Algebraic automorphic forms and Hilbert-Siegel modular forms

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1 Positive definite even unimodular lattices

Let F be a totally real number field of degree d. Let $n > 0$ be an integer such that dn is even. Then, by Minkowski-Hasse theorem, there exists a quadratic space (V_n, Q) of rank $4n$ with the following properties:

(1) (V, Q) is unramified at any non-archimedean place.

(2) (V, Q) is positive definite ai any archimedean place.

Definition 1. An algebraic automorphic form on the orthogonal group O_Q is a locally constant function on $\mathrm{O}_Q(F)\backslash\mathrm{O}_Q(\mathbb{A})$.

Put

$$(x, y)_Q = \frac{1}{2}(Q(x+y) - Q(x) - Q(y)) \qquad x, y \in V.$$

Let L be a $\mathfrak{o}$-lattie in $V(F)$. The dual lattice L^* is defined by

$$L^* = \{x \in V_n(F) \mid (x, L)_Q \subset \mathfrak{o}\}.$$

Then L is an integral lattice if and only if $L \subset L^*$. An integral lattice L is called an even lattice if

$$Q(x) \subset 2\mathfrak{o} \qquad {}^{\forall}x \in L.$$

Moreover, an integral lattice L is unimodular if $L = L^*$. By the assumption (1) and (2), there exists a positive definite even unimodular latice L_0 in $V(F)$.

Two integral lattices L_1 and L_2 are equivalent if there exists an element $g \in \mathrm{O}_Q(F)$ such that $g \cdot L_1 = L_2$. Two integral lattices L_1 and L_2 are in the same genus if there exists an element $g_v \in \mathrm{O}_Q(F_v)$ such that $g \cdot L_{1,v} = L_{2,v}$ for any finite place v. The set of positive definite even unimodular lattices in $V(F)$ form a genus. Let $\mathscr{G}$ be the set of equivalence classes in this genus. In this article, we focus on even unimodular lattices.

Choose a positive definite even unimodular lattiece L_0 in $V(F)$. Let $\mathbf{K}_0$ be the stabilezer of L_0 in $\mathrm{O}_Q(\mathbb{A})$. Then $\mathbf{K}_0$ is a maximal compact subgroup of $\mathrm{O}_Q(\mathbb{A})$. The set $\mathrm{O}_Q(\mathbb{A})/\mathbf{K}_0$ can be identified with the set of all even unimodular lattices. For $\xi \in \mathrm{O}_Q(\mathbb{A})$, let $L \subset V$ be the unique lattice such that $L_v = \xi L_{0,v}\xi^{-1}$ for any non-archimedean place v. Then L is a positive definite even unimodular lattice, and any positive definite even unimodular lattice in V is obtained in this way. Moreover, the isomorphism class $[L]$ is determined by the double coset $\mathrm{O}_Q(F)\xi\mathbf{K}_0$. Thus one can think of

$$\mathcal{G} = \mathrm{O}_Q(F)\backslash\mathrm{O}_Q(\mathbb{A})/\mathbf{K}_0.$$

In fact, the set $\mathrm{O}_Q(F)\backslash\mathrm{O}_Q(\mathbb{A})/\mathbf{K}_0$ can be identified with $\mathrm{O}_Q(F)\backslash\mathrm{O}_Q(\mathbb{A}_{\mathrm{fin}})/\mathbf{K}_{0,\mathrm{fin}}$, where $\mathbf{K}_{0,\mathrm{fin}}$ is the finite part of $\mathbf{K}_0$ and $\mathrm{O}_Q(F)$ is considered as a subgroup of $\mathrm{O}_Q(\mathbb{A}_{\mathrm{fin}})$. Choose a double coset $\mathrm{O}_Q(F)\xi\mathbf{K}_{0,\mathrm{fin}}$ corresponding to an even unimodular lattice $L \subset V$. Then the automorphism group $\mathrm{O}(L)$ can be identified with

$$\mathrm{O}_Q(F) \cap \xi\mathbf{K}_{0,\mathrm{fin}}\xi^{-1}$$

In particular, the volume of the set

$$\mathrm{O}_Q(F)\backslash\mathrm{O}_Q(F)\xi\mathbf{K}_{0,\mathrm{fin}} \simeq \xi \cdot (\mathrm{O}(L)\backslash\mathbf{K}_{0,\mathrm{fin}})$$

is equal to $E(L)^{-1}$, where $E(L)$ is the order of $\mathrm{O}(L)$.

Put

$$\mathbb{C}[\mathscr{G}] := \oplus_{L\in\mathscr{G}}\mathbb{C}\cdot[L], \qquad \mathbb{Z}[\mathscr{G}] := \oplus_{L\in\mathscr{G}}\mathbb{Z}\cdot[L].$$

Then $\mathbb{C}[\mathcal{G}]$ can be identified the the space of $\mathbf{K}_0$-invariant algegraic automorphic forms on O_Q. This correspondence is given by

$$[L] \mapsto \text{ the characteristic function on } \mathrm{O}_Q(F)\xi\mathbf{K}_0 \text{ corresponding to } L$$

Thus we identify $\mathbb{C}[\mathscr{G}]$ with $L^2(\mathrm{O}_Q(F)\backslash\mathrm{O}_Q(\mathbb{A})/\mathbf{K}_0)$.

Definition 2. Let $K, L \subset V(F)$ be even unimodular lattices. Let $\mathfrak{p}$ be a prime ideal of F. Then K is a $\mathfrak{p}$-neighbor of L if

$$L/(L\cap K) \simeq K/(L\cap K) \simeq \mathfrak{o}/\mathfrak{p}.$$

The number of $\mathfrak{p}$-neighbors of L which is isomorphic to K is denoted by $N(L, K, \mathfrak{p})$. This is determied by the isomorphism classes of K and L.

Definition 3. The operator

$$K(\mathfrak{p}) : [L] \mapsto \sum_{K \in \mathscr{G}} N(L, K, \mathfrak{p})[K]$$

on $\mathbb{Z}[\mathscr{G}]$ is called the Kneser $\mathfrak{p}$-neighbor operator. We also define the dual Kneser $\mathfrak{p}$-neighbor operator $K(\mathfrak{p})^\vee$ by

$$K(\mathfrak{p})^\vee : [L] \mapsto \sum_{K \in \mathscr{G}} N(K, L, \mathfrak{p})[K].$$

It is known that

$$\frac{N(L, K, \mathfrak{p})}{N(K, L, \mathfrak{p})} = \frac{E(L)}{E(K)}.$$

It follows that $K(\mathfrak{p})$ and $K(\mathfrak{p})^\vee$ are conjugate. Here, we work with the dual $\mathfrak{p}$-neighbor operator $K(\mathfrak{p})^\vee$. This convension is different from [2], [8], or [5].

Let $\mathcal{H} = \mathcal{H}(\mathbf{K}_0 \backslash \mathrm{O}_Q(\mathbb{A})/\mathbf{K}_0)$ be the Hecke algebra on $\mathbf{K}_0 \backslash \mathrm{O}_Q(\mathbb{A})/\mathbf{K}_0$. Then $\mathcal{H}$ acts on $L^2(\mathrm{O}_Q(F) \backslash \mathrm{O}_Q(\mathbb{A})/\mathbf{K}_0)$ as Hecke operators. The dual Kneser $\mathfrak{p}$-neighbor operator $K(\mathfrak{p})^\vee$ can be considered as a Hecke operator. Let $f = \sum_{[L]} c_L[L] \in \mathbb{C}[\mathscr{G}]$ be a Hecke eigenform with $\mathfrak{p}$-Satake parameter $\{\beta_{\mathfrak{p},1}^{\pm 1}, \ldots, \beta_{\mathfrak{p},2n}^{\pm 1}\}$. Then the eigenvalue of f with respect to $K(\mathfrak{p})^\vee$ is given by

$$q_{\mathfrak{p}}^{2n-1} \sum_{i=1}^{2n} (\beta_{\mathfrak{p},i} + \beta_{\mathfrak{p},i}^{-1}).$$

2 Theta functions

Let $m \geq 1$ is an integer. For $L \in \mathscr{G}$, we define a theta function $\theta_L^{(m)}(Z)$ by

$$\theta_L^{(m)}(Z) = \sum_{x \in L^m} \mathbf{e}\big(\mathrm{tr}((x, x)Z)\big).$$

Then $\theta_L^{(m)}(Z) \in M_{2n}(\Gamma_m[\mathfrak{d}^{-1}, \mathfrak{d}])$. Here, $\mathfrak{d}$ is the different of F and

$$\Gamma_m[\mathfrak{d}^{-1}, \mathfrak{d}] := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(F) \,\middle|\, \begin{matrix} A \in \mathrm{M}_m(\mathfrak{o}), \ B \in \mathrm{M}_m(\mathfrak{d}^{-1}) \\ C \in \mathrm{M}_m(\mathfrak{d}), \ D \in \mathrm{M}_m(\mathfrak{o}) \end{matrix} \right\}.$$

For $f = \sum_{L \in \mathscr{G}} c_L \cdot L \in \mathbb{C}[\mathscr{G}]$, we set

$$\Theta^{(m)}(f) = \sum_{L \in \mathscr{G}} \frac{c_L}{E(L)} \theta_L^{(m)}(Z).$$

(This convension is also different from [2], [8], and [5].) For a Hecke eigenvector $f \in \mathbb{C}[\mathscr{G}]$, the degree $\deg f$ is defined by

$$\deg f = \min\{m \mid \Theta^{(m)}(f) \neq 0\}.$$

Let $f \in \mathbb{C}[\mathscr{G}]$ be a Hecke eigenvector with $\deg f = m_0$. By the theory of theta correspondence, one can prove

- For $m \geq m_0$, $\Theta^{(m)}(f) \in M_{2n}(\Gamma_m[\mathfrak{d}^{-1}, \mathfrak{d}])$ is a Hecke eigenform.
- We have $\Theta^{(m_0)}(f) \in S_{2n}(\Gamma_{m_0}[\mathfrak{d}^{-1}, \mathfrak{d}])$.
- For $m > m_0$, $\Theta^{(m)}(f)$ is orthogonal with $S_{2n}(\Gamma_m[\mathfrak{d}^{-1}, \mathfrak{d}])$ with respect to the Petersson inner product.

Suppose that $f \in \mathbb{C}[\mathscr{G}]$ is a Hecke eigenform such that $\deg f = m < 2n$. Let the $\mathfrak{p}$-Satake parameter of $\Theta^{(m)}(f) \in S_{2n}(\Gamma_m[\mathfrak{d}^{-1}, \mathfrak{d}])$ be

$$\{\beta_{1,\mathfrak{p}}^{\pm 1}, \ldots, \beta_{m,\mathfrak{p}}^{\pm 1}\}.$$

Then the $\mathfrak{p}$-Satake parameter of $f \in \mathbb{C}[\mathscr{G}]$ is given by

$$\{1, \beta_{1,\mathfrak{p}}^{\pm 1}, \ldots, \beta_{m,\mathfrak{p}}^{\pm 1}\} \cup \{q_{\mathfrak{p}}^{\pm j} \ (0 \leq j \leq 2n - m - 1)\}.$$

Here, $q_{\mathfrak{p}}$ is the order of the residue field of $\mathfrak{p}$.

3 Niemeirer lattices

In this section, we take $F = \mathbb{Q}$. A Niemeier lattice is a positive definite even unimodular lattice of rank 24. There are 24 isomorphism classes of Niemeier lattices. They are classified by the root system formed by vectors of norm 2. (See Conway and Sloan [3].)

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
$\emptyset$	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	D_4^6	A_6^4
L_9	L_{10}	L_{11}	L_{12}	L_{13}	L_{14}	L_{15}	L_{16}
$A_7^2D_5^2$	A_8^3	$A_9^2D_6$	D_6^4	$A_{11}D_7E_6$	E_6^4	A_{12}^2	D_8^3
L_{17}	L_{18}	L_{19}	L_{20}	L_{21}	L_{22}	L_{23}	L_{24}
$A_{15}D_9$	$D_{10}E_7^2$	$A_{17}E_7$	D_{12}^2	A_{24}	$D_{16}E_8$	E_8^3	D_{24}

The order $E(L)$ of the automorphism group $O(L)$ can be found in Conway-Sloan [3].

	$E(L)$			$E(L)$
L_1	15570572852330496000		L_2	31522712171959008000000
L_3	312927932591898624000000		L_4	437599241673834240000000
L_5	180674574584719324741632		L_6	52278522738634063872000
L_7	1196560426451890500000		L_8	8361079854908571648000
L_9	2700612462901377024000		L_{10}	225800767686574080000
L_{11}	106690862731906252800		L_{12}	19144966823230248000
L_{13}	8082641116053504000		L_{14}	373503391765504000
L_{15}	834785957117952000		L_{16}	156983146327507500
L_{17}	33307587016704000		L_{18}	4134535541136000
L_{19}	3483146354688000		L_{20}	67271626831500
L_{21}	4173688995840		L_{22}	271057837050
L_{23}	63804560820		L_{24}	24877125

The dual Kneser neighbor operator $K(2)$ and the eigenvectors were calculated by Nebe and Venkov [8]. Let f_i $(i = 1, 2, \ldots, 24)$ be the eigenvectors.

The degree of f_i is defined by

$$n_i = \min\{n \,|\, \Theta^{(n)}(\mathtt{d}_i) \neq 0\}.$$

Nebe-Venkov [8] and Chnevier-Lannes [2] determined the degrees:

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}
0	1	2	3	4	4	5	5	6	6	6	7
f_{13}	f_{14}	f_{15}	f_{16}	f_{17}	f_{18}	f_{19}	f_{20}	f_{21}	f_{22}	f_{23}	f_{24}
8	7	8	7	8	8	9	9	10	10	11	12

Put $F_i = \Theta^{(n_i)}(\mathtt{d}_i)$.

Recall that

$$\dim_{\mathbb{C}} S_{2k}(\mathrm{SL}_2(\mathbb{Z})) = 1, \qquad 2k = 12,\ 16,\ 18,\ 20,\ 22.$$

Let

$$\phi_{2k} = \sum_{n=1}^{\infty} a_{2k}(n)\mathbf{e}(nz) \in S_{2k}(\mathrm{SL}_2(\mathbb{Z})), \qquad (2k = 12,\ 16,\ 18,\ 20,\ 22)$$

be the normalized Hecke eigenform. ϕ_{12} is also denoted by $\Delta(\tau)$. For a prime l, there exists a l-adic representation $\boldsymbol{\rho}_{2k} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$ such that

$$L(s, \phi_{2k}) = \prod_p \det(1 - \rho_{2k}(\mathrm{Frob}_p) \cdot p^{-s})$$

up to bad Euler factors.

$f_1 = \sum_{i=1}^{24}$ is a constant function on $\mathrm{O}_Q(\mathbb{A})$ and $\Theta^{(n)}$ is the Siegel Eisenstein series for any n by Siegel's main theorem. $F_2 = \Theta^{(1)}(f_2) \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is equal to $\Delta(z)$ up to constant. $F_3 = \Theta^{(2)}(f_3) \in S_{12}(\mathrm{Sp}_2(\mathbb{Z}))$ is the Saito-Kurokawa lift of ϕ_{22}. It follows that

$$L(s, F_3, \mathrm{st}) = \zeta(s) \prod_{10 \le i \le 11} L(s+i, \phi_{22}),$$

$F_5 = \Theta^{(4)}(f_5) \in S_{12}(\mathrm{Sp}_4(\mathbb{Z}))$ is the DII lift of ϕ_{20} to degree $S_{12}(\mathrm{Sp}_4(\mathbb{Z}))$. Hence we have

$$L(s, F_5, \mathrm{st}) = \zeta(s) \prod_{8 \le i \le 11} L(s+i, \phi_{20}),$$

$F_4 = \Theta^{(3)}(f_4) \in S_{12}(\mathrm{Sp}_4(\mathbb{Z}))$ is the Miyawaki lift of $\Delta(z)$ with respect to F_4. Hence we have

$$L(s, F_4, \mathrm{st}) = L(s, \Delta, \mathrm{st}) \prod_{9 \le i \le 10} L(s+i, \phi_{20})$$

$F_{24} = \Theta^{(12)}(f_{24}) \in S_{12}(\mathrm{Sp}_{12}(\mathbb{Z}))$ is the DII lift of $\Delta(z)$ to degree 12, which is investigated in Borcherds-Freitag-Weissauer [1].

Let $\rho_{j,k}$ be the holomorphic representation of $\mathrm{GL}_2(\mathbb{C})$ given by $\rho_{j,k} = \mathrm{Sym}^j \otimes \det^k$. The highset weight of $\rho_{j,k}$ is $(j+k, k)$. Let $S_{j,l}(\mathrm{Sp}_2(\mathbb{Z}))$ be the space of modular form of vector weight $\rho_{j,k}$. For a Hecke eigenform $\phi \in S_{j,k}(\mathrm{Sp}_2(\mathbb{Z}))$, the spin L-function has a functional equation

$$\begin{aligned} \Lambda(s, \phi, \mathrm{spin}) =& \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k+2)L(s, \phi, \mathrm{spin}), \\ \Lambda(2k+j-2-s, \phi, \mathrm{spin}) =& (-1)^k \Lambda(s, \phi, \mathrm{spin}). \end{aligned}$$

This is proved by Schmidt [9]. For a prime l, there exists a l-adic representation $\boldsymbol{\rho}_{j,k} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\bar{\mathbb{Q}}_l)$ such that

$$L(s, \phi_{j,k}, \mathrm{spin}) = \prod_p \det(1 - \rho_{j,k}(\mathrm{Frob}_p) \cdot p^{-s})$$

up to bad Euler factors. The eigenvalue $\phi_{i,j}$ with respect to the Hecke operator $T(p)$ is denoted by $\tau_{j,k}(p)$.

By Tsusima's dimension formula [10], we have $\dim_{\mathbb{C}} S_{j,k}(\mathrm{Sp}_2(\mathbb{Z})) = 1$ for

$$(j, k) = (4, 10),\ (6, 8),\ (8, 8),\ (12, 6).$$

Let $\phi_{i,k}$ be a generator of $S_{j,k}(\mathrm{Sp}_2(\mathbb{Z}))$ for $(j, k) = (4, 10),\ (6, 8),\ (8, 8),\ (12, 6)$.

Note that

$$\zeta(s) \prod_{10 \le i \le 11} L(s + i, \phi_{12,6}, \mathrm{spin}) \prod_{8 \le i \le 9} L(s + i, \phi_{18})$$

has a gamma factor

$$\Gamma_{\mathbb{R}}(s) \prod_{6 \le i \le 11} \Gamma_{\mathbb{C}}(s + i),$$

which is the same as the gamma factor of the standard L-function of $S_{12}(\mathrm{Sp}_6(\mathbb{Z}))$. By the Arthur endoscopic classification, one can show that there exists a Hecke eigenform $F \in S_{12}(\mathrm{Sp}_6(\mathbb{Z}))$ such that

$$L(s, F, \mathrm{st}) = \zeta(s) \prod_{10 \le i \le 11} L(s + i, \phi_{12,6}, \mathrm{spin}) \prod_{8 \le i \le 9} L(s + i, \phi_{18}).$$

comparing the Satake parameter, we have F is equal to F_{10} up to a non-zero constant. By a similar argument, we have

$$\begin{aligned}
L(s, F_{15}, \mathrm{st}) =& \zeta(s) \prod_{10 \le i \le 11} L(s + i, \phi_{8,8}, \mathrm{spin}) \prod_{6 \le i \le 9} L(s + i, \phi_{16}), \\
L(s, F_{19}, \mathrm{st}) =& L(s, \Delta, \mathrm{st}) \prod_{9 \le i \le 10} L(s + i, \phi_{6,8}, \mathrm{spin}) \prod_{7 \le i \le 8} L(s + i, \phi_{16}) \prod_{5 \le i \le 6} L(s + i, \phi_{12}), \\
L(s, F_{21}, \mathrm{st}) =& \zeta(s) \prod_{10 \le i \le 11} L(s + i, \phi_{4,10}, \mathrm{spin}) \prod_{8 \le i \le 9} L(s + i, \phi_{18}) \prod_{4 \le i \le 7} L(s + i, \Delta).
\end{aligned}$$

In this way, we have the following list.

$$
\begin{aligned}
L(s,F_3,\mathrm{st}) =& \zeta(s) \prod_{10\le i\le 11} L(s+i,\phi_{22}),\\
L(s,F_4,\mathrm{st}) =& L(s,\Delta,\mathrm{st}) \prod_{9\le i\le 10} L(s+i,\phi_{20}),\\
L(s,F_5,\mathrm{st}) =& \zeta(s) \prod_{8\le i\le 11} L(s+i,\phi_{20}),\\
L(s,F_6,\mathrm{st}) =& \zeta(s) \prod_{10\le i\le 11} L(s+i,\phi_{22}) \prod_{8\le i\le 9} L(s+i,\phi_{18}),\\
L(s,F_7,\mathrm{st}) =& L(s,\Delta,\mathrm{st}) \prod_{9\le i\le 10} L(s+i,\phi_{20}) \prod_{7\le i\le 8} L(s+i,\phi_{16}),\\
L(s,F_8,\mathrm{st}) =& L(s,\Delta,\mathrm{st}) \prod_{7\le i\le 10} L(s+i,\phi_{18}),\\
L(s,F_9,\mathrm{st}) =& \zeta(s) \prod_{10\le i\le 11} L(s+i,\phi_{22}) \prod_{6\le i\le 9} L(s+i,\phi_{16}),\\
L(s,F_{10},\mathrm{st}) =& \zeta(s) \prod_{10\le i\le 11} L(s+i,\phi_{12,6},\mathrm{spin}) \prod_{8\le i\le 9} L(s+i,\phi_{18}),\\
L(s,F_{11},\mathrm{st}) =& \zeta(s) \prod_{6\le i\le 11} L(s+i,\phi_{18}),\\
L(s,F_{12},\mathrm{st}) =& L(s,\Delta,\mathrm{st}) \prod_{5\le i\le 10} L(s+i,\phi_{16}),\\
L(s,F_{14},\mathrm{st}) =& L(s,\Delta,\mathrm{st}) \prod_{9\le i\le 10} L(s+i,\phi_{20}) \prod_{7\le i\le 8} L(s+i,\phi_{16}) \prod_{5\le i\le 6} L(s+i,\Delta),\\
L(s,F_{16},\mathrm{st}) =& L(s,\Delta,\mathrm{st}) \prod_{7\le i\le 10} L(s+i,\phi_{18}) \prod_{5\le i\le 6} L(s+i,\Delta),\\
L(s,F_{13},\mathrm{st}) =& \zeta(s) \prod_{4\le i\le 11} L(s+i,\phi_{16}),\\
L(s,F_{17},\mathrm{st}) =& \zeta(s) \prod_{8\le i\le 11} L(s+i,\phi_{20}) \prod_{4\le i\le 7} L(s+i,\Delta),\\
L(s,F_{18},\mathrm{st}) =& \zeta(s) \prod_{10\le i\le 11} L(s+i,\phi_{22}) \prod_{8\le i\le 9} L(s+i,\phi_{18}) \prod_{4\le i\le 7} L(s+i,\Delta),\\
L(s,F_{15},\mathrm{st}) =& \zeta(s) \prod_{10\le i\le 11} L(s+i,\phi_{8,8},\mathrm{spin}) \prod_{6\le i\le 9} L(s+i,\phi_{16}),\\
L(s,F_{20},\mathrm{st}) =& L(s,\Delta,\mathrm{st}) \prod_{9\le i\le 10} L(s+i,\phi_{20}) \prod_{3\le i\le 8} L(s+i,\Delta),\\
L(s,F_{19},\mathrm{st}) =& L(s,\Delta,\mathrm{st}) \prod_{9\le i\le 10} L(s+i,\phi_{6,8},\mathrm{spin}) \prod_{7\le i\le 8} L(s+i,\phi_{16}) \prod_{5\le i\le 6} L(s+i,\phi_{12}),\\
L(s,F_{22},\mathrm{st}) =& \zeta(s) \prod_{10\le i\le 11} L(s+i,\phi_{22}) \prod_{2\le i\le 9} L(s+i,\Delta),\\
L(s,F_{21},\mathrm{st}) =& \zeta(s) \prod_{10\le i\le 11} L(s+i,\phi_{4,10},\mathrm{spin}) \prod_{8\le i\le 9} L(s+i,\phi_{18}) \prod_{4\le i\le 7} L(s+i,\Delta),\\
L(s,F_{23},\mathrm{st}) =& L(s,\Delta,\mathrm{st}) \prod_{i=1}^{10} L(s+i,\Delta),\\
L(s,F_{24},\mathrm{st}) =& \zeta(s) \prod_{i=0}^{11} L(s+i,\Delta).
\end{aligned}
$$

Now we look at f_{18} and f_{21}. The coefficients of f_{18} and f_{21} are as follows:

	f_{18}	f_{21}
L_1	-497296800	-10443232800
L_2	4598528	133745920
L_3	-1339173	-47191815
L_4	1079296	47645696
L_5	-979625	-59665625
L_6	-1587744	-62532000
L_7	18238464	181232640
L_8	5882107	454089125
L_9	-1874432	304192000
L_{10}	42770511	-1585714725
L_{11}	-52307360	-6844516000
L_{12}	-33873920	-775168000
L_{13}	43287552	18627840000
L_{14}	1733363712	-100776960000
L_{15}	-1236612377	89553839375
L_{16}	456902656	67945830400
L_{17}	5926176256	-486566080000
L_{18}	-22766026752	113799168000
L_{19}	8836315488	-270161892000
L_{20}	100908408832	139639808000
L_{21}	149286312175	12525735096875
L_{22}	-817169633280	45429576192000
L_{23}	8013000038400	-64332092160000
L_{24}	-873155271532544	-4104432876544000

We follow the argument of Chenevier-Lannes [2]. The standard L-function of $F_{18} \in S_{12}(\mathrm{Sp}_8(\mathbb{Z}))$ is associated to the l-adic Galois representation

$$\mathbf{1} + (\chi^{-10} + \chi^{-11})\boldsymbol{\rho}_{22} + (\chi^{-8} + \chi^{-9})\boldsymbol{\rho}_{18} + (\chi^{-4} + \chi^{-5} + \chi^{-6} + \chi^{-7})\boldsymbol{\rho}_{12}$$

Here, $\chi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Q}_l^\times$ is the cyclotomic character. It follows that the eigenvalue $\mathrm{ev}_{18}K(p)^\vee$ of f_{18} with respect to $K(p)^\vee$ is equal to

$$p^{11}\Big(1 + (p^{-10} + p^{-11})a_{22}(p) + (p^{-8} + p^{-9})a_{18}(p) + (p^{-4} + p^{-5} + p^{-6} + p^{-7})a_{12}(p) \\ + p^{-4} + p^{-3} + p^{-2} + p^{-1} + 1 + p + p^2 + p^3 + p^4\Big)$$

Similarly, The standard L-function of $F_{21} \in S_{12}(\mathrm{Sp}_{11}(\mathbb{Z}))$ is associated to

$$\mathbf{1} + (\chi^{10} + \chi^{11})\boldsymbol{\rho}_{4,10} + (\chi^{8} + \chi^{9})\boldsymbol{\rho}_{18} + (\chi^{4} + \chi^{5} + \chi^{6} + \chi^{7})\boldsymbol{\rho}_{12}$$

It follows that the eigenvalue $\mathrm{ev}_{21}K(p)^\vee$ of f_{21} with respect to $K(p)^\vee$ is equal to

$$p^{11}\Big(1+(p^{-10}+p^{-11})\tau_{4,10}(p)+(p^{-8}+p^{-9})a_{18}(p)+(p^{-4}+p^{-5}+p^{-6}+p^{-7})a_{12}(p) + p^{-2}+p^{-1}+1+p+p^2\Big)$$

By an explicit calculation, we have

$$f_{18}-2\cdot f_{21}\in 41\mathbb{Z}[\mathscr{G}].$$

Hence we have

$$\mathrm{ev}_{18}K(p)^\vee \equiv \mathrm{ev}_{21}K(p)^\vee \bmod 41.$$

It follows that

$$(p+1)(\tau_{4,10}(p)-a_{22}(p)-p^{13}-p^{8})\equiv 0 \bmod 41.$$

Put $l=41$. By the argument as above, we have

$$(1+\bar{\chi})\big(\bar{\boldsymbol{\rho}}_{4,14}-(\bar{\boldsymbol{\rho}}_{22}+\bar{\chi}^{13}+\bar{\chi}^{8})\big)=0$$

in the Grothendieck group of mod l Galois representations with coefficient $\mathbb{F}_l$. Here, bar means the reduction mod l. After a little argument, one can show

$$\bar{\boldsymbol{\rho}}_{4,14}=\bar{\boldsymbol{\rho}}_{22}+\bar{\chi}^{13}+\bar{\chi}^{8}.$$

It follows that

$$\tau_{4,10}(p)\equiv a_{22}(p)+p^{13}+p^{3} \bmod 41$$

for $p\neq 41$. This is a special case of the Harder conjecture.

4 Positive definite even unimodular lattices of rank 8 over $\mathbb{Q}(\sqrt{2})$

Now, we set $F=\mathbb{Q}(\sqrt{2})$. By the result of Hsia and Hung [4], there are six isomorphism classes of positive definite even unimodular lattices of rank 8 over F. Let $\mathscr{G}$ be the set of isomorphism classes. They are labeled as

$$\mathcal{G}=\{E_8, 2\varDelta_4', \varDelta_8, 2D_4, 4\varDelta_2, \emptyset\}.$$

- The order $E(L)=\sharp\mathrm{O}(L)$ of the automorphism group of L is as follows.

L	E_8	$2\mathit{\Delta}_4'$	$\mathit{\Delta}_8$	$2D_4$	$4\mathit{\Delta}_2$	$\emptyset$
$E(L)$	$2^{14}{\cdot}3^5{\cdot}5^2{\cdot}7$	$2^{17}{\cdot}3^4$	$2^{15}{\cdot}3^2{\cdot}5{\cdot}7$	$2^{14}{\cdot}3^3$	$2^{18}{\cdot}3$	$2^{14}{\cdot}3^2{\cdot}5{\cdot}7$

- The Kneser $\mathfrak{q}$-neighbor matrix $N(L,K,\mathfrak{q})$ is as follows:

$L \setminus K$	E_8	$2\mathit{\Delta}_4'$	$\mathit{\Delta}_8$	$2D_4$	$4\mathit{\Delta}_2$	$\emptyset$
E_8	0	0	135	0	0	0
$2\mathit{\Delta}_4'$	0	18	36	0	81	0
$\mathit{\Delta}_8$	2	35	28	70	0	0
$2D_4$	0	0	3	96	36	0
$4\mathit{\Delta}_2$	0	6	0	64	49	16
$\emptyset$	0	0	0	0	105	30

- The coefficients of eigenvectors $f_i \in \mathbb{C}[\mathscr{G}]$ of the dual Kneser $\mathfrak{q}$-neighbor operator $K(\mathfrak{q})^\vee$ are given by

	E_8	$2\mathit{\Delta}_4'$	$\mathit{\Delta}_8$	$2D_4$	$4\mathit{\Delta}_2$	$\emptyset$
f_1	1	1	1	1	1	1
f_2	135	36	-30	3	-8	14
f_3	-14175	-216	840	81	-304	840
f_4	-135	-36	-58	-3	8	30
f_5	$5775-525\sqrt{73}$	$-88+104\sqrt{73}$	560	$-81-13\sqrt{73}$	$16+16\sqrt{73}$	560
f_6	$5775+525\sqrt{73}$	$-88-104\sqrt{73}$	560	$-81+13\sqrt{73}$	$16-16\sqrt{73}$	560

- The eigenvalue of f_i $(i=1,\dots,6)$ with respect to $K(\mathfrak{q})^\vee$:

f_1	f_2	f_3	f_4	f_5	f_6
135	-30	-8	58	$33+3\sqrt{73}$	$33-3\sqrt{73}$

Note that these eigenvalues are distinct.

5 Hecke eigenforms for $S_4(\Gamma_1[\mathfrak{d}^{-1},\mathfrak{d}])$ and $S_6(\Gamma_1[\mathfrak{d}^{-1},\mathfrak{d}])$

Let $k \geq 1$ be an integer. For an integral ideal $\mathfrak{b}$ of F, we put $\sigma_k(\mathfrak{b}) = \sum_{\mathfrak{a}|\mathfrak{b}} \mathfrak{N}(\mathfrak{a})^k$. The Eisenstein series $G_{2k}(z) \in M_{2k}(\Gamma_1[\mathfrak{d}^{-1},\mathfrak{d}])$ is defined by

$$G_{2k}(z) = 2^{-d}\zeta_F(1-2k) + \sum_{\xi \in \mathfrak{o} \cap F_+^\times} \sigma_{2k-1}((\xi))\mathbf{e}(\xi z) \in M_{2k}(\Gamma[\mathfrak{d}^{-1},\mathfrak{d}]).$$

Then

$$\bigoplus_{k \geq 0} M_{2k}(\Gamma_1[\mathfrak{d}^{-1},\mathfrak{d}]) = \mathbb{C}[G_2, G_4, G_6].$$

In particular, we have

$$\dim S_4(\Gamma_1[\mathfrak{d}^{-1},\mathfrak{d}])=1, \qquad \dim S_6(\Gamma_1[\mathfrak{d}^{-1},\mathfrak{d}])=2.$$

Let

$$\phi_4(Z)=44G_4(Z)-\frac{5}{6}G_2(Z)^2=\mathbf{q}-2\mathbf{q}^{2-\sqrt{2}}-4\mathbf{q}^2+\cdots, \qquad \mathbf{q}^\xi:=\mathbf{e}(\xi z)$$

be a normalized Hecke eigenform of $S_4(\Gamma_1[\mathfrak{d}^{-1},\mathfrak{d}])$.

The L-function $L(s,\phi_4)$ of $\phi_4(Z)=\sum_{\xi\in\mathfrak{o}\cap F_+^\times}a(\xi)\mathbf{q}^\xi$ is defined by

$$L(s,\phi_4)=\prod_{\mathfrak{p}}(1-a(\varpi_\mathfrak{p})q_\mathfrak{p}^{-s}+q_\mathfrak{p}^{3-2s})^{-1}.$$

Here, $\varpi_\mathfrak{q}$ is a positive definite generator of the ideal $\mathfrak{q}$. Then the functional equation of ϕ_4 is given by

$$\Lambda(4-s,\phi_4)=\Lambda(s,\phi_4), \qquad \Lambda(s,\phi_4)=8^{-s}\Gamma_\mathbb{C}(s)^2L(s,\phi_4).$$

Put $\mathfrak{q}=(\sqrt{2})$. Then the $\mathfrak{q}$-Satake parameter of ϕ_4 is determined by

$$2^{3/2}(\beta_\mathfrak{q}+\beta_\mathfrak{q}^{-1})=a(\varpi_\mathfrak{q})=-2.$$

It follows that the $\{\beta_\mathfrak{q},\beta_\mathfrak{q}^{-1}\}=\{\mathbf{e}(3/8),\mathbf{e}(5/8)\}$. The standard L-fuction $L(s,\phi_4,\mathrm{st})$ is given by

$$L(s,\phi_4,\mathrm{st})=\prod_{\mathfrak{q}}(1-q_\mathfrak{q}^{-s})(1-\beta_\mathfrak{q}q_\mathfrak{q}^{-s})(1-\beta_\mathfrak{q}^{-1}q_\mathfrak{q}^{-s}).$$

There are two normalized Hecke eigenforms $\{\phi_6^+,\phi_6^-\}$ for $S_6(\Gamma_1[\mathfrak{d}^{-1},\mathfrak{d}])$.

$$\begin{aligned}\phi_6^\pm(Z)=&\frac{-48240G_2(Z)G_4(Z)+2824320G_2(Z)^3-7G_6(Z)}{1560}\\ &\pm\frac{\sqrt{73}(14160G_2(Z)G_4(Z)-470400G_2(Z)^3-7G_6(Z))}{1560}\\ =&\mathbf{q}+(-1\pm\sqrt{73})\mathbf{q}^{2-\sqrt{2}}+\cdots.\end{aligned}$$

The L-function $L(s,\phi_6)$ has a functional equation

$$\Lambda(6-s,\phi_6)=\Lambda(s,\phi_6), \qquad \Lambda(s,\phi_6)=8^{-s}\Gamma_\mathbb{C}(s)^2L(s,\phi_6).$$

The $\mathfrak{q}$-Satake parameter of $\phi_6^\pm$ is also determined by

$$2^{5/2}(\gamma_\mathfrak{q}^\pm+(\gamma_\mathfrak{q}^\pm)^{-1})=-1\pm\sqrt{73}.$$

The standard L-fuction $L(s, \phi_6^{\pm}, \mathrm{st})$ is given by

$$L(s, \phi_4, \mathrm{st}) = \prod_{\mathfrak{q}} (1 - q_{\mathfrak{q}}^{-s})(1 - \gamma_{\mathfrak{q}}^{\pm} q_{\mathfrak{q}}^{-s})(1 - (\gamma_{\mathfrak{q}}^{\pm})^{-1} q_{\mathfrak{q}}^{-s}).$$

6 Hilbert-Siegel modular form arising from $\mathbb{C}[\mathscr{G}]$

- The degree of $f_i \in \mathbb{C}[\mathscr{G}]$ is given by

$$\deg f_1 = 0, \quad \deg f_2 = 4, \quad \deg f_4 = 1, \quad \deg f_5 = \deg f_6 = 2.$$

Here, the degree $\deg f_i$ of f_i is defined by

$$\deg f_i = \min\{m \mid \Theta^{(m)}(f_i) \neq 0\}.$$

We have

(1) $\Theta^{(4)}(f_2)$ is a DII lift of ϕ_4 to $S_4(\Gamma^{(4)}[\mathfrak{d}^{-1}, \mathfrak{d}])$.
(2) $\Theta^{(1)}(f_4)$ is equal to ϕ_4 up to a non-zero constant.
(3) $\Theta^{(3)}(f_3)$ is a Miyawaki lift of ϕ_4 to $S_4(\Gamma^{(3)}[\mathfrak{d}^{-1}, \mathfrak{d}])$ with respect to $\Theta^{(4)}(f_2)$.
(4) $\Theta^{(2)}(f_5)$ (resp. $\Theta^{(2)}(f_6)$) is a DII lift of ϕ_6^+ (resp. to ϕ_6^-) to $S_4(\Gamma^{(2)}[\mathfrak{d}^{-1}, \mathfrak{d}])$.

The standard L-functions are given by

$$\begin{aligned}
L(s, \Theta^{(4)}(f_2), \mathrm{st}) =& \zeta_F(s) \prod_{i=1}^{4} L(s + 4 - i, \phi_4), \\
L(s, \Theta^{(1)}(f_4), \mathrm{st}) =& L(s, \phi_4, \mathrm{st}), \\
L(s, \Theta^{(3)}(f_3), \mathrm{st}) =& L(s, \phi_4, \mathrm{st}) \prod_{i=1}^{2} L(s + 3 - i, \phi_4), \\
L(s, \Theta^{(2)}(f_5), \mathrm{st}) =& \zeta_F(s) \prod_{i=1}^{2} L(s + 4 - i, \phi_6^+), \\
L(s, \Theta^{(2)}(f_6), \mathrm{st}) =& \zeta_F(s) \prod_{i=1}^{2} L(s + 4 - i, \phi_6^-).
\end{aligned}$$

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