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Three-way preconcept and two forms of approximation operators

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Abstract Three-way decisions, as a better way than two-way decisions, has played an important role in many fields. As an extension of semiconcept, preconcept constitutes a new approach for data analysis. In contrast to preconcept, formal concept or semiconcept are too conservative about dealing with data. Hence, we want to further apply three-way decisions to preconcept. In this work, we introduce three-way preconcept by an example. This new notion combines preconcept with the assistant of three-way decisions. After that, we attain a generalized double Boolean algebra consisting of three-way preconcept. Furthermore, we give two form operators, approximation operators from lattice and set equivalence relation approximation operators, respectively. Finally, we present a conclusion with some summary and future issues that need to be addressed.

Keywords Preconcept · Approximation operator · Three-way decisions · Lattice theory

1 Introduction

Wille proposed Formal Concept Analysis (FCA) [1], another commonly used term is concept lattice theory, based on lattice theory. The formal concept is an essential element of FCA which consists of a pair of sets that say extent and intent. The underlying notion of "formal concept" evolved early in the philosophical theory of concept, which still plays a pivotal role in data analysis until today [2]. The set of all formal concepts in a formal context forms a complete lattice, called concept lattice, which is the most important structure in FCA. In 2002, Düntsch [3] proposed property-oriented concept lattice and discussed related properties. On the other hand, Yao [4] obtained a new notion, called object-oriented concept and pointed that there is an isomorphism between object-oriented concept lattice and formal concept lattice under the idea of the lattice. In algebraic structure, surveys such as that conducted by Yang [5] showed many properties in object-oriented concept lattice. By now, FCA has developed into an efficient tool for attribute reduction [6, 7] and granular computing [8]. If only consider extent or intent, in some sense, it is a more reasonable choice. Thus, as an extension of formal concept, semiconcept has been first considered in 1991 by Wille [9]. Wille pointed out some properties of semiconcept operators and proved that semiconcept algebra is a double Boolean algebra [10]. Mao [11] researched approximation operators in semiconcept in 2019, this study has provided new insights into characterizing semiconcept by a new idea with RS. Therefore, FCA has been formally enriched by introducing the notion of semiconcept.

Pawlak [12] proposed Rough Set (RS) in 1982 based on equivalence relation. According to RS, a set can be approximated by a lower approximation set and an upper approximation set. Recently, this method has been viewed as a key factor in knowledge representation [13, 14], and also, RS can be applied to forecasting models [15] and decision models in real life by reducing attribute, we can obtain a better decision than the original [16].

Three-way decisions proposed by Yao [17] has been applied into various fields successfully and this method is fast becoming a key instrument for making decisions [18, 19]. For example, in a war, the wounded are divided

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into immediate treatment, no treatment, and further diagnosis. Three-way decisions is an extension of two-way decisions model with an added third option [20, 21]. Two-way decisions yields values of 1 and 0, namely totally certain, which represent "yes" and "no", makes decisions more reasonable and close to reality. The discussion of concept lattice promotes the formation and development of three-way object-oriented concept. Furthermore, Yao proposed the role of three-way decisions in granular computing [22] and RS [21]. Three-way decisions not only applied in complete context but also applied in incomplete context [23]. Hence, three-way concept analysis as a combination of FCA and three-way decisions has been rapid development in data analysis [24, 25].

As an extension of the formal concept and the semiconcept, the preconcept is a new concept proposed by Wille in 2004 [26]. In 2006, Wille given the basic theorem on preconcept lattice [27]. Preconcept is weaker than semiconcept conditions, so more information can be found in a given information system. From another perspective, preconcept are the basis of semiconcept and formal concept. By filtering among the known preconcept, all semiconcept and then all formal concept can be obtained. For example, formal concept analysis, especially preconcept analysis, plays an important role in studying the classification of family members or the similarity of species. If we get preconcept, on the one hand, we get more information, and on the other hand, if we need to get more rigorous semiconcept or formal concept, we just need to constantly sift through these preconcept to get the final result.

However, consider both preconcept and three-way decisions had been largely under explored domain, separate consideration of them may lead to imperfect data analysis. If there is no combination of three-way decisions, in many real contexts, the information we consider will be incomplete. For example, when considering the similarities between humans and gorillas, as a classic preconcept, the common attribute they have is that they can walk and survive on land, but the two-way preconcept can not be fully reflected in the attribute of whether they have wings. If we apply the three-way decisions to the preconcept, we will consider attributes that we do not have in common. At this time, it will be reflected if the human and the gorilla have no wings at the same time. This is equivalent to increasing the credibility of the similarity between humans and gorillas, thereby increasing the breadth of information extraction.

Hence, to obtain both positive and negative information, this paper will consider preconcept combining with the three-way decisions. First of all, we define three-way preconcept (3WPC for simply). Afterward, we will find that 3WPC in a formal context can form a completely distributive lattice, and further, the set of all 3WPC forms a generalized double Boolean algebra. After that, we combine RS with 3WPC to obtain two forms of approximate operators in order to characterize 3WPC.

The contributions of this paper can be summarized as follows: The first section will briefly review the knowledge points such as semiconcept and three-way formal concept; The second section begins by laying out the notion of 3WPC and looks at generalized double Boolean algebra properties in 3WPC; Section three is concerned to characterize 3WPC by two forms of approximate operators. We conclude this article and leave room for our future research studies in the last section.

2 preliminaries

This section will review some definitions and properties that we need later on. For more detail, preconcept is seen [26] and double Boolean algebra is seen [28].

2.1 Poset and formal concept

Definition 1 [29]**.** A binary relation $\leqslant$ on a set S, which satisfies follows properties called *partial order relation*:
For all $a, b, c \in S$ we have:

(P1) : $a \leqslant a$.

(P2) : $a \leqslant b$ and $b \leqslant a$ imply that $a = b$.

(P3) : $a \leqslant b$ and $b \leqslant c$ imply that $a \leqslant c$.

$(S, \leqslant)$ called *partially ordered set* (simply *poset*) if $\leqslant$ satisfy P1, P2, P3, another commonly used terms are reflexivity, antisymmetry, transitivity, respectively.

Definition 2 [12]**.** Let U be the universe, $X \subseteq U$, $[x]_R$ is the equivalence class of x. The *lower approximations* and *upper approximations* can be presented in an equivalent form as shown below:

$$\underline{R}X = \{x \in U \mid [x]_R \subseteq X\},$$
$$\overline{R}X = \{x \in U \mid [x]_R \cap X \neq \varnothing\}.$$

Definition 3 [2]**.** A *formal context* is a triple $\mathbb{K} := (G, M, R)$, where G,M are sets of objects and properties respectively and $R \subseteq G \times M$. gRm indicates object g has property m. For $A \subseteq G$ and $B \subseteq M$,

$$A^* := \{m \in M \mid gRm, \forall g \in A\},$$
$$B^* := \{g \in G \mid gRm, \forall m \in B\}.$$

A concept of $\mathbb{K}$ is defined to be a pair (A, B) where $A \subseteq G$, $B \subseteq M$, $A^* = B$ and $B^* = A$. A is called *extent* and B is *intent* of the concept (A, B). The set of all concepts of $\mathbb{K}$ is denoted by $\mathfrak{B}(\mathbb{K})$.

For concepts (A_1, B_1) and (A_2, B_2) in $\mathbb{K}$ can be defined order as:

$$(A_1, B_1) \leqslant (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 (\Leftrightarrow B_2 \subseteq B_1).$$

$(\mathfrak{B}(\mathbb{K}), \leqslant)$ forms a complete lattice called the *concept lattice* of $\mathbb{K}$.

2.2 Semiconcept And Preconcept

Definition 4 [10]**.** A *semiconcept* of a formal context $\mathbb{K} := (G, M, R)$ is defined as a pair (A, B) with $A \subseteq G$ and $B \subseteq M$ where $A = B^*$ or $B = A^*$.

According to the definition of semiconcept, the concept is the specialization, considering only attributes or objects.

The following algebraic operations with $\sqcap$, $\sqcup$, $\rightharpoondown$, $\rightharpoonup$, $\bot$, $\top$ forms semiconcept algebra:

$$\begin{aligned}
(A_1, B_1) \sqcap (A_2, B_2) &:= (A_1 \cap A_2, (A_1 \cap A_2)^*) \\
(A_1, B_1) \sqcup (A_2, B_2) &:= ((B_1 \cap B_2)^*, B_1 \cap B_2) \\
\rightharpoondown (A, B) &:= (G \backslash A, (G \backslash A)^*) \\
\rightharpoonup (A, B) &:= ((M \backslash B)^*, M \backslash B) \\
\top &:= (G, \varnothing) \\
\bot &:= (\varnothing, M)
\end{aligned}$$

Definition 5 [26]**.** A *preconcept* of a formal context $\mathbb{K} := (G, M, R)$ is defined as a pair (A, B) with $A \subseteq G$ and $B \subseteq M$ where $A \subseteq B^*$ or $B \subseteq A^*$. The set of all preconcepts of $\mathbb{K}$ is denoted by $\mathfrak{H}(\mathbb{K})$.

2.3 Three-Way Formal Concept

In [30], R^c represents the set of all the dissatisfying relation R, and gives two negative operators as follows:

$$A^{\overline{*}} := \{m \in M \mid gR^c m, \forall g \in A\},$$
$$B^{\overline{*}} := \{g \in G \mid gR^c m, \forall m \in B\}.$$

Remark 1. Given two sets, $A \subseteq M, B \subseteq M$. If $A = \varnothing$ or $B = \varnothing$, the natural definitions of the two operators are as follows:

$$A^* = \varnothing^* = M, B^* = \varnothing^{\overline{*}} = M$$

Definition 6 [30]**.** Let $\mathbb{K} = (G, M, R)$ be a formal context. A pair $(X, (A, B))$ of an object subset $X \subseteq G$ and two attribute subsets $A, B \subseteq M$ is called an *object-induced three-way concept*, for short, an *OE-concept*, of (G, M, R), if and only if $X^* = A$and $X^{\overline{*}} = B$ and $A^* \cap B^{\overline{*}} = X$. X is called the *extension* and (A, B) is called the *intension* of the OE-concept $(X, (A, B))$.

Given two OE-concept $(X, (A, B))$ and $(Y, (C, D))$, [30] defined a partial order as follows:

$$(X, (A, B)) \leqslant (Y, (C, D)) \Leftrightarrow X \subseteq Y \Leftrightarrow (C, D) \subseteq (A, B).$$

Lemma 1. Let $\mathbb{K} = (G, M, R)$ be a formal context. Then the following statements are hold:
[(1)]$X \subseteq X^{**}$ and $A \subseteq A^{**}$ $X \subseteq X^{\overline{**}}$ and $A \subseteq A^{\overline{**}}$ $X \subseteq Y \Rightarrow Y^* \subseteq X^*$ and $A \subseteq B \Rightarrow B^* \subseteq A^*$ $X \subseteq Y \Rightarrow Y^{\overline{*}} \subseteq X^{\overline{*}}$ and $A \subseteq B \Rightarrow B^{\overline{*}} \subseteq A^{\overline{*}}$ $X^* = X^{***}$ and $A^* = A^{***}$ $X^{\overline{*}} = X^{\overline{***}}$ and $A^{\overline{*}} = A^{\overline{***}}$ $X \subseteq A^* \Leftrightarrow A \subseteq X^*$ $X \subseteq A^{\overline{*}} \Leftrightarrow A \subseteq X^{\overline{*}}$ $(X \cup Y)^* = X^* \cap Y^*$ and $(A \cup B)^* = A^* \cap B^*$ $(X \cup Y)^{\overline{*}} = X^{\overline{*}} \cap Y^{\overline{*}}$ and $(A \cup B)^{\overline{*}} = A^{\overline{*}} \cap B^{\overline{*}}$ $(X \cap Y)^* \supseteq X^* \cup Y^*$ and $(A \cap B)^* \supseteq A^* \cup B^*$ $(X \cap Y)^{\overline{*}} \supseteq X^{\overline{*}} \cup Y^{\overline{*}}$ and $(A \cap B)^{\overline{*}} \supseteq A^{\overline{*}} \cup B^{\overline{*}}$

2.4 Double Boolean Algebra

Wille found preconcept can construct a generalized double Boolean algebra with some operators.

Definition 7 [28]**.** A *generalized double Boolean algebra* $(A, \sqcup, \sqcap, \rightharpoondown, \rightharpoonup, \top, \bot, \vee, \wedge, \overline{\top}, \underline{\bot})$ is an abstract algebra which satisfies the following properties: For any $x, y, z \in A$, where $\vee$ and $\wedge$ is defined as $x \vee y = \rightharpoondown (\rightharpoondown x \sqcap \rightharpoondown y)$

$(1a)\ (x \sqcap x) \sqcap y = x \sqcap y,$	$(1b)\ (x \sqcup x) \sqcup y = x \sqcup y$
$(1a)\ (x \sqcap x) \sqcap y = x \sqcap y,$	$(1b)\ (x \sqcup x) \sqcup y = x \sqcup y$
$(2a)\ x \sqcap y = y \sqcap x,$	$(2b)\ x \sqcup y = y \sqcup x$
$(3a)\ (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z),$	$(3b)\ (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
$(4a)\ x \sqcap (x \sqcup y) = x \sqcap x,$	$(4b)\ x \sqcup (x \sqcap y) = x \sqcup x$
$(5a)\ x \sqcap (x \vee y) = x \sqcap x,$	$(5b)\ x \sqcup (x \wedge y) = x \sqcup x$
$(6a)\ x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z),$	
$(6b)\ x \sqcup (y \wedge z) = (x \sqcup y) \vee (x \sqcup z)$	
$(7a)\ \rightharpoondown\rightharpoondown (x \sqcap y) = x \sqcap y,$	$(7b)\ \rightharpoonup\rightharpoonup (x \sqcup y) = x \sqcup y$
$(8a)\ \rightharpoondown (x \sqcap x) = \rightharpoondown x,$	$(8b)\ \rightharpoonup (x \sqcup x) = \rightharpoonup x$
$(9a)\ x \sqcap \rightharpoondown x = \bot,$	$(9b)\ x \sqcup \rightharpoonup x = \top$
$(10a)\ \rightharpoondown \bot = \top \sqcap \top,$	$(10b)\ \rightharpoonup \top = \bot \sqcup \bot$
$(11a)\ \rightharpoondown \top = \bot,$	$(11b)\ \rightharpoonup \bot = \top$
$(12a)\ x_{\sqcap\sqcup\sqcap} = x_{\sqcap\sqcup},$	$(12b)\ x_{\sqcup\sqcap\sqcup} = x_{\sqcup\sqcap}$

and $x \wedge y = \rightharpoonup (\rightharpoonup x \sqcup \rightharpoonup y)$. $x_{\sqcap} = x \sqcap x$ and $x_{\sqcup} = x \sqcup x$ is defined for every term x.

3 Three-Way Preconcept

According to the definition of preconcept, combining with three-way decisions, we can define three-way preconcept, which is referred to as 3WPC.

Definition 8. Let $\mathbb{K} = (G, M, R)$ be a formal context. Let $X \subseteq G$ and $A \subseteq M$. Then a pair $(X, (A, B))$ is called a *3WPC* if $X \subseteq A^* \cap B^{\overline{*}}(\Leftrightarrow A \subseteq X^*$and $B \subseteq X^{\overline{*}})$. The set of all 3WPC in $\mathbb{K}$ is denoted by $\mathfrak{R}(\mathbb{K})$.

Remark 2. If $X \subseteq A^* \cap B^{\overline{*}}$ holds, we obtain $X \subseteq A^*$, according to le1 1(4), $A \subseteq X^*$ is hold. Similarly, $B \subseteq X^*$ is correct since $X \subseteq B^*$. Therefore, 3WPC 8 is well-defined.

Table 1: A formal context.

	Male(Ma)	Female(Fe)	Old	Young
Father(Fa)	*		*	
Mother(Mo)		*	*	
Son(So)	*			*
Daughter(Da)		*		*

Example 1. Let $\mathbb{K}_1 = (G, M, R)$ be the formal context in [26], where $G = \{\text{Fa}, \text{Mo}, \text{So}, \text{Da}\}, M = \{\text{Ma}, \text{Fe}, \text{Old}, \text{Young}\}$ and R is shown as tab1 2. If $X = \{\text{Fa}\}$ and $A = \{\text{Old}\}, B = \{\text{Young}\}$, Then owing to the definition of operators, we can get $X \subseteq A^* \cap B^{\overline{*}}$. If $X = \{\text{Mo}\}$ and $A = \{\text{Old}\}, B = \{\text{Young}\}$, we also receive $X \subseteq A^* \cap B^{\overline{*}}$. So according to 3WPC 8, we obtain that both $(\text{Fa}, (\text{Old}, \text{Young}))$ and $(\text{Mo}, (\text{Old}, \text{Young}))$ are 3WPC.

Definition 9. We define binary relations $\leqslant$ in $\mathfrak{R}(\mathbb{K})$ as follows:

(1) $(A, B) \leqslant_1 (C, D) \Leftrightarrow A \subseteq C$ and $B \subseteq D$.

(2) $(X, (A, B)) \leqslant_2 (Y, (C, D)) \Leftrightarrow X \subseteq Y$ and $(C, D) \leqslant_1 (A, B)$.

Example 2. Let $\mathbb{K}_1$ be in example1 1, and we know $(\text{Fa}, (\text{Old}, \text{Young})), (\{\text{Fa}, \text{Mo}\}, (\text{Old}, \text{Young})) \in 3WPC$. According to above definition, $(\text{Fa}, (\text{Old}, \text{Young})) \leqslant_2 (\{\text{Fa}, \text{Mo}\}, (\text{Old}, \text{Young}))$.

1[illegible]. **Lemma 2** *Let $\mathbb{K} = (G, M, R)$ be a formal context. The following statements hold in $\mathfrak{R}(\mathbb{K})$:*

1. *The binary relation $\leqslant_2$ in $\mathfrak{R}(\mathbb{K})$ is a partial order relation.*
2. *$(\mathfrak{R}(\mathbb{K}), \leqslant_2)$ is a poset.*

Proof 1. First of all, it is clear that the binary relation $\leqslant_1$ is a partial order relation. To prove: $\leqslant_2$ is a partial order relation.
  (a) Since $X \subseteq X, (A, B) \geqslant_2 (A, B)$, by twopartial 9(2), we receive $(X, (A, B)) \leqslant_1 (X, (A, B))$. So we obtain that $\leqslant_1$ satisfies reflexivity.
  (b) If $(X, (A, B)) \leqslant_2 (Y, (C, D))$, and if $(Y, (C, D)) \leqslant_2 (X, (A, B))$. Then according to twopartial 9(2), we obtain $X = Y, (A, B) = (C, D)$, and $(X, (A, B)) = (Y, (C, D))$ is hold and $\leqslant_2$ satisfies antisymmetry.
  (c) If $(X_1, (A_1, B_1)) \leqslant_2 (X_2, (A_2, B_2))$ and $(X_2, (A_2, B_2)) \leqslant_2 (X_3, (A_3, B_3))$. Hence, $X_1 \subseteq X_2 \subseteq X_3, (A_1, B_2) \geqslant_1 (A_2, B_2) \geqslant_1 (A_3, B_3)$ holds. So, according to twopartial 9, $(X_1, (A_1, B_1)) \leqslant_2 (X_3, (A_3, B_3))$ is hold. This means that $\leqslant_2$ satisfies transitivity.
2. Since binary relation $\leqslant_2$ is a partial order relation, we receive $(\mathfrak{R}(\mathbb{K}), \leqslant_2)$ is a poset.

Definition 10. Let $\mathbb{K} = (G, M, R)$ be a formal context. And $X_i \subseteq G, A_i \subseteq M, B_i \subseteq M$ for $i = 1, 2$. Then we define the following operators $\sqcap, \sqcup, \rightharpoondown, \rightharpoonup, \perp, \top, \vee, \wedge, \overline{\top}, \underline{\perp}$ in $\mathfrak{R}(\mathbb{K})$, respectively:

$$(X_1, (A_1, B_1)) \sqcap (X_2, (A_2, B_2)) := (X_1 \cap X_2, ((X_1 \cap X_2)^*, (X_1 \cap X_2)^{\overline{*}}))$$

$$(X_1, (A_1, B_1)) \sqcup (X_2, (A_2, B_2)) := ((A_1 \cap A_2)^* \cap (B_1 \cap B_2)^{\overline{*}}, (A_1 \cap A_2, B_1 \cap B_2))$$

$$\rightharpoondown (X, (A, B)) := (G \backslash X, ((G \backslash X)^*, (G \backslash X)^{\overline{*}}))$$

$$\rightharpoonup (X, (A, B)) := ((M \backslash A)^* \cap (M \backslash B)^{\overline{*}}, ((M \backslash A), (M \backslash B)))$$

$$(X_1, (A_1, B_1)) \vee (X_2, (A_2, B_2)) :=\rightharpoondown (\rightharpoondown (X_1, (A_1, B_1)) \sqcap \rightharpoondown (X_2, (A_2, B_2)))$$

$$(X_1, (A_1, B_1)) \wedge (X_2, (A_2, B_2)) :=\rightharpoonup (\rightharpoonup (X_1, (A_1, B_1)) \sqcup \rightharpoonup (X_2, (A_2, B_2)))$$

$$\top := (G, (\varnothing, \varnothing))$$

$$\perp := (\varnothing, (M, M))$$

$$\overline{\top} :=\rightharpoondown \perp$$

$$\underline{\perp} :=\rightharpoonup \top$$

Theorem 1 *Let $\mathbb{K} = (G, M, R)$ be a formal context. $X_i \subseteq G, A_i \subseteq M, B_i \subseteq M$ for $i \in I$, I is an index set. If $(X_i, (A_i, B_i)) \in \mathfrak{R}(\mathbb{K})$. Then $inf_{i \in I}(X_i, (A_i, B_i))$ and $sup_{i \in I}(X_i, (A_i, B_i))$ exists and the following statements are hold:*

$$inf_{i \in I}(X_i, (A_i, B_i)) = (\bigcap_{i \in I} X_i, (\bigcup_{i \in I} A_i, \bigcup_{i \in I} B_i))$$

$$sup_{i \in I}(X_i, (A_i, B_i)) = (\bigcup_{i \in I} X_i, (\bigcap_{i \in I} A_i, \bigcap_{i \in I} B_i))$$

Proof At first, we illustrate correct of inf in infsup 1 as follows:

[(1)]Since $(X_i, (A_i, B_i)) \in \mathfrak{R}(\mathbb{K})$, we receive $X_i \subseteq A_i^* \cap B_i^{\overline{*}}$. So $\bigcap_{i \in I} X_i \subseteq (\bigcup_{i \in I} A_i)^* \cap (\bigcup_{i \in I} B_i)^{\overline{*}}$ holds. According to 3WPC 8, $(\bigcap_{i \in I} X_i, (\bigcup_{i \in I} A_i, \bigcup_{i \in I} B_i)) \in \mathfrak{R}(\mathbb{K})$ is tenable. $\bigcap_{i \in I} X_i \subseteq X_i$ and $\bigcup_{i \in I} A_i \supseteq A_i, \bigcup_{i \in I} B_i \supseteq$

B_i holds. If $(X,(A,B)) \leqslant_2 (X_i,(A_i,B_i))$, we obtain $X \subseteq \bigcap_{i\in I} X_i$ and $A \supseteq \bigcup_{i\in I} A_i, B \supseteq \bigcup_{i\in I} B_i$. Therefore, $inf_{i\in I}(X_i,(A_i,B_i)) = (\bigcap_{i\in I} X_i, (\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i))$ exists.

Secondly, we illustrate correct of *sup* in infsup 1 as follows:

[(1)]Since $(X_i,(A_i,B_i)) \in \mathfrak{R}(\mathbb{K})$, we receive $X_i \subseteq A_i^* \cap B_i^{\overline{*}}$. Then $\bigcup_{i\in I} X_i \subseteq \bigcup_{i\in I} A_i^* \subseteq (\bigcap_{i\in I} A_i)^*$, similarly, $\bigcup_{i\in I} X_i \subseteq (\bigcap_{i\in I} B_i)^{\overline{*}}$. So $\bigcup_{i\in I} X_i \subseteq (\bigcap_{i\in I} A_i)$ $(\bigcap_{i\in I} B_i)^{\overline{*}}$ holds. According to 3WPC 8, $(\bigcup_{i\in I} X_i, (\bigcap_{i\in I}$ $\mathfrak{R}(\mathbb{K})$ is tenable. $\bigcup_{i\in I} X_i \supseteq X_i$ and $\bigcap_{i\in I} A_i \subseteq A_i, \bigcap_{i\in I} B$ B_i holds. If $(X,(A,B)) \geqslant_2 (X_i,(A_i,B_i))$, we obtain $X \supseteq \bigcup_{i\in I} X_i$ and $A \subseteq \bigcap_{i\in I} A_i, B \subseteq \bigcap_{i\in I} B_i$. Therefore, $sup_{i\in I}(X_i,(A_i,B_i)) = (\bigcup_{i\in I} X_i, (\bigcap_{i\in I} A_i, \bigcap_{i\in I} B_i)$ exists.

Example 3. Let $\mathbb{K}_1$ be in example1 1. If $X_1 = (\text{Fa}, (\text{Old}, \text{Yo}$ $(\{\text{Fa}, \text{Mo}\}, (\text{Old}, \text{Young})), X_3 = (\text{Mo}, (\text{Old}, \text{Young}))$, then owing to infsup 1, we obtain $inf_{i\in\{1,2,3\}}(X_i,(A_i,B_i)) = (\varnothing, (\text{Old}, \text{Young}))$ and $sup_{i\in\{1,2,3\}}(X_i,(A_i,B_i)) = (\{\text{Fa}, \text{Mo}\}$ respectively.

3. **Theorem 2** *Let $\mathbb{K} = (G, M, R)$ be a formal context. Then $(\mathfrak{R}(\mathbb{K}), \leqslant_2)$ is a distributive complete lattice, and isomorphic to a concept lattice.*

Proof Obviously, $(\mathfrak{R}(\mathbb{K}), \leqslant_2)$ is complete lattice by infsup 1. Therefore, we only need to proof correct of distributive as follows:

$$
\begin{aligned}
&(X_1,(A_1,B_1)) \bigwedge [(X_2,(A_2,B_2)) \bigvee (X_3,(A_3,B_3))] \\
=&(X_1,(A_1,B_1)) \bigwedge (X_2 \cup X_3, (A_2 \cap A_3, B_2 \cap B_3)) \\
=&(X_1 \cap (X_2 \cup X_3), (A_1 \cup (A_2 \cap A_3), B_1 \cup (B_2 \cap B_3))) \\
=&((X_1 \cap X_2) \cup (X_1 \cap X_3), ((A_1 \cup A_2) \cap (A_1 \cup A_3), \\
&(B_1 \cup B_2) \cap (B_1 \cup B_3))) \\
=&[(X_1,(A_1,B_1)) \bigwedge (X_2,(A_2,B_2))] \bigvee [(X_1,(A_1,B_1)) \bigwedge \\
&(X_3,(A_3,B_3))]
\end{aligned}
$$

According to lattice theory, a complete lattice L is isomorphic to concept lattice $\mathfrak{B}(L, L, \leqslant)$. Therefore, $(\mathfrak{R}(\mathbb{K}), \leqslant_2)$ is a distributive complete lattice, and isomorphic to concept lattice $\mathfrak{B}(\mathfrak{R}(\mathbb{K}), \mathfrak{R}(\mathbb{K}), \leqslant)$.

Example 4. Give a formal context as follows:

Table 2: A small formal context.

	Male(Ma)	Female(Fe)
Father(Fa)	*	
Mother(Mo)		*

Let the preconcept lattice be L. According to [1], since L do not have a sublattice isomorphic to M_3, N_5, we attain L is a distributive lattice. For simply, let Ma be 1, Fe be 2, we get 3WPCLATTICE 1. And we can receive L is isomorphic to formal context in 3WPCcon 2. The 3WPCLattice 3 delegates 3WPCcon 2 concept lattice by using *Lattice Miner Platform* 1.4.

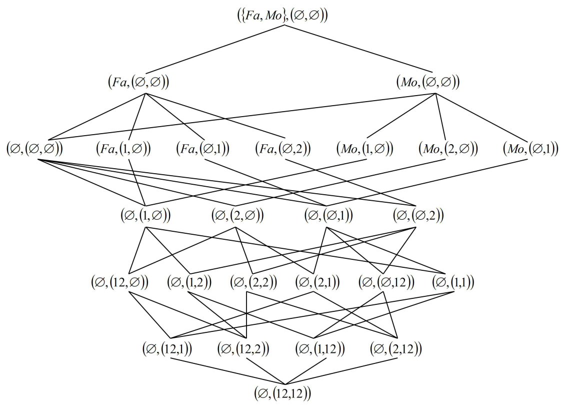

Fig. 1: 3WPC

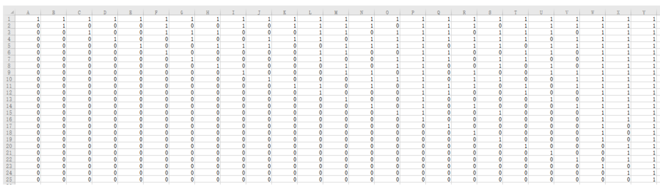

Fig. 2: A context isomorphic to 3WPC

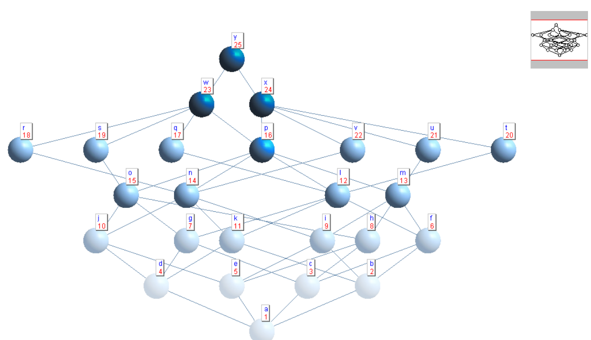

Fig. 3: The lattice of 3WPC

Lemma 3 *Let $\mathbb{K} = (G, M, R)$ be a formal context and x, y, z be 3WPC. Then the following properties are correct in $\mathfrak{R}(\mathbb{K})$ for x, y, z:*

$(1a)\ (x \sqcap x) \sqcap y = x \sqcap y,$ $(1b)\ (x \sqcup x) \sqcup y = x \sqcup y$
$(1a)\ (x \sqcap x) \sqcap y = x \sqcap y,$ $(1b)\ (x \sqcup x) \sqcup y = x \sqcup y$
$(2a)\ x \sqcap y = y \sqcap x,$ $(2b)\ x \sqcup y = y \sqcup x$
$(3a)\ (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z),$ $(3b)\ (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
$(4a)\ x \sqcap (x \sqcup y) = x \sqcap x,$ $(4b)\ x \sqcup (x \sqcap y) = x \sqcup x$
$(5a)\ x \sqcap (x \vee y) = x \sqcap x,$ $(5b)\ x \sqcup (x \wedge y) = x \sqcup x$
$(6a)\ x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z),$
$(6b)\ x \sqcup (y \wedge z) = (x \sqcup y) \vee (x \sqcup z)$
$(7a)\ \rightharpoondown\rightharpoondown (x \sqcap y) = x \sqcap y,$ $(7b)\ \rightharpoonup\rightharpoonup (x \sqcup y) = x \sqcup y$
$(8a)\ \rightharpoondown (x \sqcap x) = \rightharpoondown x,$ $(8b)\ \rightharpoonup (x \sqcup x) = \rightharpoonup x$
$(9a)\ x \sqcap \rightharpoondown x = \bot,$ $(9b)\ x \sqcup \rightharpoonup x = \top$
$(10a)\ \rightharpoondown \bot = \top \sqcap \top,$ $(10b)\ \rightharpoonup \top = \bot \sqcup \bot$
$(11a)\ \rightharpoondown \top = \bot,$ $(11b)\ \rightharpoonup \bot = \top$
$(12a)\ x_{\sqcap\sqcup\sqcap} = x_{\sqcap\sqcup},$ $(12b)\ x_{\sqcup\sqcap\sqcup} = x_{\sqcup\sqcap}$

Therefore, $(\mathfrak{R}(\mathbb{K}), \sqcap, \sqcup, \rightharpoondown, \rightharpoonup, \vee, \wedge)$ is generalized double Boolean algebra.

Proof We will prove this lemma by le1 1 and notion 4 as follows. For convenience, let $x = (X_1, (A_1, B_1)), y = (X_2, (A_2, B_2)), z = (X_3, (A_3, B_3))$
(1a) According to the definition of operator $\sqcap$, we obtain $(x \sqcap x) \sqcap y = ((X_1, (A_1, B_2)) \sqcap (X_1, (A_1, B_1))) \sqcap (X_2, (A_2, B_2)) = (X_1 \cap X_2, ((X_1 \cap X_2)^*, (X_1 \cap X_2)^{\overline{*}})) = x \sqcap y$. Hence, $(x \sqcap x) \sqcap y = x \sqcap y$ holds.

(1b) According to the definition of operator $\sqcup$, we receive $(x \sqcup x) \sqcup y = (A_1^* \cap B_1^{\overline{*}}, (A_1, B_1)) \sqcup (X_2, (A_2, B_2)) = ((A_1 \cap B_2)^* \cap (B_1 \cap B_2)^{\overline{*}}) = x \sqcup y$. Thus, $(x \sqcup x) \sqcup y = x \sqcup y$ is right.

(2a) Owing to notion 4, we have $x \sqcap y = (X_1 \cap X_2, ((X_1 \cap X_2)^*, (X_1 \cap X_2)^{\overline{*}})) = y \sqcap x$. Therefore, this item holds.

(2b) Owing to notion 4, we get $x \sqcup y = ((A_1 \cap A_2)^* \cap (B_1 \cap B_2)^{\overline{*}}, (A_1 \cap A_2, B_1 \cap B_2)) = y \sqcup x$, we confirm correct of this item.

(3a) Similar to the proof of item (1a), we know $x \sqcap (y \sqcap z) = (X_1, (A_1, B_1)) \sqcap (X_2 \cap X_3, ((X_2 \cap X_3)^*, (X_2 \cap X_3)^{\overline{*}})) = (X_1 \cap X_2 \cap X_3, ((X_1 \cap X_2 \cap X_3)^*, (X_1 \cap X_2 \cap X_3)^{\overline{*}}))$. But $(x \sqcap y) \sqcap z = (X_1 \cap X_2, ((X_1 \cap X_2)^*, (X_1 \cap X_2)^{\overline{*}})) \sqcap (X_3, (A_3, B_3)) = (X_1 \cap X_2 \cap X_3, ((X_1 \cap X_2 \cap X_3)^*, (X_1 \cap X_2 \cap X_3)^{\overline{*}}))$. Therefore, this item holds.

(3b) Similar to the proof of item (1b), we receive $(x \sqcup y) \sqcup z = (X_1, (A_1, B_1)) \sqcup ((A_2 \cap A_3)^* \cap (B_2 \cap B_3)^{\overline{*}}, (A_2 \cap A_3, B_2 \cap B_3)) = ((A_1 \cap A_2 \cap A_3)^* \cap (B_1 \cap B_2 \cap B_3)^{\overline{*}}, (A_1 \cap A_2 \cap A_3, B_1 \cap B_2 \cap B_3)) = x \sqcup (y \sqcup z)$. Hence, this item is right.

(4a) With the definition of operators $\sqcap$ and $\sqcup$, we obtain $x \sqcap (x \sqcup y) = (X_1, (A_1, B_1)) \sqcap ((A_1 \cap A_2)^* \cap (B_1 \cap B_2)^{\overline{*}}, (A_1 \cap A_2, B_1 \cap B_2))$. Since $X_1 \subseteq A_1^* \cap B_1^{\overline{*}}$, $X_1 \cap (A_1 \cap A_2)^* \cap (B_1 \cap B_2)^{\overline{*}} = X_1$ holds. But $x \sqcap (x \sqcup y) = (X_1, (X_1^*, X_1^{\overline{*}}))$. Thus, we receive $x \sqcap (x \sqcup y) = x \sqcap x$.

(4b) Similar to the above proof, we find $A_1 \subseteq X_1^*$ and $B_1 \subseteq X_1^{\overline{*}}$. It proves that left hand side $= (X_1, (A_1, B_1)) \sqcup (X_1 \sqcap X_2, ((X_1 \cap X_2)^*, (X_1 \cap X_2)^{\overline{*}})) = ((A_1 \cap (X_1 \cap X_2)^*)^* \cap (B_1 \cap (X_1 \cap X_2)^{\overline{*}})^{\overline{*}}, (A_1 \cap (X_1 \cap X_2)^*, B_1 \cap (X_1 \cap X_2)^{\overline{*}})) = (A_1^* \cap B_1^{\overline{*}}, (A_1, B_1))$. But right hand side $= (A_1^* \cap B_1^{\overline{*}}, (A_1, B_1))$. Therefore, this item holds.

(5a) By notion 4, we receive $x \sqcap (x \vee y) = (X_1, (A_1, B_1)) \sqcap \rightharpoondown (\rightharpoondown x \sqcap \rightharpoondown y) = (X_1, (A_1, B_1)) \sqcap \rightharpoondown ((X_1^c, (X_1^{c*}, X_1^{c\overline{*}})) \sqcap (X_2^c, (X_2^{c*}, X_2^{c\overline{*}}))) = (X_1, (A_1, B_1)) \sqcap (X_1 \cup X_2, ((X_1 \cup X_2)^*, (X_1 \cup X_2)^{\overline{*}})) = (X_1, (X_1^*, X_1^{\overline{*}})) = x \sqcap x$

(5b) By notion 4, we know $x \sqcap (x \wedge y) = (X_1, (A_1, B_1)) \sqcup \rightharpoonup (\rightharpoonup x \sqcup \rightharpoonup y) = (X_1, (A_1, B_1)) \sqcup \rightharpoonup ((A_1^{c*} \cap B_1^{c\overline{*}}, (A_1^c, B_1^c)) \sqcup (A_2^{c*} \cap B_2^{c\overline{*}}, (A_2^c, B_2^c))) = (A_1^* \cap B_1^{\overline{*}}, (A_1, B_1)) = x \sqcup x$

(6a) Owing to notion 4, left hand side $= (X_1, (A_1, B_1)) \sqcap \rightharpoondown (\rightharpoondown y \sqcap \rightharpoondown z) = ((X_1 \cap X_2) \cup (X_1 \cap X_3), (((X_1 \cap X_2) \cup (X_1 \cap X_3))^*, ((X_1 \cap X_2) \cup (X_1 \cap X_3))^{\overline{*}}))$. But right hand side $= (X_1 \cap X_2, ((X_1 \cap X_2)^*, (X_1 \cap X_2)^{\overline{*}})) \vee (X_1 \cap X_3, ((X_1 \cap X_3)^*, (X_1 \cap X_3)^{\overline{*}}))$. Then correct of (6a) is obvious.

(6b) Similar to (6a), left hand side $= (X_1, (A_1, B_1)) \sqcup \rightharpoonup (\rightharpoonup (X_2, (A_2, B_2)) \sqcup \rightharpoonup (X_3, (A_3, B_3))) = (X_1, (A_1, B_1)) \sqcup \rightharpoonup ((A_2^{c*} \cap B_2^{c\overline{*}}, (A_2^c, B_2^c)) \sqcup (A_3^{c*} \cap B_3^{c\overline{*}}, (A_3^c, B_3^c))) = (X_1, (A_1, B_1)) \sqcup \rightharpoonup ((A_2^c \cap A_3^c)^* \cap (B_2^c \cap B_3^c)^{\overline{*}}, (A_2^c \cap A_3^c, B_2^c \cap B_3^c)) = (A_1 \cap (A_2 \cup A_3))^* \cap (B_1 \cap (B_2 \cup B_3))^{\overline{*}}, (A_1 \cap (A_2 \cup A_3), B_1 \cap (B_2 \cup B_3)))$. And right hand side $= [(X_1, (A_1, B_1)) \sqcup (X_2, (A_2, B_2))] \wedge [(X_1, (A_1, B_1)) \sqcup (X_3, (A_3, B_3))]$. Thus, we can confirm correct of this item.

(7a) We receive $\rightharpoondown\rightharpoondown (x \sqcap y) = \rightharpoondown\rightharpoondown (X_1 \cap X_2, ((X_1 \cap X_2)^*, (X_1 \cap X_2)^{\overline{*}})) = (X_1 \cap X_2, ((X_1 \cap X_2)^*, (X_1 \cap X_2)^{\overline{*}}))$ and $x \sqcap y = (X_1 \sqcap X_2, ((X_1 \cap X_2)^*, (X_1 \cap X_2)^{\overline{*}}))$. This illustrates correct of (7a).

(7b) Similar to the proof of item (7a), we get $\rightharpoonup\rightharpoonup (x \sqcup y) = x \sqcup y$ immediately.

(8a) Since $\rightharpoondown (x \sqcap x) = \rightharpoondown ((X_1, (A_1, B_1)) \sqcap (X_1, (A_1 B_1))) = (X_1^c, (X_1^*, X_1^{\overline{*}})) = \rightharpoondown x$. Therefore, we confirm correct of this item.

(8b) We get $\rightharpoonup (x \sqcup x) = \rightharpoonup ((X_1, (A_1, B_1)) \sqcup (X_1, (A_1, B_1))) = \rightharpoonup (A_1^* \cap B_1^{\overline{*}}, (A_1, B_1)) = (A_1^{c*} \cap B_1^{c\overline{*}}, (A_1^c, B_1^c))$ and we also obtain $\rightharpoonup x = (A_1^{c*} \cap B_1^{c\overline{*}}, (A_1^c, B_1^c))$. So, this item is correct.

(9a) By notion 4, we receive $x \sqcap \rightharpoondown x = (X_1, (A_1, B_1)) \sqcap \rightharpoondown (X_1, (A_1, B_1) = (\varnothing, (\varnothing^*, \varnothing^{\overline{*}})) = (\varnothing, (M, M)) = \bot$. Hence, we get correct of this item.

(9b) Since $x\sqcup\rightharpoonup x = (X_1,(A_1,B_1))\sqcup(A_1^{c*}\cap B_1^{c\overline{*}},(A_1^c,B_1^c))$ = $(\varnothing^*\cap\varnothing^{\overline{*}},(\varnothing,\varnothing)) = (G,(\varnothing,\varnothing)) = \top$ holds by notion 4, we obtain (9b).

(10a) We receive $\rightharpoonup\bot=\rightharpoonup(\varnothing,(M,M)) = (G,(G^*,G^{\overline{*}}))$. But $\top\sqcap\top = (G,(\varnothing,\varnothing))\sqcap(G,(\varnothing,\varnothing)) = (G,(G^*,G^{\overline{*}}))$. So, this item is correct.

(10b) Similar to the proof of item (10a), we get $\rightharpoonup\top=\rightharpoonup(G,(\varnothing,\varnothing)) = (M^*\cap M^{\overline{*}},(M,M)) = \bot\sqcup\bot$. Therefore, $\rightharpoonup\top = \bot\sqcup\bot$ holds.

(11a) Similar to proof of item (10a), we get $\rightharpoonup\bot=\rightharpoonup(G,(\varnothing,\varnothing)) = (\varnothing,(M,M)) = \top$. Thus, we confirm correct of $\rightharpoonup\bot = \top$.

(11b) Similar to proof of item (11b), we get $\rightharpoonup\bot=\rightharpoonup(\varnothing,(M,M)) = (\varnothing^*\cap\varnothing^{\overline{*}},(\varnothing,\varnothing)) = (G,(\varnothing,\varnothing)) = \bot$. So, we receive correct of this item.

(12a) Since $x_{\sqcap\sqcup\sqcap} = [(X_1,(A_1,B_1))\sqcap(X_1,(A_1,B_1))]_{\sqcup\sqcap} = (X_1,(X_1^*,X_1^{\overline{*}}))_{\sqcup\sqcap} = (X_1^{**}\cap X_1^{\overline{**}},(X_1^*,X_1^{\overline{*}}))_{\sqcap} = (X_1^{**}\cap X_1^{\overline{**}},((X_1^{**}\cap X_1^{\overline{**}})^*,(X_1^{**}\cap X_1^{\overline{**}})^{\overline{*}}))$. And $x_{\sqcap\sqcup} = ((X_1,(A_1,B_1))\sqcap (X_1,(A_1,B_1)))_{\sqcup} = (X_1,(X_1^*,X_1^{\overline{*}}))_{\sqcup} = (X_1^{**}\cap X_1^{\overline{**}},((X_1^{**}\cap X_1^{\overline{**}})^*,(X_1^{**}\cap X_1^{\overline{**}})^{\overline{*}}))$. Therefore, (12a) holds.

(12b) Since $x_{\sqcup\sqcap\sqcup} = [(X_1,(A_1,B_1))\sqcup(X_1,(A_1,B_1))]_{\sqcap\sqcup} = (A_1^*\cap B_1^{\overline{*}},(A_1,B_1))_{\sqcap\sqcup} = (A_1^*\cap B_1^{\overline{*}},((A_1^*\cap B_1^{\overline{*}})^*,(A_1^*\cap B_1^{\overline{*}})^{\overline{*}}))$, and $x_{\sqcup\sqcap} = ((X_1,(A_1,B_1))\sqcup(X_1,(A_1,B_1)))_{\sqcap} = (A_1^*\cap B_1^{\overline{*}},(A_1,B_1))_{\sqcap}$. Thus we obtain $x_{\sqcup\sqcap\sqcup} = x_{\sqcup\sqcap}$.

4 Two Forms Of Approximation Operators

4.1 Approximation Operators From Lattice

Definition 11. Let $\mathbb{K} = (G,M,R)$ be a formal context, and $X,X_i\subseteq G, A,B,A_i,B_i\subseteq M$, $i\in I$, with I is an index set. Then give four operators as follows:

$$l(X,(A,B)) = \{(X_i,(A_i,B_i)) \mid \forall(X_i,(A_i,B_i))\in 3WPC \text{ and } X_i\subseteq X, A_i\supseteq A, B_i\supseteq B\}$$

$$h(X,(A,B)) = \{(X_i,(A_i,B_i)) \mid \forall(X_i,(A_i,B_i))\in 3WPC \text{ and } X_i\supseteq X, A_i\subseteq A, B_i\subseteq B\}$$

$$L(X,(A,B)) = \bigvee l(X,(A,B))$$

$$H(X,(A,B)) = \bigwedge h(X,(A,B))$$

Remark 3. We should illustrate lhoperators 11 for two parts:

firstly, we decipher l,h operators are well-defined.

[(1)]According to definition, $\varnothing\subseteq X, G\supseteq B$ and $(\varnothing,(G,G))\in 3WPC$. Thus, l is well-defined.
Similarly, $G\supseteq X,\varnothing\subseteq B$, and $(G,(\varnothing,\varnothing))\in 3WPC$ holds. Therefore, h is well-defined.

secondly, we illustrate L,H operators are well-defined.

[(1)]Owing to the definition, we receive $L(X,(A,B)) = (\bigcup_{i\in I}X_i,(\bigcap_{i\in I}A_i,\bigcap_{i\in I}B_i))$.
Similarly, $H(X,(A,B)) = (\bigcap_{i\in I}X_i,(\bigcup_{i\in I}A_i,\bigcup_{i\in I}B_i))$.

Example 5. Let $\mathbb{K} = (G,M,R)$ be a formal context. If $X = \{\text{Fa},\text{Mo}\}, A = \text{Old}, B = \text{Young}$. Then according to lhoperators 11, we receive $l(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young})) = \{(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young})),(\text{Fa},(\text{Old},\text{Young})),(\text{Mo},(\text{Old},\text{Young})),(\text{Fa},(\{\text{Ma},\text{Old}\},\text{Young})),(\text{Fa},(\text{Old},\{\text{Fe},\text{Young}\})),(\text{Fa},(\{\text{Ma},\text{Old}\},\{\text{Fe},\text{Young}\})),(\text{Mo},(\{\text{Fe},\text{Old}\},\text{Young})),(\text{Mo},(\text{Old},\{\text{Ma},\text{Young}\})),(\text{Mo},(\{\text{Fe},\text{Old}\},\{\text{Ma},\text{Young}\}))\}$. Similarly, we obtain $h(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young})) = \{(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Yo}$ $\text{Mo}\},(\varnothing,\varnothing)),(\{\text{Fa},\text{Mo}\},(\text{Old},\varnothing)),(\{\text{Fa},\text{Mo},\text{So}\},(\varnothing,\varnothing)),(\{\text{Fa},\text{Mo},\text{Da}\},(\varnothing,\varnothing)),(\{\text{Fa},\text{Mo},\text{So},\text{Da}\},(\varnothing,\varnothing))\}$. Owing to definitions, $L(X,(A,B)) = (\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young})), H(X,(A,$ $(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young}))$.

1. **Theorem 3** *Let $\mathbb{K} = (G,M,R)$ be a formal context. Then the following statements are hold:*

(1)$L(X,(A,B))\leqslant_2(X,(A,B))\leqslant_2 H(X,(A,B))$

(2)$(X,(A,B))\in 3WPC\Leftrightarrow L(X,(A,B)) = (X,(A,B))$

(3)$(X,(A,B))\in 3WPC\Leftrightarrow H(X,(A,B)) = (X,(A,B))$

Proof We proof LHtheorem 3 (1)-(3) step by step:
(1) Owing to lhoperators 11, we obtain $\bigcup_{i\in I}X_i\subseteq X\subseteq\bigcap_{i\in I}X_i$, $\bigcap_{i\in I}A_i\supseteq A\supseteq\bigcup_{i\in I}A_i$ and $\bigcap_{i\in I}B_i\supseteq B\supseteq\bigcup_{i\in I}B_i$.

(2) For the forward implication. Since $(X,(A,B))\in 3WPC$, $(X,(A,B))\in l(X,(A,B))$ holds. But $L(X,(A,B)) = \bigvee l(X,(A,B))$, we get $x\leqslant_2(X,(A,B))$ with the help of $\forall x\in l(X,(A,B))$, which deciphers $L(X,(A,B)) = (X,(A,B))$.

For the backward implication. From the $L(X,(A,B)) = (X,(A,B))$ that we know $(\bigcup_{i\in I}X_i,(\bigcap_{i\in I}A_i,\bigcap_{i\in I}B_i)) = (X,(A,B))$. Whence, $(X,(A,B))\in 3WPC$.

(3) For the forward implication. Since $(X,(A,B))\in 3WPC$, $(X,(A,B))\in h(X,(A,B))$ holds. According to $H(X,(A,B)) = \bigwedge h(X,(A,B))$, we receive $x\geqslant_2(X,(A,B)),\forall x\in h(X,(A,B))$. It illustrates $H(X,(A,B)) = (X,(A,B))$.

For the backward implication. From the $H(X,(A,B)) = (X,(A,B))$ that we know $(\bigcap_{i\in I}X_i,(\bigcup_{i\in I}A_i,\bigcup_{i\in I}B_i)) = (X,(A,B))$. Whence, $(X,(A,B))\in 3WPC$.

Example 6. Let X,A,B be in lhoex 5, then we receive $L(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young}))\leqslant_2(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young}))\leqslant_2 H(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young}))$. And $(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young}))\in 3WPC\Leftrightarrow L(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young})) = (\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young}))$. Similarly, $(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young}))\in 3WPC\Leftrightarrow H(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Yo}$ $(\{\text{Fa},\text{Mo}\},(\text{Old},\text{Young}))$ holds.

4.2 Set Equivalence Relation Approximation Operators

Definition 12. Let $\mathbb{K} = (G, M, R)$ be a formal context. The operators $\underline{r}, \overline{r}, \underline{R}, \overline{R}$ are defined respectively as follows:

$$\underline{r}(X,(A,B)) = \{([x]_R,(C,D)) \mid [x]_R \cap X \neq \varnothing, C \supseteq A, D \supseteq B, \forall([x]_R,(C,D)) \in 3WPC\}$$

$$\overline{r}(X,(A,B)) = \{([x]_R,(C,D)) \mid [x]_R \cap X \neq \varnothing, C \subseteq A, D \subseteq B, \forall([x]_R,(C,D)) \in 3WPC\}$$

$$\underline{R}(X,(A,B)) = (\bigcup_{i\in I}[x]_R \cap X, (\bigcap_{i\in I} C, \bigcap_{i\in I} D))$$

$$\overline{R}(X,(A,B)) = (\bigcup_{i\in I}[x]_R \cap X, (\bigcup_{i\in I} C, \bigcup_{i\in I} D))$$

$i \in I$, I is an index set. $[x]_R$ denotes $R-$ equivalence class, xRy if and only if $x^* = y^*$ and $x^{\overline{*}} = y^{\overline{*}}$.

Remark 4. Speaking universally, binary relation R is an equivalence relation since it satisfies following statements:

[(1)]$x^* = x^*$ and $x^{\overline{*}} = x^{\overline{*}}$, therefore, xRx holds.

If xRy, we get $x^* = y^*$ and $x^{\overline{*}} = y^{\overline{*}}$. Thus, xRy and yRx are one and the same thing.

If xRy and yRz, similar to above item, we receive xRz.

In general terms, (1)(2)(3) illustrate R is an equivalence relation.

Example 7. Let $\mathbb{K}_1$ be in example1 1. If we assume $X = \{\text{Fa}\}$, $A = \{\text{Ma}, \text{Old}\}$ and $B = \{\text{Fe}, \text{Young}\}$. Thus, $x = \{\text{Fa}\}$ and C can be the set $\{\text{Ma}, \text{Old}\}, \{\text{Ma}, \text{Fe}, \text{Old}\}, \{\text{Ma}, \text{Old}, \text{Young}\}, \{\text{Ma}, \text{Fe}, \text{Old}, \text{Young}\}$. Similarly, D can be the set $\{\text{Fe}, \text{Young}\}, \{\text{Fe}, \text{Ma}, \text{Young}\}, \{\text{Fe}, \text{Young}, \text{Old}\}, \{\text{Fe}, \text{Ma}, \text{Young}, \text{Old}\}$. Therefore, $\underline{r}(X,(A,B)) = (\text{Fa}, (\{\text{Ma}, \text{Old}\}, \{\text{Fe}, \text{Young}\}))$. And if we consider $\overline{r}(X,(A,B))$, C can be the set $\varnothing, \{\text{Ma}\}, \{\text{Old}\}, \{\text{Ma}, \text{Old}\}$, D can be the set $\varnothing, \{\text{Fe}\}, \{\text{Young}\}, \{\text{Fe}, \text{Young}\}$. So that $\overline{r}(X,(A,B)) = \{(\text{Fa},(\varnothing,\varnothing)), (\text{Fa},(\varnothing,\text{Fe})), (\text{Fa},(\varnothing,\text{Young})), (\text{Fa},(\varnothing,\{\text{Fe}, \text{Young}\})), (\text{Fa},(\text{Ma},\varnothing)), (\text{Fa},(\text{Ma},\text{Fe})), (\text{Fa},(\text{Ma},\text{Young})), (\text{Fa},(\text{Ma},\{\text{Fe},\text{Young}\})), (\text{Fa},(\text{Old},\varnothing)), (\text{Fa},(\text{Old},\text{Fe})), (\text{Fa},(\text{Old},\text{Young})), (\text{Fa},(\text{Old},\{\text{Fe},\text{Young}\})), (\text{Fa},(\{\text{Ma},\text{Old}\},\varnothing)), (\text{Fa},(\{\text{Ma},\text{Old}\},\text{Fe})), (\text{Fa},(\{\text{Ma},\text{Old}\},\text{Young})), (\text{Fa},(\{\text{Ma},\text{Old}\},\{\text{Fe},\text{Young}\}))\}$.

Lemma 4. Let $\mathbb{K} = (G, M, R)$ be a formal context. Then $\forall X \subseteq G, (X,(\varnothing,\varnothing)) \in 3WPC$

3. *Proof* Since $X \subseteq \varnothing^* \cap \varnothing^{\overline{*}} = G$ holds, we confirm correct of feikong 4.

Example 8. Let $\mathbb{K} = (G, M, R)$ be a formal context. If $X = \{\text{Fa}, \text{So}\}$, then we obtain $(\{\text{Fa}, \text{So}\}, (\varnothing, \varnothing)) \in 3WPC$.

Lemma 5. Let $\mathbb{K} = (G, M, R)$ be a formal context. Then $\overline{R}(X,(A,B)) \in 3WPC$.

Proof We only need to proof $\bigcup_{i\in I}[x]_R \cap X \subseteq (\bigcap_{i\in I} C_i)^* \cap (\bigcap_{i\in I} D_i)^{\overline{*}}$. Since $\forall C_i, D_i$, $C_i \subseteq (\bigcap_{i\in I} C_i)^*$ and $D_i \subseteq (\bigcap_{i\in I} D_i)^{\overline{*}}$ hold. Therefore, $\forall [x]_R \cap X \subseteq C_i^* \cap D_j^{\overline{*}} (\exists i, j) \subseteq (\bigcap_{i\in I} C_i)^* \cap (\bigcap_{i\in I} D_i)^{\overline{*}}$. Whence, $\bigcup_{i\in I}[x]_R \cap X \subseteq (\bigcap_{i\in I} C_i)^* \cap (\bigcap_{i\in I} D_i)^{\overline{*}}$

Example 9. Let X, A, B be in suanziex 7, we will receive $\overline{R}(X,(A,B)) \in 3WPC$.

Remark 5. In case that $x = \varnothing$, which means sample is empty, we do not think it makes sense. So in all of the following discussions, we are going to say that X is not an empty set. Furthermore, $\overline{r}(X,(A,B))$ is well-defined since feikong 4. But $\underline{r}(X,(A,B))$ could be an empty set since following example:

Example 10. Let X, A, B be Fa, $\{\text{Ma}, \text{Old}\}, \{\text{Ma}, \text{Young}\}$, respectively. We obtain $\underline{r}(\text{Fa}, (\{\text{Ma}, \text{Old}\}, \{\text{Ma}, \text{Young}\})) = \varnothing$ since $(\text{Fa}, (\{\text{Ma}, \text{Old}\}, \{\text{Ma}, \text{Young}\})) \notin 3WPC$.

Speaking universally, we have following theorem to determine whether $\underline{r}(X,(A,B))$ is an empty set.

Theorem 4 *Let $\mathbb{K} = (G, M, R)$ be a formal context. Then the following statement holds:*

$$\underline{r}(X,(A,B)) = \varnothing \Leftrightarrow ([x]_R,(A,B)) \notin 3WPC, \forall x \in X$$
$$\Leftrightarrow (x,(A,B)) \notin 3WPC, \forall x \in X.$$

Proof Firstly, we illustrate $([x]_R,(A,B)) \notin 3WPC \Leftrightarrow (X,(A,B)) \notin 3WPC, \forall x \in X$. For the forward implication, if exists $(x_0,(A,B)) \in 3WPC$, we get $x_0 \subseteq A^* \cap B^{\overline{*}}$, whence, $x_0 \subseteq A^*$ and $x_0 \subseteq B^{\overline{*}}$ hold. Therefore, we have $[x]_R^* \supseteq A$ and $[x]_R^{\overline{*}} \supseteq B$ since $x_0^* \supseteq A^{**} \supseteq A, x_0^{\overline{*}} \supseteq B^{\overline{**}} \supseteq B$ and $[x_0]_R^* = x_0^*, [x_0]_R^{\overline{*}} = x_0^{\overline{*}}$. And we receive $[x_0]_R \subseteq [x_0]_R^{**} \subseteq A^*$ and $[x_0]_R \subseteq [x_0]_R^{\overline{**}} \subseteq B^{\overline{*}}$, which means $[x_0]_R \subseteq A^* \cap B^{\overline{*}}$, in which case $([x_0]_R,(A,B)) \in 3WPC$. For the backward implication, if exists $([x_0]_R,(A,B)) \in 3WPC$, then we obtain $x_0 \subseteq [x_0]_R \subseteq A^* \cap B^{\overline{*}}$, in which case it contradicts the given condition.

Secondly, we decipher $\underline{r}(X,(A,B)) = \varnothing \Leftrightarrow (x,(A,B)) \notin 3WPC, \forall x \in X$. For the forward implication, if exists $x_0 \in X$ that satisfies $(x_0,(A,B)) \in 3WPC$, then we can get $(x_0,(A,B)) \in \underline{r}(X,(A,B))$ by definition of $\underline{r}$. Therefore, it is contradictory to condition $\underline{r}(X,(A,B)) = \varnothing$. For the backward implication, if $\underline{r}(X,(A,B)) \neq \varnothing$, which means there is some $x_0 \in X, C \supseteq A, D \supseteq B$ that makes $([x_0]_R,(C,D)) \in 3WPC$. Thus, $[x_0]_R \subseteq C^* \cap D^{\overline{*}} \subseteq A^* \cap B^{\overline{*}}$ which means $(x_0,(A,B)) \in 3WPC$.

Theorem 5 *Let $\mathbb{K} = (G, M, R)$ be a formal context. Then the following statements are hold:*

$$\underline{r}(X,(A,B)) \neq \varnothing \Leftrightarrow (x_0,(A,B)) \in 3WPC, \exists x_0 \in X$$
$$\underline{r}(X,(A,B)) \neq \varnothing \Leftrightarrow ([x_0]_R,(A,B)) \in 3WPC, \exists x_0 \in X$$

Proof Similar to equivalencetheorem 4, we can obtain correct of equivalenceprop 5.

Example 11. Let $\mathbb{K}_1$ be in example1 1. If $X = \text{Fa}, A = \{\text{Ma}, \text{Old}\}, B = \{\text{Fe}, \text{Young}\}$, according to equivalence-prop 5, we know $\underline{r}(X,(A,B)) \neq \varnothing$ since $(\text{Fa}, (\{\text{Ma}, \text{Old}\}, \{\text{Fe}, \text{Young}\})) \in 3WPC$.

Theorem 6 *Let $\mathbb{K} = (G, M, R)$ be a formal context. If $(X,(A,B)) \in 3WPC$, then $\underline{R}(X,(A,B)) = \overline{R}(X,(A,B)) = (X,(A,B))$.*

Proof Since $(X,(A,B)) \in 3WPC$, $X \subseteq A^* \cap B^{\overline{*}}$ holds. Therefore, $\forall x \in X, x \subseteq A^* \cap B^{\overline{*}}$ induce $[x]^*_R = x^* \supseteq A^{**} \supseteq A$, which means $[x]_R \subseteq [x]^{**}_R \subseteq A^*$. Similarly, we have $[x]_R \subseteq B^{\overline{*}}$. Speaking universally, $([x]_R,(A,B)) \in 3WPC$ since $[x]_R \subseteq A^* \cap B^{\overline{*}}$ holds, and we receive $([x]_R,(A,B)) \in \overline{r}, \underline{r}$. $\forall([x]_R,(C,D)) \in \underline{r}(X,(A,B)), C \supseteq A$ and $D \supseteq B$. Thus, owing to $\bigcap_{i\in I} C = A, \bigcap_{i\in I} D = B$ and $\bigcup_{i\in I}[x]_R \cap X = X$, $\underline{R}(X,(A,B)) = (X,(A,B))$ is correct. On the other hand, $C \subseteq A, D \subseteq B$ since $\forall([x]_R,(C,D)) \in \overline{r}(X,(A,B))$ holds. Whence, $\bigcup_{i\in I} C = A, \bigcup_{i\in I} D = B$ and $\bigcup_{i\in I}[x]_R \cap X = X$ hold. In general terms, $\overline{R}(X,(A,B)) = (X,(A,B))$ is correct.

Example 12. Let X, A, B be in suanziex 7. According to rRdef 12, $\underline{R}(X,(A,B)) = (\text{Fa}, (\{\text{Ma}, \text{Old}\}, \{\text{Fe}, \text{Young}\}))$ and $\overline{R}(X,(A,B)) = (\text{Fa}, (\{\text{Ma}, \text{Old}\}, \{\text{Fe}, \text{Young}\}))$. Thus, owing to zhudingli1 6, $(X,(A,B)) \in 3WPC \Rightarrow \overline{R}(X,(A,B)) = \underline{R}(X,(A,B)) = (X,(A,B))$.

Theorem 7 *Let $\mathbb{K} = (G, M, R)$ be a formal context. If $\underline{r}(X,(A,B)) \neq \varnothing, \underline{R}(X,(A,B)) = (X,(A,B))$, then $(X,(A,B)) \in 3WPC$ is correct.*

Proof Owing to Rprop 5, we receive $\underline{R}(X,(A,B)) \in 3WPC$. But $\underline{R}(X,(A,B)) = (X,(A,B))$, therefore, $(X,(A,B)) \in 3WPC$ is correct.

Example 13. Let X, A, B be in suanziex 7. According to rRdef 12, $\underline{R}(X,(A,B)) = (\text{Fa}, (\{\text{Ma}, \text{Old}\}, \{\text{Fe}, \text{Young}\})) = (X,(A,B))$. Therefore, owing to zhudingli2 7, $\underline{R}(X,(A,B)) = (X,(A,B)) \Rightarrow (X,(A,B)) \in 3WPC$.

Those bevy of results completely characterizes the following proposition, and omitting the proof.

Proposition 1 *Let $\mathbb{K} = (G, M, R)$ be a formal context. If $\underline{r}(X,(A,B)) \neq \varnothing$, then the following statement is hold:*

$$\underline{R}(X,(A,B)) = \overline{R}(X,(A,B)) = (X,(A,B))$$
$$\Leftrightarrow (X,(A,B)) \in 3WPC$$

Example 14. Let X, A, B be in suanziex 7. According to rRdef 12, $\underline{R}(X,(A,B)) = (\text{Fa}, (\{\text{Ma}, \text{Old}\}, \{\text{Fe}, \text{Young}\}))$ and $\overline{R}(X,(A,B)) = (\text{Fa}, (\{\text{Ma}, \text{Old}\}, \{\text{Fe}, \text{Young}\}))$. Thus, owing to zhudingli1 6, $\overline{R}(X,(A,B)) = \underline{R}(X,(A,B)) = (X,(A,B)) \Leftrightarrow (X,(A,B)) \in 3WPC$.

Remark 6. If you need to stay in a hotel in real life, you may need to consider many factors, including the distance, the price, the size of the room and so on. Some factors need to be satisfied, while others need not be satisfied. However, it is difficult to satisfy or not satisfy these factors at the same time in practice, so the most important factors need to be selected for consideration, which makes the object and the object satisfying the attribute are not equal, but included in the relationship. Therefore, the upper approximation operator and the lower approximation operator play a very important role in the practical application. We do not need to require an accurate preconcept, but only need to work out the upper approximation operator or the lower approximation operator according to the actual demand, and select the suitable object from the set that satisfies.

5 Conclusion

In order to study semiconcept or formal concept in the context of given information, we introduce 3WPC, since either a semiconcept or a formal concept can be viewed as being generated by a preconcept. In a formal context $\mathbb{K} = (G, M, R)$, 3WPC is the combining of three-way decisions and preconcept. After that, we attain $(\mathfrak{R}(\mathbb{K}), \sqcap, \sqcup, \neg, \lrcorner, \vee, \wedge)$ is generalized double Boolean algebra, which is weaker than semiconcept. Besides, we construct two forms of approximation operators, approximation operators from lattice and set equivalence relation approximation operators respectively, which can characterize $\mathfrak{R}(\mathbb{K})$. In nature, the similarities between two species should be considered not only in terms of what they have in common but in combination with what they do not. This can reduce the probability of misjudgment. Therefore, 3WPC which combines three-way decisions is better than preconcept. However, 3WPC makes the actual search process cumbersome while obtaining more information. How to find a quick and efficient algorithm to generate all 3WPC is the first thing we need to do. How to apply in more practical contexts, such as the context of incomplete information also requires more discussion. In the future, we hope to attain accuracy measures and other properties in 3WPC. Furthermore, we will examine the preconcept in the context of incomplete information and some of its properties.

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Compliance with ethical standards

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