12.2

Vectors

12.1 Three Dimensional Coordinate Systems (Review)

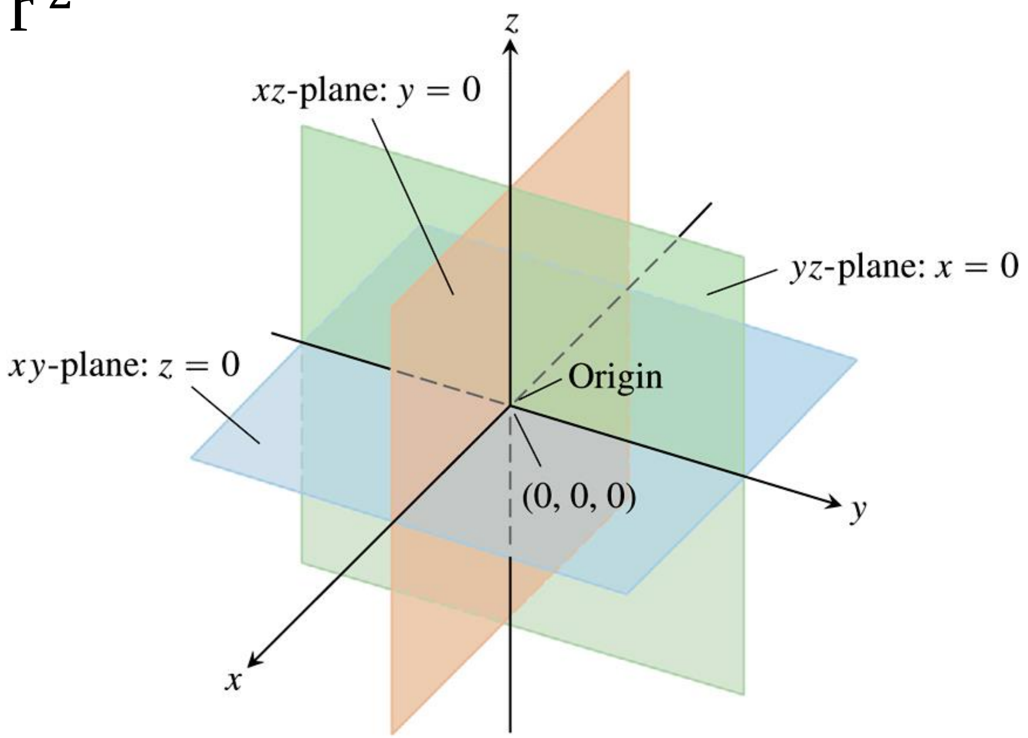
Distance between points: $P_1 : (x_1, y_1, z_1)$ $\quad P_2 : (x_2, y_2, z_2)$

$$d = |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of a sphere

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

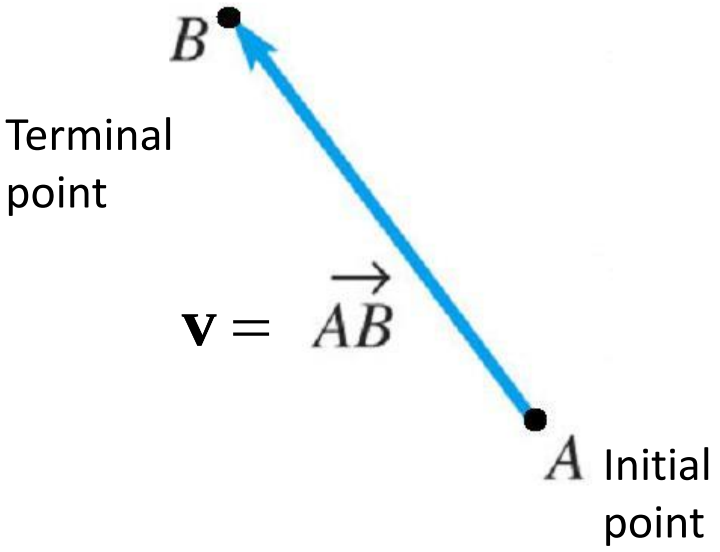
Center (a,b,c) radius r



12.2 Vectors

Quantities like displacement, velocity, and force involve both magnitude and direction (unlike quantities like mass or time).

To represent these quantities we use a **vector** represented by a directed line segment (arrow)



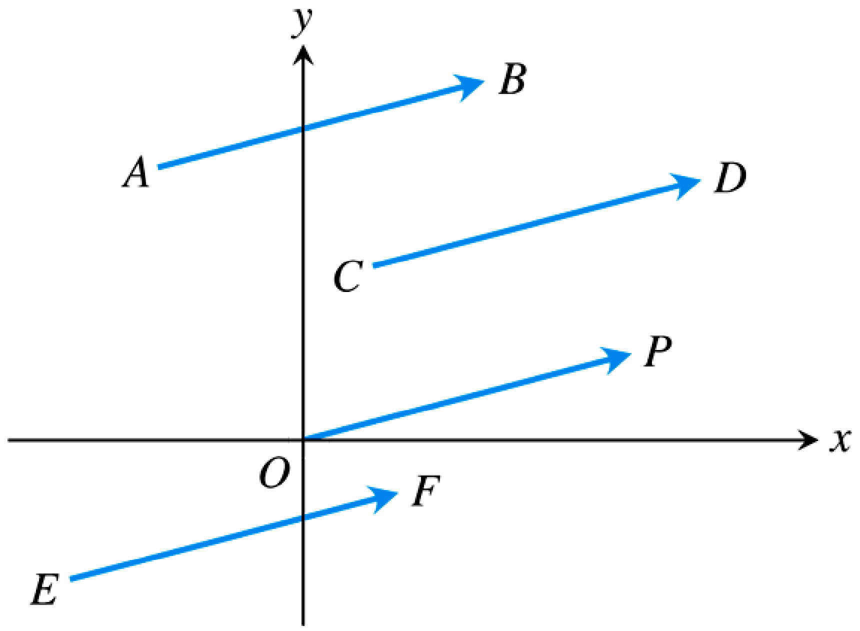
The magnitude of a vector is represented by $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

or sometimes by $|\overrightarrow{AB}|$

We also call it the length of **v**

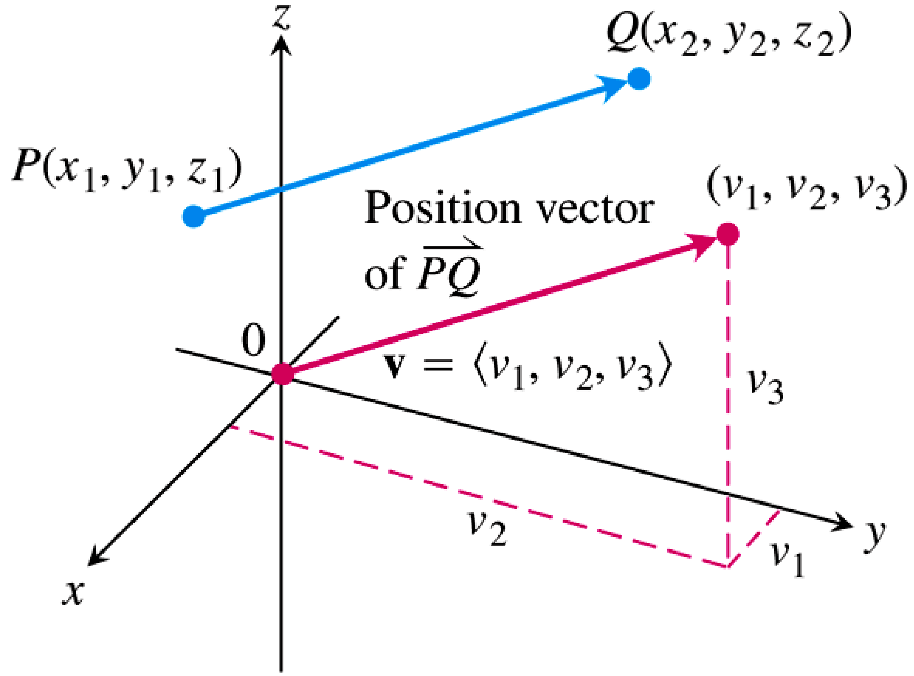
Any other vector that has the same magnitude and direction is called an equivalent or equal vector

$$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$$



A vector in standard position has its initial point at the origin.

The directed line segment $\overrightarrow{PQ}$ and v are equivalent (same).



We can multiply a vector v by a real number c.

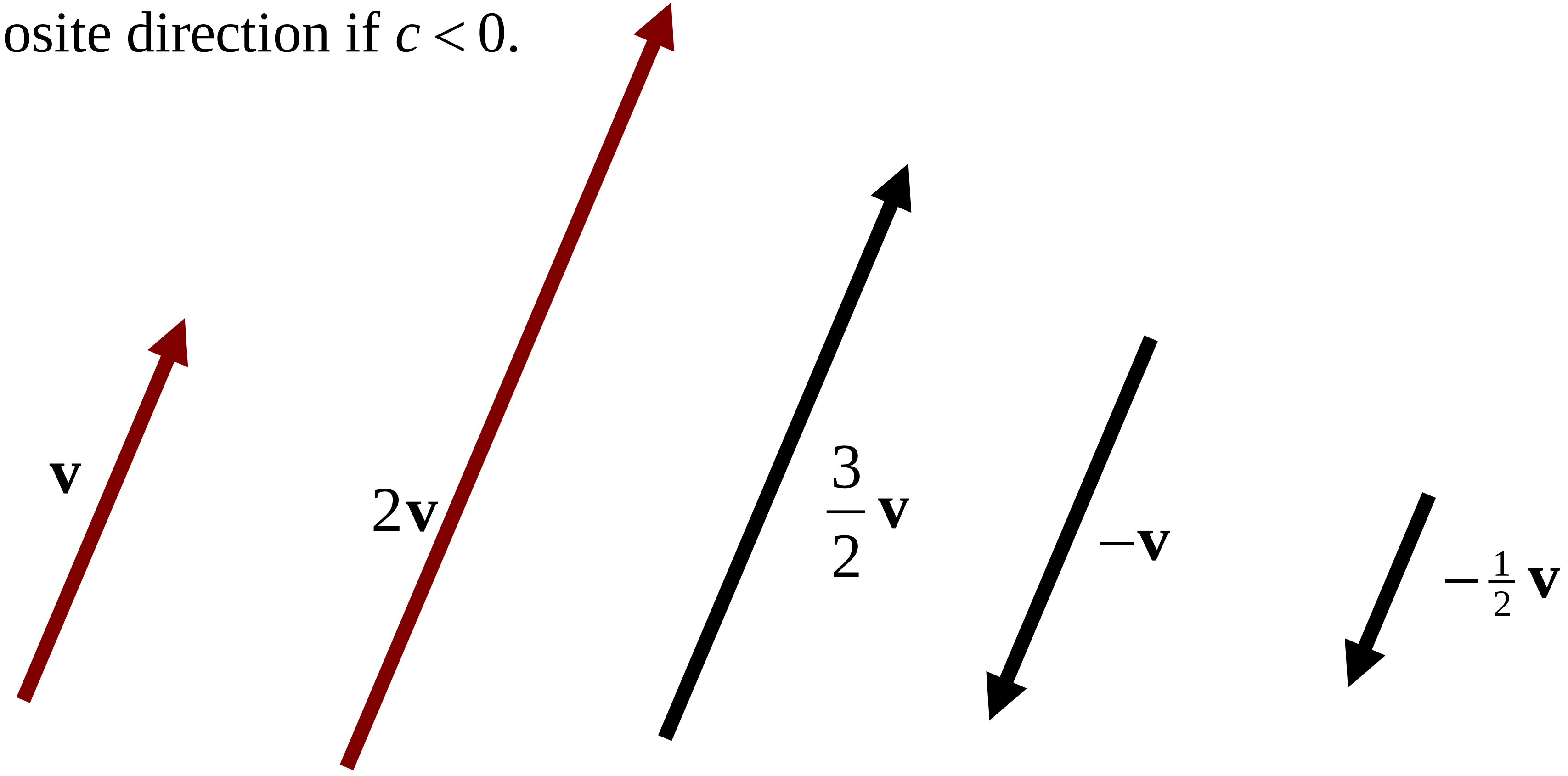
This is called **scalar multiplication**

and denoted by cv.

$c\mathbf{v}$ has magnitude $|c|$ times the magnitude of $\mathbf{v}$: $|c\mathrm{v}| = |c|\ |\mathrm{v}|$

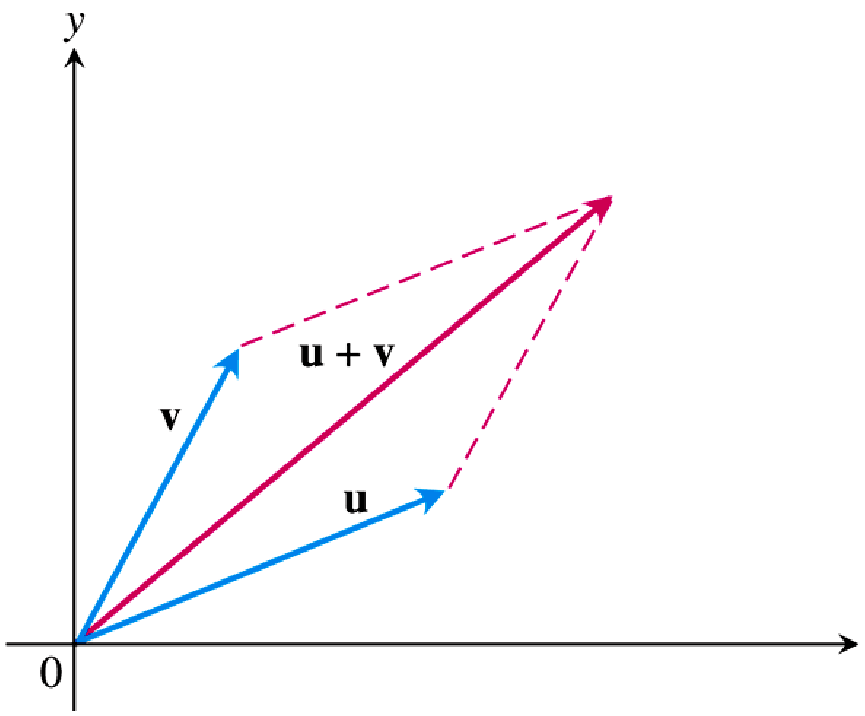
and points in the same direction as v if $c > 0$

or opposite direction if $c < 0$.

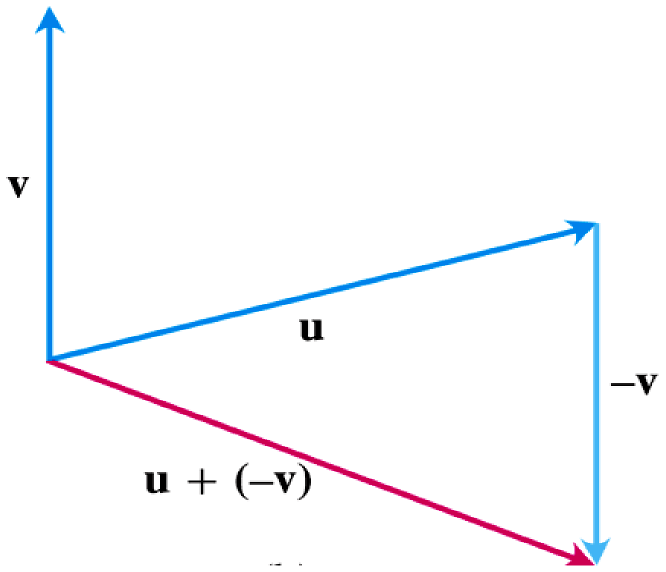


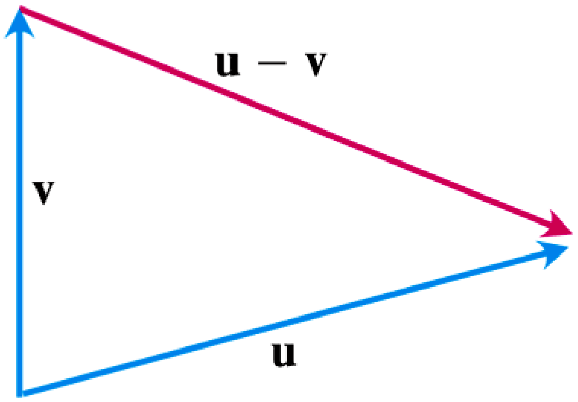
We can add a vector $\mathbf{v}$ to another vector $\mathbf{u}$.

This is called **vector addition**, $\mathbf{v}+\mathbf{u}$



Vector subtraction $\mathbf{v}-\mathbf{u}$ is just vector addition in disguise $\mathbf{v}+(-\mathbf{u})$





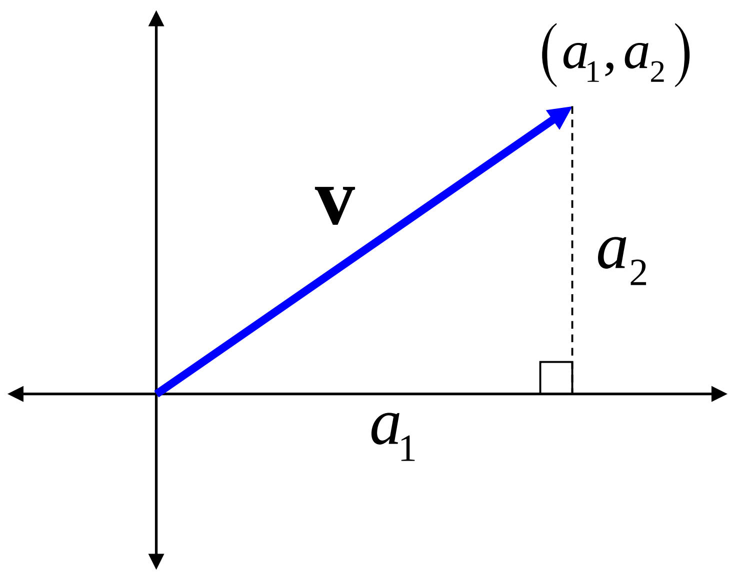
Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors and a, b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

It is much simpler to study vectors algebraically:

$\boxed{\mathbf{v} = \langle a_1, a_2 \rangle}$ or $\boxed{\mathbf{v} = \langle a_1, a_2, a_3 \rangle}$



Standard unit (i.e. length 1) vectors

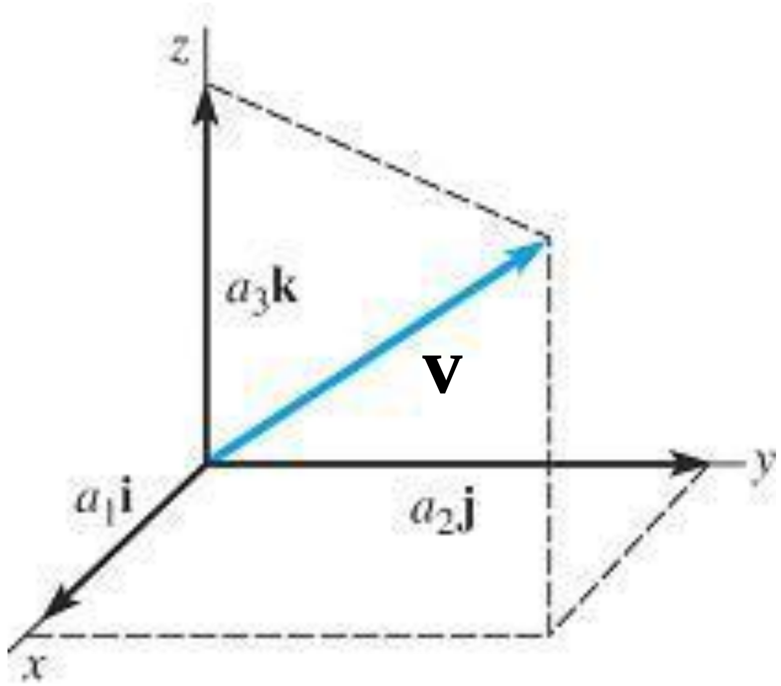
$\mathbf{i} = \langle 1, 0, 0 \rangle \quad \mathbf{j} = \langle 0, 1, 0 \rangle \quad \mathrm{k} = \langle 0, 0, 1 \rangle$

$\boxed{\mathbf{v} = \langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}}$

$a_1 =$ $\mathbf{i}$ component of $\mathbf{v}$

$a_2 =$ $\mathbf{j}$ component of $\mathbf{v}$

$a_3 =$ $\mathbf{k}$ component of $\mathbf{v}$



Scalar Multiplication:

$$\mathbf{v} = \langle a_1, a_2, a_3 \rangle \text{ scaled by a factor } c$$

$$c\mathbf{v} = \langle ca_1, ca_2, ca_3 \rangle \qquad (\text{multiply each component by } c)$$

Vector Addition:

$$\mathbf{v} = \langle a_1, a_2, a_3 \rangle \text{ added to } \mathbf{u} = \langle b_1, b_2, b_3 \rangle$$

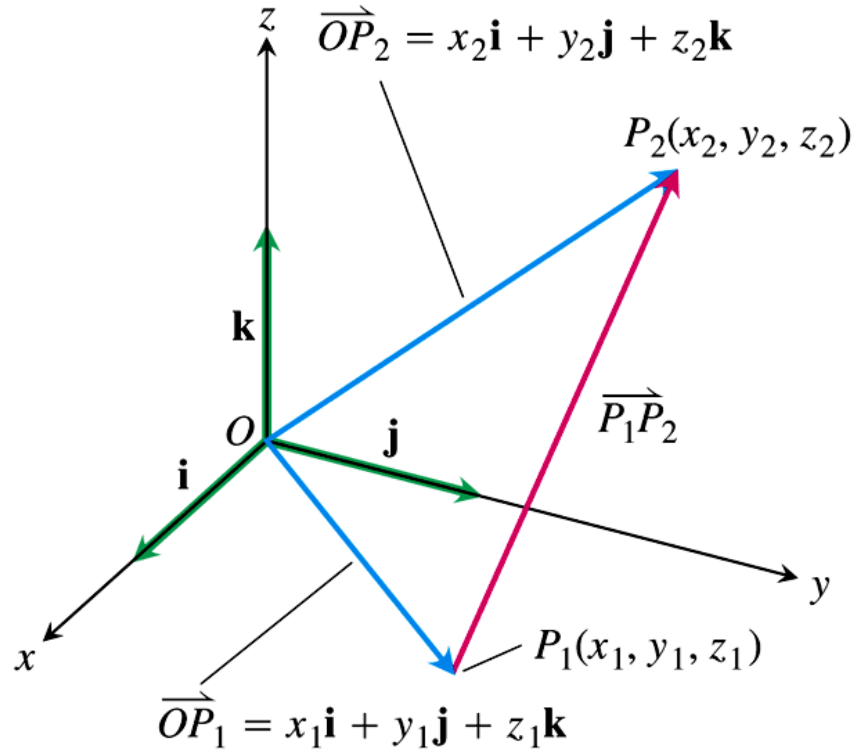
$$\mathbf{v} + \mathbf{u} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \qquad (\text{add componentwise})$$

Length of the vector **v**: $|\mathbf{v}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Unit vector **u** in the direction of **v**: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \left(\dfrac{1}{|\mathbf{v}|}\right)\langle a_1, a_2, a_3 \rangle = \left\langle \dfrac{a_1}{|\mathbf{v}|}, \dfrac{a_2}{|\mathbf{v}|}, \dfrac{a_3}{|\mathbf{v}|} \right\rangle$

Vector u = $\overrightarrow{P_1P_2}$ from a point $P_1 = (x_1, y_1, z_1)$ to a point $P_2 = (x_2, y_2, z_2)$:

$$\boxed{\overrightarrow{P_1P_2} = P_2 - P_1 = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}}$$



⊙ Find the component form and magnitude of the vector **v** with the initial point $(3,2,0)$ and terminal point $(4,4,2)$.

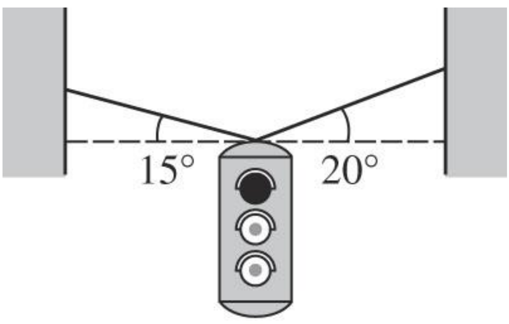
⊙ Find a unit vector in the direction of **v**.

$$\mathbf{v} = \langle 4-3, 4-2, 2-0 \rangle = \langle 1, 2, 2 \rangle$$

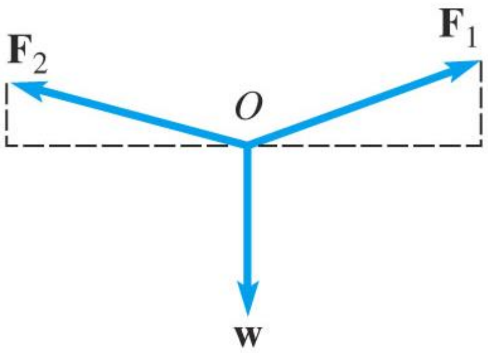
$$|\mathbf{v}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{3}\right)\langle 1, 2, 2 \rangle = \left\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right\rangle$$

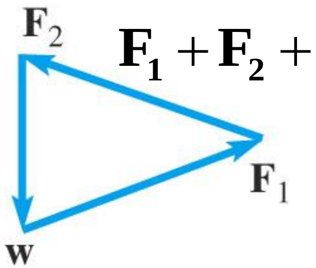
$$\text{check:} \quad |\mathrm{u}| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1$$



(a)



(b)



$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{w} = \mathbf{0}$$

(c)

A 200 lb. traffic light supported by two cables hangs in equilibrium. As shown in figure (b), let the weight of the light be represented by **w** and the forces in the two cables by $\mathbf{F}_1$ and $\mathbf{F}_2$. Determine the magnitude of $\mathbf{F}_1$ and $\mathbf{F}_2$.

As shown in figure (c), the forces can be arranged to form a triangle.

Equilibrium implies that the sum of the forces is $\mathbf{0}$: $\mathbf{F}_1 + \mathbf{F}_2 + \mathrm{w} = 0$

$$\mathbf{F}_1 = \left\langle \|\mathbf{F}_1\| \cos\left(20^\circ\right), \|\mathbf{F}_1\| \sin\left(20^\circ\right) \right\rangle$$

$$\mathbf{w} = \langle 0, -200 \rangle$$

$$\mathbf{F}_2 = \left\langle -\|\mathbf{F}_2\| \cos\left(15^\circ\right), \|\mathbf{F}_2\| \sin\left(15^\circ\right) \right\rangle$$

x coordinates cancel: $\|\mathbf{F}_1\| \cos\left(20^\circ\right) + -\|\mathbf{F}_2\| \cos\left(15^\circ\right) = 0$ or $\|\mathbf{F}_1\| = \dfrac{\|\mathbf{F}_2\| \cos\left(15^\circ\right)}{\cos\left(20^\circ\right)}$

y coordinates cancel: $\|\mathbf{F}_1\| \sin\left(20^\circ\right) + \|\mathbf{F}_2\| \sin\left(15^\circ\right) - 200 = 0$

$$\|\mathbf{F}_2\| \left[\cos\left(15^\circ\right) \tan\left(20^\circ\right) + \sin\left(15^\circ\right) \right] = 200$$

$$\|\mathbf{F}_2\| = \frac{200}{\cos\left(15^\circ\right) \tan\left(20^\circ\right) + \sin\left(15^\circ\right)} \approx \mathbf{327.66\ lbs.}$$

$$\|\mathbf{F}_1\| \approx \mathbf{336.81\ lbs.}$$

12.3

The Dot Product

We can add two vectors, what about multiplying two vectors?

Since adding two vectors yields another vector where the corresponding components are added, will the same work for multiplication?

No, the product of two vectors yielding another vector where the corresponding components are multiplied is meaningless.

There are actually two vector products that yield meaningful results but neither of these give a new vector using component-wise multiplication.

12.3 **Dot** Product

12.4 **Cross** Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Note: The dot product of two vectors is a **number** (scalar) not a vector.

Example:

(a) $\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(-3)$
$= -6 - 4 + 3 = -7$

(b) $\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1$

Properties of the Dot Product

If **u**, **v**, and **w** are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0.$

Check property 4:

$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle \quad |\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

$$\mathbf{u} \cdot \mathbf{u} = \langle u_1, u_2, u_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle = u_1^2 + u_2^2 + u_3^2 = |\mathbf{u}|^2$$

or $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

Are the following expression meaningful?

$(a)\quad (\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$ $\qquad$ $\text{scalar}\cdot\text{vector}$

$(b)\quad (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ $\qquad$ $(\text{scalar})\text{vector}$

$(c)\quad |\mathbf{u}|(\mathbf{v}\cdot\mathbf{w})$ $\qquad$ $(\text{scalar})(\text{scalar})$

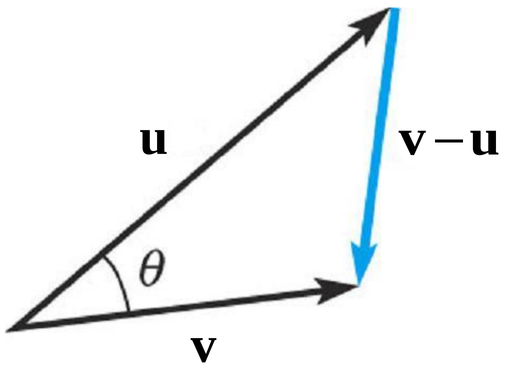
$(d)\quad \mathbf{u}\cdot(\mathbf{v}+\mathbf{w})$ $\qquad$ $\text{vector}\cdot\text{vector}$

$(e)\quad \mathbf{u}\cdot\mathbf{v}+\mathbf{w}$ $\qquad$ $\text{scalar}+\text{vector}$

$(f)\quad |\mathbf{u}|\cdot(\mathbf{v}\cdot\mathbf{w})$ $\qquad$ $\text{scalar}\cdot\text{scalar}$

Find the angle between two vectors :

Let **u** and **v** be nonzero vectors, then



<u>Law of Cosines:</u>

$$|\mathbf{v}-\mathbf{u}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$$

$$|\mathbf{v}-\mathbf{u}|^2 = (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{u}) = (\mathbf{v}-\mathbf{u})\cdot\mathbf{v} - (\mathbf{v}-\mathbf{u})\cdot\mathbf{u}$$

$$= \mathbf{v}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{u} = |\mathbf{v}|^2 + |\mathbf{u}|^2 - 2\mathbf{u}\cdot\mathbf{v}$$

hence $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$

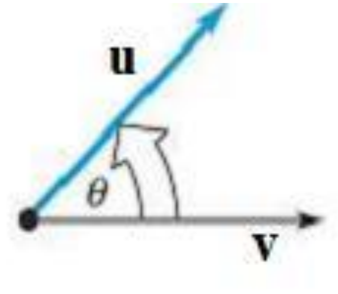
$$\boxed{\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}}$$

$$\boxed{\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}} \qquad 0 \le \theta \le \pi$$

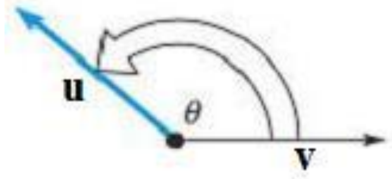
Alternate form of dot product: $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$

$\Rightarrow \mathbf{u}\cdot\mathbf{v}$ and $\cos\theta$ will always have the same sign

If $0 < \theta < \frac{\pi}{2}$ (θ acute) then $\cos\theta > 0$ or $\mathbf{u}\cdot\mathbf{v} > 0$



If $\frac{\pi}{2} < \theta < \pi$ (θ obtuse) then $\cos\theta < 0$ or $\mathbf{u}\cdot\mathbf{v} < 0$



$\theta = \frac{\pi}{2}$ (θ right angle) if $\cos(\theta)=0$ or $\mathbf{u}\cdot\mathbf{v} = 0$

$$\boxed{\mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal if } \mathbf{u}\cdot\mathbf{v} = 0}$$

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

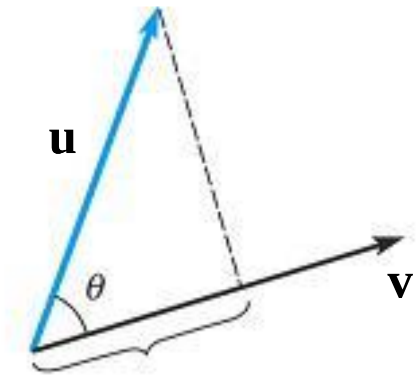
$$\mathbf{u} \cdot \mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{-4}{(3)(7)}\right) \approx 1.76 \text{ radians}.$$

vector projection of $\mathbf{u}$ onto $\mathbf{v}$



$\text{proj}_{\mathbf{v}}\mathbf{u}$

$\text{comp}_{\mathbf{v}}\mathbf{u}$ The component of **u** in the direction of **v**

is $|\mathbf{u}|\cos\theta$, which is (up to sign) the length of $\text{proj}_{\mathbf{v}}\mathbf{u}$

recall $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ hence $|\mathbf{u}|\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$ $\boxed{\text{comp}_{\mathbf{v}}\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}}$

the vector projection of $\mathbf{u}$ onto $\mathbf{v}$ has $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$ as its magnitude and goes in the same direction as $\mathbf{v}$

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|}$$

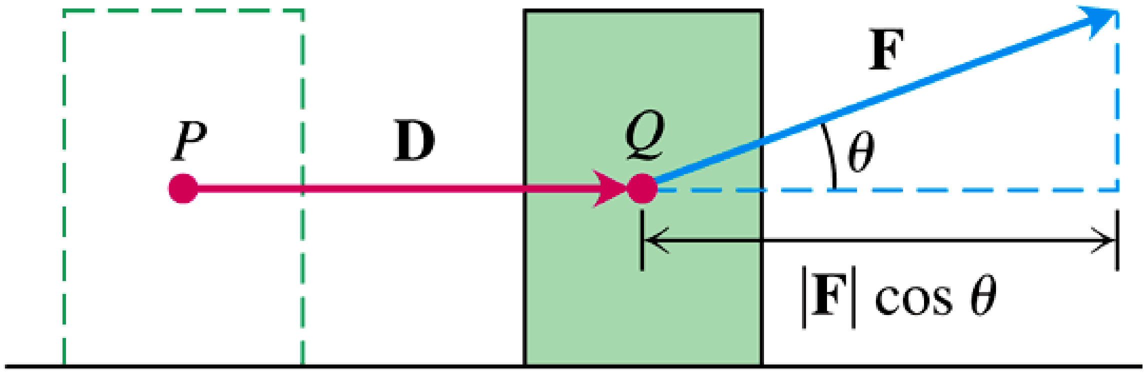
$$\boxed{\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}}$$

Work = Force x Distance

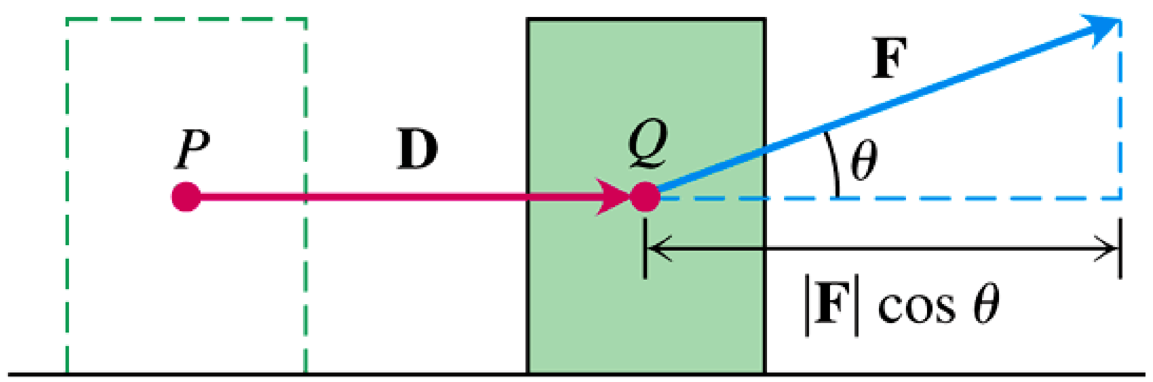
$W = |\mathbf{F}|d$ If force and motion have the same direction

If a constant force $\mathbf{F}$ applied to a body acts at an angle θ to the direction of motion, then the work done W is: $W = |\mathbf{F}|\cos\theta|\mathbf{D}|$ where $\mathbf{D}$ is the displacement vector

Using the dot product we have:



$$\boxed{W = \mathbf{F}\cdot\mathbf{D}}$$



A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a 20° angle with the horizontal. Find the work done in Pulling the wagon 50 feet.

$$\mathbf{F} = \left\langle 25\cos\left(20^\circ\right), 25\sin\left(20^\circ\right)\right\rangle \qquad \mathbf{D} = \left\langle 50, 0\right\rangle$$

$$W = \mathbf{F}\cdot\mathbf{D} \quad = 1250\cos\left(20^\circ\right) \quad \approx 1175 \text{ ft. lbs. } \left(\text{or } 1593 \text{ joules}\right)$$