank

7. $$h(x) = \frac{2}{x+2} + \frac{4}{x^2+5} - \frac{18}{(x^2+5)(x+2)}, \qquad x \geqslant 0$$

(a) Show that $h(x) = \dfrac{2x}{x^2+5}$ **(4)**

(b) Hence, or otherwise, find $h'(x)$ in its simplest form. **(3)**

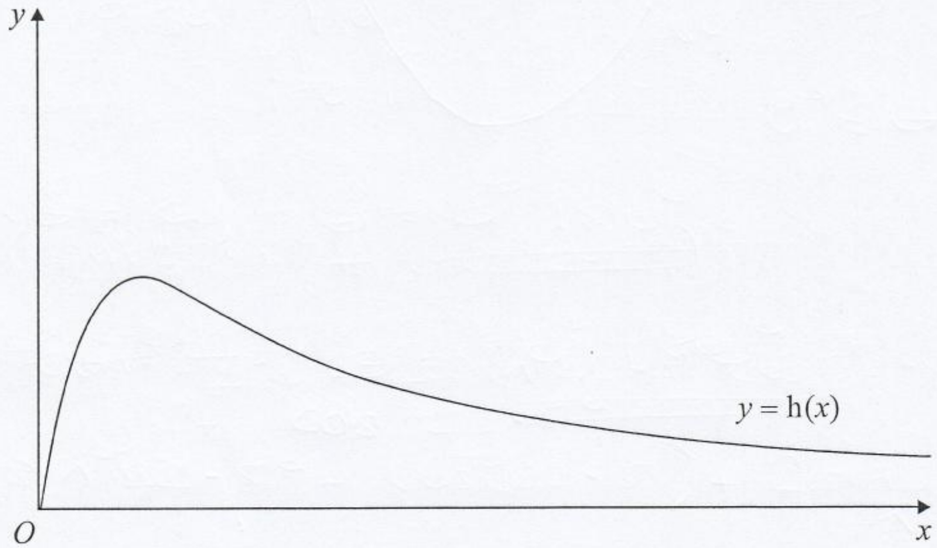


Figure 2

Figure 2 shows a graph of the curve with equation $y = h(x)$.

(c) Calculate the range of $h(x)$. **(5)**

a) $$h(x) = \frac{2(x^2+5)+4(x+2)-18}{(x^2+5)(x+2)}$$

$$= \frac{2x^2+10+4x+8-18}{(x^2+5)(x+2)}$$

$$= \frac{2x^2+4x}{(x^2+5)(x+2)} = \frac{2x\cancel{(x+2)}}{(x^2+5)\cancel{(x+2)}}$$

$$= \frac{2x}{x^2+5}$$ as required

C3 JAN 2013

7b) $h(x) = \dfrac{2x}{x^2+5}$

$u = 2x \qquad v = x^2+5$

$\dfrac{du}{dx} = 2 \qquad \dfrac{dv}{dx} = 2x$

$h'(x) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \dfrac{2(x^2+5) - 2x \times 2x}{(x^2+5)^2}$

$h'(x) = \dfrac{2x^2+10-4x^2}{(x^2+5)^2} = \underline{\underline{\dfrac{-2x^2+10}{(x^2+5)^2}}}$

c) Max point of curve when

$h'(x) = 0$

$0 = \dfrac{-2x^2+10}{(x^2+5)^2}$

$-2x^2+10 = 0$

$2x^2 = 10$

$x^2 = 5$

$x = \sqrt{5}$ as $x \geqslant 0$

don't need $-\sqrt{5}$

Minimum when $x = 0$, $h(x) = 0$

Max when $x = \sqrt{5}$, $h(x) = \dfrac{2\sqrt{5}}{5+5}$

$= \dfrac{2\sqrt{5}}{10} = \dfrac{\sqrt{5}}{5}$

range of $h(x)$ $\underline{\underline{0 \leq y \leq \dfrac{\sqrt{5}}{5}}}$

C3 June 2013

Leave blank

7. The function f has domain $-2 \leqslant x \leqslant 6$ and is linear from $(-2, 10)$ to $(2, 0)$ and from $(2, 0)$ to $(6, 4)$. A sketch of the graph of $y = \mathrm{f}(x)$ is shown in Figure 1.

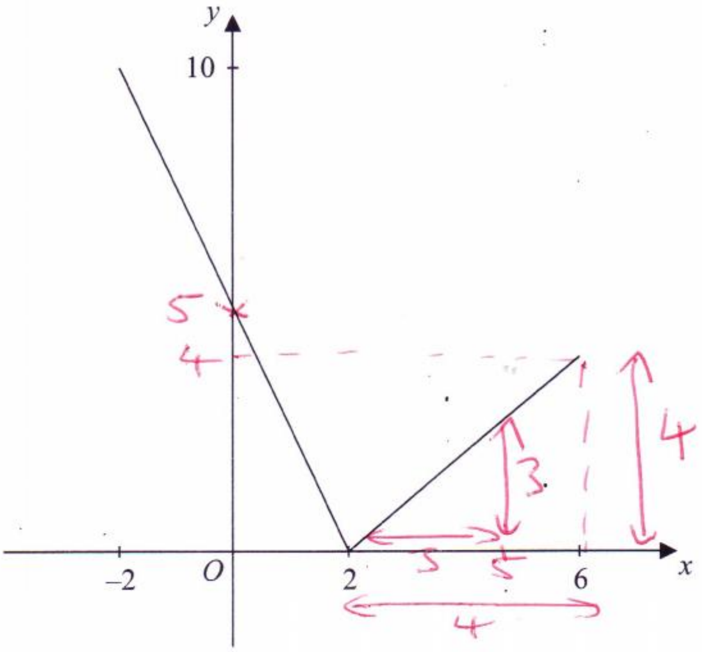


Figure 1

(a) Write down the range of f.

(1)

(b) Find ff(0).

(2)

The function g is defined by

$$\mathrm{g} : x \rightarrow \frac{4 + 3x}{5 - x}, \qquad x \in \mathbb{R}, \qquad x \neq 5$$

(c) Find $\mathrm{g}^{-1}(x)$

(3)

(d) Solve the equation $\mathrm{gf}(x) = 16$

(5)

a) $0 \leq y \leq 10$

b) $f(0) = 5$

from diagram $f(5) = 3$

using similar triangles

P 4 3 0 1 6 A 0 2 4 3 2

C3 June 2013

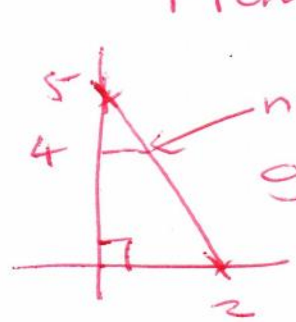
7c) $y = \dfrac{4+3x}{5-x}$

$$y(5-x) = 4+3x$$
$$5y - xy = 4+3x$$
$$5y - 4 = xy + 3x$$
$$5y - 4 = x(y+3)$$
$$\frac{5y-4}{y+3} = x$$
$$g^{-1}(x) = \frac{5x-4}{x+3}$$

d) $gf(x) = 16$

$$\frac{4+3f(x)}{5-f(x)} = 16$$
$$4+3f(x) = 16(5-f(x))$$
$$4+3f(x) = 80 - 16f(x)$$
$$3f(x) + 16f(x) = 80-4$$
$$19f(x) = 76$$
$$f(x) = \frac{76}{19} = 4$$

From diagram $f(6) = 4 \therefore \underline{\underline{x=6}}$



need to find x value

gradient $= -\frac{5}{2}$

equation of line $y = -\frac{5}{2}x + 5$

solve when $y = 4$

$$4 = -\frac{5}{2}x + 5$$
$$\frac{5}{2}x = 5-4, \quad \frac{5}{2}x = 1$$
$$x = \underline{\underline{\frac{2}{5}}}$$