Integrable Lattice Equations and Discrete Painlevé Equations

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II Similarity Reduction to Discrete Painlevé Equations and Garnier Systems.
(collaborations with V. Papageorgiou, A. Walker and A. Tongas)

What are discrete Painlevé Equations?

The common consensus (not a strict definition!) is that they are *second order (first degree) non-autonomous ordinary difference equations which are "integrable" in some specific sense*.

"difference" can have several meanings:

- *finite-difference*, in the sense of a recurrence relation, or as a dynamical map of iterates, or as a birational transformation of algerbraic varieties;
- *analytic difference*, in the sense of a delay-type equation for functions of continuous variable, or even as a functional relation.

"Integrable" can also have several meanings:

- an algorithmic property that is believed to be the discrete analogue of the Painlevé property, such as singularity confinement, limited growth criteria, criteria in terms of algebraic entropy or Nevanlinna order, etc. ;
- structural properties, such as the existence of an isomonodromic deformation problem (Lax pair), existence of Bäcklund-Schlesinger transformations;
- the existence of an exact solution scheme, such as inverse problem associated with the isomonodromic deformation scheme, or Riemann-Hilbert scheme, and/or the connection with the resolution problem of singularities of the spaces of initial values.

Origins of dPs:

Discrete Painlevé eqations; origins:

dPs have emerged from various approaches:

♯ semiclassical orthogonal polynomials (J A Shohat, 1936, G Freud, 1976);
♯ simplicial gravity and random matrix models (Bessis et al. , 1980, Fokas, Its& Kitaev, 1991);
♯ similarity reduction on the lattice (FWN, V Papageorgiou, 1991);
♯ Bäcklund-Schlesinger transformations of continuous P eqs (Grammaticos, Ramani & Fokas, 1991);
♯ Singularity confinement (V Papageorgiou et al.; Ramani, Grammaticos & Hietarinta, 1991);
♯ Affine Weyl groups (Noumi & Yamada, 1998);
♯ Algebraic geometry of rational surfaces (Sakai, 1999).

Most of the pre-1990 papers dealt with dPI, the first "new" example (dPII) came independently from the similarity reduction (FWN, Papageorgiou), approach and unitary matrix models (Perival & Shevitz).

Recently the Garnier systems revived in interest (K. Okamoto since the 1980s, B. Dubrovin and M. Mazzocco more recently). The first discrete version of a higher Garnier system, through similarity reduction, was presented on Islay (A. Walker & FWN, 1999). Very recently H. Sakai constructed a q-deformation of the Garnier system.

Some of the discrete Painlevé eqs. (dP's):

$$\text{dPI}: \quad x_{n+1}+x_n+x_{n-1}=\frac{\zeta_n}{x_n}+a$$

$$\text{dPII}: \quad x_{n+1}+x_{n-1}=\frac{\zeta_n x_n+a}{1-x_n^2}$$

$$\text{qPIII}: \quad x_{n+1}x_{n-1}=\frac{(x_n+a)(x_n+b)}{(cq^n x_n+1)(dq^n x_n+1)}$$

$$\text{dPIV}: \quad (x_{n+1}+x_n)(x_n+x_{n-1})=\frac{(x_n^2-a^2)(x_n-b^2)}{(x_n+\zeta_n)^2-c^2}$$

$$\text{qPV}: \quad (x_{n+1}x_n-1)(x_nx_{n-1}-1)=\eta_n\theta_n\frac{(x_n+a)(x_n+a^{-1})(x_n+b)(x_n+b^{-1})}{(x_n+\eta_n)(x_n+\theta_n)}$$

$$\text{qPVI}: \frac{(x_{n+1}x_n-\eta_n\eta_{n+1})(x_nx_{n-1}-\eta_n\eta_{n-1})}{(x_{n+1}x_n-1)(x_nx_{n-1}-1)}=\frac{(x_n-\eta_n a)(x_n-\eta_n/a)(x_n-\eta_n b)(x_n-\eta_n/b}{(x_n-c)(x_n-1/c)(x_n-d)(x_n-1/d)}$$

(where $\zeta_n=\alpha n+\beta,\ \eta_n=\gamma q^n,\ \theta_n=\delta q^n$, α,β,γ,δ,a,b and c being constants).

[1] Shohat (1936); [2] Bessis (1979); [3] FWN & V. Papageorgiou, (1990); [4] Periwal & Shevitz, (1990); [5] Ramani, Grammaticos & Hietarinta, (1991); [6] Grammaticos & Ramani, (1999).

Similar coalescence pattern as for continuous Painlevé eqs:

$$dPVI \rightarrow dPV \begin{array}{c}\nearrow \ \text{dPIII} \ \searrow \\ \\ \searrow \ \text{dPIV} \ \nearrow\end{array} dPII \rightarrow dPI$$

"Asymmetric" discrete Painlevé equations:

$$\text{asym}-\text{dPII}: \quad \begin{cases} x_{n+1}+x_n &= \dfrac{(an+b)y_n+c}{1-y_n^2} \ , \\[2ex] y_n+y_{n-1} &= \dfrac{(an+b)x_n+c}{1-x_n^2} \ , \end{cases}$$

$$\text{asym}-\text{dPIII}: \quad \begin{cases} x_{n+1}x_n &= \dfrac{(y_n-q^n a)(y_n-q^n b)}{(y_n-c)(y_n-d)} \ , \\[2ex] y_n y_{n-1} &= \dfrac{(x_n-q^n\alpha)(x_n-q^n\beta)}{(x_n-\gamma)(x_n-\delta)} \ , \end{cases}$$

$$\text{with} \quad \frac{\alpha\beta}{\gamma\delta} = q\frac{ab}{cd} \ ,$$

Alternative versions of dP's:

$$\text{alt}-\text{dPI}: \quad \frac{n+1}{x_{n+1}+x_n}+\frac{n}{x_n+x_{n-1}} = n+a+bx_n^2$$

$$\text{alt}-\text{dPII}: \quad \frac{n+1}{x_{n+1}x_n+1}+\frac{n}{x_n x_{n-1}+1} = n+a+b\left(x_n-\frac{1}{x_n}\right)$$

[6] Jimbo & Miwa (1982); [7] Fokas, Grammaticos, Ramani (1991); [8] Jimbo & Sakai (1996).

Sakai's Classification

H. Sakai (1999) achieved a classification of *discrete as well as continuous* Painlevé equations based on the theory of rational surfaces, associating the singularity structure of the spaces of initial conditions with *affine Weyl groups*.

Key idea: singularity confinement can be recognised as the mechanism of blowing-up/blowing-down of singularities.

In the most generic case he found an elliptic discrete Painlevé equation (with $W(E_8^{(1)})$-symmetry). In projective coordinates this equation reads:

$$[X:Y:Z] \;\mapsto\; [\widetilde{X}:\widetilde{Y}:\widetilde{Z}] = P_{457}\circ P_{789}\circ P_{456}\circ P_{123}([X:Y:Z])$$

in which thee P_{ijk} are quadratic maps involving an elliptic dependence on parameters θ_i,θ_j,θ_k from a collection of 9 parameters θ_1,. . . ,θ_9, and where the action on the parameters is according to shifts induced by the Weyl group element:

$$w_3\circ w_2\circ w_4\circ w_3\circ w_1\circ w_2\circ w_5\circ w_4\circ w_3\circ w_6\circ w_5\circ w_4\circ w_8\circ w_3\circ w_2\circ w_1\circ w_7\circ w_6\circ w_5$$
$$\circ w_4\circ w_3\circ w_2\circ w_0\circ w_7\circ w_6\circ w_5\circ w_4\circ w_3\circ w_8\circ w_3\circ w_4\circ w_5\circ w_6\circ w_7\circ w_0\circ w_2\circ w_3\circ w_4$$
$$\circ w_5\circ w_6\circ w_7\circ w_1\circ w_2\circ w_3\circ w_8\circ w_4\circ w_5\circ w_6\circ w_3\circ w_4\circ w_5\circ w_2\circ w_1\circ w_3\circ w_4\circ w_2\circ w_3\circ$$

The construction results from the blowing-up/blowing-down of 9 points in general position on a general cubic.

It was shown that most of the known discrete (and continuous) Painlevé equations arise as degeneracies/coalescences of this equation.

A description of the ell-dP was given by M. Noumi and his group, in terms of a complicated system of bilinear equations, and special solutions have been found by K. Kajiwara et al. in terms of modular hypergeometric functions.

Singularity Confinement of O△E's

Consider discrete PI:

$$x_{n+1} + x_n + x_{n-1} = \frac{a_n}{x_n} + b_n$$

and try to specify a_n, b_n s.t. singularities do not persist.

Consider initial data, s.t. $x_n = 0$. Then a singularities will occur according to the following pattern:

$$\begin{array}{ccccccccccc}
\mathrm{x}_{n-1} & & \mathrm{x}_{n} & & \mathrm{x}_{n+1} & & \mathrm{x}_{n+2} & & \mathrm{x}_{n+3} & & \mathrm{x}_{n+4} \\
\circ & \longrightarrow & \circ & \longrightarrow & \circ & \longrightarrow & \circ & \longrightarrow & \circ & \longrightarrow & \circ \\
\mathrm{f} & & 0 & & \infty & & \infty & & 0 & & ?
\end{array}$$

Iteration scheme taking $x_{n-1} = f$, $x_n = \varepsilon$ as initial data:

$$\begin{aligned}
x_{n+1} &= \frac{a_n}{\varepsilon} + b_n - f - \varepsilon \\
x_{n+2} &= -\frac{a_n}{\varepsilon} + b_{n+1} - b_n + f + \frac{a_{n+1}}{a_n}\varepsilon + \mathcal{O}(\varepsilon^2) \\
x_{n+3} &= b_{n+2} - b_{n+1} + \left(1 - \frac{a_{n+1}}{a_n} - \frac{a_{n+2}}{a_n}\right)\varepsilon + \mathcal{O}(\varepsilon^2) \\
x_{n+4} &= \frac{a_{n+3}}{x_{n+3}} + \frac{a_n}{\varepsilon} + b_{n+3} - b_{n+2} + b_n - f + \mathcal{O}(\varepsilon)
\end{aligned}$$

The only way to cancel the singular term (as $\varepsilon \to \infty$) is by imposing:

$b_{n+2} = b_{n+1}$, $a_{n+3} - a_{n+2} - a_{n+1} + a_n = 0$ $\Rightarrow$ $b_n = \beta =$ const., $a_n = \alpha n + \gamma(-1)^n$.

The singularity confinement criterium has been proven very powerful in finding new *non-autonomous* "integrable" OΔE's (i.e. discrete Painlevé equations), by de-autonomising known integrable autonomous mappings (in the QRT family).
(B. Grammaticos, A. Ramani and J. Hietarinta, 1991).

A starting point has been the QRT(R. Quispel, J. Roberts, C. Thompson) family of mappings of plane:

$$\widetilde{x} = \frac{f_1(y) - f_2(y)x}{f_2(y) - f_3(y)x} \quad , \quad \widetilde{y} = \frac{g_1(\widetilde{x}) - g_2(\widetilde{x})y}{g_2(\widetilde{x}) - g_3(\widetilde{x})y}$$

in which $f_1, f_2, f_3, g_1, g_2, g_3$ fourth-order polynomials.

The QRT mapping leaves invariant a parameter-family of biquadratic curves (in general position) foliating the plane:

$$(\alpha_0 + K\alpha_1)x^2y^2 + (\beta_0 + K\beta_1)x^2y + (\gamma_0 + K\gamma_1)x^2 + (\delta_0 + K\delta_1)xy^2 + (\epsilon_0 + K\epsilon_1)xy$$
$$+(\zeta_0 + K\zeta_1)x + (\kappa_0 + K\kappa_1)y^2 + (\lambda_0 + K\lambda_1)y + (\mu_0 + K\mu_1) = 0$$

Furthermore, this mapping is *measure-preserving*

In particular in the case of a symmetric biquadratic:

$$\widetilde{x} = \frac{f_1(x) - \underset{\sim}{x}\, f_2(x)}{f_2(x) - \underset{\sim}{x}\, f_3(x)} \ ,$$

What are Garnier systems?

We could consider them as higher order analogues of the Painlevé VI equation.

Painlevé VI

R. Fuchs (1905): isomonodromic deformation of the linear differential equation:

$$\frac{d^2y}{dx^2} = p(x)y \quad , \quad p(x) = \frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{(x-t)^2} + \frac{\alpha}{x} + \frac{\beta}{x-1} + \frac{\gamma}{x-t} + \frac{3}{4(x-\lambda)^2} + \frac{\varepsilon}{x-\lambda}$$

with regular singularities at $0, 1, t, \infty$ and apparent singularity at λ, $(\alpha+\beta+\gamma+\varepsilon = 0)$. t is assumed a moving singularity.

Fuchs investigated the conditions on $\lambda = \lambda(t)$ such that the monodromy of the differential eqn. is preserved $\Rightarrow$ $\quad y_t = Ay + By_x$,

coefficients A, B follow from the consistency of the overdetermined linear system (*Lax pair*).

This leads to the following nonlinear ODE for λ (Painlevé VI):

$$\frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}$$
$$+2\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[\kappa - (a+\frac{1}{4})\frac{t}{\lambda^2} + (b+\frac{1}{4})\frac{t-1}{(\lambda-1)^2} - c\frac{t(t-1)}{(\lambda-t)^2}\right]$$

Remark: One year later (1906) this equation was incorporated by M. Gambier in the famous list of the six Painlevé transcendental equations. The equation possesses the celebrated *Painlevé property*, i.e. the general solution is meromorphic (solution single-valued around all "moving" singularities, as a consequence of initial values).

(Isomonodromic) Garnier systems

R. Garnier (1912) generalised the derivation by R. Fuchs by extending the number of singularities. Thus, he investigated the isomonodromic deformation of the linear differential equation:

$$\frac{d^2y}{dx^2} = p(x)y \quad ,$$
$$p(x) = \sum_{l=1}^{n}\left[\frac{c_l}{(x-t_l)^2} + \frac{\alpha_l}{x(x-1)(x-t_l)}\right] + \frac{c_{n+1}}{x^2} + \frac{c_{n+2}}{(x-1)^2} + \frac{c_{n+3}}{x(x-1)}$$
$$+ \sum_{j=1}^{n}\left[\frac{3}{4(x-\lambda_j)^2} + \frac{\beta_j}{x(x-1)(x-\lambda_j)}\right] \quad .$$

There are $n+3$ regular singularities, namely at $0, 1, \infty$ and t_i, $(i = 1, \ldots, n)$, and there are "apparent" singularities at the λ_j, $(j = 1, \ldots, n)$. Considering the t_i as moving singularities by supplementing with a set of deformation equation:

$$\frac{\partial y}{\partial t_i} = A_i y + B_i \frac{\partial y}{\partial x} \quad , \quad (i = 1, \ldots, n) \ ,$$

Garnier derives the following overdetermined system of differential equations:

Garnier system:

$$\frac{\varphi'(t_i)(t_i-\lambda_j)}{\psi(t_i)}\frac{\partial\lambda_j}{\partial t_i}-\frac{\varphi'(t_k)(t_k-\lambda_j)}{\psi(t_k)}\frac{\partial\lambda_j}{\partial t_k}=\frac{t_i-t_k}{(\lambda_j-t_i)(\lambda_j-t_k)}\frac{\varphi(\lambda_j)}{\psi'(\lambda_j)}\ ,$$

$$\begin{aligned}\frac{\partial^2\lambda_j}{\partial t_i^2}=&\frac{1}{2}\left(\frac{\varphi'(\lambda_j)}{\varphi(\lambda_j)}-\frac{\psi''(\lambda_j)}{2\psi'(\lambda_j)}\right)\left(\frac{\partial\lambda_j}{\partial t_i}\right)^2-\left(\frac{\varphi''(t_i)}{2\varphi'(t_i)}-\frac{\psi'(t_i)}{\psi(t_i)}\right)\frac{\partial\lambda_j}{\partial t_i}\\&+\frac{1}{2}\sum_{\substack{l=1\\l\neq j}}^{n}\frac{\varphi(\lambda_j)\psi'(\lambda_l)(\lambda_l-t_i)^2}{\varphi(\lambda_l)\psi'(\lambda_j)(\lambda_j-t_i)^2(\lambda_j-\lambda_l)}\left(\frac{\partial\lambda_l}{\partial t_i}\right)^2\\&-\sum_{\substack{l=1\\l\neq j}}^{n}\frac{\lambda_j-t_i}{(\lambda_l-t_i)(\lambda_l-\lambda_j)}\frac{\partial\lambda_j}{\partial t_i}\frac{\partial\lambda_l}{\partial t_i}+2\frac{\psi^2(t_i)}{\varphi'^2(t_i)(\lambda_j-t_i)^2}\frac{\varphi(\lambda_j)}{\psi'(\lambda_j)}\\&\times\left[\sum_{k=1}^{n+3}\left(c_k+\frac{3}{4}\right)-2+\sum_{\substack{k=1\\k\neq i}}^{n+2}\frac{\varphi'(t_k)}{\psi(t_k)}\frac{c_k+\frac{1}{4}}{\lambda_j-t_k}+\frac{\varphi'(t_i)}{\psi(t_i)}\frac{c_i}{\lambda_j-t_i}\right]\ ,\end{aligned}$$

in which

$$\varphi(x)\equiv x(x-1)\prod_{l=1}^{n}(x-t_l)\quad,\quad\psi(x)\equiv\prod_{j=1}^{n}(x-\lambda_j)\ .$$

Lagrange structure

The system of Garnier ODEs for any chosen independent variable t_i (fixing i) has a natural Lagrange structure:

$$\mathfrak{L}_i = \frac{1}{2}\sum_{j=1}^{n}(\lambda_j - t_i)\frac{\varphi'(t_i)\psi'(\lambda_j)}{\varphi(\lambda_j)\psi(t_i)}\left(\frac{\partial\lambda_j}{\partial t_i}\right)^2 +$$

$$+2\left[\left(\sum_{l=1}^{n+3}\left(c_l+\frac{3}{4}\right)-2\right)\frac{\psi(t_i)}{\varphi'(t_i)} - \sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n+2}\left(c_l+\frac{1}{4}\right)\frac{\varphi'(t_l)\psi(t_i)}{\psi(t_l)\varphi'(t_i)(\lambda_j-t_i)} - \sum_{j=1}^{n}\frac{c_i}{\lambda_j - t_i}\right] ,$$

where we set $t_{n+1} = 0$, $t_{n+2} = 1$, $t_{n+3} = \infty$, with the Euler Lagrange equations

$$\frac{\delta\mathfrak{L}_i}{\delta\lambda_j} = 0 \quad \text{with} \quad \mathfrak{L}_i = \mathfrak{L}_i(\lambda_1,\ldots,\lambda_n;\dot\lambda_1,\ldots,\dot\lambda_n;t_i) \quad , \quad \text{with} \quad \dot\lambda_j = \partial\lambda_j/\partial t_i \; .$$

In the case $n = 1$, setting $\lambda_1 = \lambda$, $t_1 = t$, we have the Lagrangian

$$\mathfrak{L}(\lambda,\dot\lambda,t) = \frac{1}{2}\frac{t(t-1)}{\lambda(\lambda-1)(t-\lambda)}\dot\lambda^2 + 2\left[\alpha\frac{t-\lambda}{t(t-1)} - \frac{\beta}{(t-1)\lambda} + \frac{\gamma}{t(\lambda-1)} - \frac{\delta}{\lambda - t}\right]$$

with the identifications: $\alpha = c_1 + c_2 + c_3 + c_4 + 1$, $\beta = c_2 + 1/4$, $\gamma = c_3 + 1/4$, $\delta = c_1$. The PDE aspect of the Garnier system: the Lagrangian $\mathfrak{L}$ obey the following closure relation

$$\frac{\partial\mathfrak{L}_i}{\partial t_j} = \frac{\partial\mathfrak{L}_j}{\partial t_i} \quad , \quad \forall i,j = 1,\ldots,n \; .$$

The Garnier system and its Hamiltonian structures were investigated by K. Okamoto(and collaborators) in the 1980s.

Similarity reduction

M.J. Ablowitz and H. Segur discovered in 1976 that Painlevé transcendents arise from symmetry reductions of integrable nonlinear evolution equations (soliton equations).

Example MKdV equation:

$$\bar{v}_t = \bar{v}_{xxx} - 6\bar{v}^2\bar{v}_x$$

MKdV is invariant under scaling:

$$x \mapsto \rho x \quad , \quad t \mapsto \rho^3 t \quad , \quad \bar{v} \mapsto \rho^{-1}\bar{v}$$

and scaling-invariant solutions can be ontained by setting

$$\bar{v}(x,t) = t^{-1/3}V(\xi) \quad , \quad \xi = \frac{x}{t^{1/3}} \quad \text{(similarity variable).}$$

Inserting this form for $\bar{v}$ into the MKdV we obtain an ODE (Painlevé II):

$$V'' + \frac{1}{3}\xi V - 2V^3 = \bar{\mu} \qquad , \qquad \bar{\mu} \quad \text{integration constant.}$$

Alternatively, we can consider the "potential" MKdV eq.:

$$v_t = v_{xxx} - 3\frac{v_x v_{xx}}{v}$$

where $\bar{v} = \partial_x \ln v$, and impose on the equation a compatible constraint:

$$\mu v = x v_x + 3t v_t \qquad \Rightarrow \qquad \bar{v} + x\bar{v}_x + 3t\bar{v}_t = 0$$

which we call *similarity constraint*, and obtain PII in the form:

$$\bar{v}_{xx} = 2\bar{v}^3 - \frac{x}{3t}\bar{v} + \frac{\mu}{3t} \ ,$$

without having to integrate.

Main examples of quadrilateral lattices to be considered here:

1. Lattice Modified Korteweg-de Vries (MKdV) equation:

$$p(v\widehat{v} - \widetilde{v}\widehat{\widetilde{v}}) = q(v\widetilde{v} - \widehat{v}\widehat{\widetilde{v}})$$

where v is scalar;

2. Lattice Modified Boussinesq (MBSQ) system:

$$\frac{p\widehat{v} - q\widetilde{v}}{\widehat{\widetilde{v}}} = \frac{p\widetilde{w} - q\widehat{w}}{w} = \frac{p\widetilde{v}\widehat{w} - q\widehat{v}\widetilde{w}}{v\widehat{\widetilde{w}}}$$

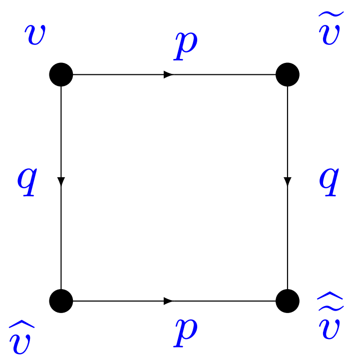
which is a 2-component system in v, w.

Recall the notation:

$$v := v_{n,m} \quad , \quad \widetilde{v} = v_{n+1,m}$$

$$\widehat{v} := v_{n,m+1} \quad , \quad \widehat{\widetilde{v}} = v_{n+1,m+1}$$

Schematically:



Here p, q are *lattice parameters*

lattice shifts: $v \xrightarrow{p} \widetilde{v}$
$v \xrightarrow{q} \widehat{v}$

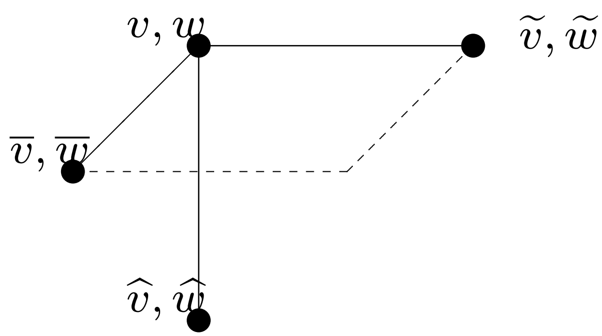
Both systems obey the *multidimensional consistency property* explained in Part I.

Integrability of the lattice BSQ system

The MBSQ system can be rewritten as a coupled system:

$$\frac{p\widehat{v}-q\widetilde{v}}{\widehat{\widetilde{v}}}=\frac{p\widetilde{w}-q\widehat{w}}{w}=\frac{p\widetilde{v}\widehat{w}-q\widehat{v}\widetilde{w}}{v\widehat{\widetilde{w}}} \quad\Leftrightarrow\quad \frac{p\widetilde{v}w+qv\widehat{\widetilde{w}}}{\widehat{\widetilde{v}}\widetilde{w}}=\frac{q\widehat{v}w+pv\widehat{\widetilde{w}}}{\widehat{\widetilde{v}}\widehat{w}}=\frac{p^2\widetilde{v}-q^2\widehat{v}}{p\widehat{v}-q\widetilde{v}}$$

In terms of this system the compatibility is easily checked imposing on each vertex initial data for both v and w.



$$\widehat{\widetilde{\overline{v}}}=v\frac{p^2q\overline{v}\widetilde{w}+q^2r\widetilde{v}\widehat{w}+r^2p\widehat{v}\overline{w}-q^2p\overline{v}\widehat{w}-r^2q\widetilde{v}\overline{w}-p^2r\widehat{v}\widetilde{w}}{p^2q\widetilde{v}\widehat{w}+q^2r\widehat{v}\overline{w}+r^2p\overline{v}\widetilde{w}-q^2p\widehat{v}\widetilde{w}-r^2q\overline{v}\widehat{w}-p^2r\widetilde{v}\overline{w}}$$

$$\widehat{\widetilde{\overline{w}}}=w\frac{p^2q\widehat{v}\overline{w}+q^2r\overline{v}\widetilde{w}+r^2p\widetilde{v}\widehat{w}-q^2p\widetilde{v}\overline{w}-r^2q\widehat{v}\widetilde{w}-p^2r\overline{v}\widehat{w}}{p^2q\widetilde{v}\widehat{w}+q^2r\widehat{v}\overline{w}+r^2p\overline{v}\widetilde{w}-q^2p\widehat{v}\widetilde{w}-r^2q\overline{v}\widehat{w}-p^2r\widetilde{v}\overline{w}}$$

which is independent of the way in which it was calculated.

Lax pair for the lattice MBSQ system

Derived in the same way as before (i.e. by exploiting the consistency of the system on the multi-dimensional lattice), but now in terms of two variables. By writing the equations in terms of a third "virtual" direction

$$\frac{p\overline{v} - k\widetilde{v}}{\widetilde{\overline{v}}} = \frac{p\widetilde{w} - k\overline{w}}{w} = \frac{p\widetilde{v}\overline{w} - k\overline{v}\widetilde{w}}{v\widetilde{\overline{w}}}$$

solving for $\widetilde{\overline{v}}$ and $\widetilde{\overline{w}}$ and setting:

$$\overline{v} = \frac{f}{h} \quad , \quad \overline{w} = \frac{g}{h}$$

we obtain the Lax form:

$$\widetilde{\boldsymbol{\phi}} = L\boldsymbol{\phi} = \begin{pmatrix} p & 0 & -k\widetilde{v} \\ -k\widetilde{w}/v & p\widetilde{v}/v & 0 \\ 0 & -k/w & p\widetilde{w}/w \end{pmatrix} \boldsymbol{\phi}$$

with $\boldsymbol{\phi} = (f, g, h)^T$. Changing $p \to q$, $\widetilde{\cdot} \to \widehat{\cdot}$ we get obviously the other part M of the Lax pair.

Easy to verify that the compatibility

$$\widehat{L}M = \widetilde{M}L \quad \Rightarrow \quad \text{lattice MBSQ}$$

Interplay Discrete $\longleftrightarrow$ Continuous

The lattice systems we consider here admit a role reversal:
lattice *parameters* $p, q, \dots$ $\leftrightarrow$ lattice *variables* $n, m, \dots$

D∆E: Differential-difference MKdV equation

$$-p\frac{\partial}{\partial p}\ln v = n\frac{\widetilde{v}-\underset{\sim}{v}}{\widetilde{v}+\underset{\sim}{v}} := na \quad , \quad -q\frac{\partial}{\partial q}\ln v = m\frac{\widehat{v}-\underset{\frown}{v}}{\widehat{v}+\underset{\frown}{v}} := mb$$

Here shifts $v \to \underset{\sim}{v}$ and $v \to \underset{\frown}{v}$ are the reverse to the shifts $v \to \widetilde{v}$ and $v \to \widehat{v}$ respectively.

The D∆Es can be simultaneously imposed on the same function:

$$v = v_{n,m,h,\dots}(p, q, r, \dots)$$

i.e. it also obeys the lattice MKdV:

P∆E: Partial difference MKdV equation

$$p\left(v\widehat{v} - \widetilde{v}\widehat{\widetilde{v}}\right) = q\left(v\widetilde{v} - \widehat{v}\widehat{\widetilde{v}}\right)$$

Furthermore, we have in terms of the lattice parameters only a system of PDEs given by:

PDE: "Generating" PDE for the MKdV system:

$$\partial_s\partial_t \ln v = \frac{nm}{4st(s-t)}\left[t(1-a)(1+b)Y - s(1+a)(1-b)\frac{1}{Y}\right] \ ,$$

$$2st\partial_s\partial_t \ln Y = ns\partial_s\left[(1-Y)\frac{2tY+(s-tY)(1+a)}{(t-s)Y}\right]$$

$$-mt\partial_t\left[(1-Y)\frac{2s-(s-tY)(1+b)}{(t-s)Y}\right] .$$

where $t=p^2$, $s=q^2$. This coupled system, constructed in [FWN, A. Hone & N. Joshi,2000], constitutes a fourth order second degree PDE which encodes the entire hierarchy of MKdV equations.

Thus, we have *fully consistent system of equations* comprising three types of equations, all compatible discrete as well as continuous.

P△E ↔ D△E ↔ PDE

The consistency not only among itself but also compatible with the lattice MKdV equation:

$$\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial q}\right) = \frac{\partial}{\partial q}\left(\frac{\partial v}{\partial p}\right) \quad , \quad \widetilde{\left(\frac{\partial v}{\partial q}\right)} = \frac{\partial \widetilde{v}}{\partial q} \quad , \quad \widetilde{\left(\frac{\partial v}{\partial p}\right)} = \frac{\partial \widetilde{v}}{\partial p} \quad , \quad \ldots$$

Continuous Lax system MKdV

Consistency-around-the-cube allows one to obtain the discrete Lax pair from the lattice equation itself. From the DΔE in terms of lattice parameter we can achieve the same for the continuous part: DΔE:

$$-p\frac{\partial}{\partial p}\ln v = n\frac{\widetilde{v}-\underset{\sim}{v}}{\widetilde{v}+\underset{\sim}{v}} \quad \Rightarrow \quad -p\frac{\partial}{\partial p}\ln \overline{v} = n\frac{\widetilde{\overline{v}}-\underset{\sim}{\overline{v}}}{\widetilde{\overline{v}}+\underset{\sim}{\overline{v}}}$$

by applying an auxiliary lattice shift $v \to \overline{v}$ (associated with lattice parameter k) to the equation. From the lattice equation (in terms of $\widetilde{\ }$ and $\overline{\ }$) we can solve:

$$\widetilde{\overline{v}} = v\frac{p\overline{v}-k\widetilde{v}}{p\widetilde{v}-k\overline{v}} \quad , \quad \underset{\sim}{\overline{v}} = v\frac{k\underset{\sim}{v}+p\overline{v}}{p\underset{\sim}{v}+k\overline{v}}$$

and inserting this into the above differential equation for $\overline{v}$ we get:

$$-p\frac{\partial}{\partial p}\ln \overline{v} = n\frac{2pk(\overline{v}^2-\widetilde{v}\underset{\sim}{v})-(p^2+k^2)(\widetilde{v}-\underset{\sim}{v})\overline{v}}{(p^2-k^2)(\widetilde{v}+\underset{\sim}{v})\overline{v}}$$

which is a Riccati equation for $\overline{v}$. Thus, inserting again $\overline{v} = f/g$ and splitting the result into two linear equations we obtain:

$$p\frac{\partial}{\partial p}\begin{pmatrix} f \\ g \end{pmatrix} = \frac{n}{(p^2-k^2)(\widetilde{v}+\underset{\sim}{v})}\begin{pmatrix} (p^2+k^2)\widetilde{v}+(p^2-k^2)\underset{\sim}{v} & , & 2pk\widetilde{v}\underset{\sim}{v} \\ 2pk & , & 2p^2\underset{\sim}{v} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}$$

If we write this relation as $p\partial_p\phi = N\phi$, the DΔE follows from the consistency condition

$$p\frac{\partial}{\partial p}L = \widetilde{N}L - LN$$

Continuous Lax system MBSQ

For the MBSQ system we have the following system of compatible DΔEs:

$$p\frac{\partial}{\partial p}\ln v = n\left(1-\frac{3\widetilde{v}w}{\Gamma}\right) \quad , \quad -p\frac{\partial}{\partial p}\ln w = n\left(1-\frac{3v\underset{\sim}{w}}{\Gamma}\right) ,$$

where $\Gamma \equiv \widetilde{v}w + v\underset{\sim}{w} + \underset{\sim}{v}\widetilde{w}$, and similar for derivatives w.r.t q. Shifting in an auxiliary direction (associated with spectral parameter k) and using the relation:

$$\overline{\Gamma} = (p^3-k^3)\frac{\overline{v}\,\overline{w}w}{(p\widetilde{w}-k\overline{w})(k^2\overline{v}w + p^2 v\underset{\sim}{w} + kp\underset{\sim}{v}\overline{w})}\Gamma ,$$

we obtain two coupled Riccati equations for $p\partial_p\overline{v}$ and $p\partial_p\overline{w}$ which can then be linearised substituting $v = f/h$, $w = g/h$. This leads to the following linear equation:

$$p\frac{\partial}{\partial p}\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \left[\begin{pmatrix} p\partial_p \ln v & 0 & 0 \\ 0 & p\partial_p \ln w & 0 \\ 0 & 0 & 0 \end{pmatrix} + \right.$$
$$\left. + \frac{3np}{(p^3-k^3)\Gamma}\begin{pmatrix} p^2\widetilde{v}w & k^2\widetilde{v}\underset{\sim}{v} & pk\widetilde{v}v\underset{\sim}{w} \\ pk\widetilde{w}w & p^2\widetilde{w}\underset{\sim}{v} & k^2\widetilde{w}v\underset{\sim}{w} \\ k^2 w & pk\underset{\sim}{v} & p^2 v\underset{\sim}{w} \end{pmatrix}\right]\begin{pmatrix} f \\ g \\ h \end{pmatrix} ,$$

A similar linear relation holds interchanging n and m and p and q and interchanging the reverse lattice shifts.

The continuous part of the linear system allows one to derive the relevant monodromy problems for the similarity reductions on the lattice.

Continuum limits on the lattice MKdV and lattice MBSQ:

MKdV :

$$\partial_\xi(v_{n+1}v_n) = p(v_{n+1}^2 - v_n^2)$$

$$p\partial_\tau \ln v_N = \frac{v_{N+1} - v_{N-1}}{v_{N+1} + v_{N-1}}$$

$$V_t = V_{xxx} - 2V_x^3$$

where $V = \ln v$.

MBSQ :

$$\begin{cases} \partial_\xi(v_{n+1}w_n) = p(v_{n+1}w_{n+1} - v_n w_n) \\ v_n\partial_\xi(w_n w_{n+1}) - w_n w_{n+1}\partial_\xi v_n = p(v_n w_{n+1}^2 - w_n^2 v_{n+1}) \end{cases}$$

$$\begin{cases} p\partial_\tau \ln v_n = \dfrac{v_{n-1}w_{n+1} + v_n w_{n-1} - 2v_{n+1}w_n}{v_{n-1}w_{n+1} + v_n w_{n-1} + v_{n+1}w_n} \\ p\partial_\tau \ln w_n = \dfrac{2w_{n-1}v_n - w_n v_{n+1} - w_{n+1}v_{n-1}}{v_{n-1}w_{n+1} + v_n w_{n-1} + v_{n+1}w_n} \end{cases}$$

$$\begin{cases} V_{yy} + \frac{1}{3}V_{xxxx} - 2V_{xx}V_y - 2V_x^2 V_{xx} = 0 \\ W_{yy} + \frac{1}{3}W_{xxxx} + 2W_{xx}W_y - 2W_x^2 W_{xx} = 0 \end{cases}$$

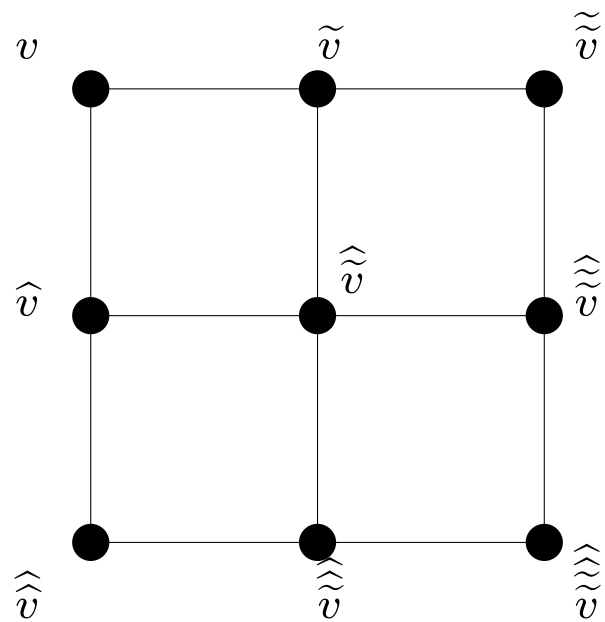
where $V = \ln v$, $W = \ln w$.

Lattice Modified BSQ:

$$\left(\frac{p^2\widehat{\widetilde{v}}-q^2\widehat{\widehat{v}}}{p\widehat{\widehat{v}}-q\widehat{\widetilde{v}}}\right)\frac{\widehat{\widehat{\widetilde{v}}}}{\widehat{v}}-\left(\frac{p^2\widetilde{\widetilde{v}}-q^2\widehat{\widetilde{v}}}{p\widehat{\widetilde{v}}-q\widetilde{\widetilde{v}}}\right)\frac{\widehat{\widetilde{\widetilde{v}}}}{\widetilde{v}}=p\left(\frac{v}{\widetilde{v}}-\frac{\widehat{\widehat{\widetilde{v}}}}{\widehat{\widehat{\widetilde{\widetilde{v}}}}}\right)-q\left(\frac{v}{\widehat{v}}-\frac{\widehat{\widetilde{\widetilde{v}}}}{\widehat{\widehat{\widetilde{\widetilde{v}}}}}\right)$$

[FWN,H.W. Capel, V. Papageorgiou & G.R.W. Quispel,1992], which involves a configuration on a 9-point stencil

$$f(v,\widehat{v},\widetilde{v},\widetilde{\widetilde{v}},\widehat{\widetilde{v}},\widehat{\widehat{v}},\widehat{\widetilde{\widetilde{v}}},\widehat{\widehat{\widetilde{v}}},\widehat{\widehat{\widetilde{\widetilde{v}}}};p,q)=0$$



So far, no classification result for such equations exist.

Continuum limits of the lattice MBSQ equation:

$$\partial_\xi^2 \ln(v_{n+1}v_nv_{n-1}) =$$
$$= \left(\frac{{v'}^2_{n+1}}{v^2_{n+1}} - \frac{{v'}^2_{n-1}}{v^2_{n-1}} + \frac{v'_{n+1}}{v_{n+1}}\frac{v'_n}{v_n} - \frac{v'_n}{v_n}\frac{v'_{n-1}}{v_{n-1}}\right) + 3p\left(\frac{v'_{n+1}v_n}{v^2_{n+1}} - \frac{v'_{n-1}}{v_n}\right)$$
$$+ p^2\left(\frac{v_n}{v_{n-1}} - \frac{v^2_{n-1}}{v^2_n} + \frac{v^2_n}{v^2_{n+1}} - \frac{v_{n+1}}{v_n}\right)$$

where $v'_n = \partial_\xi v_n$

$$\partial_\tau\left(\frac{v_{n+1}}{v_{n-1}}\frac{(p^2\dot{v}_n - 2pv_n)}{(p\dot{v}_n + v_n)}\right) = \frac{v_{n+1}}{v_{n+2}}\left(p\frac{\dot{v}_{n+1}}{v_{n+1}} + 1\right) - \frac{v_{n-2}}{v_{n-1}}\left(p\frac{\dot{v}_{n-1}}{v_{n-1}} + 1\right)$$

Potential MBSQ equation:

$$q_{tt} + \frac{1}{3}q_{xxxx} + 2q_tq_{xx} - 2q_x^2q_{xx} = 0 \ .$$

Similarity reduction

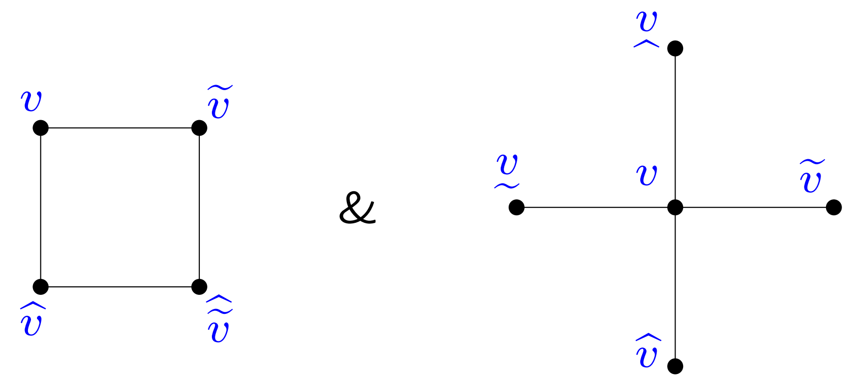
As in the continuous case we would like to find the natural symmetries of the lattice system and use them to obtain reductions on the solutions. This can be obtained by finding the appropriate *similarity constraints*.

Definition [FWN,1996]: A symmetry of a given PΔE is a pair consisting of the original lattice equation and an additional linear or nonlinear PΔE (called the symmetry constraint) such that *there exist localized configurations of lattice points carrying initial data that allow a global solution of the system consisting of this pair of PΔE's*.

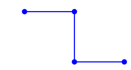
In the case of quadrilateral lattices this can be achieved by considering the pair of equations associated with a 4-point and 5-point schemes:

In this case the constraints may be taken of the elementary form:

i.e. an equation corresponding to a configuration of vertices forming a cross. Thus we are led to a coupled system:

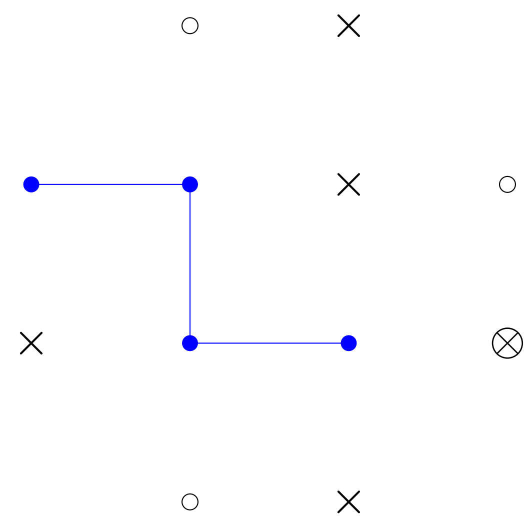


$$f(v,\widetilde{v},\widehat{v},\widehat{\widetilde{v}})=0 \qquad \& \qquad g(v,\widetilde{v},\underset{\sim}{v},\widehat{v},\underset{\frown}{v})=0$$

The iteration scheme leading to global solutions on the lattice start from *initial data on localised configurations of vertices*, typically of the form: 

However, in general the iteration scheme is nontrivial as there is the possibility of <u>multivalued determinations</u>: *after some initial steps the vertices can be calculated via different routes, leading to potential inconsistencies*.

This is indicated in the following diagram:



$\times$: points that are calculated with the lattice equation
$\circ$: points that are calculated with the similarity constraint
$\otimes$: points at which the evaluation can be multi-valued

Similarity reduction for the lattice MKdV system

$$\text{function} \quad f: \qquad pv\widehat{v} + q\widehat{v}\widehat{\widetilde{v}} = qv\widetilde{v} + p\widetilde{v}\widehat{\widetilde{v}}$$

$$\text{function} \quad g: \qquad n\frac{\widetilde{v} - \utilde{v}}{\widetilde{v} + \utilde{v}} + m\frac{\widehat{v} - \underset{\frown}{v}}{\widehat{v} + \underset{\frown}{v}} = \mu - \lambda(-1)^{n+m}$$

Statement: the above similarity constraints are compatible with the corresponding lattice equations in the sense explained above.

How to implement the similarity constraint and obtain an explicit reduction: P∆E → O∆E ?

Write the constraint as:

$$na + mb = \mu - \nu \quad , \quad \nu \equiv \lambda(-1)^{n+m} \quad \text{with}: \quad a \equiv \frac{\widetilde{v} - \utilde{v}}{\widetilde{v} + \utilde{v}} \quad , \quad b \equiv \frac{\widehat{v} - \underset{\frown}{v}}{\widehat{v} + \underset{\frown}{v}}$$

Backshift MKdV:

$$p\underset{\frown}{v}v + qv\widetilde{v} = q\underset{\frown}{v}\underset{\frown}{\widetilde{v}} + p\underset{\frown}{\widetilde{v}}\widetilde{v}$$

Adding this to MKdV above and rearranging we get:

$$(px + q)\widetilde{b} + px = (pX - q)b + pX$$

in terms of *reduced variables* $x = v/\widehat{\widetilde{v}}$, $X = \widetilde{v}/\widehat{v}$ which are fractionally linearly related: $X = (px + q)/(qx + p)$.

Observing that $a = (X - \underset{\sim}{x})/(X + \underset{\sim}{x})$, we can combine the different relations to derive the following closed-form difference equation for x in terms of the $\widetilde{\ }$ shift only:

$$(n+1)(r+x)(1+rx)\frac{\widetilde{x}-x+r(1-x\widetilde{x})}{\widetilde{x}+x+r(1+x\widetilde{x})} - n(1-r^2)x\frac{x-\underset{\sim}{x}+r(1-x\underset{\sim}{x})}{x+\underset{\sim}{x}+r(1+x\underset{\sim}{x})} =$$
$$= \mu r(1+2rx+x^2) + \lambda(-1)^{n+m}(r+2x+rx^2) - mr(1-x^2)\ ,$$

in which $r = q/p$, m, μ and λ are parameters of the equation. This is a 4-parameter *discrete Painlevé* (dP) equation. [FWN, A. Ramani, B. Grammaticos, Y. Ohta, 2001]

Connection with PVI

Using the DΔEs:

$$-p\frac{\partial}{\partial p}\ln v = na = n\frac{\widetilde{v}-\underset{\sim}{v}}{\widetilde{v}+\underset{\sim}{v}} \quad , \quad -q\frac{\partial}{\partial q}\ln v = mb = m\frac{\widehat{v}-\underset{\frown}{v}}{\widehat{v}+\underset{\frown}{v}}$$

and the previous relations for the shift $\widetilde{b}$:

$$(px+q)\widetilde{b} + px = (pX-q)b + pX$$

we can derive also:

$$n\frac{\partial a}{\partial q} = m\frac{\partial b}{\partial p} = \frac{mnpq}{p^2-q^2}\left[(1-a)(1+b)X - (1+a)(1-b)\frac{1}{X}\right]$$

In addition we have:

$$\begin{aligned}\mu+\nu+p\frac{\partial}{\partial p}\log X &= n\widehat{a}+m\widetilde{b}\\ &+\frac{(pX-q)(qX-p)}{(q^2-p^2)X}(mb-na)+\frac{pq}{q^2-p^2}(n+m)\left(X-\frac{1}{X}\right)\end{aligned}$$

Using the simil. constr. we can eliminate b from the 1st order differential system leading to:

$$
\begin{aligned}
&p(p^2-q^2)^2 X(qX-p)(pX-q)\frac{\partial^2 X}{\partial p^2} = \\
&= \frac{1}{2}p(p^2-q^2)^2\left[pq(3X^2+1)-2(p^2+q^2)X\right]\left(\frac{\partial X}{\partial p}\right)^2 + \\
&\quad + (q^2-p^2)\left[2p^2X(pX-q)(qX-p)+(q^2-p^2)^2X^2\right]\frac{\partial X}{\partial p} \\
&\quad + \frac{1}{2}q\left[(\alpha X^2-\beta)(pX-q)^2(qX-p)^2\right. \\
&\qquad\qquad \left. +(p^2-q^2)X^2\left((\gamma-1)(qX-p)^2-(\delta-1)(pX-q)^2\right)\right]
\end{aligned}
$$

with the identifications:

$$
\begin{aligned}
\alpha &= (\mu-\nu+m-n)^2 \quad , \quad \beta = (\mu-\nu-m+n)^2 \ , \\
\gamma &= (\mu+\nu-m-n-1)^2 \quad , \quad \delta = (\mu+\nu+m+n+1)^2 \ ,
\end{aligned}
$$

Writing the equation for X in terms of $t = p^2$ (taking w.l.o.g. $q = 1$) and identifying $w(t) = pX(p)$ we get:

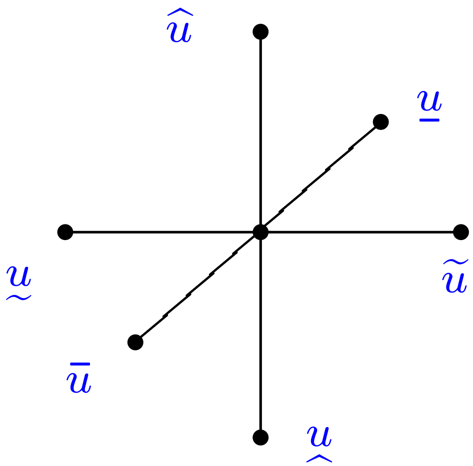
$$
\begin{aligned}
\frac{d^2 w}{dt^2} = \quad & \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right)\left(\frac{dw}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{w-t}\right)\frac{dw}{dt} \\
& + \frac{w(w-1)(w-t)}{8t^2(t-1)^2}\left(\alpha - \beta\frac{t}{w^2} + \gamma\frac{t-1}{(w-1)^2} - (\delta-4)\frac{t(t-1)}{(w-t)^2}\right)
\end{aligned}
$$

with the previous identification of the parameters α,β,γ,δ. This is the famous Painlevé VI equation, [R. Fuchs, 1905].

Beyond PVI: multi-dimensional similarity reduction

Embed the 2D lattice equation into 3-dimensional lattice, using CAC

$$
\begin{aligned}
p(v\widehat{v} - \widetilde{v}\widehat{\widetilde{v}}) &= q(v\widetilde{v} - \widehat{v}\widehat{\widetilde{v}}) \\
q(v\overline{v} - \widehat{v}\widehat{\overline{v}}) &= r(v\widehat{v} - \overline{v}\widehat{\overline{v}}) \\
r(v\widetilde{v} - \overline{v}\widetilde{\overline{v}}) &= p(v\overline{v} - \widetilde{v}\widetilde{\overline{v}})
\end{aligned}
$$

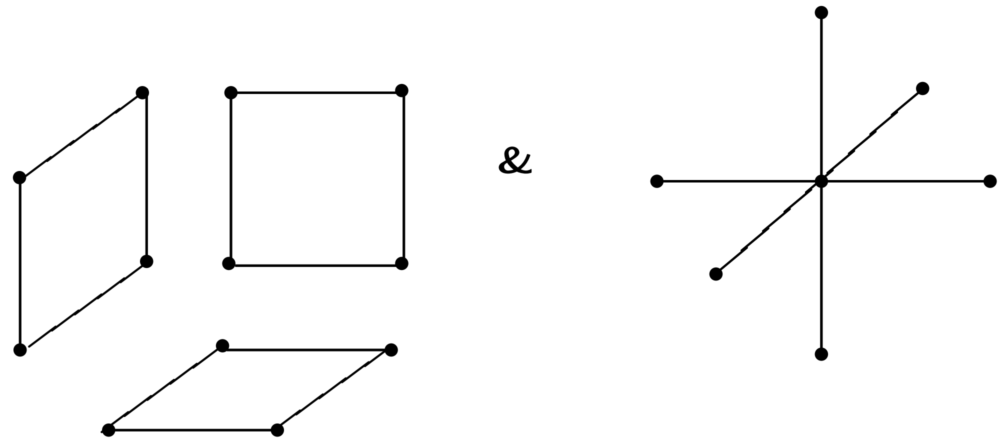


similarity constraint: $g(\widetilde{u}, \underset{\sim}{u}, \widehat{u}, \underset{\frown}{u}, \bar{u}, \underline{u}) = 0$

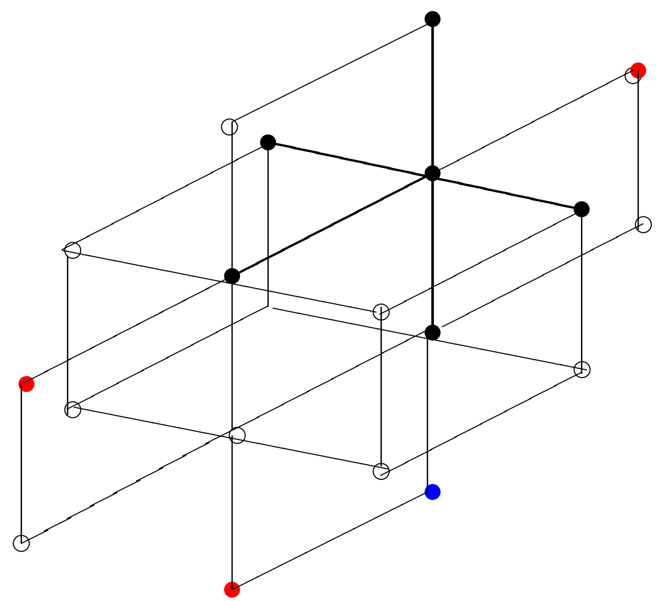
function g:

$$
n\underbrace{\frac{\widetilde{v} - \underset{\sim}{v}}{\widetilde{v} + \underset{\sim}{v}}}_{\equiv a} + m\underbrace{\frac{\widehat{v} - \underset{\frown}{v}}{\widehat{v} + \underset{\frown}{v}}}_{\equiv b} + l\underbrace{\frac{\bar{v} - \underline{v}}{\bar{v} + \underline{v}}}_{\equiv c} = \mu - \lambda(-1)^{n+m+l} \; .
$$

Check of consistency between lattice equation and similarity constraint



consistency diagram:



Explicit reduction to O$\triangle$E system

Introduce quantities:

$$x = \frac{v}{\widehat{\widetilde{v}}} \quad , \quad y = \frac{v}{\overline{\widetilde{v}}} \quad , \quad X = \frac{\widetilde{v}}{\widehat{v}} \quad , \quad Y = \frac{\widetilde{v}}{\overline{v}} \ .$$

Lattice MKdV $\quad \Rightarrow$:

$$X = \frac{px+q}{qx+p} \quad , \quad Y = \frac{py+r}{ry+p} \ ,$$

and:

$$\widetilde{b} = \frac{(pX-q)b + p(X-x)}{px+q} \quad , \quad \widetilde{c} = \frac{(pY-r)c + p(Y-y)}{py+r} \ .$$

Furthermore, by the definitions:

$$a = \frac{\widetilde{v} - \underset{\sim}{v}}{\widetilde{v} + \underset{\sim}{v}} = \frac{X - \underset{\sim}{x}}{X + \underset{\sim}{x}} = \frac{Y - \underset{\sim}{y}}{Y + \underset{\sim}{y}} \ .$$

Using similarity constraint

$$na + mb + lc = \mu - \nu \quad , \quad (n+1)\widetilde{a} + m\widetilde{b} + l\widetilde{c} = \mu + \nu \quad , \quad \nu = \lambda(-1)^{n+m+l} \ ,$$

we obtain the system:

$$\begin{pmatrix} (q^2-p^2)X & (pX-q)(qX-p) \\ m(r^2-p^2)Y & -m(pY-r)(rY-p) \end{pmatrix} \begin{pmatrix} \widetilde{b} \\ b \end{pmatrix} = \begin{pmatrix} pq(X^2-1) \\ * \end{pmatrix}$$

where $* = [\mu + \nu - (n+1)\widetilde{a}]p(r-pY) - (\mu - \nu - na)(pY-r)(rY-p) - prl(Y^2-1)$.

Eliminating b we get a fourth order system together with:

$$\frac{qX\underset{\sim}{X} - p(X - \underset{\sim}{X}) - q}{qX\underset{\sim}{X} + p(X - \underset{\sim}{X}) + q} = \frac{rY\underset{\sim}{Y} - p(Y - \underset{\sim}{Y}) - r}{rY\underset{\sim}{Y} + p(Y - \underset{\sim}{Y}) + r} \ .$$

$$F(X, Y, \widetilde{X}, \widetilde{Y}) = 0 \quad , \quad G(\underset{\sim}{X}, X, \widetilde{X}, \widetilde{\widetilde{X}}, Y, \widetilde{Y}) = 0$$

with 6 parameters: $m, l, q/p, r/p, \mu, \nu$.

A corresponding differential system in terms of the lattice parameters (as independent variables) was given in [FWN,A Walker, 2001].

The other route: similarity reduction of the lattice MBSQ

Similarity constraints:

$$\begin{aligned}\left(p\frac{\partial}{\partial p}+q\frac{\partial}{\partial q}\right)\ln v &= \alpha_1+\alpha_2+\beta_1\omega^{-(n+m)}+\beta_2\omega^{-2(n+m)}\\ -\left(p\frac{\partial}{\partial p}+q\frac{\partial}{\partial q}\right)\ln w &= \alpha_1\omega^{-1}+\alpha_2\omega^{-2}+\beta_1\omega^{-(n+m+1)}+\beta_2\omega^{-2(n+m+1)}\end{aligned}$$

(where $\omega=\exp(2\pi i/3)$), in which we have to substitute:

$$p\frac{\partial}{\partial p}\ln v = n\left(1-\frac{3\widetilde{v}w}{\widetilde{v}w+v\underset{\sim}{w}+\underset{\sim}{v}\widetilde{w}}\right)\quad,\quad -p\frac{\partial}{\partial p}\ln w = n\left(1-\frac{3v\underset{\sim}{w}}{\widetilde{v}w+v\underset{\sim}{w}+\underset{\sim}{v}\widetilde{w}}\right)$$

$$q\frac{\partial}{\partial q}\ln v = m\left(1-\frac{3\widehat{v}w}{\widehat{v}w+v\underset{\frown}{w}+\underset{\frown}{v}\widehat{w}}\right)\quad,\quad -q\frac{\partial}{\partial q}\ln w = m\left(1-\frac{3v\underset{\frown}{w}}{\widehat{v}w+v\underset{\frown}{w}+\underset{\frown}{v}\widehat{w}}\right)$$

The similarity constraints are slight generalisations of constraints found several years ago [FWN,1996].

Statement: These similarity constraints are consistent with the lattice MBSQ system in the sense explained earlier.

The aim now is to derive an explicit reduction to an O∆E in terms of one discrete variable, say n, where the other discrete variable m becomes just a parameter.

Introduce the quantities:

$$X=\frac{\widehat{v}}{\widetilde{v}}\quad,\quad Y=\frac{\widehat{w}}{\widetilde{w}}\quad,\quad \Xi=\frac{3\widehat{v}w}{\widehat{v}w+v\underset{\frown}{w}+\underset{\frown}{v}\widehat{w}}\quad,\quad \Upsilon=\frac{3v\underset{\frown}{w}}{\widehat{v}w+v\underset{\frown}{w}+\underset{\frown}{v}\widehat{w}}\quad.$$

We can express

$$\xi \equiv \frac{3\widetilde{v}w}{\widetilde{v}w + v\underset{\sim}{w} + \underset{\sim}{v}\widetilde{w}} \quad , \quad \eta \equiv \frac{3v\underset{\sim}{w}}{\widetilde{v}w + v\underset{\sim}{w} + \underset{\sim}{v}\widetilde{w}}$$

entirely in terms of X and Y, namely:

$$\begin{aligned} \xi &= 3/\left[1 + \frac{X}{Y}\left(\frac{r\underset{\sim}{Y} - \underset{\sim}{X}}{r\underset{\sim}{X} - 1}\right) + X\left(\frac{r - \underset{\sim}{Y}}{r\underset{\sim}{X} - 1}\right)\right] \\ \eta &= 3/\left[1 + \frac{1}{X}\left(\frac{r\underset{\sim}{X} - 1}{r - \underset{\sim}{Y}}\right) + \frac{1}{Y}\left(\frac{r\underset{\sim}{Y} - \underset{\sim}{X}}{r - \underset{\sim}{Y}}\right)\right] \end{aligned}$$

where $r = p/q$. Furthermore, it can be shown that the quantities Ξ and Υ obey the shift relations:

$$\Xi = \frac{X(r^2Y - 1)\widetilde{\widetilde{\Xi}}}{(rX - 1)\,(rY - X)} + \frac{X(\widetilde{\widetilde{\Upsilon}} - 3)}{rY - X} \quad , \quad \Upsilon = \frac{\widetilde{\widetilde{\Xi}}}{1 - rX} + \frac{r\widetilde{\widetilde{\Upsilon}}}{r - Y} \ .$$

Introducing from the similarity constraints the explicit n-dependent quantities:

$$\begin{aligned} \zeta_1 &= n + m - \alpha_1 - \alpha_2 - \beta_1\omega^{-(n+m)} - \beta_2\omega^{-2(n+m)} \\ \zeta_2 &= n + m - \alpha_1\omega^{-1} - \alpha_2\omega^{-2} - \beta_1\omega^{-(n+m+1)} - \beta_2\omega^{-2(n+m+1)} \ , \end{aligned}$$

we can use the similarity constraints:

$$n\xi + m\Xi = \zeta_1 \quad , \quad n\eta + m\Upsilon = \zeta_2 \ ,$$

to derive a coupled system of two second order non-autonomous OΔEs for the variables X and Y, namely ....

Discrete Garnier system:

$$\zeta_1 - n\xi = \frac{X(r^2Y-1)\left(\widetilde{\zeta}_1-(n+1)\widetilde{\xi}\right)}{(rX-1)(rY-X)} + \frac{X\left(\widetilde{\zeta}_2-(n+1)\widetilde{\eta}-3m\right)}{rY-X}$$

$$\zeta_2 - n\eta = \frac{\widetilde{\zeta}_1-(n+1)\widetilde{\xi}}{1-rX} + \frac{r\left(\widetilde{\zeta}_2-(n+1)\widetilde{\eta}\right)}{r-Y}$$

Inserting the expressions for ξ and η we obtain a coupled system of two second order OΔEs for X, Y with six free parameters: α_1, α_2, β_1, β_2, r, m.

Properties of the discrete Garnier system:

• The discrete Garnier system derives from a 3×3 matrix isomonodromic deformation problem of the Schlesinger type, which can be recovered from the Lax representations of both the continuous as discrete MBSQ systems.

• It is conjectured that this discrete system constitutes the superposition formulae for the Bäcklund-Schlesinger transformations of some continuous Garnier/Schlesinger system. In other words: it is expected to form a system of nonlinear *contiguous* relations for that continuous Garnier system.

• There is a connected system of two coupled ODEs associated with the continuous flows (either in terms of p or in q). We believe that these have a direct relation with the first higher order Garnier system, and we interpret it as the next member in a PVI hierarchy.

Generating PDE for the BSQ system

This is a coupled system [A. Tongas & FWN, 2005] :

$$(\widehat{U}-\widetilde{U})_{ts}=\left[\frac{U_sW_{ts}-W_sU_{ts}}{\Delta}(\widehat{U}-\widetilde{U})\right]_t-\left[\frac{U_tW_{ts}-W_tU_{ts}}{\Delta}(\widehat{U}-\widetilde{U})\right]_s\ ,$$
$$(\widehat{V}-\widetilde{V})_{ts}=\left[\frac{U_sW_{ts}-W_sU_{ts}}{\Delta}(\widehat{V}-\widetilde{V})\right]_t-\left[\frac{U_tW_{ts}-W_tU_{ts}}{\Delta}(\widehat{V}-\widetilde{V})\right]_s\ ,$$

in which $\Delta=W_sU_t-W_tU_s$, and where we have to substitute the relations:

$$\widehat{U}-\widetilde{U}\ =\ \frac{nU_s+mU_t}{\Delta}+\frac{s-t}{\Delta^2}\left[2U_sU_tW_{ts}-(U_sW_t+U_tW_s)\,U_{ts}\right]\ ,$$
$$\widehat{V}-\widetilde{V}\ =\ \frac{nW_s+mW_t}{\Delta}-\frac{s-t}{\Delta^2}\left[2W_sW_tU_{ts}-(U_sW_t+U_tW_s)\,W_{ts}\right]\ .$$

The resulting system can be derived as the Euler-Lagrange equations for the 2-field system in terms of $U=U(t,s)$, $W=W(t,s)$:

$$\begin{aligned}\mathcal{L}[U,W]\ &=\ \frac{t-s}{\Delta^2}(W_sU_{ts}-U_sW_{ts})(W_tU_{ts}-U_tW_{ts})\\ &\quad+\frac{n}{\Delta}(W_sU_{ts}-U_sW_{ts})+\frac{m}{\Delta}(W_tU_{ts}-U_tW_{ts})\ ,\end{aligned}$$

in which $\Delta=W_sU_t-W_tU_s$, and where $n=n(t)$,$m=m(s)$ are functions of one of the independent variables s, t only.

Remarks:

• $\mathcal{L}$ is invariant under $\mathbf{SL}(3,\mathbb{C})$ acting projectively on (U,W)

$$U \longmapsto \frac{\alpha_1 U + \alpha_2 W + \alpha_3}{\gamma_1 U + \gamma_2 W + \gamma_3}$$

$$W \longmapsto \frac{\beta_1 U + \beta_2 W + \beta_3}{\gamma_1 U + \gamma_2 W + \gamma_3}$$

• The system of PDEs, for $n = n(t).\ m = m(s)$ reduces to the Ernst-Maxwell-Weyl (EMW) system of equations from General Relativity.

• The underlying compatible discrete system (lattice BSQ) gives rise to both an auto- as well as hetero-BT for the EMW equations.

• Symmetry reduction, leading to solutions invariant along the orbits of the vector field $X = t\partial_t + s\partial_s + 2\alpha_1 U\partial_U + 2\alpha_2 W\partial_W$ leading to

$$U(t,s) = (ts)^{\alpha_1} F_1(\tau) \quad , \quad W(t,s) = (ts)^{\alpha_2} F_2(\tau) \quad , \quad \tau = t/s \ ,$$

leads to the similarity solution giving rise, after two integrations, to the following couples 2nd order system of ODEs, involving 6 free parameters $(n, m, \alpha_1, \alpha_2, s_1, s_2)$:

$$
\begin{aligned}
&\frac{\alpha_1\,\alpha_2}{\alpha_1-\alpha_2}(\mathcal{G}_1-\mathcal{G}_2)(Q_1Q_2'-Q_1'Q_2)+\big((\tau-1)\mathcal{G}_1-(\tau+1)\big)Q_1' \\
&-\frac{{\alpha_1}^2}{\tau}\left((\tau-1)(\mathcal{G}_1-1)^2-4\mathcal{G}_1+\frac{\tau}{\alpha_1}(\mathcal{G}_1-1)+\frac{2\tau}{\alpha_1}\frac{(n+m)}{(\tau-1)}\right)Q_1 \\
&-\frac{\alpha_1\,\alpha_2}{\tau}\left((\mathcal{G}_2-1)^2-\frac{4\mathcal{G}_2}{\tau-1}-\frac{\alpha_1(\mathcal{G}_1-\mathcal{G}_2)^2}{\alpha_1-\alpha_2}\right)Q_1Q_2 \\
&=s_1-\frac{{\alpha_1}^2}{\tau}\left((\mathcal{G}_1-1)^2-\frac{4\mathcal{G}_1}{\tau-1}\right){Q_1}^2
\end{aligned}
$$

(together with same equation with $1\leftrightarrow 2$), introducing new dependent variables:

$$
\mathcal{G}_i(\tau)=\frac{\tau}{\alpha_i}\frac{F_i'(\tau)}{F_i(\tau)}\,,\qquad i=1,2\,,
$$

and making the substitutions:

$$
\begin{aligned}
Q_1=&\frac{m+n\tau}{2\alpha_1(\mathcal{G}_1-\mathcal{G}_2)}+\frac{(m-n\tau)\mathcal{G}_2}{2\alpha_1(\mathcal{G}_1-\mathcal{G}_2)}+\frac{(\tau-1)\alpha_2\mathcal{G}_1\mathcal{G}_2(1-{\mathcal{G}_2}^2)}{2\alpha_1(\mathcal{G}_1-\mathcal{G}_2)^2}- \\
&\frac{(\tau-1)(\alpha_1-\alpha_2)({\mathcal{G}_2}^2-1)}{2\alpha_1(\mathcal{G}_1-\mathcal{G}_2)^2}+\frac{(\tau-1){\mathcal{G}_1}^2({\mathcal{G}_2}^2-1)}{2(\mathcal{G}_1-\mathcal{G}_2)^2} \\
&+\frac{\tau(\tau-1)({\mathcal{G}_2}^2-1)\mathcal{G}_1'}{2\alpha_1(\mathcal{G}_1-\mathcal{G}_2)^2}-\frac{\tau(\tau-1)(\mathcal{G}_1\mathcal{G}_2-1)\mathcal{G}_2'}{2\alpha_1(\mathcal{G}_1-\mathcal{G}_2)^2}\,,
\end{aligned}
$$

(Q_2 follows from Q_1 by interchange $1\leftrightarrow 2$).

Conclusions:

♭ The approach to obtain discrete Painlevé and Garnier type systems from reduction on the 2D lattice is useful to obtain the necessary information (isomonodromic deformation problems).

The Painlevé reduction arises by staying within the 2D lattice, whereas the higher-order systems arise from two different scenarios:
• Either start from a quadrilateral scalar equation (e.g. MKdV) and embed the system in a higher-dimensional lattice, coupling the lattice equations through the similarity constraint;
• Or start from higher-order/multicomponent system (e.g.) and stay in the two-dimensional lattice.

♭ Both scenarios have given rise to coupled second order systems of O∆Es with 6 free parameters, believed to be higher-oredr versions of discrete PVI.

♯ It seems that various hierarchies exist of (discrete) PVI exist, which may coincide at the $N = 1$ and $N = 2$ level, but that bifurcate for $N > 2$.

♭ Multidimensional consistency of the original lattice equations is a key ingredient in these constructions.

♯ It remains to be shown that proper continuum limits of these systems give rise to the corresponding well-known Garnier systems, but in any event the reduction of the corresponding *generating PDEs* lead similarly to coupled second-order systems with 6 free parameters, generalising PVI.

Other approaches to similarity reduction:

- reductions of q-KP, leading to q-Painlevé equations (M. Noumi et al., 2001);
- periodic reductions from non-autonomous versions of the lattice equations (B. Grammaticos et al., 2004);
- construction of a q-deformation of the Garnier system by H. Sakai (2005).

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