NAME (please print): ______________________________

# FINAL EXAM (SAMPLE 1)

1. Which of these PDEs is the heat equation, the wave equation, or the Laplace's equation? Write the name of the equation next to it.

- $u_{tt} = 2u_{xx}$,
- $u_t = 3u_{xx} + 3$,
- $2u_t = 3u_{xx}$,
- $2u_{xx} + 2u_{yy} = 0$.

2. (a) How many initial data, and how many boundary data do we need to prescribe for the heat equation

$$u_t = ku_{xx}, \quad x \in (0, L), t > 0,$$

to have a well-defined initial boundary value problem?

(b) List at least one type of boundary data for the heat equation that can be prescribed at the end points?

3. State the principle of superposition for the non-homogeneous problems $L(u) = f$, $f \neq 0$.

4. (a) Solve the following initial-boundary-value problem for the heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad x \in (0, L), t > 0, \\ u(0, t) &= 0, \\ u(L, t) &= 0, \\ u(x, 0) &= \sin \frac{2\pi x}{L} + 2 \sin \frac{3\pi x}{L}. \end{aligned}$$

(b) For the initial-boundary-value problem listed above, what is the behavior of the solution as $t \to \infty$?

5. Find the eigenvalues and the corresponding eigenfunctions of the following eigenvalue problem:

$$\begin{aligned} \frac{d^2\phi}{dx^2} &= -\lambda\phi, \quad x \in (-\pi, \pi) \\ \phi(-\pi) &= \phi(\pi), \\ \frac{d\phi}{dx}(-\pi) &= \frac{d\phi}{dx}(\pi). \end{aligned}$$

6. True of false:

(a) Solutions of the wave equation $u_{tt} = c^2 u_{xx}$ decay in time.

(b) Solutions of the heat equation $u_t = ku_{xx}$, $k > 0$, decay in time.

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7. Using D'Alambert formula, find the solution of the following initial-value problem for the wave equation:

$$\begin{aligned} u_{tt} &= 9u_{xx}, \quad x \in (-\infty, \infty),\ t > 0, \\ u(x,0) &= 3e^{-x^2}, \\ u_t(x,0) &= x^2. \end{aligned}$$

8. Find the solution of the following initial-boundary value problem for a damped spring:

$$\begin{aligned} u_{tt} &= 2u_{xx} - 2u_t, \quad x \in (0,1),\ t > 0, \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= \sin(2\pi x), \\ u_t(x,0) &= 0. \end{aligned}$$

9. What happens with the total energy $E$ of the vibrating string satisfying

$$u_{tt} = u_{xx}, \quad x \in (0,L), t > 0$$

if $\frac{\partial u}{\partial x}(0,t) = 0$ and $u(L,t) = 0$?

10. Solve the Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the following boundary conditions:

$$\frac{\partial u}{\partial x}(0,y) = 0, u(L,y) = \sin\left(\frac{2\pi y}{H}\right), u(x,0) = 0, u(x,H) = 0.$$

11. Are solutions of $\left\{ \begin{array}{rcll} \triangle u &=& 0 & \text{in } \Omega \\ u &=& f(x) & \text{on } \partial\Omega \end{array} \right\}$ unique? Prove your statement.

12. Solve the Laplace's equations inside a circular disk of radius 1, with the prescribed boundary data equal to:

$$u(1,\theta) = 2 + \sin(\theta)\cos(\theta), \quad -\pi \le \theta < \pi.$$

13. When do we say that a given PDE problem is well-posed?

1. 
- $u_{tt} = 2u_{xx}$ - wave equation
- $u_t = 3u_{xx} + 3$ - heat equation
- $2u_t = 3u_{xx}$ - heat equation
- $2u_{xx} + 2u_{yy} = 0$ - Laplace's equation

2. (a) We need 1 initial datum, and 2 boundary data.
(b) $u(0) = 0$, $u(L) = f(t)$

3. Let $u_1$ and $u_2$ be two homogeneous solutions, i.e.,
$L(u_1) = 0$ and $L(u_2) = 0$
and let $u_p$ be a particular solution of $L(u) = f$. Then

$$\boxed{u = c_1 u_1 + c_2 u_2 + u_p}, \quad c_1, c_2 \text{ - constants}$$

is another particular solution of $L(u) = f$.

4. (a) $\begin{cases} u_t = u_{xx} \\ u(0,t) = 0, u(L,t) = 0 \\ u(x,0) = \sin\frac{2\pi x}{L} t + 2\sin\frac{3\pi x}{L} \end{cases}$

SOLUTION: $u(x,t) = \phi(x) G(t) \Rightarrow$ From PDE: $\begin{cases} G'(t)\phi(x) = G(t)\phi''(x) \quad / \div G\phi \\ \frac{G'(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda, \ \lambda > 0 \end{cases}$

EIGENVALUE PROBLEM:

$$\left.\begin{array}{l} \phi''(x) = -\lambda\phi(x) \\ \phi(0) = \phi(L) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \phi_n(x) = \sin\frac{n\pi x}{L} \\ \lambda_n = \left(\frac{n\pi}{L}\right)^2 \end{array}, \ n = 1, 2, 3, \ldots$$

GENERAL SOLUTION:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cdot G_n(t).$$

SOLVE FOR $G_n(t)$: $\frac{dG}{dt} = \lambda G \Rightarrow G_n(t) = e^{-\lambda t} = e^{-\left(\frac{n\pi}{L}\right)^2 t}$

GENERAL SOLUTION: $\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-\left(\frac{n\pi}{L}\right)^2 t}}$

INITIAL DATA: $A_2 = 1$, $A_3 = 2$, $A_n = 0$ for $n \neq 2, 3$

SOLUTION: $\boxed{u(x,t) = \sin\frac{2\pi x}{L} e^{-\left(\frac{2\pi}{L}\right)^2 t} + 2\sin\frac{3\pi x}{L} e^{-\left(\frac{3\pi}{L}\right)^2 t}}$

4(b) $\lim_{t\to\infty} u(x,t)=0$.

5 $\frac{d^2\phi}{dx^2} = -\lambda\phi \Rightarrow \phi(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

$\phi(-\pi)=\phi(\pi) \Rightarrow A\cos\sqrt{\lambda}\pi - B\sin\sqrt{\lambda}\pi = A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi$

$\phi'(-\pi)=\phi'(\pi) \Rightarrow A\sqrt{\lambda}\sin\sqrt{\lambda}\pi + B\sqrt{\lambda}\cos\sqrt{\lambda}\pi = -A\sqrt{\lambda}\sin\sqrt{\lambda}\pi + B\sqrt{\lambda}\cos\sqrt{\lambda}\pi$

$\Rightarrow \boxed{2B\sin\sqrt{\lambda}\pi=0}$ and $\boxed{2A\sqrt{\lambda}\sin\sqrt{\lambda}\pi=0} \Rightarrow \sin\sqrt{\lambda}\pi=0 \Rightarrow \boxed{\lambda=n^2}$

$$\boxed{\phi_n(x) = A_n\cos nx + B_n\sin nx}$$

6 (a) FALSE; (b) TRUE.

7 D'Alambert: $\boxed{u(x,t) = \frac{1}{2}\left(f(x-ct)+f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$

$c=3$: $u(x,t) = \frac{1}{2}\left[3e^{-(x+3t)^2} + 3e^{-(x-3t)^2}\right] + \frac{1}{6}\int_{x-3t}^{x+3t} s^2\,ds$

$= \frac{1}{2}\left[3e^{-(x+3t)^2} + 3e^{-(x-3t)^2}\right] + \frac{1}{6}\left[\frac{s^3}{3}\right]_{x-3t}^{x+3t}$

$= \frac{1}{2}\left[3e^{-(x+3t)^2} + 3e^{-(x-3t)^2}\right] + \frac{1}{6}\left[\frac{(x+3t)^3}{3} - \frac{(x-3t)^3}{3}\right]$

8 $\begin{cases} u_{tt} = 2u_{xx} - 2u_t, & x\in(0,1),\ t>0 \\ u(0,t)=u(1,t)=0 \\ u(x,0)=\sin(2\pi x) \\ u_t(x,0)=0 \end{cases}$

$u(x,t) = h(t)\phi(x) \Rightarrow$ FROM PDE: $\begin{cases} h''\phi = 2\phi''h - 2\phi h' \quad / \div 2h\phi \\ \frac{h''}{2h} = \frac{\phi''}{\phi} - \frac{h'}{h} = -\lambda \\ \frac{h''}{2h} + \frac{h'}{h} = \frac{\phi''}{\phi} = -\lambda \end{cases}$

- EIGENVALUE PROBLEM:

$\frac{\phi''}{\phi} = -\lambda \Rightarrow \left.\begin{cases} \phi'' = -\lambda\phi \\ \phi(0)=\phi(1)=0 \end{cases}\right\} \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 = (n\pi)^2$

$\phi_n = \sin(n\pi x)$

- $h''(t) + 2h'(t) + 2\lambda h(t) = 0$

$\alpha^2 + 2\alpha + 2\lambda = 0 \Rightarrow \alpha_{1,2} = \frac{-2\pm\sqrt{4-8\lambda}}{2} = -1\pm\sqrt{1-2\lambda}$

③

$$h_n(t) = e^{-t}\left(A_n \cos\sqrt{1-2\lambda_n}\, t + B_n \sin\sqrt{1-2\lambda_n}\, t\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left\{A_n \cos\sqrt{1-2\left(\tfrac{n\pi}{1}\right)^2}\, t + B_n \sin\sqrt{1-2(n\pi)^2}\, t\right\} e^{-t}$$

$$u(x,0) = \sin 2\pi x \Rightarrow \boxed{n=2,\ A_2 = 1,\ A_n = 0,\ \forall n \neq 2}$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left\{-\sqrt{1-2(n\pi)^2}\, A_n \sin\sqrt{1-2\lambda_n}\, t + B_n\sqrt{1-2(n\pi)^2} \cos\sqrt{\phantom{x}}\, t\right\} e^{-t} +$$
$$+ \sin(n\pi x)\left\{ A_n \cos\sqrt{\phantom{x}}\, t + B_n \sin\sqrt{\phantom{x}}\, t\right\}(-e^{-t})$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin(n\pi x)\left\{B_n\sqrt{1-2(n\pi)^2} + A_n\right\} = 0$$

$$\Rightarrow \boxed{B_2 = -\frac{A_2}{\sqrt{1-2(2\pi)^2}} = -\frac{A_2}{\sqrt{1-8\pi^2}},\quad B_n = 0,\ n \neq 2}$$

$$\boxed{u(x,t) = \sin(2\pi x)\left\{\cos\sqrt{1-8\pi^2}\, t - \frac{1}{\sqrt{1-8\pi^2}} \sin\sqrt{1-8\pi^2}\, t\right\} e^{-t}}$$

⑨ $u_{tt} = u_{xx} \;\Big/ \cdot u_t \;\Big/ \cdot \int_0^L dx \Rightarrow$

$$\int_0^L u_{tt} u_t\, dx = \int_0^L u_{xx} u_t\, dx \Rightarrow \frac{1}{2}\int_0^L \frac{\partial}{\partial t}(u_t)^2 = -\int_0^L \underbrace{u_x u_{xt}}_{\frac{1}{2}\frac{\partial}{\partial t}(u_x)^2} dx + \left[u_x u_t\right]_0^L$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_0^L (u_t)^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^L (u_x)^2 dx = (u_x u_t)(x=L) - (u_x u_t)(x=0)$$

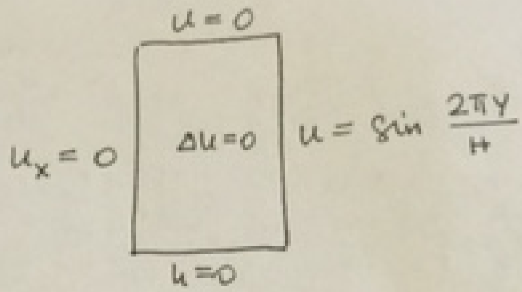
Since $u(L,t) = 0 \Rightarrow \underline{u_t(L,t) = 0}$. **AND** Since $\underline{u_x(0,t) = 0}$ we have

$\frac{1}{2}\frac{d}{dt}\int_0^L (u_t)^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^L (u_x)^2 dx = 0$. Thus, $\frac{dE(t)}{dt} = 0 \Rightarrow E(t)$ is constant

$\Rightarrow$ ENERGY IS CONSERVED.

10

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u_x(0,y) = 0 \\ u(L,y) = \sin\left(\frac{2\pi y}{H}\right) \\ u(x,0) = 0 \\ u(x,H) = 0 \end{cases}$$




$$u(x,y) = h(x)\phi(y) \Rightarrow \frac{h''}{h} = -\frac{\phi''}{\phi} = \lambda, \quad \lambda > 0$$

EIGENVALUE PROBLEM:

$$\left\{\begin{array}{l} \phi'' = -\lambda\phi \\ \phi(y=0) = \phi(y=H) = 0 \end{array}\right\} \Rightarrow \phi_n(y) = \sin\frac{n\pi y}{H}, \quad \lambda_n = \left(\frac{n\pi}{H}\right)^2$$

$n = 1, 2, 3, \dots$

PROBLEM FOR $h$:

$$h''(x) = \lambda h(x) \Rightarrow h(x) = A\cosh\sqrt{\lambda}\,x + B\sinh\sqrt{\lambda}\,x$$

GENERAL SOLUTION:

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{H}\left(A_n\cosh\frac{n\pi x}{H} + B_n\sinh\frac{n\pi x}{H}\right)}$$

use: $u(x=0,y) = 0$: $\Rightarrow \quad u(0,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{H} A_n = 0 \Rightarrow \boxed{A_n = 0, \forall n}$

$$\Rightarrow u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{H} \cdot B_n \sinh\frac{n\pi x}{H}$$

use: $u(L,y) = \sin\left(\frac{2\pi y}{H}\right)$: $\Rightarrow \quad \boxed{B_2 = 1, \; B_n = 0, \; n \neq 2}$

$$\Rightarrow \boxed{u(x,y) = \sin\frac{2\pi y}{H} \cdot \sinh\frac{2\pi x}{H}}$$

11 Yes, solutions of $\begin{Bmatrix} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{Bmatrix}$ are unique.

Proof: Suppose $u_1$ and $u_2$ are two solutions. Then:

$$\begin{Bmatrix} \Delta u_1 = 0 & \text{in } \Omega \\ u_1 = f & \text{on } \partial\Omega \end{Bmatrix} \text{ AND } \begin{Bmatrix} \Delta u_2 = 0 & \text{in } \Omega \\ u_2 = f & \text{on } \partial\Omega \end{Bmatrix}.$$

Let $w := u_1 - u_2$. The function $w$ satisfies:

$$\begin{Bmatrix} \Delta w = \Delta u_1 - \Delta u_2 = 0 & \text{in } \Omega \\ w = f - f = 0 & \text{on } \partial\Omega \end{Bmatrix}$$

From the maximum principle: $w \leq \max_{\partial\Omega} w = 0$

From the minimum principle: $\min_{\partial\Omega} w \leq w$

Therefore:

$$0 \leq w \leq 0$$

and so $w \equiv 0$ everywhere in $\Omega$. Therefore, $u_1$ must be equal to $u_2$, <u>and so there is only one solution to</u> $\begin{Bmatrix} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{Bmatrix}$ $\square$

12

$$\boxed{\Delta u = 0} \text{ in } \Omega = \{(r,\theta);\, 0 \leq r < 1,\, -\pi \leq \theta < \pi\}$$
$$\boxed{u = 2 + \sin\theta \cdot \cos\theta} \text{ on } \partial\Omega = \{(1,\theta),\, -\pi \leq \theta < \pi\}$$

<u>SOLUTION:</u>

Recall: $\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

Separation of variables: $\boxed{u(r,\theta) = G(r)\,\phi(\theta)}$

Plug into PDE $\Rightarrow$

$$\boxed{\frac{r}{G(r)}\left(r\,G'(r)\right)' = -\frac{\phi''(\theta)}{\phi(\theta)} = \lambda > 0}$$

Need boundary data at $\theta = -\pi$, $\theta = \pi$, $r = 0$, $r = 1$.
Periodic boundary conditions are prescribed at $\theta = -\pi$, $\theta = \pi$:

$$\boxed{\begin{aligned} u(r,-\pi) &= u(r,\pi) \\ \frac{\partial u}{\partial \theta}(r,-\pi) &= \frac{\partial u}{\partial \theta}(r,\pi) \end{aligned}} \quad (*)$$

At $r=0$ we want the solution to be bounded:

$$\boxed{|u(0,\theta)| < \infty}$$

At $r=1$ we have our prescribed boundary data:

$$\boxed{u = 2 + \sin\theta \cdot \cos\theta = 2 + \frac{1}{2}\sin(2\theta)}$$

Now we can solve the problem.

- EIGENVALUE PROBLEM: $\phi''(\theta) = -\lambda\phi(\theta)$

From $(*)$ we have: $\begin{cases} \phi(-\pi) = \phi(\pi) \\ \phi'(-\pi) = \phi'(\pi) \end{cases}$

Solution: $\phi(\theta) = A\cos\sqrt{\lambda}\,\theta + B\sin\sqrt{\lambda}\,\theta$

Boundary data: $\left.\begin{aligned} \phi(-\pi) &= A\cos\sqrt{\lambda}\pi - B\sin\sqrt{\lambda}\pi \\ \phi(\pi) &= A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi \end{aligned}\right\} =$

$\phi(-\pi) = \phi(\pi)$ implies $\boxed{2B\sin\sqrt{\lambda}\pi = 0}$

Boundary data: $\left.\begin{aligned} \phi'(-\pi) &= A\sqrt{\lambda}\sin\sqrt{\lambda}\pi + B\sqrt{\lambda}\cos\sqrt{\lambda}\pi \\ \phi'(\pi) &= -A\sqrt{\lambda}\sin\sqrt{\lambda}\pi + B\sqrt{\lambda}\cos\sqrt{\lambda}\pi \end{aligned}\right\} =$

$\phi'(-\pi) = \phi(\pi)$ implies $\boxed{2A\sqrt{\lambda}\sin\sqrt{\lambda}\pi = 0}$

$\Longrightarrow \boxed{\sin\sqrt{\lambda}\pi = 0}$ which implies $\sqrt{\lambda_n} = n$ or $\boxed{\lambda_n = n^2}$

Solution (eigenvalues and eigenfunctions):

$$\boxed{\phi_n(\theta) = A_n\cos n\theta + B_n\sin n\theta, \quad \lambda_n = n^2, \quad n = 0,1,2,\dots}$$

we need to solve:

- Solution for $G_n(r)$: $r^2 G''(r) + r G'(r) = n^2 G(r)$

Look for Solution: $G(r) = r^\alpha$. Plug into ODE to obtain:

$$\alpha(\alpha-1) r^2 r^{\alpha-2} + \alpha r r^{\alpha-1} = n^2 r^\alpha$$

or $$\alpha(\alpha-1) r^\alpha + \alpha r^\alpha = n^2 r^\alpha$$

or $$\left(\alpha(\alpha-1) + \alpha - n^2\right) r^\alpha = 0$$

Thus: $\alpha(\alpha-1)+\alpha-n^2=0 \Rightarrow \boxed{\alpha = \pm n}$

Therefore: $\boxed{G_n(r) = C_1 r^n + C_2 r^{-n}}$, $n = 1,2,3,\ldots$

This is a good general solution when $n \neq 0$ (we get 2 linearly independent solutions $r^n$ and $r^{-n}$).
When $n=0$, both $r^n = r^0 = 1$, $r^{-n} = r^0 = 1$ are the same.
This is why we solve the ODE for $G$ directly when $n=0$:

$$r^2 G''(r) + r G'(r) = 0$$

or $$\boxed{r\left(r G'(r)\right)' = 0} \quad \Big| \div r$$

or $$\left(r G'(r)\right)' = 0 \Rightarrow r G'(r) = \text{Const} = C$$

$$\Rightarrow \quad G'(r) = \frac{C}{r} \Rightarrow \boxed{G_0(r) = C \ln r + D}, \quad C, D\text{-constants}$$

From $|u(0,\theta)| < \infty \Rightarrow$ $C_2 = 0$ in $G_n(r) = C_1 r^n + C_2 r^{-n}$, $n = 1,2,\ldots$
$C = 0$ in $G_0(r) = C \ln r + D$

Thus: $G_n(r) = r^n$ for $n = 1,2,3,\ldots$
$G_0(r) = 1$ for $n = 0$

GENERAL SOLUTION: $\boxed{u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos n\theta + B_n \sin n\theta\right) r^n}$

From boundary data: $u(1,\theta) = 2 + \frac{1}{2}\sin 2\theta \Rightarrow \boxed{A_0 = 2,\ B_2 = \frac{1}{2},\ A_n = 0}$ $n \neq 0, 2$

SOLUTION: $\boxed{u(r,\theta) = 2 + \frac{1}{2}\sin 2\theta\, r^2}$

(13) A PDE problem is well-posed if:

(1) IT HAS AT LEAST ONE SOLUTION (EXISTENCE)

(2) THE SOLUTION IS UNIQUE (UNIQUENESS)

(3) THE SOLUTION DEPENDS CONTINUOUSLY ON NON-HOMOGENEOUS DATA.