Integrable Lattice Equations and Discrete Painlevé Equations

The Painlevé Equations and Monodromy Problems
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Outline:

I Integrability of Lattice Equations and Continuous counterparts: Special Solutions
(work with J. Atkinson, J. Hietarinta);

II Similarity Reduction to Discrete Painlevé Equations and Garnier Systems
(work with A. Tongas, A. Walker).

Some History:

Recent:

- M. Ablowitz & F. Ladik (1976): integrable difference scheme (motivated by numerics)
- R. Hirota (1977): bilinear approach
- FWN, R. Quispel, H. Capel (1982): direct linearisation and connection with Bäcklund transfs. and Bianchi permutability conditions.
- Since 1990: lots of activity, in particular:
  SIDE (Symmetries and Integrability of Difference Equations), (1994-present)

Pre-history:

Padé approximants and convergence acceleration algorithms:
Wynn (1956,1966), Cordellier (1979)
Frobenius (1881), Padé (1892)

Differential Geometry: Bianchi (1899)

Motivation:

The theory of integrable nonlinear evolutions equations (soliton theory) has a perfect parallel in the realm of discete systems and difference equations.

The resulting systems (PΔEs, OΔEs, dynamical mappings) not only share many of the key integrability properties with the continuous systems, but *they seem actually much richer and reveal the true nature of integrable systems*.

The Painlevé equations have emerged from the study of soliton systems as special (similarity) solutions (M. Ablowitz & H. Segur, 1976; H. Flaschka and A. Newell, 1979; many others), and this connection has proved extremely fruitful in the development of solution methods for those equations.

Thus, it seems natural to pursue similar connections in the case of difference analogues of the Painlevé equations, and indeed one of the first sources of discrete Painlevé equations has been the application of similarity reduction of integrable lattice equations (FWN & V. Papageorgiou, 1991).

These lectures will be a review to this particular approach to find, and study, discrete Painlevé equations, and in order to do this we need to apprehend first the key integrability features of the integrable lattice equations.

Key integrability features of P△E's on the 2D lattice

The class of *quadrilateral* P△Es has the following canonical form:

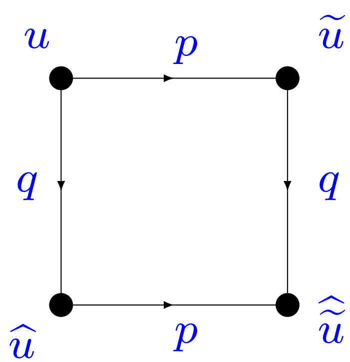
$$f(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; p, q) = 0$$

where we adopt the canonical notation of verices along an elementary plaquette on a rectangular lattice:

$$u := u_{n,m} \qquad , \qquad \widetilde{u} = u_{n+1,m}$$

$$\widehat{u} := u_{n,m+1} \qquad , \qquad \widehat{\widetilde{u}} = u_{n+1,m+1}$$

Schematically:



Here a, b are *lattice parameters*

lattice shifts: $u \xrightarrow{p} \widetilde{u}$
$u \xrightarrow{q} \widehat{u}$

Where f is *linear* in each of the 4 vertices.

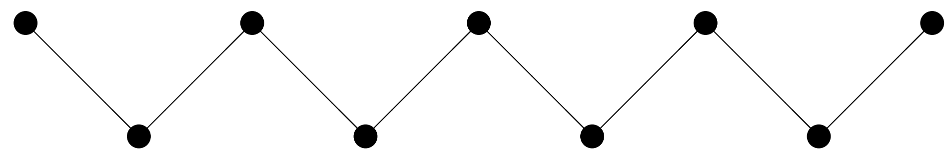
Here p, q are parameters of the equation, related to the lattice spacing. The lattice parameters allow us to take continuum limits.

Questions to be addressed:

1. What *initial value problems* (IVPs) on the lattice can be imposed?

2. What are the key aspects of *integrability*?

Clearly, if the equation $f = 0$ can be solved uniquely at each vertex of the plaquette we can define initial value problems on configurations like:



If the value of the dependent variable can be solved uniquely at each vertex of the plaquette, we have the analogue of a *hyperbolic* PDE: only the initial values contribute to a given vertex that are in the corresponding "lightcone"

• It turns out that already the class of equations which are *linear in each vertex* is considerably rich. The general form, *linear in each vertex*, and having D_4 symmetry:

$$
\begin{aligned}
&k_0 u\widehat{u}\widetilde{u}\widehat{\widetilde{u}} - k_1\left(u\widehat{u}\widetilde{u} + u\widehat{u}\widehat{\widetilde{u}} + u\widetilde{u}\widehat{\widetilde{u}} + \widehat{u}\widetilde{u}\widehat{\widetilde{u}}\right) + k_2\left(\widehat{u}\widetilde{u} + u\widehat{\widetilde{u}}\right) - k_3\left(u\widetilde{u} + \widehat{u}\widehat{\widetilde{u}}\right) \\
&-k_4\left(u\widehat{u} + \widetilde{u}\widehat{\widetilde{u}}\right) + k_5\left(u + \widetilde{u} + \widehat{u} + \widehat{\widetilde{u}}\right) + k_6 = 0\ .
\end{aligned}
$$

Many integrable cases of this general form were studied in the past two decades, but an integrable example of the generic case was only found by V.Adler in 1997.

Special Examples:

Lattice Korteweg-de Vries (KdV) equation:

$$(p - q + \widehat{u} - \widetilde{u})(p + q + u - \widehat{\widetilde{u}}) = p^2 - q^2$$

Lattice Modified KdV (MKdV) equation:

$$p(v\widehat{v} - \widetilde{v}\widehat{\widetilde{v}}) = q(v\widetilde{v} - \widehat{v}\widehat{\widetilde{v}})$$

Lattice Schwarzian KdV (SKdV) equation:

$$\frac{(z - \widehat{z})(\widetilde{z} - \widehat{\widetilde{z}})}{(z - \widetilde{z})(\widehat{z} - \widehat{\widetilde{z}})} = \frac{p^2}{q^2}$$

The lattice parameters p, q allow us to take continuum limits.

Example # 1: Consider $z_{n,m} = z_n(\xi_0 + m/q)$ and expand in Taylor series around $\xi = \xi_0 + m/q$ as follows:

$$\widehat{z} = z_{n,m+1} = z_n(\xi) + \frac{1}{q} z_n'(\xi) + \cdots \quad , \quad \widehat{\widetilde{z}} = z_{n+1,m+1} = z_{n+1}(\xi) + \frac{1}{q} z_{n+1}'(\xi) + \cdots$$

(where $z' = dz/d\xi$) we obtain the differential-difference equation:

$$z_n' z_{n+1}' = p^2 (z_n - z_{n+1})^2$$

Example # 2: A more subtle limit is obtained by first doing a change of variables on the lattice:

$$z_{n,m} = z_{n+m}(\tau_0 + (p-q)m) = z_N(\tau)$$

keeping $N = n+m$ fixed. The lattice SKdV quation is then rewritten as

$$(p-\delta)^2 \frac{(z_N(\tau) - z_{N+1}(\tau+\delta))(z_{N+1}(\tau) - z_{N+2})(\tau+\delta)}{(z_N(\tau) - z_{N+1}(\tau))(z_{N+1}(\tau+\delta) - z_{N+2}(\tau+\delta))} = p^2$$

where expanding in the variable $\delta = p - q$ as follows:

$$z_N(\tau+\delta) = z_N(\tau) + \delta \dot{z}_N(\tau) + \cdots$$

(where $\dot{z} = dz/d\tau$) leads to the following differential-difference equation in leading order:

$$\dot{z}_N = \frac{2}{p} \frac{(z_{N+1} - z_N)(z_N - z_{N-1})}{z_{N+1} - z_{N-1}}$$

Next, we can perform a second continuum limit on the remaining discrete variable (associated witht he remaing parameter p). This is even more subtle and requires higher-order expansions in the variable $1/p$. Omitting details we mention this gives us eventually a PDE of the form:

$$z_t = z_{xxx} - \frac{3}{2}\frac{z_{xx}^2}{z_x} = z_x \{z, x\}$$

which is the "Schwarzian KdV equation". Invariance under Möbius transformations:

$$z \quad \mapsto \quad Z = \frac{\alpha z + \beta}{\gamma z + \delta}$$

Similar continuum limits on the lattice KdV and lattice MKdV yield the following:

<u>KdV :</u>

$$\partial_\xi(u_{n+1}+u_n) = p^2 - (p+u_n-u_{n+1})^2$$

$$\partial_\tau u_N = \frac{2p}{2p+u_{N-1}-u_{N+1}} - 1$$

$$u_t = u_{xxx} + 3u_x^2$$

<u>MKdV :</u>

$$\partial_\xi(v_{n+1}v_n) = p(v_{n+1}^2 - v_n^2)$$

$$p\partial_\tau \log v_N = \frac{v_{N+1}-v_{N-1}}{v_{N+1}+v_{N-1}}$$

$$v_t = v_{xxx} - 3\frac{v_x v_{xx}}{v}$$

<u>"Miura" relations between the lattice equations:</u>

$$p(z-\widetilde{z}) = v\widetilde{v} \quad , \quad q(z-\widehat{z}) = v\widehat{v}$$

$$p - q + \widehat{u} - \widetilde{u} = \frac{p\widetilde{v} - q\widehat{v}}{v}$$

$$p + q + u - \widehat{\widetilde{u}} = \frac{pv + q\widehat{\widetilde{v}}}{\widetilde{v}}$$

Higher order lattice systems: Boussinesq family

[FWN, V Papageorgiou, H W Capel & G R W Quispel, 1992]

Lattice Schwarzian BSQ:

$$\frac{(\widehat{\widehat{\widetilde{\widetilde{z}}}}-\widehat{\widehat{\widetilde{z}}})(\widehat{\widehat{z}}-\widehat{\widetilde{z}})(\widehat{z}-z)}{(\widehat{\widehat{\widetilde{\widetilde{z}}}}-\widehat{\widetilde{\widetilde{z}}})(\widetilde{\widetilde{z}}-\widehat{\widetilde{z}})(\widetilde{z}-z)}=\frac{p^3(\widehat{\widehat{\widetilde{z}}}-\widehat{\widehat{z}})(\widehat{\widetilde{z}}-\widehat{z})-q^3(\widehat{\widehat{\widetilde{z}}}-\widehat{\widetilde{z}})(\widehat{\widehat{z}}-\widehat{z})}{q^3(\widehat{\widetilde{\widetilde{z}}}-\widetilde{\widetilde{z}})(\widehat{\widetilde{z}}-\widetilde{z})-p^3(\widehat{\widetilde{\widetilde{z}}}-\widehat{\widetilde{z}})(\widetilde{\widetilde{z}}-\widetilde{z})}$$

Lattice Modified BSQ:

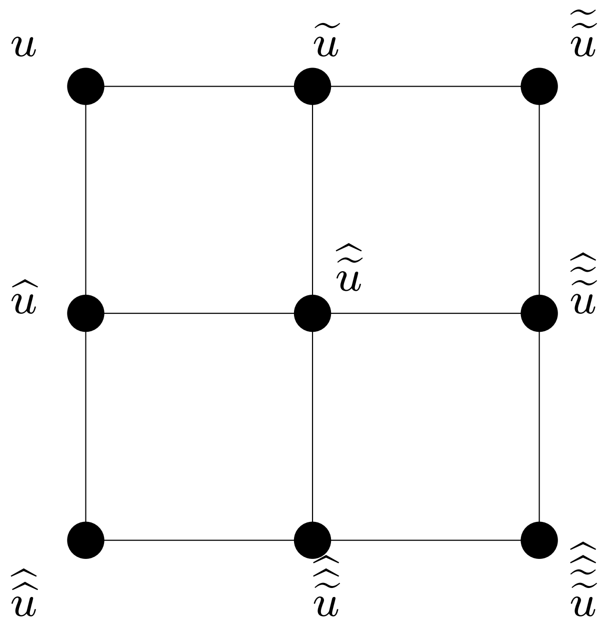
$$\left(\frac{p^2\widehat{\widetilde{v}}-q^2\widehat{\widehat{v}}}{p\widehat{\widehat{v}}-q\widehat{\widetilde{v}}}\right)\frac{\widehat{\widehat{\widetilde{v}}}}{\widehat{v}}-\left(\frac{p^2\widetilde{\widetilde{v}}-q^2\widehat{\widetilde{v}}}{p\widehat{\widetilde{v}}-q\widetilde{\widetilde{v}}}\right)\frac{\widehat{\widetilde{\widetilde{v}}}}{\widetilde{v}}=p\left(\frac{v}{\widetilde{v}}-\frac{\widehat{\widehat{\widetilde{v}}}}{\widehat{\widehat{\widetilde{\widetilde{v}}}}}\right)-q\left(\frac{v}{\widehat{v}}-\frac{\widehat{\widetilde{\widetilde{v}}}}{\widehat{\widehat{\widetilde{\widetilde{v}}}}}\right)$$

Lattice BSQ:

$$\begin{aligned}\frac{p^3-q^3}{p-q+\widehat{\widetilde{u}}-\widetilde{\widetilde{u}}}-\frac{p^3-q^3}{p-q+\widehat{\widehat{u}}-\widehat{\widetilde{u}}} \quad &= \quad (p-q+\widehat{\widehat{\widetilde{u}}}-\widehat{\widetilde{\widetilde{u}}})(2p+q+\widehat{u}-\widehat{\widehat{\widetilde{\widetilde{u}}}})\\ &\quad -(p-q+\widehat{u}-\widetilde{u})(2p+q+u-\widehat{\widetilde{\widetilde{u}}})\end{aligned}$$

Configuration of the lattice equations: 9-point stencil

$$f(u,\widehat{u},\widetilde{u},\widetilde{\widetilde{u}},\widehat{\widetilde{u}},\widehat{\widehat{u}},\widehat{\widetilde{\widetilde{u}}},\widehat{\widehat{\widetilde{u}}},\widehat{\widehat{\widetilde{\widetilde{u}}}};{\color{red}p,q})=0$$



So far, no classification result for such equations exist.

Key Integrability features of P△Es # 1: Singularity Confinement

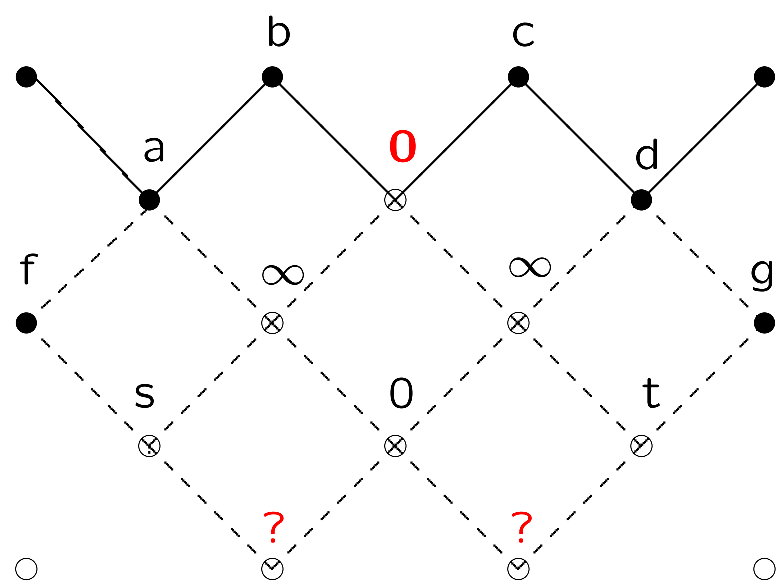
V. Papageorgiou, B. Grammaticos and A. Ramani, (1991): Observation that in integrable cases *singularities induced by initial data do not propagate*.

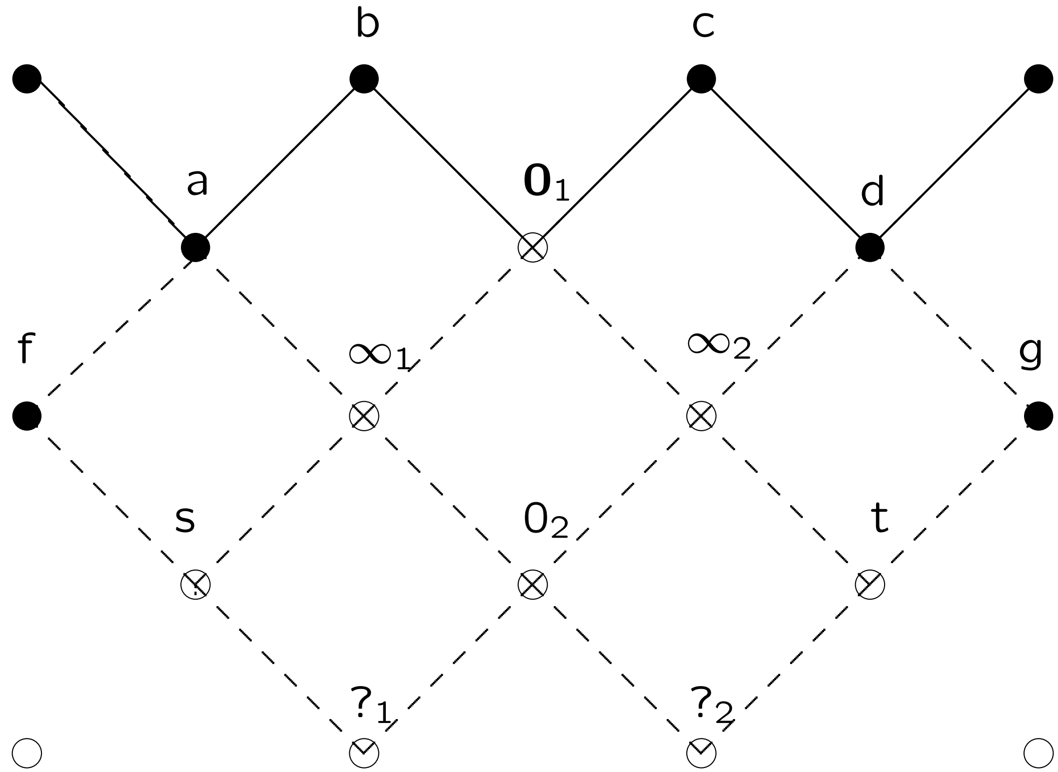
Example: Lattice KdV equation

$$w_{n+1,m+1} - w_{n,m} = \frac{1}{w_{n+1,m}} - \frac{1}{w_{n,m+1}}$$

i.e. *rational map* under initial value problems on the lattice.

Question: How do initial data progress, when one hits a singularity?





Calculating the various iteration precisely, starting from: $0_1 = \varepsilon$ (small) we find:

$$\infty_1 = b + \frac{1}{\varepsilon} - \frac{1}{a} \qquad , \qquad s = a + \frac{1}{\infty_1} - \frac{1}{f}$$

$$\infty_2 = c - \frac{1}{\varepsilon} + \frac{1}{d} \qquad , \qquad t = d - \frac{1}{\infty_1} + \frac{1}{g}$$

$$0_2 = \varepsilon + \frac{1}{\infty_2} - \frac{1}{\infty_1} \;=\; -\varepsilon + \left(b - c - \frac{1}{a} - \frac{1}{d}\right)\varepsilon^2 + \ldots$$

Then in the next step we find finite values:

$$\begin{aligned}
?_1 &= \infty_1 + \frac{1}{0_2} - \frac{1}{s} = c + \frac{1}{d} - \frac{1}{a - 1/f} + \mathcal{O}(\varepsilon) \\
?_2 &= \infty_2 - \frac{1}{0_2} + \frac{1}{t} = b - \frac{1}{a} + \frac{1}{d + 1/g} + \mathcal{O}(\varepsilon)
\end{aligned}$$

Remark: Note that if the lattice equation is deformed, e.g. by taking:

$$w_{n+1,m+1} - w_{n,m} = \frac{1}{w_{n+1,m}} - \frac{\lambda}{w_{n,m+1}}$$

wit $\lambda \neq 1$, this fine cancellation no longer happens, and singularities will again occur at $?_1$ and $?_2$, and will persist throughout! In that case the singularities are no longer confined to a finite number of iteration steps, and we conclude that the corresponding map is not integrable any more.

Key integrability features # 2: Multidimensional consistency

Take the example of the lattice MKdV equation:

$$p(v\widehat{v} - \widetilde{v}\widehat{\widetilde{v}}) = q(v\widetilde{v} - \widehat{v}\widehat{\widetilde{v}})$$

or written out in the long way:

$$p\left(v_{n,m}v_{n,m+1} - v_{n+1,m}v_{n+1,m+1}\right) = q\left(v_{n,m}v_{n+1,m} - v_{n,m+1}v_{n+1,m+1}\right)$$

Let us reconsider the role of the parameters p, q (the "lattice parameter"). Normally, they are considered to be fixed when solving the equation on the lattice. However, there is more to it:

Statement: the PΔE represents a *compatible parameter-family* of partial difference equations which can be embedded *consistently* in a multidimensional lattice on which the evolution is well-posed.
[FWN, AJ Walker, Glasgow Math J 43A (2001) 109]

Meaning: to each value of the lattice parameters p, q we can associate a discrete variable corresponding to a direction in a multidimensional lattice, s.t. the solution u can be considered as a *function* on this multidimensional lattice;

$$v = v_{n,m,h,\dots} = v(n,m,h,\dots;p,q,r,\dots)$$

with

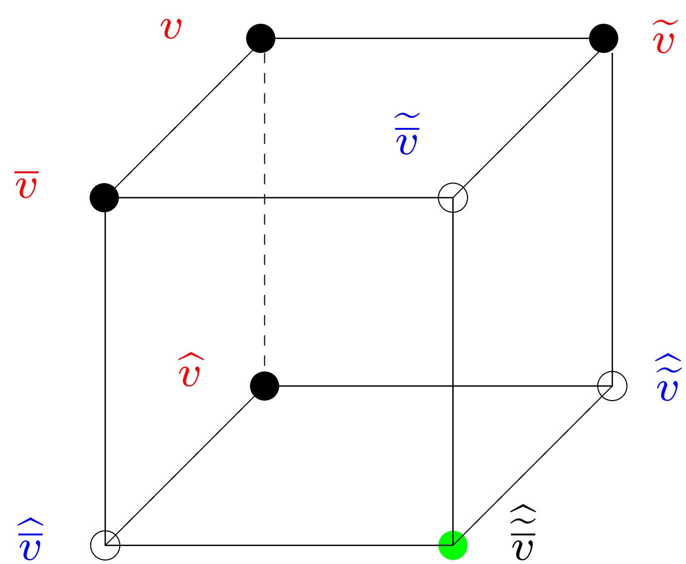
$$\widetilde{v} := v_{n+1,m,h} \quad , \quad \widehat{v} := v_{n,m+1,h} \quad , \quad \overline{v} := v_{n,m,h+1}$$

Embedding the lattice SKdV equation in a three-dimensional lattice:

$$\begin{aligned}
p(v\widehat{v}-\widetilde{v}\widehat{\widetilde{v}}) &= q(v\widetilde{v}-\widehat{v}\widehat{\widetilde{v}}) \\
q(v\overline{v}-\widehat{v}\widehat{\overline{v}}) &= r(v\widehat{v}-\overline{v}\widehat{\overline{v}}) \\
r(v\widetilde{v}-\overline{v}\widetilde{\overline{v}}) &= p(v\overline{v}-\widetilde{v}\widetilde{\overline{v}})
\end{aligned}$$

These equations are consistent if the evaluations along the cube are independent of the way of calculating the final point.

Given : $v \;,\; \widetilde{v} \;,\; \widehat{v} \;,\; \overline{v} \;\Rightarrow\; \widehat{\widetilde{v}} \;,\; \widehat{\overline{v}} \;,\; \widetilde{\overline{v}} \;\Rightarrow\; \widehat{\widetilde{\overline{v}}}$



In fact:

$$\widehat{\widetilde{\overline{v}}} = \frac{(p^2-q^2)r\widetilde{v}\widehat{v} + (q^2-r^2)p\widehat{v}\overline{v} + (r^2-p^2)q\overline{v}\widetilde{v}}{(p^2-q^2)r\overline{v} + (q^2-r^2)p\widetilde{v} + (r^2-p^2)q\widehat{v}}$$

Lax pair for lattice MKdV

Lax pair: overdetermined *linear* system, the consistency condition of wich yields the *nonlinear* equation under consideration.

Derivation of the Lax pair. Idea: exploit the consistency of the lattice equation on the multidimensional lattice.

• Fix a "virtual" direction on the 3D lattice, e.g. the direction associated with lattice shift $v \to \overline{v}$ and lattice parameter k. This leads to the equations:

$$p(v\overline{v} - \widetilde{v}\widetilde{\overline{v}}) = k(v\widetilde{v} - \overline{v}\widetilde{\overline{v}})$$
$$q(v\overline{v} - \widehat{v}\widehat{\overline{v}}) = k(v\widehat{v} - \overline{v}\widehat{\overline{v}})$$

• Consider the shifted object $\overline{v} := V$ to be a new dependent variable of the two main lattice variables n, m, and solve for the dynamics:

$$\widetilde{V} = \frac{(k\widetilde{v} - pV)v}{kV - p\widetilde{v}} \quad , \quad \widehat{V} = \frac{(k\widehat{v} - qV)v}{kV - q\widehat{v}}$$

in which $\widetilde{v}$ and $\widehat{v}$ represent the "physical" shifts on the lattice.

• Linearise these two fractional linear equations by the substitution $V = f/g$ and separate into linear system of eqs. for f and g:

$$\begin{pmatrix} \widetilde{f} \\ \widetilde{g} \end{pmatrix} = \begin{pmatrix} -p & k\widetilde{v} \\ k/v & -p\widetilde{v}/v \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \quad , \quad \begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix} = \begin{pmatrix} -q & k\widehat{v} \\ k/v & -q\widehat{v}/v \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

where in each an arbitrary "splitting" factor is chosen such that the determinants of the 2×2 matrices are of the form $\widetilde{a}/a$, respectively $\widehat{a}/a$ (up to a constant).

Thus, taking $\varphi \equiv (f,g)^T$ we obtain an overdetermined linear system (Lax pair) of the form:

$$\widetilde{\boldsymbol{\phi}} = L\boldsymbol{\phi} \quad , \quad \widehat{\boldsymbol{\phi}} = M\boldsymbol{\phi} \ ,$$

the compatibility of which arising from

$$(\widetilde{\boldsymbol{\phi}})\widehat{\ } = (\widehat{\boldsymbol{\phi}})\widetilde{\ } \quad \Rightarrow \quad \widehat{L}M = \widetilde{M}L$$

leads back to the lattice MKdV in terms of the lattice directions given by the shifts $v \mapsto \widetilde{v}$, $v \mapsto \widehat{v}$.

$$\begin{tikzcd}
\boldsymbol{\phi} \arrow[r, "\boldsymbol{L}"] \arrow[d, dotted, "\boldsymbol{M}"'] & \widetilde{\boldsymbol{\phi}} \arrow[d, "\widetilde{\boldsymbol{M}}"] \\
\widehat{\boldsymbol{\phi}} \arrow[r, dotted, "\widehat{\boldsymbol{L}}"'] & \widehat{\widetilde{\boldsymbol{\phi}}}
\end{tikzcd}$$

Remark: The lattice parameter k plays the role of a *spectral parameter*. Note that it is not present in the nonlinear system!

Lax pair $\quad \Rightarrow \quad$ *isospectral deformation problem*

This can be used effectively to solve the IVPs for the nonlinear lattice equation under appropriate boundary conditions.

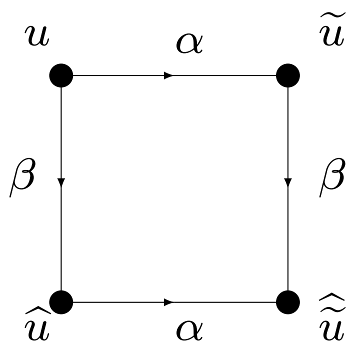
Classification Result

V Adler, A Bobenko and Yu Suris (2002) considered recently the classification problem of *quadrilateral lattices* integrable in the sense of "consistency around the cube".

General form:

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; \alpha, \beta) = 0$$

such that Q is <u>linear</u> in each vertex-variable u,$\widetilde{u}$, $\widehat{u}$,$\widehat{\widetilde{u}}$.



Further restrictions:

Linearity: Q multilinear in each vertex.

Symmetry: Invariance of Q under the group D_4 of symmetries of the square:

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; \alpha, \beta) = 0 \quad \Leftrightarrow \quad Q(u, \widehat{u}, \widetilde{u}, \widehat{\widetilde{u}}; \beta, \alpha) = 0 \quad \Leftrightarrow \quad Q(\widetilde{u}, u, \widehat{\widetilde{u}}, \widehat{u}; \alpha, \beta) = 0$$

Tetrahedron Condition: Impose that the evaluation of the point on the 3-dim. cube given by $\widehat{\widetilde{\overline{u}}}$ is actually *independent of* u.

Results:
The (exhaustive) list contains three classes of equations:

List Q:

1. $$\alpha(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-\beta(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})=\delta^2\alpha\beta(\beta-\alpha)$$

2. $$\begin{aligned}&\alpha(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-\beta(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})\\&+\alpha\beta(\alpha-\beta)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})=\alpha\beta(\alpha^2-\alpha\beta+\beta^2)\end{aligned}$$

3. $$\begin{aligned}&\beta(\alpha^2-1)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-\alpha(\beta^2-1)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})\\&-(\alpha^2-\beta^2)(\widehat{u}\widetilde{u}+u\widehat{\widetilde{u}})=\delta^2(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)/(4\alpha/\beta)\end{aligned}$$

4. Adler system:

$$\begin{aligned}&A[(u-b)(\widehat{u}-b)-(a-b)(c-b)]\,[(\widetilde{u}-b)(\widehat{\widetilde{u}}-b)-(a-b)(c-b)]+\\&+B[(u-a)(\widehat{u}-a)-(b-a)(c-a)]\,[(\widehat{u}-a)(\widehat{\widetilde{u}}-a)-(b-a)(c-a)]=ABC(a-b)\end{aligned}$$

with lattice parameters $(a,A)=(\wp(\alpha),\wp'(\alpha))$, $(b,B)=(\wp(\beta),\wp'(\beta))$, $(c,C)=(\wp(\beta-\alpha),\wp'(\beta-\alpha))$, on the Weierstrass elliptic curve

$$\Gamma=\{(x,X)|X^2=4x^3-g_2x-g_2\}$$

Other cases:

List H:

1.

$$(\widetilde{u}-\widehat{u})(u-\widehat{\widetilde{u}})=\alpha-\beta$$

2.

$$(\widetilde{u}-\widehat{u})(u-\widehat{\widetilde{u}})-(\alpha-\beta)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})=\alpha^2-\beta^2$$

3.

$$\alpha(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-\beta(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})=\delta(\alpha^2-\beta^2)$$

List A:

1.

$$\alpha(u+\widehat{u})(\widetilde{u}+\widehat{\widetilde{u}})-\beta(u+\widetilde{u})(\widehat{u}+\widehat{\widetilde{u}})=\delta^2\alpha\beta(\alpha-\beta)$$

2.

$$(\alpha^2-\beta^2)(u\widetilde{u}\widehat{u}\widehat{\widetilde{u}}+1)=\alpha(\beta^2-1)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})-\beta(\alpha^2-1)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})$$

Further Developments:

- Yang-Baxter mappings from quadrilateral lattice equations (V. Papageorgiou, A. Tongas, A. Veselov, 2005): derivation of quadrirational maps of the type

$$R : (u, v) \quad \mapsto \quad (x, y)$$

by special choices of variables on the lattice (YB Variables) such that the set-theoretic YB equation holds:

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}$$

This is intimately related to the CAC property of the quadrilateral lattices.

- Reductions to finite-dimensional mappings via initial value problems on the lattice (V. Papageorgiou, FWN, R. Quispel, H. Capel, V. Enolskii);

- Symmetries and conservation laws (P. Hydon, D. Levi and P. Winternitz, R. Quispel)

- Higher-dimensional lattices: lattice Kadomtsev-Petviashvili (KP): (E. Date, T. Miwa & M. Jimbo, FWN et al. in 1980s; J. Nimmo & W. Schief, B. Konopelchenko, and many others in 1990s);

- Difference geometry: discrete surface theory (A. Bobenko and U. Pinkall, A. Doliwa and P. Santini et al. , W. Schief and B. Konopelchenko).

Similarity reduction on the lattice will be discussed in Lecture II.

Seed & Soliton Solutions (work with J. Atkinson & J. Hietarinta)

• In some of the cases of lattices of KdV and BSQ type an inverse scheme is known leading in particular to soliton type solutions on the lattice. However, for many of the new cases in the ABS list no solutions were so far constructed.

The cubic consistency of a given quadrilateral equation means that given one solution, u,

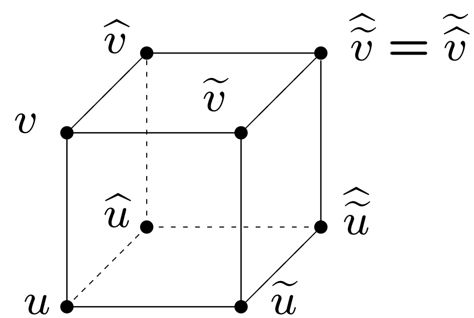
$$\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\tilde{u},\hat{u},\hat{\tilde{u}})=0\ ,$$

the pair of ordinary difference equations for v

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u,\widetilde{u},v,\widetilde{v})=0,\quad \mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u,\widehat{u},v,\widehat{v})=0,$$

are compatible. Moreover, if v satisfies this system it is also a solution of the original equation: We say that this forms an *auto Bäcklund transformation* (auto-BT). The new solution v may depend not only on u but also on the Bäcklund parameter $\mathfrak{l}$ and on one integration constant.

Diagrammatically:



Main Example: Adler's equation (Q4) in Jacobi form (Hietarinta, 2004)

$$\boxed{\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) - q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) - r(\widetilde{u}\widehat{u}+u\widehat{\widetilde{u}}) + pqr(1+u\widetilde{u}\widehat{u}\widehat{\widetilde{u}})}$$

parametrised in terms of Jacobi elliptic functions

$$\begin{array}{lclclc} p=\sqrt{k}\,\mathrm{sn}(\alpha;k) & , & q=\sqrt{k}\,\mathrm{sn}(\beta;k) & , & r=\sqrt{k}\,\mathrm{sn}(\alpha-\beta;k) & , \\ P=\mathrm{sn}'(\alpha;k) & , & Q=\mathrm{sn}'(\beta;k) & , & R=\mathrm{sn}'(\alpha-\beta;k) & , \end{array}$$

i.e. $\mathfrak{p}=(p,P)$, $\mathfrak{q}=(q,Q)$ and $\mathfrak{r}=(r,R)$ lie on the Jacobi type elliptic curve:

$$\Gamma: \quad X^2 = x^4+1-\left(k+\frac{1}{k}\right)x^2 \quad ,$$

with modulus k. Note that in this case parameters are related through the group law on the Jacobi type curve $\mathfrak{r}=\mathfrak{p}*\mathfrak{q}$ defined by:

$$r = \frac{Qp-Pq}{1-p^2q^2} = \frac{p^2-q^2}{Qp+Pq} \quad ,$$

$$R = \frac{p^2+q^2-r^2(1+p^2q^2)}{2pq} = \frac{[PQ+(k+k^{-1})pq](1+p^2q^2)-2pq(p^2+q^2)}{(1-p^2q^2)^2} \ .$$

For Adler's equation in Jacobi form we can identify the elementary solution

$$u=\sqrt{k}\,\mathrm{sn}(\xi_0+n\alpha+m\beta;k),$$

where ξ_0 is an arbitrary constant. This can be verified directly using standard Jacobi elliptic function identities.

However, *this solution does not generate a nontrivial BT chain.* Thus, there is a problem of finding appropriate seed solutions for the BTs!!

applying the auto-BT we obtain: two solutions which we label $\overline{u}$ and $\underline{u}$, given by

$$\overline{u} = \sqrt{k}\,\mathrm{sn}(\xi_0 + n\alpha + m\beta + \lambda; k), \qquad \underline{u} = \sqrt{k}\,\mathrm{sn}(\xi_0 + n\alpha + m\beta - \lambda; k),$$

where λ is the uniformising variable associated with $\mathfrak{l}$, i.e. $\mathfrak{l} = (l, L) = (\sqrt{k}\,\mathrm{sn}(\lambda; k), \mathrm{sn}'(\lambda; k)$ which is a trivial extension of the original solution.

We call such a seed a *non-germinating seed*.

To obtain a *germinating seed* we proceed as follows: we search for stationary solutions of the auto-BT obtained by solving the defining equations of the seed solution:

$$\boxed{\mathcal{Q}_{\mathfrak{p},\mathfrak{t}}(u, \widetilde{u}, u, \widetilde{u}) = 0, \quad \mathcal{Q}_{\mathfrak{q},\mathfrak{t}}(u, \widehat{u}, u, \widehat{u}) = 0}$$

A solution of this system is a fixed point of the auto-BT, i.e. it is constant in the lattice direction associated with the Bäcklund parameter which we label $\mathfrak{t} = (t, T) = (\sqrt{k}\ \mathrm{sn}(\theta; k), \mathrm{sn}'(\theta; k))$.

Note: It is not obvious that the above system of eqs. is compatible, i.e. that such fixed-points exist. The observation here is that for Adler's equation such a solution does indeed exist (for each given point $\mathfrak{t}$ on the parameter curve, and provides us with a germinating seed for the BT.

Remark: In fact this solution coincides with the non-germinating at the special value of the parameter $\mathfrak{t} = (0, 1)$ which is unit of the Abelian group of the curve.

To find such solutions let us analyse the solution of the first equation: It turns out that the resulting biquadratic is of the form

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{t}}(u,\widetilde{u},u,\widetilde{u}) = tpr(u^2\widetilde{u}^2+1) - t(u^2+\widetilde{u}^2) + 2(p-r)u\widetilde{u} = t\,\mathcal{H}_{\mathfrak{p}_\theta}(u,\widetilde{u}) \ ,$$

by making the identifications

$$r = \frac{pT-tP}{1-p^2t^2} \quad , \quad \mathfrak{p}_\theta = (p_\theta, P_\theta) \quad , \quad p_\theta^2 = pr \quad , \quad P_\theta = \frac{p-r}{t} \ .$$

with $\mathcal{H}_\mathfrak{p}$ defined by:

$$\mathcal{Q}\mathcal{Q}_{\widehat{u}\widetilde{u}} - \mathcal{Q}_{\widehat{u}}\mathcal{Q}_{\widetilde{u}} = tr\mathcal{H}_\mathfrak{p} \quad \text{with} \quad \mathcal{H}_\mathfrak{p}(u,\widetilde{u}) = p^2(1+u^2\widetilde{u}^2) - (u^2+\widetilde{u}^2) + 2Pu\widetilde{u} \ .$$

The remarkable obervation here is that the discriminant of the biquadratic $\mathcal{H}_{\mathfrak{p}_\theta}$ factorises as follows:

$$\boldsymbol{\Delta} = 4[(p-r)^2u^2 - t^2(pru^2-1)(pr-u^2)] = 4t^2pr\left(u^4+1-2\frac{1-T}{t^2}u^2\right) \ ,$$

using the relation:

$$2prT = p^2 + r^2 - t^2(1-p^2r^2) \ .$$

This suggests the introduction of a deformed curve, the *seed curve*, given by

$$\Gamma_\theta : \ X^2 = \mathcal{R}_\theta(x) \equiv x^4 + 1 - 2\frac{1-T}{t^2}x^2$$

where the label θ refers to the uniformising variable associated with the point $\mathfrak{t}$ on the original curve.

Furthermore, the deformed lattice parameter $\mathfrak{p}_\theta$ lies on the deformed curve: $\mathfrak{p}_\theta = (p_\theta, P_\theta) \in \Gamma_\theta$.

In fact, using the relation above, the new curve Γ_θ depends only on the chosen point $\mathfrak{t}$ on the curve, and not on the lattice parameter $\mathfrak{p}$ associated with the shift $u \mapsto \widetilde{u}$.

Thus, precisely a similar analysis holds for the solution of the companion equation $\mathcal{Q}_{\mathfrak{q},\mathfrak{t}}(u, \widehat{u}, u, \widehat{u}) = 0$ and it leads to the same deformed curve with deformed lattice parameter $\mathfrak{q}_\theta = (q_\theta, Q_\theta)$ defined by

$$q_\theta^2 = qr' \quad , \quad r' = \frac{qT - tQ}{1 - q^2 t^2} \quad \text{and} \quad Q_\theta = \frac{q - r'}{t} \ .$$

In terms of Jacobi elliptic functions on the original curve with modulus k we have:

$$\mathfrak{p} = (p, P) = (\sqrt{k}\,\mathrm{sn}(\alpha;k), \mathrm{sn}'(\alpha;k)) \quad , \quad \mathfrak{q} = (q, Q) = (\sqrt{k}\,\mathrm{sn}(\alpha;k), \mathrm{sn}'(\alpha;k)) \ ,$$

the θ-deformed curve carries a different modulus k_θ as given by

$$k_\theta + \frac{1}{k_\theta} = 2\frac{1 - \mathrm{sn}'(\theta;k)}{k\,\mathrm{sn}^2(\theta;k)}$$

where θ is the parameter of the point $\mathfrak{t} = (\sqrt{k}\,\mathrm{sn}(\theta;k), \mathrm{sn}'(\theta;k))$. Thus, the deformed lattice parameters $\mathfrak{p}_\theta$ and $\mathfrak{q}_\theta$ are parametrised by

$$\mathfrak{p}_\theta = (\sqrt{k_\theta}\,\mathrm{sn}(\alpha_\theta;k_\theta), \mathrm{sn}'(\alpha_\theta;k_\theta)) \quad , \quad \mathfrak{q}_\theta = (\sqrt{k_\theta}\,\mathrm{sn}(\beta_\theta;k_\theta), \mathrm{sn}'(\beta_\theta;k_\theta))$$

where α_θ and β_θ are to solved from:

$$\begin{aligned} k_\theta\,\mathrm{sn}^2(\alpha_\theta;k_\theta) &= k\ \mathrm{sn}(\alpha-\theta;k)\mathrm{sn}(\alpha;k) \ , \\ \mathrm{sn}'(\alpha_\theta;k_\theta) &= \frac{\mathrm{sn}(\alpha;k) - \mathrm{sn}(\alpha-\theta;k)}{\mathrm{sn}(\theta;k)} \ , \end{aligned}$$

and similar equations for β_θ, replacing α by β in the above formulae.

Now we turn to the actual solution of fixed point equations, leading to the germinating seed. In fact, By solving the biquadratic $t\mathcal{H}_{\mathfrak{p}_\theta}(u,\widetilde{u})=0$, which leads to a simple quadratic equation in terms of $\widetilde{u}$, the solution of which will be denoted by u_θ, since the discriminant $\Delta = 4t^2p_\theta^2\mathcal{R}_\theta(u_\theta)$ leads naturally to define a point $\mathfrak{u}_\theta=(u_\theta,U_\theta)$ on the deformed curve Γ_θ.

Thus, we obtain the *seed map* $\mathfrak{u}_\theta \mapsto \widetilde{\mathfrak{u}}_\theta$ (associated with the shift $u\mapsto \widetilde{u}$) which leaves the curve Γ_θ invariant:

$$\widetilde{u}_\theta=\frac{P_\theta u_\theta+p_\theta U_\theta}{1-p_\theta^2u_\theta^2}\quad,\quad \widetilde{U}_\theta=\frac{\left(P_\theta U_\theta-(k_\theta+k_\theta^{-1})p_\theta u_\theta\right)(1+p_\theta^2u_\theta^2)+2p_\theta u_\theta(p_\theta^2+u_\theta^2)}{(1-p_\theta^2u_\theta^2)^2}\ ,$$

Since this map is nothing but the (Abelian) group action on the deformed curve, it is evident that a similar map $\mathfrak{u}_\theta\widehat{\mathfrak{u}}_\theta$ but now associated with the shift $u\mapsto\widehat{u}$, with lattice parameter $\mathfrak{q}$ instead of $\mathfrak{p}$, commutes with the map above:

$$\widehat{u}_\theta=\frac{Q_\theta u_\theta+q_\theta U_\theta}{1-q_\theta^2u_\theta^2}\quad,\quad \widehat{U}_\theta=\frac{\left(Q_\theta U_\theta-(k_\theta+k_\theta^{-1})q_\theta u_\theta\right)(1+q_\theta^2u_\theta^2)+2q_\theta u_\theta(q_\theta^2+u_\theta^2)}{(1-q_\theta^2u_\theta^2)^2}\ .$$

Since these map commute, they can be simultaneously solved and lead to the *germinating seed solution*:

$$\boxed{u_\theta=\sqrt{k_\theta}\,\mathrm{sn}(\xi_\theta;k_\theta)\quad,\quad \text{with}\quad \xi_\theta=\xi_{\theta,0}+n\alpha_\theta+m\beta_\theta}$$

in terms of the Jacobi sn function with modulus k_θ,

The new solution can be used as starting point for application of Bäcklund transformations, will be used in section 3 to generate 1-soliton solutions for the Jacobi form of Adler's equation.

Group-theoretical Explanation:

The "Jacobi quadrilateral"

$$\mathcal{Q}_{p,q,r}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) - q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) - r(\widetilde{u}\widehat{u}+u\widehat{\widetilde{u}}) + pqr(1+u\widetilde{u}\widehat{u}\widehat{\widetilde{u}}) \ .$$

in terms of which the Jacobi form of Adler's system is defined, can also be considered as an object depending on three parameters (p,q,r) (so far unrelated), which generates a canonical biquadratic through:

$$\mathcal{Q}\mathcal{Q}_{\widehat{\widetilde{u}u}} - \mathcal{Q}_{\widehat{u}}\mathcal{Q}_{\widetilde{u}} = p^2qr(1+u^2\widetilde{u}^2) - qr(u^2+\widetilde{u}^2) + (p^2-q^2-r^2+p^2q^2r^2)u\widetilde{u} \ .$$

If p, q, r are related through the group law on the curve Γ (with modulus k) then as a consequence of the identity

$$2qrP = p^2(1+q^2r^2) - q^2 - r^2 \quad , \qquad \Rightarrow \qquad \mathcal{Q}\mathcal{Q}_{\widehat{\widetilde{u}u}} - \mathcal{Q}_{\widehat{u}}\mathcal{Q}_{\widetilde{u}} = qr\mathcal{H}_{\mathfrak{p}}$$

and can then be written as $\mathcal{H}_{\mathfrak{p}}(u,\widetilde{u}) = \gamma H(s,u,\widetilde{u})$ in terms of the canonical <u>tri</u>quadratic on the curve given by:

$$H(x,y,z) = \frac{1}{\sqrt{k}}\left(1+x^2y^2+y^2z^2+z^2x^2\right) - \sqrt{k}\,\left(x^2+y^2+z^2+x^2y^2z^2\right) - 2\left(k-\frac{1}{k}\right)xyz \ ,$$

identifying

$$p^2 = \frac{1-ks^2}{k-s^2} \quad , \quad P = \frac{(k-1/k)s}{s^2/\sqrt{k}-\sqrt{k}} \quad , \quad \gamma = \left(\sqrt{k}-\frac{s^2}{\sqrt{k}}\right)^{-1} \ ,$$

which forms a rational realisation of the elliptic curve $P^2 = \mathcal{R}(p)$ in terms of a new parameter s.

Note that for the triquadratic H we have the factorisation property:

$$H_x^2 - 2HH_{xx} = 4\mathcal{R}(y)\mathcal{R}(z) \quad , \quad \mathcal{R}(x) \equiv x^4 + 1 - \left(k + \frac{1}{k}\right) x^2 \ ,$$

whereas the condition $\mathcal{H}_{\mathfrak{p}}(u, \widetilde{u}) = 0$ implies that the map $u \mapsto \widetilde{u}$ amounts to a shift on this elliptic curve (Chasles correspondence). This translates into the condition that $H(p, q, r) = 0$ is satisfied when $\mathfrak{p} * \mathfrak{q} * \mathfrak{r} = \mathfrak{s}$, with $\mathfrak{s} = (1/\sqrt{k}, 0)$ which is the map of the unit of the curve in the realisation above.

The emergence of the seed solution and corresponding deformed parameters $\mathfrak{p}_\theta$, $\mathfrak{q}_\theta$ can be illustrated by the following commuting diagram:

$$\begin{tikzcd}
\mathcal{Q}_{\mathfrak{p},\mathfrak{t}}(u, \widetilde{u}, v, \widetilde{v}) = 0 \arrow[r, "\mathcal{Q}\mathcal{Q}_{\widetilde{u},\widetilde{v}} - \mathcal{Q}_{\widetilde{u}}\mathcal{Q}_{\widetilde{v}}"] \arrow[d, "\mathrm{proj}_{v \mapsto u}"'] & \mathcal{H}_{\mathfrak{p}}(u, v) = 0 \arrow[d, "\mathrm{proj}_{v \mapsto u}"] \\
\mathcal{H}_{\mathfrak{p}_\theta}(u, \widetilde{u}) = 0 \arrow[r, "\mathcal{H}_{\widetilde{u}}^2 - 2\mathcal{H}\mathcal{H}_{\widetilde{u},\widetilde{u}}"'] & \mathcal{H}_{\mathfrak{p}}(u, u) = \mathcal{R}_\theta(u)
\end{tikzcd}$$

defining the correspondence between between the old parameters $\mathfrak{p}$, $\mathfrak{q}$ and the deformed ones, which we indicate by the symbol $\delta_\theta(\mathfrak{p}, \mathfrak{p}_\theta)$.

One-Soliton Solution

We will illustrate that the canonical seed solution u_θ *germinates* –by applying the BT to it and thus obtaining the one-soliton solution of Adler's equation.

We need to solve the set of simultaneous O∆Es in v:

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u_\theta,\widetilde{u}_\theta,v,\widetilde{v})=0,\quad \mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u_\theta,\widehat{u}_\theta,v,\widehat{v})=0,$$

which define the BT $u_\theta \mapsto v$ with Bäcklund parameter $\mathfrak{l}$, where

$$u_\theta(n,m)=\sqrt{k_\theta}\,\mathrm{sn}(\xi_\theta(n,m);k_\theta)\quad,\quad \xi_\theta(n,m)=\xi_{\theta,0}+n\alpha_\theta+m\beta_\theta\ .$$

This seed solution can be covariantly extended in the lattice direction associated with the parameter $\mathfrak{l}=(l,L)=(\sqrt{k}\,\mathrm{sn}(\lambda;k),\mathrm{sn}'(\lambda;k))$, by letting the initial value $\xi_{\theta,0}$ depend on the lattice variable (with lattice shift denoted by $\bar{\ }$) as:

$$\overline{\xi}_\theta=\xi_\theta+\lambda_\theta\quad\Rightarrow\quad \overline{u}_\theta=\sqrt{k_\theta}\,\mathrm{sn}(\xi_\theta+\lambda_\theta;k_\theta)\ ,$$

and where λ_θ is the deformed BT parameter, related to $\mathfrak{l}$ by the same correspondence $\delta_\theta(\mathfrak{l},\mathfrak{l}_\theta)$ as before.

Thus, we have the following set of equations satisfied by these BT-shifted seed solutions:

$$\begin{aligned}
&\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u_\theta,\widetilde{u}_\theta,\overline{u}_\theta,\widetilde{\overline{u}}_\theta)=0\ , &\qquad &\mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u_\theta,\widehat{u}_\theta,\overline{u}_\theta,\widehat{\overline{u}}_\theta)=0\ ,\\
&\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u_\theta,\widetilde{u}_\theta,\underline{u}_\theta,\widetilde{\underline{u}}_\theta)=0\ , &\qquad &\mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u_\theta,\widehat{u}_\theta,\underline{u}_\theta,\widehat{\underline{u}}_\theta)=0\ ,
\end{aligned}$$

which are in fact discrete Riccati equations of the form:

$$v\widetilde{v}+a\widetilde{v}+bv+c=0$$

in terms of the third and fourth arguments in these quadrilaterals.

Lemma: If v_1 and v_2 are two given independent solutions of a Riccati equation of the form above, i.e. $v\widetilde{v}+a\widetilde{v}+bv+c=0$, then the linear combination

$$v=\frac{v_1-\rho v_2}{1-\rho}$$

is a solution of the same equation, provided ρ obeys the following linear homogeneous first order difference equation;

$$\widetilde{\rho}=-\frac{v_2\widetilde{v}_1+a\widetilde{v}_1+bv_2+c}{v_1\widetilde{v}_2+a\widetilde{v}_2+bv_1+c}\,\rho\ .$$

We will use for the Riccati equations coming from the quadrilaterals, which are of the form above by identifying:

$$a=-\frac{ru+l\widetilde{u}}{p(1+rlu\widetilde{u})}\quad,\quad b=-\frac{r\widetilde{u}+lu}{p(1+rlu\widetilde{u})}\quad,\quad c=\frac{rl+u\widetilde{u}}{1+rlu\widetilde{u}}\ ,$$

with $r=(pL-lP)/(1-p^2l^2)$, and $u=u_\theta$, taking $v_1=\overline{u}_\theta$ and $v_2=\underline{u}_\theta$.

Because the auto-BT share their particular solutions, $\overline{u}_\theta$ and $\underline{u}_\theta$, a similar substitution holds for the equations in terms of the other lattice shift (associated with $\mathfrak{q}$), and thus these substitutions reduce both equations simultaneously.

$$\widetilde{\rho}=\left(\frac{p_\theta l-l_\theta p}{p_\theta l+l_\theta p}\right)\left(\frac{1-l_\theta\overline{p}_\theta u_\theta\widetilde{u}_\theta}{1+l_\theta\underline{p}_\theta u_\theta\widetilde{u}_\theta}\right)\rho,\qquad \widehat{\rho}=\left(\frac{q_\theta l-l_\theta q}{q_\theta l+l_\theta q}\right)\left(\frac{1-l_\theta\overline{q}_\theta u_\theta\widehat{u}_\theta}{1+l_\theta\underline{q}_\theta u_\theta\widehat{u}_\theta}\right)\rho,$$

We take the above linear equations as the defining equations for the *plane-wave factor* ρ (i.e. a discrete elliptic analogue of the exponential function).

where we mildly abuse notation by introducing the modified parameters

$$\overline{p}_\theta = \sqrt{k_\theta}\ \mathrm{sn}(\alpha_\theta + \lambda_\theta; k_\theta), \quad \underline{p}_\theta = \sqrt{k_\theta}\ \mathrm{sn}(\alpha_\theta - \lambda_\theta; k_\theta),$$
$$\overline{q}_\theta = \sqrt{k_\theta}\ \mathrm{sn}(\beta_\theta + \lambda_\theta; k_\theta), \quad \underline{q}_\theta = \sqrt{k_\theta}\ \mathrm{sn}(\beta_\theta - \lambda_\theta; k_\theta)$$

(evn though p_θ and q_θ do not depend on lattice shifts).

The compatibility of the system for ρ can be verified directly, i.e. specifically $\widetilde{\widehat{\rho}} = \widehat{\widetilde{\rho}}$, and arises as a consequence of the following remarkable identity for the Jacobi sn function:

$$\left(\tfrac{1-k^2\mathrm{sn}(\lambda)\mathrm{sn}(\alpha+\lambda)\mathrm{sn}(\xi)\mathrm{sn}(\xi+\alpha)}{1+k^2\mathrm{sn}(\lambda)\mathrm{sn}(\alpha-\lambda)\mathrm{sn}(\xi)\mathrm{sn}(\xi+\alpha)}\right)\left(\tfrac{1-k^2\mathrm{sn}(\lambda)\mathrm{sn}(\beta+\lambda)\mathrm{sn}(\xi+\alpha)\mathrm{sn}(\xi+\alpha+\beta)}{1+k^2\mathrm{sn}(\lambda)\mathrm{sn}(\beta-\lambda)\mathrm{sn}(\xi+\alpha)\mathrm{sn}(\xi+\alpha+\beta)}\right) =$$
$$= \left(\tfrac{1-k^2\mathrm{sn}(\lambda)\mathrm{sn}(\beta+\lambda)\mathrm{sn}(\xi)\mathrm{sn}(\xi+\beta)}{1+k^2\mathrm{sn}(\lambda)\mathrm{sn}(\beta-\lambda)\mathrm{sn}(\xi)\mathrm{sn}(\xi+\beta)}\right)\left(\tfrac{1-k^2\mathrm{sn}(\lambda)\mathrm{sn}(\alpha+\lambda)\mathrm{sn}(\xi+\beta)\mathrm{sn}(\xi+\alpha+\beta)}{1+k^2\mathrm{sn}(\lambda)\mathrm{sn}(\alpha-\lambda)\mathrm{sn}(\xi+\beta)\mathrm{sn}(\xi+\alpha+\beta)}\right).$$

The one-soliton for the Jacobi form of Adler's equation, which we denote v_1, is thus given by

$$\boxed{v_1 = \tfrac{\sqrt{k_\theta}}{1-\rho}\left(\mathrm{sn}(\xi_\theta - \lambda_\theta; k_\theta) - \rho\ \mathrm{sn}(\xi_\theta + \lambda_\theta; k_\theta)\right), \qquad \xi_\theta = \xi_{\theta,0} + n\alpha_\theta + m\beta_\theta}$$

with ρ defined in terms of the earlier equations.

Once we have a seed and the 1-soliton solution, we can next use the permutability condition of BTs (which is once again a version of the original quadrilateral lattice equation) to obtain 2-soliton solutions, etc.

Compatible Continuous Systems

The Adler system allows in a similar way as before to take continuum limits.

Straight continuum limits: Limit that $\mathfrak{q} \to (0,1)$, implying $\beta \to 0$ Setting $\mathrm{sn}(\beta) = \epsilon \to 0$, we have $q = \sqrt{k}\epsilon$, $\widehat{u} \to u + \sqrt{\epsilon} u_x + \ldots$, we obtain

$$\boxed{p u_x \widetilde{u}_x = \sqrt{k}\,\mathcal{H}_{\mathfrak{p}}(u, \widetilde{u})}$$

which is in the form of the Bäcklund transformation of Krichever-Novikov (KN) equation in the form

$$u_t = u_{xxx} - \frac{3}{2u_x}\left(u_{xx}^2 - u^4 - 1 + \left(k + \frac{1}{k}\right) u^2\right)$$

Eliminating the derivative of the BT by composing it with a similar form with parameter $\mathfrak{q}$, $q u_x \widehat{u}_x = \sqrt{k}\,\mathcal{H}_{\mathfrak{q}}(u, \widehat{u})$ gives us an equation of the form

$$q^2 \mathcal{H}_{\mathfrak{p}}(u, \widetilde{u}) \mathcal{H}_{\mathfrak{p}}(\widehat{u}, \widehat{\widetilde{u}}) - p^2 \mathcal{H}_{\mathfrak{q}}(u, \widehat{u}) \mathcal{H}_{\mathfrak{q}}(\widetilde{u}, \widehat{\widetilde{u}}) = 0$$

which factors in the form $Q_{\mathfrak{p},\mathfrak{q}}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) \mathcal{Q}_{\mathfrak{p},-\mathfrak{q}}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}})$.

Skew continuum limit: Limit that $\mathfrak{q} \to \mathfrak{p}$, implying that $\beta \to \alpha$. Setting $\beta = \alpha + \epsilon$, we have $q \sim p + \epsilon\sqrt{k}P$, $Q \sim P + \epsilon\sqrt{k}\,p(p^2 - k - 1/k)$, and $r \sim -\epsilon\sqrt{k}$. Taking the skew limit

$$u \to \widetilde{u} + \epsilon \widetilde{u}_\tau + \ldots$$

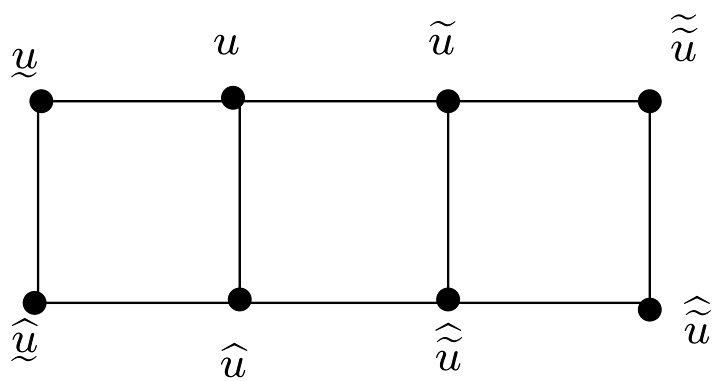
we obtain:

$$pu_\tau = \sqrt{k}\,\frac{Pu(\widetilde{u} + \underset{\sim}{u}) + p^2 - u^2 - \widetilde{u}\,\underset{\sim}{u}(1 - p^2u^2)}{\widetilde{u} - \underset{\sim}{u}}$$

Full continuum limit: This is more involved and produces the KN equation in leading order.

Note that the seed and soliton solutions can be easily extended to the continuum limits, and this yields the first nontrivial *explicit* solutions of the KN equation!

Furthermore, the τ differential-difference flow is related to an equation given by R. Yamilovseveral years ago. It provides a *compatible* flow with the original lattice equation, which diagrammatically follows the following computation:



Conclusions

We have shown that there is a rich class of lattice equations coming from fairly simple assumptions (multilinearity, covariance, D_4 symmetry, etc.) having as a key integrability criterium the multidimensional consistency (CAC) property, which can be algorithmically verified.

A full list (modulo some assumptions), resembling somewhat the Painlevĺist of second order transcendental ODEs, was obtained by Adler et al.including as top equation Adler's elliptic lattice. All aother equations in the list can be obtained by degeneration (coalescence).

The CAC property not only leads to (isospectral) Lax pairs of these equations, but also to an effective solution method as we have demonstrated

Some Recent Literature:

1. V. Adler, Int. Math Res. Notices, # 1 (1998).
2. F.W. Nijhoff, Phys. Lett. A297 (2002).
3. A. Bobenko and Yu. Suris, IRMN (2002).
4. V. Adler, A. Bobenko and Yu. Suris, Commun. Math. Phys. (2002).
5. V. Adler and Yu. Suris, *Q4*, IMRN (2004).
6. J. Hietarinta, Journ, Nonl. Math. Phys. (2005).
7. J. Atkinson, J. Hietarinta and F.W. Nijhoff, in preparation.