I_0 AND COMBINATORICS AT λ^+

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Abstract. We investigate the compatibility of I_0 with various combinatorial principles at λ^+, which include the existence of λ^+-Aronszajn trees, square principles at λ, the existence of good scales at λ, stationary reflections for subsets of λ^+, diamond principles at λ and the Singular Cardinal Hypothesis at λ. We also discuss whether these principles can hold in $L(V_{\lambda+1})$.

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1. Introduction

Axiom $I_0(\lambda)$ is the assertion that there is an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ such that $\text{crit}(j) < \lambda$. It was first proposed and studied by Woodin in the early 80's and by Laver in the 90's. For the introductory material on this axiom and its connection with other rank-into-rank axioms, we refer the readers to [10].

Although it is stronger than the existence of supercompact cardinals in consistency strength, the statement $I_0(\lambda)$ only implies the existence of $<\lambda$-supercompact cardinals, there are a fair number of statements that follow from supercompactness but are independent of $I_0(\lambda)$. The theme of this paper is to present some examples of this sort in the area of combinatorics at λ^+. In this context, λ is an ω-limit of very strong large cardinals, for instance, limit of $<\lambda$-supercompact cardinals.

Key words and phrases. Axiom I_0, λ^+-Aronszajn tree, square, weak square, stationary reflection, good scales, diamond, λ-continuum hypothsis, Generic absoluteness, λ-good.

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Let φ be a combinatorical principle at λ^+. In this paper, we investigate the compatibility of $I_0(\lambda)$ axiom with various φ's over the base theory $\Gamma = \mathsf{ZFC} + I_0(\lambda)$. We ask three questions:

- Is φ consistent with Γ?
- Is $\neg\varphi$ consistent with Γ?
- Is φ true in $L(V_{\lambda+1})$?

The combinatorial principles investigated in this paper include

(1) the existences of (special) λ^+-Aronszajn tree and of λ^+-Suslin tree; (see §2.1 and §2.2)
(2) the $\square_\lambda$ and the $\square_\lambda^*$ principles; (see §2.1 and §2.2)
(3) the existence of (good, very good) scale at λ^+; (see §2.3)
(4) stationary reflection at λ^+; (see §3)
(5) the $\diamondsuit_{\lambda^+}$ principle; (see §4)
(6) GCH (as well as SCH) at λ; (see §4)

In the discussion of the compatibility of these principles with Γ, we also look into their independence with respect to ZFC plus the following stronger form of I_0-type assertion:

> There is an elementary embedding
> $$j : L_{\omega\cdot 2+1}(V_{\lambda+1}^\sharp, V_{\lambda+1}) \to L_{\omega\cdot 2+1}(V_{\lambda+1}^\sharp, V_{\lambda+1})$$
> with $\mathrm{crit}(j) < \lambda$.

We are unable to answer the question regarding stationary reflection at λ^+ in $L(V_{\lambda+1})$, due to the lack of choice in this model. We include a scenario (see Theorem 3.3) where it could be true in $L(V_{\lambda+1})$, although it is unknown if that setting is even compatible with I_0. Our discussion regarding the generalized continuum hypothesis at λ (see Theorem 4.4) assumes a stronger form of Generic absoluteness. To apply it, we need to show that Gitik's one-extender-based Prikry forcing is λ-good. For that we extend the idea in [14], introduce two rank notions and develop in §5 a systematic analysis on the ranks of (finite parts of) conditions in Gitik's forcing.

Notations. We write I_0 for the statement $\exists\lambda\ I_0(\lambda)$. For two cardinals $\kappa < \lambda$, κ regular, we write $E^\kappa_\lambda = \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \kappa\}$, and similarly write $E^{>\kappa}_\lambda$, $E^{\leqslant\kappa}_\lambda$ to denote the obvious sets. If C is a set of ordinals, we use $\lim(C)$ to denote the set of limit ordinals of C.

2. λ^+-Aronszajn tree, good scales at λ and $\square_\lambda$

κ-tree is a tree on κ of size κ whose every level has size $<\kappa$. A *κ-Aronszajn tree* is a κ-tree that has no cofinal branch of length κ.

2.1. There are no λ^+-Aronszajn trees and $\square_\lambda$-sequences in $L(V_{\lambda+1})$.

Under ZFC, there is an ω_1-Aronszajn tree, however this is not true under the axiom of determinacy. Being more precise, assuming $\mathsf{AD}^{L(\mathbb{R})}$, there is no ω_1-Aronszajn tree in $L(\mathbb{R})$, while it may exist in V, if AC is assumed there. In this section, we show that a similar situation occurs at λ^+, assuming $I_0(\lambda)$.

Theorem 2.1 (ZFC)**.** *Assume $I_0(\lambda)$. There is no λ^+ Aronszajn tree in $L(V_{\lambda+1})$.*

Proof. The reason there is no λ^+-Aronszajn tree in $L(V_{\lambda+1})$ is the same as that of the nonexistence of ω_1-Aronszajn tree in $L(\mathbb{R})$ under $\mathsf{AD}^{L(\mathbb{R})}$. First, note that $(\lambda^+)^V = (\lambda^+)^{L(V_{\lambda+1})}$, so a λ^+-tree in $L(V_{\lambda+1})$ is also a λ^+-tree in V. We show that such a tree can not be a λ^+-Aronszajn tree.

By a theorem of Woodin (see [17], 1.B.5), $I_0(\lambda)$ implies that

$$L(V_{\lambda+1}) \models \lambda^+ \text{ is a measurable cardinal.}$$

Assuming towards a contradiction that there is a λ^+-Aronszajn tree T in $L(V_{\lambda+1})$. Let $\pi : L[T] \to M \cong \mathrm{Ult}(L[T], \mu \cap L[T])$ be the ultrapower embedding induced by a λ^+-complete measure μ on λ^+. Then $\pi(T)$ is a $\pi(\lambda^+)$-Aronszajn tree in M. Notice that $\mathrm{crit}(\pi) = \lambda^+$, we have $T = \pi``T \subset \pi(T)$ and $\pi(\lambda^+) > \lambda^+$. Any node at the $\lambda^+$-th level of $\pi(T)$ is a cofinal branch of $\pi``T = T$. Thus there can be no λ^+-Aronszajn tree in $L(V_{\lambda+1})$. □

The same argument gives us a similar result regarding the square principle, which is due to Jensen [9].

Definition 1. Let λ be an uncountable cardinal. A $\square_\lambda$*-sequence* is sequence $\langle C_\alpha : \alpha < \lambda^+, \alpha \in \lim(\lambda^+)\rangle$ such that for all $\alpha < \lambda^+$,

1. $C_\alpha \subseteq \alpha$ is closed and unbounded in α,
2. $\mathrm{otp}\, C_\alpha \leqslant \lambda$,
3. For all $\beta \in \lim(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$.

We say $\square_\lambda$ *holds* if there exists a $\square_\lambda$-sequence.

Theorem 2.2. *Assume $I_0(\lambda)$. Then $L(V_{\lambda+1}) \models \neg\square_\lambda$.*

Proof. Assume not, and let $\bar{C} = \langle C_\alpha : \alpha < \lambda^+, \alpha \in \lim(\lambda^+)\rangle$ be a $\square_\lambda$-sequence in $L(V_{\lambda+1})$. Let μ be a λ^+-complete ultrafilter that witnesses the measurability of λ^+ in $L(V_{\lambda+1})$. Let $\pi : L[\bar{C}] \to M \cong \mathrm{Ult}(L[\bar{C}], \mu \cap L[\bar{C}])$ be the induced elementary embedding. Then $\pi(\bar{C})$ is a $\square_{\pi(\lambda^+)}$-sequence in M. Since every C_α, $\alpha < \lambda^+$, has ordertype $\leqslant \lambda$ in $L[\bar{C}]$, every member of $\pi(\bar{C})$ has ordertype $\leqslant \pi(\lambda) = \lambda$, as $\mathrm{crit}(\pi) = \lambda^+$. Let C_{λ^+} be the λ^+-th element of $\pi(\bar{C})$. On the one hand, $\mathrm{otp}(C_{\lambda^+}) = \lambda$ by elementarily, the definition of $\bar{C}$, and the fact that $\mathrm{crit}(\pi) = \lambda^+$; on the other hand, as a member of $\square_{\pi(\lambda^+)}$-sequence, C_{λ^+} is a closed unbounded subset of λ^+. This is a contradiction! □

Remark. Although $\square_\lambda$ implies the existence of a λ^+-Aronszajn tree (see Exercise IV.1C and the proof of Theorem IV.2.4, [5]), this does not enable us to conclude the failure of $\square_\lambda$ in $L(V_{\lambda+1})$ from Theorem 2.1, as the construction of a λ^+-Aronszajn tree uses λ^+-DC, which fails in $L(V_{\lambda+1})$.

2.2. **λ^+-Aronszajn trees and $\square_\lambda$ in V.** The two theorems above say that $I_0(\lambda)$ pushes λ^+-Aronszajn trees as well as $\square_\lambda$-sequences, if exist, out of $L(V_{\lambda+1})$, but it does not necessarily eliminate their existence in V. Next we show that given the consistency of $I_0(\lambda)$ for some λ, it is possible to produce a model with both $I_0(\lambda)$ and a λ^+-Suslin tree. A κ-Suslin tree is a κ-Aronszajn tree with no antichain of size κ.

Theorem 2.3 (ZFC)**.** *Assume $I_0(\lambda)$. Then there is a model in which $I_0(\lambda)$ holds and there is a special λ^+-Aronszajn tree, even furthermore a λ^+-Suslin tree.*

Proof. To produce a special λ^+-Aronszajn tree, we need the following "weak square" principle, $\square^*_\lambda$, due to Jensen (see [2], §5):

> There exist $\langle C_\alpha : \alpha < \lambda^+, \alpha \text{ limit}\rangle$ such that each C_α is a nonempty set of club subsets of α, $|C_\alpha| \leqslant \lambda$, and for all limit $\alpha < \lambda^+$, all $C \in C_\alpha$ and all $\beta \in \lim(C)$, $\mathrm{otp}(C) \leqslant \lambda$ and $C \cap \beta = C_\beta$.

Jensen showed that $\square^*_\lambda$ is equivalent to the existence of a special λ^+-Aronszajn tree. Our approach is to force a weak square sequence. In fact, the standard forcing $\mathbb{P}_\lambda$ due to Jensen for adding a square sequence will do. For the detail of $\mathbb{P}_\lambda$, one can read Cummings' handbook article ([3], 6.6). Another relevant point is that this forcing is $<\lambda^+$-strategically closed, therefore it adds no new subsets of λ, preserves cardinals and cofinalities up to λ^+. Let $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be a witness embedding for $I_0(\lambda)$. Then the same elementary embedding witnesses $I_0(\lambda)$ in the generic extension.

To get a λ^+-Suslin tree, we need before applying the forcing $\mathbb{P}_\lambda$ over a ground model that satisfies GCH at λ, namely $2^\lambda = \lambda^+$. This is not difficult to achieve, as we may first force $2^\lambda = \lambda^+$ then force a square sequence, i.e. use $\mathrm{Coll}(\lambda^+, 2^\lambda) * \dot{\mathbb{P}}_\lambda$, where $\dot{\mathbb{P}}_\lambda$ is the $\mathrm{Coll}(\lambda^+, 2^\lambda)$-name of $\mathbb{P}_\lambda$. Note that this Levy collapse is a $<\lambda^+$-closed forcing, so this two-step iterated forcing poset does not change $V_{\lambda+1}$ and therefore the $L(V_{\lambda+1})$ of the models before and after applying this forcing are the same, hence the same elementary embedding j witnesses $I_0(\lambda)$ in the generic extension.

Let κ be a regular uncountable cardinal $< \lambda$. One can produce a $\square_\lambda$-sequence $\bar{D} = \langle D_\alpha \mid \alpha < \lambda^+\rangle$ and a stationary set $S \subseteq E^\kappa_{\lambda^+}$ such that $S \cap \lim(D_\alpha) = \varnothing$ for all $\alpha < \lambda^+$. The proof that such $\bar{D}$ and S exist can be found in [2], the paragraph prior to 4.2. By a result of Shelah ([12], see also Theorem 2.2 of [2]), if $2^{<\lambda} = \lambda$ and GCH holds at λ, then $\diamondsuit_{\lambda^+}(T)$ holds for every stationary $T \subset E^{>\omega}_{\lambda^+}$. Thus we have a $\diamondsuit_{\lambda^+}(S)$-sequence. Then by Jensen's argument (see [2], 4.2), a λ^+-Suslin tree can be constructed from the $\square_\lambda$-sequence $\bar{D}$ and that $\diamondsuit_{\lambda^+}(S)$-sequence. $\square$

Next we show that under suitable assumptions, $I_0(\lambda)$ is not compatible with the existence of λ^+-Aronszajn trees. For that we need a theorem in I_0 theory.

Theorem (Cramer [1])**.** *Assume there is an elementary embedding*

$$j : L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1}) \to L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1})$$

with $\mathrm{crit}(j) < \lambda$. *Then there is a* $\bar{\lambda} < \lambda$ *such that* I_0 *holds at* $\bar{\lambda}$, *namely, there is an elementary embedding* $\bar{j} : L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1})$ *with* $\mathrm{crit}(\bar{j}) < \bar{\lambda}$.

By a result of Shelah (see [4] Fact 2.10), if there is a supercompact κ and λ is a cardinal such that $\mathrm{cf}(\lambda) < \kappa < \lambda$, then $\square^*_\lambda$ fails (in fact, the proof just needs κ to be λ^+-supercompact). Under the hypothesis in Cramer's theorem, it is easy to arrange a $\bar{\lambda}$ so that $\bar{\lambda} > \mathrm{crit}(j)$, i.e. $\bar{\lambda} > \kappa$ for some $\kappa < \lambda$ that is $<\lambda$-supercompact. In particular, this κ is $\bar{\lambda}^+$-supercompact, so we have $\square^*_{\bar{\lambda}}$ fails and consequently that there is no special $\bar{\lambda}^+$-Aronszajn tree. The elimination of the adjective "special" follows from a careful examination of Cramer's proof of the theorem.

Theorem 2.4 (ZFC)**.** *Assume there is an elementary embedding*

$$j : L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1}) \to L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1})$$

with $\mathrm{crit}(j) < \lambda$. *Then there is a* $\bar{\lambda} < \lambda$ *such that* $I_0(\bar{\lambda})$ *holds and there is no* $\bar{\lambda}^+$*-Aronszajn tree.*

Proof. In [11], Magidor and Shelah show that if λ is a singular limit of strongly compact cardinals, then λ^+ carries no Aronszajn trees. For our purpose, it suffices to have λ being a limit of λ^+-strongly compact cardinals. Let $\bar\lambda$ be as in Cramer's theorem. From Cramer's proof, there is an inverse limit $(J, \vec{j})$ such that $\bar\lambda = \lambda_J$. Let $\vec{j} = \langle j_n : n < \omega\rangle$, then $\bar\lambda = \lim_{n<\omega} \mathrm{crit}(j_n)$. Here each j_n is an $I_0(\lambda)$ embedding, thus each $\mathrm{crit}(j_n)$ is a $<\lambda$-strongly compact. Thus $\bar\lambda$ is a limit of $\bar\lambda^+$-strongly compact cardinals. Then by Magidor-Shelah's theorem, there is no $\bar\lambda^+$-Aronszajn tree. $\square$

In fact, we have also shown that

Theorem 2.5 (ZFC). *(1) $\mathrm{Con}(I_0(\lambda))$ implies $\mathrm{Con}(I_0(\lambda) + \square_\lambda)$, and hence implies $\mathrm{Con}(I_0(\lambda) + \square^*_\lambda)$.*

(2) Assume there is an elementary embedding

$$j : L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1}) \to L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1})$$

with $\mathrm{crit}(j) < \lambda$. Then there is a $\bar\lambda < \lambda$ such that $I_0(\bar\lambda)$ holds and $\square_{\bar\lambda}$ fails.

The proof of 1 is in the proof of Theorem 2.3. The proof of 2 is essentially included in the paragraph following Cramer's Theorem on 4, where it is argued that under the same hypothesis, the weak square $\square^*_{\bar\lambda}$ fails for some $\bar\lambda < \lambda$.

Remark. Notice that the forcing that adds a $\square_\lambda$-sequence adds no new subsets of λ, and by the definability of the sharp, $V^\sharp_{\lambda+1}$ is absolute, therefore the embedding

$$j : L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1}) \to L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1})$$

remains to be elementary on the $L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1})$ of the generic extension. Therefore Cramer's hypothesis is consistent with the existence of a $\square_\lambda$-sequence, therefore a special λ^+-Aronszajn tree, as well as a λ^+-Suslin tree.

Therefore $\square_\lambda$ is in some sense independence of I_0 axiom, as well as the stronger variation in the hypothesis of Theorem 2.4. The same hold for the existence of λ^+-Aronszajn tree and λ^+-Suslin tree. More precisely,

Corollary 1 (ZFC). *Let $\Gamma(\lambda)$ denote the assertion that there exists an elementary embedding*

$$j : L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1}) \to L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1})$$

with $\mathrm{crit}(j) < \lambda$. Let φ be one the following statements. Assume that $\exists\lambda\,\Gamma(\lambda)$ is consistent. Then both $\exists\lambda\,(\Gamma(\lambda) + \varphi(\lambda))$ and $\exists\lambda\,(\Gamma(\lambda) + \neg\varphi(\lambda))$ are consistent, where $\varphi(\lambda)$ is one of the following statements.

(1) there is a $\square_\lambda$-sequence.
(2) there is a $\square^*_\lambda$-sequence.
(3) there is a (special) λ^+-Aronszajn tree.
(4) there is a λ^+-Suslin tree.

Contrast Corollary 1 with Solovay's theorem (see [15, 16]) regarding the incompatibility of square principle with supercompact cardinals, more precisely: If $\kappa \leqslant \lambda$ and κ is λ^+-supercompact, then $\square_\lambda$ fails.

2.3. **Good scales at λ.** Next we discuss good scales at λ. We are going to show that there is no (very) good scale at λ in $L(V_{\lambda+1})$ and to add the assertion of its existence to the list in Corollary 1. In this paper, as λ is a singular cardinal of countable cofinality, we consider only the set $\prod_{i<\omega}\kappa_i$, where $\bar{\kappa} = \langle \kappa_i : i < \omega\rangle$ is a sequence of regular cardinals such that $\lambda = \sup_{i<\omega}\kappa_i$, and the ideal I on ω that consists of all finite subsets of ω. Given $f, g \in \prod_i \kappa_i$, $f <_I g$ if and only if $\omega \backslash \{i \mid f(i) < g(i)\} \in I$. *A scale of length α in $\prod_i\kappa_i/I$* is a $<_I$-increasing sequence $\langle f_i : i < \alpha\rangle$ in $\prod_i\kappa_i$ which is cofinal in $\prod_i\kappa_i$ under the relation $<_I$. *A scale for λ* is a pair $(\bar{\kappa}, \bar{f})$, where $\bar{f}$ is a scale of length λ^+ in $\prod_i\kappa_i/I$. As λ is singular, a basic fact of PCF theory is that, there exists a scale for λ.

Definition 2. (1) Suppose $(\bar{\kappa}, \bar{f})$ is a scale for λ. A point $\alpha < \lambda^+$ is *good for* $(\bar{\kappa}, \bar{f})$ iff there is an $A \subset \alpha$ unbounded in α and $i < \omega$ such that

$$\forall \alpha, \beta \in A \forall j > i\, (\alpha < \beta \rightarrow f_\alpha(j) < f_\beta(j)).$$

(2) Let $\langle g_i : i < \beta\rangle$ be a $<_I$-increasing sequence in $\prod_i\kappa_i$ and $g \in \prod_i\kappa_i$. g is an *exact upper bound (eub)* for $\langle g_i : i < \beta\rangle$ if $g_i <_I g$ for every $i < \beta$ and for any $h \in \prod_i\kappa_i$, $h <_I g \Rightarrow h \leqslant_I g_i$ for some $i < \beta$.

By Shelah's PCF theory, the set of good points in a scale for λ is a stationary subset of λ^+. This set is determined by the sequence $\bar{\kappa}$ modulo the nonstationary ideal on λ^+.

Definition 3. A scale $(\bar{\kappa}, \bar{f})$ for λ is *good* if except a nonstationary subset of λ^+ every point of uncountable cofinality is good for $\bar{f}$.

A scale $(\bar{\kappa}, \bar{f})$ for λ is *very good* if for every limit $\alpha < \lambda^+$ such that $\mathrm{cf}(\alpha) > \omega$, there is a $C \subseteq \alpha$ club in α and an integer $m < \omega$ such that for all $n > m$, $\langle f_\beta(n) : \beta \in C\rangle$ is strictly increasing.

Theorem 2.6 (ZFC)**.** *Assume $I_0(\lambda)$. There is no (good, very good) scale at λ in $L(V_{\lambda+1})$.*

Proof. It suffices to show that there is no scale at λ in $L(V_{\lambda+1})$. Suppose otherwise and let $(\bar{\kappa}, \bar{f})$ be a scale for λ in $L(V_{\lambda+1})$. Let μ be a λ^+-complete ultrafilter that witnesses the measurability of λ^+ in $L(V_{\lambda+1})$. Let

$$\pi : L[\bar{\kappa}, \bar{f}] \to M \cong \mathrm{Ult}(L[\bar{\kappa}, \bar{f}], \mu \cap L[\bar{\kappa}, \bar{f}])$$

be the induced elementary embedding. Since $L[\bar{\kappa}, \bar{f}] \models \forall \alpha < \beta\, (f_\alpha <_I f_\beta)$, by elementarity, $f_\alpha <_I \pi(\bar{f})(\lambda^+)$ in M, for every $\alpha < \lambda^+$. Since $<_I$ is absolute, that is also true in $L(V_{\lambda+1})$. But then $\bar{f}$ is not a scale in $L(V_{\lambda+1})$. Contradiction! $\square$

Similar to the situation of $\square_\lambda$, we have

Theorem 2.7. *(1) Assume $I_0(\lambda)$. Then there is a model of ZFC + $I_0(\lambda)$, in which there is a (very) good scale at λ.*

(2) Assume there is an elementary embedding

$$j : L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1}) \to L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1})$$

with $\mathrm{crit}(j) < \lambda$. Then there is a $\bar{\lambda} < \lambda$ such that $I_0(\bar{\lambda})$ holds and there is no good scale at $\bar{\lambda}$.

Proof. 1 follows Theorem 2.5-1 and a theorem of Cummings, Foreman and Magidor (see [4] Theorem 3.1): If λ is singular and $\kappa < \lambda$, then $\square_{\lambda,\kappa}$[1] implies that there is a very good scale at λ. $\square_\lambda$ implies $\square_{\lambda,\kappa}$, therefore in the model obtained by adding a $\square_\lambda$-sequence, there is a very good scale at λ.

For 2, we need a theorem of Shelah (see [13], or [2] Theorem 18.1): If there is a κ such that $\mathrm{cf}(\lambda) < \kappa < \lambda$ and κ is λ^+-supercompact, then there is no good scale at λ. By the discussion in the paragraph following Cramer's Theorem on page 4, one can arrange $I_0(\bar\lambda)$ for some $\bar\lambda > \kappa = \mathrm{crit}(j)$, but κ is $<\lambda$-supercompact, in particular $\bar\lambda^+$-supercompact, therefore, there is no good scale at $\bar\lambda$. □

Corollary 2. *The assertion that "there is a (very) good scale at λ" can be added to the list in Corollary 1.*

3. Stationary reflection at λ^+

Let κ be an uncountable regular cardinal. Let S be a stationary subset of κ. S *reflects at* α if $\alpha < \kappa$, $\mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α. *Stationary Reflection Principle for* T, where $T \subseteq \kappa$ is stationary, says that for every stationary $S \subseteq T$, S reflects at some $\alpha < \kappa$.

In this section, we show that I_0 is compatible with either side of the Stationary Reflection Principle. Let $\otimes_{\lambda^+}$ denote the Stationary Reflection Principle for λ^+.

Theorem 3.1 (ZFC)**.** *Assume there is an elementary embedding*

$$j : L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1}) \to L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1})$$

with $\mathrm{crit}(j) < \lambda$. *Then there is a* $\bar\lambda < \lambda$ *such that* I_0 *holds at* $\bar\lambda$ *and* $\otimes_{\bar\lambda^+}$ *is true.*

Proof. As before (see page 4, after Cramer's Theorem), this hypothesis yields $\kappa, \bar\lambda$ such that $\kappa < \bar\lambda < \lambda$ and κ is $\bar\lambda^+$-supercompact. Then it follows from the standard argument that the Stationary Reflection Principle for $\bar\lambda^+$ is true: Fix a stationary $S \subseteq \bar\lambda^+$. Let $\pi : V \to M$ be an embedding witnessing the $\bar\lambda^+$-supercompactness of κ. We claim that

Claim. $\pi``S$ is a stationary subset of $\gamma = \sup \pi``S = \sup \pi``\bar\lambda^+$ in M.

Let C be a closed and unbounded subset of γ in M. Since $\pi``\bar\lambda^+$ is $\kappa$-closed, i.e. closed under supremum of $<\kappa$-sequences, $\pi``\bar\lambda^+ \cap C$ is a κ-closed and unbounded subset of γ. Pull it back, $D = \pi^{-1}``(\pi``\bar\lambda^+ \cap C)$ is a κ-closed and unbounded subset of λ^+. Then we have $S \cap D \neq \varnothing$. And then $\pi``S \cap C \neq \varnothing$. Thus $\pi``S$ is stationary in γ.

Since $\pi``S \subseteq \pi(S) \cap \gamma$, we have

$$M \models \exists\gamma < \pi(\lambda^+)\, (\pi(S) \text{ reflects at } \gamma).$$

By elementarity, $V \models S$ reflects at some $\alpha < \bar\lambda^+$. □

It is well known that $\square_\kappa$ implies that the Stationary Reflection Principle fails for every stationary $T \subseteq \kappa^+$ ([4], Theorem 1). So one can obtain the failure of $\otimes_{\bar\lambda^+}$ by forcing a square sequence. As discussed in the proof of Theorem 2.3, that forcing is $<\lambda^+$-strategically closed, it preserves $I_0(\lambda)$, therefore we have both $I_0(\lambda)$ and $\neg\otimes_{\lambda^+}$ in the generic extension. One can also force directly a non-reflecting stationary

[1] The definition of $\square_{\lambda,\kappa}$ is irrelevant to our proof, we refer the reader to Cummings [2] for details.

subset of λ^+. One can find such a forcing in Cummings' handbook article [3], 6.5. That forcing is λ^+-strategically closed, therefore adds no new subsets of λ. Thus in $V[G]$, we also have both $I_0(\lambda)$ and $\neg\otimes_{\lambda^+}$.

Theorem 3.2 (ZFC). *Assume $I_0(\lambda)$ is consistent. Then so is $I_0(\lambda) + \neg\otimes_{\lambda^+}$.*

Corollary 3. *The assertion $\otimes_{\lambda^+}$ can be added to the list in Corollary 1.*

The question left is that

- Assuming $I_0(\lambda)$, is it true that $L(V_{\lambda+1}) \models \otimes_{\lambda^+}$?

Our first attempt is to try the trick we did in the proofs for the nonexistence of λ^+-Aronszajn tree (see Theorem 2.1) and the existence of $\square_{\lambda^+}$-sequences (see Theorem 2.2) in $L(V_{\lambda+1})$. However, the $\otimes_{\lambda^+}$ case is subtle. Its negation is the following statement

$$\exists S \notin \mathscr{I}_{\lambda^+} \forall \alpha \in E^{>\omega}_{\lambda^+} \exists C_\alpha (C_\alpha \text{ is club in } \alpha \wedge S \cap \alpha \cap C_\alpha = \varnothing).$$

Here $\mathscr{I}_{\lambda^+}$ denote the nonstationary ideal on λ^+ and $E^{>\omega}_{\lambda^+}$ denote the set of ordinals $<\lambda^+$ with uncountable cofinalities. For each such α, let $\mathcal{C}_\alpha$ be the collection of clubs C in α such that $S \cap C \cap \alpha = \varnothing$. We would like to take the ultrapower of the structure $L(\langle \mathcal{C}_\alpha : \alpha < \lambda^+ \rangle, S)$ by a measure on λ^+. The problem is that Łos theorem fails for the ultrapower. In particular, we are not able to show that, letting i be the ultrapower map and $\langle \mathcal{D}_\beta : \beta < i(\lambda^+) \rangle = i(\langle \mathcal{C}_\alpha : \alpha < \lambda^+ \rangle)$, for each $\beta < i(\lambda^+)$, $\mathcal{D}_\beta \neq \varnothing$. Also, since λ^+-DC fails in $L(V_{\lambda+1})$, we are not able to choose, for each $\alpha < \lambda^+$, a $C_\alpha \in \mathcal{C}_\alpha$ and consider the ZFC model $L[\langle C_\alpha : \alpha < \lambda^+ \rangle, S]$.

We will obtain stationary reflection in $L(V_{\lambda+1})$ from a slightly stronger principle, which unfortunately is not known to be consistent relative to $I_0(\lambda)$.

Theorem 3.3 (ZFC). *Assume in $L(V_{\lambda+1})$, λ^+ is $V_{\lambda+1}$-supercompact*[2]. *Then $L(V_{\lambda+1}) \models \otimes_{\lambda^+}$.*

Proof. Working in $L(V_{\lambda+1})$, fix a measure μ witnessing that λ^+ is $V_{\lambda+1}$-supercompact. For each $\sigma \in \mathscr{P}_{\kappa^+}(V_{\lambda+1})$, let $M_\sigma = \mathsf{HOD}_{\sigma \cup \{\sigma\}}$ and let $M = \prod_\sigma M_\sigma/\mu$ be the μ-ultraproduct of the structures M_σ's.

Claim 3.1. Łos theorem holds for this ultraproduct.

Proof. The proof is by induction on the complexity of formulas. It's enough to show the following. Suppose $\varphi(x, y)$ is a formula such that the claim holds for φ and f is a function such that $\{\sigma \mid M_\sigma \models \exists x \varphi[x, f(\sigma)]\} \in \mu$. We show that $M \models \exists x \varphi[x, [f]_\mu]$.

Let $g(\sigma) = \{x \in \sigma \mid (\exists y \in \mathsf{OD}(x))(M_\sigma \models \varphi[y, f(\sigma)])\}$. Then $\{\sigma \mid g(\sigma)$ is a non-empty subset of $\sigma\} \in \mu$. By normality of μ, there is a fixed real x such that $\{\sigma : x \in g(\sigma)\} \in \mu$. Hence we can define $h(\sigma)$ to be the least y in $\mathsf{OD}(x)$ such that $M_\sigma \models \varphi[y, f(\sigma)]$. It's easy to see then that $M \models \varphi[[h]_\mu, [f]_\mu]$. □

For each x, let c_x be the constant function $f : \mathscr{P}_{\lambda^+}(V_{\lambda+1}) \to \{x\}$. By λ^+-completeness, it's easy to see that for each $\alpha < \lambda^+$, $\alpha = [c_\alpha]_\mu$. Also for each set x, there is some $a \in V_{\lambda+1}$ such that x is $\mathsf{OD}(a)$. In particular, if x is a set of ordinals, by fineness of μ, $\{\sigma \mid x \in M_\sigma\} \in \mu$. Also if $A \subseteq V_{\lambda+1}$, then $\{\sigma \mid A \cap \sigma \in M_\sigma\} \in \mu$.

[2] This means there is a fine, normal, λ^+-complete measure μ on $\mathscr{P}_{\kappa^+}(V_{\lambda+1})$. Fineness and completeness have standard meanings. In the context where full AC does not hold, normality is defined as follows: suppose $F : \mathscr{P}_{\kappa^+}(V_{\lambda+1}) \to \mathscr{P}_{\kappa^+}(V_{\lambda+1})$ is such that $\{\sigma : F(\sigma) \subseteq \sigma \wedge F(\sigma) \neq \varnothing\} \in \mu$, then there is some x such that $\{\sigma : x \in F(\sigma)\} \in \mu$

This implies that $A \in M$ by Łos theorem and the fact that $A = [\sigma \mapsto A \cap \sigma]_\mu$. So $V_{\lambda+1} \in M$.

Now let $S \subseteq \lambda^+$ be stationary and $S^* = [c_S]_\mu$. By the previous paragraph, in M, $S^* \cap \lambda^+ = S$ (note that $(\lambda^+)^M = \lambda^+$ because $V_{\lambda+1} \in M$) and hence $S^* \cap \lambda^+$ is stationary in M. By Łos,

$$\{\sigma \mid \exists \alpha < \lambda^+ \; M_\sigma \models S \cap \alpha \text{ is stationary}\}.$$

By normality of μ, there is some $\alpha < \lambda^+$ such that

$$\{\sigma \mid M_\sigma \models S \cap \alpha \text{ is stationary}\}.$$

Now we claim that $S \cap \alpha$ is stationary. Let $C \cap \alpha$ be club in α. By the discussion above, $\{\sigma \mid C \in M_\sigma\} \in \mu$. Fix σ such that $C \in M_\sigma$ and $M_\sigma \models$ "$S \cap \alpha$ is stationary". Now in M_σ, C is club in α, so $C \cap S \neq \varnothing$. This shows $S \cap \alpha$ is stationary. □

Remark. The proof above works also if we are in a model M of the form $L(V_{\lambda+1})[\mu]$ and $M \models \mu$ is a normal, fine, λ^+-complete measure on $\mathscr{P}_{\lambda^+}(V_{\lambda+1})$. We are optimistic that such a model can be constructed from $I_0(\lambda)$ or from its strengthenings.

4. Diamond and GCH at λ

First of all, assuming I_0, no matter whether $\diamondsuit_{\lambda^+}$ is true or not in the universe, diamond sequence can not exist in $L(V_{\lambda+1})$.

Theorem 4.1 (ZFC). *Assume I_0 holds at λ. Then in $L(V_{\lambda+1})$, $2^\lambda \neq \lambda^+$ and $\diamondsuit_{\lambda^+}$ fails.*

Proof. It is a ZF theorem that $\diamondsuit_{\lambda^+}$ yields an injective function from $\mathscr{P}(\lambda)$ into λ^+. The inverse of this injective function gives a λ^+-sequence of distinct subsets of λ. So we have $L(V_{\lambda+1}) \models \diamondsuit_{\lambda^+} \to (2^\lambda = \lambda^+)$. If $\diamondsuit_{\lambda^+}$ holds in $L(V_{\lambda+1})$ then GCH holds at λ^+. But $2^\lambda = \lambda^+$ implies that $V_{\lambda+1}$ is wellorderable in $L(V_{\lambda+1})$, this contradicts the fact that $L(V_{\lambda+1}) \models \neg\mathsf{AC}$. □

This proof utilizes the fact that GCH at λ leads to the violation of the fact that $L(V_{\lambda+1})$ is not a full choice model. Here we give another proof, which shows that both $\diamondsuit_{\lambda^+}$ and GCH at λ violates a weaker statement in $L(V_{\lambda+1})$. It is the following analog of the AD-fact that there is no ω_1-sequence of distinct reals.

Theorem 4.2 (ZFC). *Assume I_0 holds at λ. Then there is no λ^+-sequence of distinct members of $V_{\lambda+1}$ in $L(V_{\lambda+1})$.*

Proof. The key point again is that λ^+ is measurable in $L(V_{\lambda+1})$. Suppose $X = \langle x_\alpha : \alpha < \lambda^+ \rangle$ is a sequence of distinct subsets of λ. Let

$$\pi : L[X] \to M \cong \mathrm{Ult}(L[X], \mu \cap L[X])$$

be the ultrapower embedding induced by a λ^+-complete measure μ on λ^+. Then in M, $\pi(X)$ is a $\pi(\lambda^+)$-sequence of distinct subsets of λ. Every member of $\pi(X)$ is represented by a function $\lambda^+ \to \{x_\alpha \mid \alpha < \lambda^+\}$ in V, in particular, let $[f]$ be the λ^+-th element of $\pi(X)$.

Claim. f is constant on a measure one subset $A \subset \lambda^+$.

For each $\beta < \lambda$, there is a unique $i_\beta \in \{0, 1\}$ such that

$$A_\beta^{i_\beta} = \{\alpha < \lambda^+ \mid f(\alpha)(\beta) = i\}$$

is a measure one subset of λ^+. By λ^+-completeness, the set $A = \bigcap\{A_\beta^{i_\beta} \mid \beta < \lambda\}$ has measure one. Therefore for every $\alpha \in A$, $f(\alpha)(\beta) = i_\beta$.

This means that $[f]$ equals to x_α for some $\alpha < \lambda^+$, contradicting to the assumption that members of $\pi(X)$ are all distinct. □

This effectively rules out $2^\lambda \geqslant \lambda^+$ in $L(V_{\lambda+1})$, thus gives a more direct reason why $\diamondsuit_{\lambda^+}$ and GCH at λ fail in $L(V_{\lambda+1})$.

As we have discussed earlier (see the proof of Theorem 2.3), one can easily obtain $\diamondsuit_{\lambda^+}$ by forcing $2^\lambda = \lambda^+$ (using Levy collapse $\mathrm{Coll}(\lambda^+, 2^\lambda)$) without adding bounded subsets of λ, therefore preserves $2^{<\lambda} = \lambda$ and I_0 at λ. Thus we have

Theorem 4.3 (ZFC). *Assume the consistency of I_0, then the following are consistent*

(1) $\exists\lambda(I_0(\lambda) + \diamondsuit_{\lambda^+})$,
(2) $\exists\lambda(I_0(\lambda) + 2^\lambda = \lambda^+)$.

Regarding GCH, Dimonte-Friedman ([6] Corollary 3.9) sketches an argument that it is relatively consistent with I_0 that GCH holds, in particular at λ. However, there are flaws in that argument. We will remark on this after proving our next theorem. Here we present our result. We show the compatibility of $I_0(\lambda)$ with the failure of GCH at λ, and consequently, the compatibility with $\neg\mathsf{SCH}$ at λ (as λ is a singular strong limit cardinal) and with $\neg\diamondsuit_{\lambda^+}$, from a stronger form of I_0-type axiom and a strong generic absoluteness assumption. A few definitions.

Definition 4. Suppose $X \subseteq V_{\lambda+1}$.

(1) Let $\Theta_\lambda^X =_{\mathrm{def}} \{\alpha \mid L(X, V_{\lambda+1}) \models$ there is a surjective $\pi : V_{\lambda+1} \to \alpha\}$.
(2) An ordinal $\alpha < \Theta_\lambda^X$ is X*-good* if every element of $L_\alpha(X, V_{\lambda+1})$ is definable in $L_\alpha(X, V_{\lambda+1})$ from an element in $V_{\lambda+1} \cup \{X\}$.

Definition 5. Assume $j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$ is a proper elementary embedding and $\mathrm{crit}(j) < \lambda$. Let $(M_\omega, j_{0,\omega})$ be the ω-iterate of $(L(X, V_{\lambda+1}), j)$. Suppose $\alpha < \Theta_\lambda^X$ and α is X-good. We say that *Generic Absoluteness holds for X at α* if the following proposition holds:

> *Suppose $\mathbb{P} \in j_{0,\omega}(V_\lambda)$, $G \in V$ is an M_ω-generic filter for $\mathbb{P}$, and $\mathrm{cof}(\lambda) = \omega$ in M_ω. Then there is some $\alpha' \leqslant \alpha$ and $X' \subseteq V_{\lambda+1}$ such that $L_{\alpha'}(X', M_\omega[G] \cap V_{\lambda+1}) \prec L_\alpha(X, V_{\lambda+1})$.*

We refer the readers to Woodin's monograph [18] for relevant terminology and basics in I_0 theory. Recent work by S. Cramer [1] suggests the Generic Absoluteness hypothesis in the following theorem is redundant, but at the moment, we don't see how to make do without it.

Theorem 4.4 (ZFC). *Assume there is a proper elementary embedding*

$$j : L(V_{\lambda+1}^\sharp, V_{\lambda+1}) \to L(V_{\lambda+1}^\sharp, V_{\lambda+1})$$

with $\mathrm{crit}(j) < \lambda$ and GCH holds in V_λ. Suppose that $\alpha \in (\Theta_\lambda, \Theta_\lambda^{V_{\lambda+1}^\sharp})$ and α is $V_{\lambda+1}^\sharp$-good and assume that Generic Absoluteness holds for $V_{\lambda+1}^\sharp$ at α. Then it is consistent that $I_0(\lambda)$ holds and $2^\lambda > \lambda^+$.

Proof. Let M_ω be the ω-iterate of $L(V^\sharp_{\lambda+1}, V_{\lambda+1})$ by j. Then by elementarity, λ is $<j_{0,\omega}(\lambda)$-strong in M_ω and GCH holds in $j_{0,\omega}(V_\lambda)$. Pick an $\eta \in [\lambda^{++}, j_{0,\omega}(\lambda))$. Let $\mathbb{P} = \mathbb{P}_{\lambda,\eta}$ be Gitik's one-extender-based Prikry forcing (with a single extender) that changes the cofinality of λ to ω and adds η many cofinal ω-sequence in λ (see [7]). The key is to show that $\mathbb{P}$ is λ-good in M_ω, as this implies that there are M_ω-generic filters in V (see [14] Proposition 3.9 or [18] page 405). The next section is devoted to verifying this matter.

Let $G \subseteq \mathbb{P}$ be an M_ω-generic filter in V. Then $2^\lambda = \eta$ holds in $M_\omega[G]$. As $\Theta_\lambda < \alpha$, $j \restriction L_{\Theta_\lambda}(V_{\lambda+1}) \in L_\alpha(V^\sharp_{\lambda+1}, V_{\lambda+1})$. By Generic Absoluteness for $V^\sharp_{\lambda+1}$ at α, there is an $\alpha' \leqslant \alpha$ and an $X' \subseteq V_{\lambda+1}$ such that

$$L_{\alpha'}(X', M_\omega[G] \cap V_{\lambda+1}) \prec L_\alpha(V^\sharp_{\lambda+1}, V_{\lambda+1}).$$

By the definability of sharp, $X' = (M_\omega[G] \cap V_{\lambda+1})^\sharp$. Since $j \restriction L_{\Theta_\lambda}(V_{\lambda+1})$ is in $L_\alpha(V^\sharp_{\lambda+1}, V_{\lambda+1})$, there is a

$$j' \in L_{\alpha'}((M_\omega[G] \cap V_{\lambda+1})^\sharp, M_\omega[G] \cap V_{\lambda+1})$$

such that $\mathrm{dom}(j') = L_{\Theta'}(M_\omega[G] \cap V_{\lambda+1})$, where Θ' is the Θ_λ computed in $L(M_\omega[G] \cap V_{\lambda+1})$, and such that the $L(M_\omega[G] \cap V_{\lambda+1})$-ultrafilter $\mu_{j'}$ given by $X \in \mu_{j'}$ iff $j' \restriction V_\lambda \in j'(X)$ induces an elementary embedding of $L(M_\omega[G] \cap V_{\lambda+1})$ into itself. This gives us $I_0(\lambda)$ in $M_\omega[G]$. □

Remarks. 1. The GCH assumption in the theorem is not essential. Suppose $j : L(V^\sharp_{\lambda+1}, V_{\lambda+1}) \to L(V^\sharp_{\lambda+1}, V_{\lambda+1})$ is a proper elementary embedding with $\mathrm{crit}(j) < \lambda$. Relativize Dimonte-Friedman argument (see [6]) for $L(V_{\lambda+1})$, then there is a poset $\mathbb{P}$ (backward Easton forcing up to λ) such that in its generic extension $V[H]$, j can be lifted to $L(V^\sharp_{\lambda+1}, V_{\lambda+1})[H]$ and GCH holds in V_λ. According to Dimonte-Friedman ([6]), this poset is above ω, so we have

$$L(V^\sharp_{\lambda+1}, V_{\lambda+1})[H] = L(V[H]^\sharp_{\lambda+1}, V[H]_{\lambda+1}).$$

Moreover, this poset is λ^+-c.c. and is definable in

$$N = L_{\alpha'}((M_\omega[G] \cap V_{\lambda+1})^\sharp, M_\omega[G] \cap V_{\lambda+1}).$$

Notice that N and V agree on V_λ and the elementary embedding witnessing Generic Absoluteness for $V^\sharp_{\lambda+1}$ (at α), let us call it π, has critical point $\geqslant (\lambda^+)^N$. Thus π can be lifted to a $\bar{\pi} : N[H_0] \to L_\alpha(V[H]^\sharp_{\lambda+1}, V[H]_{\lambda+1})$. Again

$$N[H_0] = L_{\alpha'}((M_\omega[G][H_0] \cap V[H]_{\lambda+1})^\sharp, M_\omega[G][H_0] \cap V[H]_{\lambda+1}).$$

Therefore the generic absoluteness assumption is also preserved by $\mathbb{P}$.

2. We pointed out earlier that there are some issues with the argument Dimonte-Friedman sketched for the compatibility of I_0 with the failure of GCH at λ ([6], Corollary 3.9). To be more specific, one is that the hypothesis of their corollary, that generic absoluteness holds for all $\alpha < \Theta$, is not enough to ensure that $\pi^{-1}(j \restriction L_\alpha(V_{\lambda+1}))$, $\alpha < \Theta$, can be pieced together to form j^*. It is unclear why (the union of) the sequence $\langle \pi^{-1}(j \restriction L_\alpha(V_{\lambda+1})) : \alpha < \Theta \rangle$ is in the domain of π. The second issue is more serious: it is not clear why $j \restriction L_\alpha(V_{\lambda+1})$ falls in the range of π, and then it would make no sense to talk about $\pi^{-1}(j \restriction L_\alpha(V_{\lambda+1}))$.

3. However, the current status of generic absolutness is only up to $L_\delta(V_{\lambda+1})$, where δ is least such that $L_\delta(V_{\lambda+1}) \prec L(V_{\lambda+1})$, which is due to Cramer [1]. It is not clear at this point if generic absoluteness assumption in the hypothesis of our

theorem follows from the existence of an elementary embedding $j : L(V^\sharp_{\lambda+1}, V_{\lambda+1}) \to L(V^\sharp_{\lambda+1}, V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$.

5. The one-extender-based Prikry forcing is λ-good

5.1. **Preliminaries on λ-good forcings.** In order to apply the Generic Absoluteness Theorem, we need to ensure that their generics exist in V. For that, we use a notion of λ-goodness for posets due to Woodin (see [18]).

Definition 6. Let λ be an infinite cardinal. We say a partially ordered set $\mathbb{P}$ is *λ-good* (in V) if it adds no bounded subsets of λ and for every generic filter G and for every $A \subset \mathrm{Ord}$ in $V[G]$ and of size $< \lambda$, there is a non-$\subset$-decreasing ω-sequence $\langle A_i : i < \omega\rangle$ such that $A = \bigcup_i A_i$ and each A_i, $i < \omega$, is in V.

Below is a relativized version of Proposition 3.8 of [14], which asserts that generics for forcings that are λ-good in the ω-th iterate exist in V.

Proposition. *Assume that $j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$ is a proper elementary embedding with critical point $< \lambda$. Let $(M_\omega, j_{0,\omega})$ be the ω-iterate of $(L(X, V_{\lambda+1}), j)$. Suppose $\mathbb{P} \in j_{0,\omega}(V_\lambda)$ and $\mathbb{P}$ is λ-good in M_ω. Then there exists $G \subseteq \mathbb{P}$ in V such that G is M_ω-generic.*

Here we are only interested in the case that $X = V^\sharp_{\lambda+1}$. A useful sufficient condition for showing λ-goodness as follows (see [14]): For all

$$\mathscr{D} \subseteq \{D \subseteq \mathbb{P} \mid D \text{ is open dense in } \mathbb{P}\}$$

such that $|\mathscr{D}| < \lambda$, for any $p \in \mathbb{P}$, there are $p^\circ \leqslant_{\mathbb{P}} p$ and a nondecreasing sequence $\langle \mathscr{D}_{p,i} : i < \omega\rangle$ of subsets of $\mathscr{D}$ such that the following hold

1. $\mathscr{D} = \bigcup\{\mathscr{D}_{p,i} \mid i < \omega\}$,
2. for all $i < \omega$ such that $\mathscr{D}_{p,i} \neq \varnothing$, $\bigcap \mathscr{D}_{p,i}$ is dense below p°, i.e. for any $r \leqslant_{\mathbb{P}} p^\circ$, there exists $r' \leqslant_{\mathbb{P}} r$ such that $r' \in D$ for every $D \in \mathscr{D}_{p,i}$.

5.2. **Gitik's one extender-based Prikry forcing.** Now we describe Gitik's one-extender-based Prikry forcing and show that it is λ-good. The definitions in the next two pages are taken from §3 of Gitik's handbook article ([7]).[3] However we keep it minimal as far as it is necessary for our later arguments, for further details regarding this forcing, we refer the readers to Gitik's article.

Let λ, δ be two cardinals such that δ is a strong limit cardinal above λ and λ is $<\delta$-strong. We assume that GCH holds up to δ. Let η be a cardinal $\geqslant \lambda^{++}$. Then there is a (λ, η)-extender E and a function $f : \lambda \to \lambda$ such that $j(f)(\eta) = \lambda$, where j is the elementary embedding corresponded to E. For every $\alpha \in [\lambda, \eta)$, define a λ-complete ultrafilter U_α as follows: for $X \subseteq \lambda$,

$$X \in U_\alpha \quad \text{iff} \quad \alpha \in j(X).$$

Clearly, each U_α, $\alpha \in [\lambda, \eta)$, is normal. A relevant property is that they are P-point ultrafilters, i.e. for every $f : \lambda \to \lambda$, if f is not constant modulo U_α, then there is a $Y \in U_\alpha$ such that for every $\nu < \lambda$, $|Y \cap f^{-1}\{\nu\}| < \lambda$.

The binary relation $\leqslant_E$ defined below is a partial order on $[\lambda, \eta)$:

$$\alpha \leqslant_E \beta \quad \text{iff} \quad \alpha \leqslant \beta \wedge j_E(f)(\beta) = \alpha \text{ for some } f : \lambda \to \lambda.$$

[3] Some small modifications are made for the sake of the proof of λ-goodness.

$([\lambda, \eta), \leqslant_E)$ is a λ^{++}-directed and $\lambda \leqslant_E \alpha$ for every $\alpha \in [\lambda, \eta)$. There is a system of mappings $\pi_{\beta,\alpha} : \lambda \to \lambda$, for $\alpha, \beta \in [\lambda, \eta)$ such that $\alpha \leqslant_E \beta$, with the following properties:[4]

(1) $\langle U_\alpha, \pi_{\beta,\alpha} : \lambda \leqslant \alpha \leqslant_E \beta < \eta\rangle$ is a $\leqslant_{\text{RK}}$-commutative system of λ-complete ultrafilters, i.e.

$$\alpha \leqslant_E \beta \quad \text{iff} \quad \forall X \subseteq \lambda\, (X \in U_\alpha \leftrightarrow \pi^{-1}_{\beta,\alpha}(X) \in U_\beta).$$

(2) There is a set $\bar{X}$ such that $\bar{X} \in U_\alpha$ and $\pi_{\alpha,\alpha} \restriction \bar{X}$ = identity, for every $\alpha \in [\lambda, \eta)$.

(3) For every $\alpha, \beta, \gamma \in [\lambda, \eta)$ such that $\gamma \leqslant_E \beta \leqslant_E \alpha$, $\pi_{\alpha,\gamma}$ agrees with $\pi_{\alpha,\beta} \circ \pi_{\beta,\gamma}$ on a set $Y \in U_\alpha$.

(4) For every $\alpha, \beta, \gamma \in [\lambda, \eta)$, if $\alpha, \beta \leqslant_E \gamma$ and $\alpha < \beta$, then

$$\{\nu \in \lambda \mid \pi_{\gamma,\alpha}(\nu) < \pi_{\gamma,\beta}(\nu)\} \in U_\gamma.$$

(5) For $\alpha, \beta \in [\lambda, \eta)$, if $\alpha \leqslant_E \beta$, then $\pi_{\beta,\lambda}(\nu) = \pi_{\alpha,\lambda}(\pi_{\beta,\alpha}(\nu))$ for all $\nu \in \lambda$.

(6) For every $\alpha, \beta \in [\lambda, \eta)$, $\pi_{\alpha,\lambda}(\nu) = \pi_{\beta,\lambda}(\nu)$ for all $\nu \in \lambda$.

For $\nu \in \bar{X}$, let $\nu^* = \pi_{\alpha,\lambda}(\nu)$ for some (or equivalently, for all) $\alpha \in [\lambda, \eta)$. Then the following *weak normality* holds for U_α, $\alpha \in [\lambda, \eta)$:

(7) If $X_i \in U_\alpha$ for $i < \lambda$, then

$$\Delta^*_{i<\lambda} X_i =_{\text{def}} \{\nu \mid \forall i < \nu^* \,(\nu \in X_i)\} \in U_\alpha.$$

We say that a sequence $\langle \nu_i : i \leqslant n\rangle$, where $n > 0$ and each $\nu_i < \lambda$, is $*$-*increasing* if $\nu_0^* < \nu_1^* < \cdots < \nu_n^*$, and an ordinal $\nu < \lambda$ is *permitted* for $\langle \nu_i : i < k\rangle$ if $\nu^* > \nu_i^*$ for all $i < k$. A very important fact about members of U_α, $\alpha \in [\lambda, \eta)$, is that if $X \in U_\alpha$, then for every $\nu_0, \nu_1 \in X$ such that $\nu_0^* < \nu_1^*$, $|\{\nu \in X \mid \nu^* < \nu_0^*\}| < \nu_1^*$.

Let $(\Xi, \sqsubseteq)$ denote the tree of all finite $*$-increasing sequences of ordinals in λ, ordered by end-extension. Let f be any one of $\pi_{\beta,\alpha}$, $\alpha \leqslant_E \beta$. By property 5 and 6 on page 13, f preserves the $*$-value, namely $(f(\nu))^* = \nu^*$ for $\nu \in \lambda$. Thus such f induces a length-preserving homomorphism of Ξ into itself. Abusing the notation, we use f for the induced homomorphism as well. Below is a frequently used fact about these f's:

Fact 5.1. *Let $f = \pi_{\beta,\alpha}$ for some $\alpha \leqslant_E \beta$. Suppose $T_\alpha \subseteq \Xi$ is a U_α-tree and $T_\beta \subseteq \Xi$ is a U_β-tree. Then $T_\alpha \cap f$ “T_β is a U_α-tree and $T_\beta \cap (f^{-1})$ “T_α is a U_β-tree.*

Now we define the extender-based Prikry-like forcing $\mathbb{P}_{\lambda,\eta}$ that changes the cofinality of λ to ω and at the same time adds η many ω-sequences of ordinals that are cofinal in λ.

Definition 7. A condition $p \in \mathbb{P}_{\lambda,\eta}$ is of the form

$$\{\langle \gamma, p^\gamma\rangle \mid \gamma \in g\backslash\{\max(g)\}\} \cup \{\langle \max(g), p^{\max(g)}, T\rangle\},$$

where

(1) $g \subset [\lambda, \eta)$ has cardinality $\leqslant \lambda$, $\lambda \in g$ and g has a $\leqslant_E$-maximal element. Denote g by $\text{supp}(p)$, $\max(g)$ by $\text{mc}(p)$, T by T^p, and $p^{\max(g)}$ by p^{mc}.

(2) $p^\gamma \in \Xi$, for every $\gamma \in g$.

[4]These properties and an example of such a system can be found in Gitik [8, 7].

(3) $T \subseteq \Xi$ is a subtree with trunk p^{mc}. All splitting nodes of T are required to be in $U_{\mathrm{mc}(p)}$, i.e. for every $t \in T$ such that $t \geqslant_T p^{\mathrm{mc}}$,

$$\mathrm{succ}_T(t) =_{\mathrm{def}} \{\nu < \lambda \mid \sigma^\frown \nu \in T\} \in U_{\mathrm{mc}(p)},$$

and further that $t_1 \geqslant_T t_2 \geqslant_T p^{\mathrm{mc}} \Rightarrow \mathrm{succ}_T(t_1) \subseteq \mathrm{succ}_T(t_2)$.

(4) For every $\gamma \in \mathrm{supp}(p) \cap \mathrm{mc}(p)$, $\max(p^{\mathrm{mc}})$ is not permitted for p^γ.

(5) For every $\nu \in \mathrm{succ}_T(p^{\mathrm{mc}})$,

$$|\{\gamma \in g \mid \nu \text{ is permitted for } p^\gamma\}| < \nu^*.$$

(6) $\pi_{\mathrm{mc}(p),\lambda}(p^{\mathrm{mc}}) = p^\lambda$.[5]

We will only be concerned with subtrees of Ξ such that all its splitting nodes are in the associated ultrafilter as in item 3 above. So when we say a "*tree at* α", we refer to a subtree of Ξ with the property that all its splitting nodes are in U_α.

For a tree T and $\sigma \in T$, let $T_\sigma =_{\mathrm{def}} \{\tau \mid \sigma^\frown\tau \in T\}$. Next we define the binary relation on $\mathbb{P} = \mathbb{P}_{\lambda,\eta}$.

Definition 8. For $p, q \in \mathbb{P}$, let $p \leqslant_{\mathbb{P}} q$ iff

(1) $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$;

(2) For every $\gamma \in \mathrm{supp}(q)$, $p^\gamma \sqsupseteq q^\gamma$;

(3) $p^{\mathrm{mc}(q)} \in T^q$;

(4) For every $\gamma \in \mathrm{supp}(q)$,

$$p^\gamma \backslash q^\gamma = \pi_{\mathrm{mc}(q),\gamma}((p^{\mathrm{mc}(q)} \backslash q^{\mathrm{mc}(q)}) \restriction (|p^{\mathrm{mc}(q)}| \backslash (i_\gamma + 1))),$$

where i_γ is the largest $i < |p^{\mathrm{mc}(q)}|$ such that $p^{\mathrm{mc}(q)}(i)$ is not permitted for q^γ;

(5) $\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}$ projects $T^p_{p^{\mathrm{mc}}}$ into $T^q_{p^{\mathrm{mc}(q)}}$, namely $\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}$"$T^p_{p^{\mathrm{mc}}} \subseteq T^q_{p^{\mathrm{mc}(q)}}$;[6]

(6) For every $\gamma \in \mathrm{supp}(q)$ and $\nu \in \mathrm{succ}_{T^p}(p^{\mathrm{mc}})$, if ν is permitted for p^γ, then $\pi_{\mathrm{mc}(p),\gamma}(\nu) = \pi_{\mathrm{mc}(q),\gamma}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\nu))$.

A remark about item 5. Consider $\pi_{\beta,\alpha}$, $\alpha \leqslant_E \beta$. Note that $\pi_{\beta,\alpha}$ sends members of U_β to members of U_α. So $\pi_{\beta,\alpha}$ projects a subtree at β to a subtree at α.

Let $p, q \in \mathbb{P}_{\lambda,\eta}$, when $p \leqslant_{\mathbb{P}} q$ and for every $\gamma \in \mathrm{supp}(q)$, $p^\gamma = q^\gamma$, we say p is a *direct extension of* q and write $p \leqslant^*_{\mathbb{P}} q$. We will omit the subscript $\mathbb{P}$ in these two partial orders when it causes no confusion. Below we summerize the facts about this forcing in Gitik's article [7].

Fact. *Let* $\mathbb{P} = \mathbb{P}_{\lambda,\eta}$. *Then*

(1) $(\mathbb{P}, \leqslant)$ *is a partial order.*

(2) $(\mathbb{P}, \leqslant)$ *satisfies* λ^{++}*-c.c.*

(3) $(\mathbb{P}, \leqslant^*)$ *is* λ*-closed.*

(4) $(\mathbb{P}, \leqslant, \leqslant^*)$ *satisfies Prikry condition: For every* $p \in \mathbb{P}$ *and for every sentence* φ *in the forcing language, there is a* $q \leqslant^* p$ *such that* q *decides* φ*, i.e. either* $q \Vdash \varphi$ *or* $q \Vdash \neg\varphi$.

Below is the main theorem in §3 of Gitik's handbook article ([7]),

[5]Here it should be "$\pi_{\mathrm{mc}(p),\lambda}$"$p^{\mathrm{mc}} = p^\lambda$". But as we said earlier, from here on, we abuse the notation, write $\pi_{\beta,\alpha}$'s as functions on Ξ.

[6]In Gitik's article, it is "$\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}$ projects $T^p_{p^{\mathrm{mc}}}$ into $T^q_{q^{\mathrm{mc}}}$". This should be an error.

Theorem. *Suppose δ is a strong limit cardinal, $\lambda < \delta$ is $<\delta$-strong and η is a cardinal in $[\lambda^{++}, \delta)$. Let $\mathbb{P} = \mathbb{P}_{\lambda,\eta}$ as defined above and $G \subseteq \mathbb{P}$ be a V-generic filter. Then the following hold in $V[G]$:*

(1) $\mathrm{cof}(\lambda) = \omega$ *and* $\lambda^\omega \geqslant \eta$.
(2) *All the cardinals are preserved.*
(3) *No new bounded subsets of λ is added.*

5.3. **Gitik's forcing is λ-good.** To show that $\mathbb{P}$ is λ-good, we follow the idea in §3.5 of [14], define a notion of rank with respect to this forcing. For the rest of the section, we fix some notations. We use U_p, $\pi_{q,p}$ and $\pi_{p,\gamma}$, for $p, q \in \mathbb{P}$ such that $q \leqslant p$ and $\gamma \in [\lambda, \eta)$ such that $\gamma \leqslant_E \mathrm{mc}(p)$, to abbreviate for $U_{\mathrm{mc}(p)}$, $\pi_{\mathrm{mc}(q),\mathrm{mc}(p)}$ and $\pi_{\mathrm{mc}(p),\gamma}$, respectively. For $p \in \mathbb{P}$ and $\delta \in \mathrm{succ}_{T^p}(p^{\mathrm{mc}})$, let

$$\begin{aligned} p^- &=_{\mathrm{def}} \{\langle \gamma, p^\gamma \rangle \mid \gamma \in \mathrm{supp}(p) \cap \mathrm{mc}(p)\}, \\ t^p &=_{\mathrm{def}} p^- \cup \{\langle \mathrm{mc}(p), p^{\mathrm{mc}} \rangle\}, \\ (p)_\delta &=_{\mathrm{def}} \{\langle \gamma, (p^\gamma)_{\pi_{p,\gamma}(\delta)} \rangle \mid \gamma \in \mathrm{supp}(p) \cap \mathrm{mc}(p)\} \\ &\qquad\qquad \cup \{\langle \mathrm{mc}(p), p^{\mathrm{mc}\frown}\langle \delta \rangle, T^{\mathrm{mc}}_{p^{\mathrm{mc}\frown}\langle \delta \rangle} \rangle\}, \end{aligned}$$

where

$$(p^\gamma)_{\pi_{p,\gamma}(\delta)} = \begin{cases} p^{\gamma\frown}\pi_{p,\gamma}(\delta), & \text{if } \delta \text{ is permitted for } p^\gamma; \\ p^\gamma, & \text{otherwise.} \end{cases}$$

So $p = p^- \cup \{\langle \mathrm{mc}(p), p^{\mathrm{mc}}, T^p \rangle\}$, and using the t^p notation, p can be naturally identified as the pair $(t^p, T^p_{p^{\mathrm{mc}}})$. For a $s \in \Xi_{p^{\mathrm{mc}}}$, $(p)_s$ is recursively defined by $p_\varnothing = p$ and $p_{s\restriction i+1} = (p_{s\restriction i})_{s(i)}$ for $i < |s|$. The $(p)_\delta, (p)_s$ notations also make sense when p is of the form t^q for some $q \in \mathbb{P}$.

Definition 9. Suppose $D \subseteq \mathbb{P}$ is open. Define R^D_α on $\{t^p \mid p \in \mathbb{P}\}$ as follows:

- Let $H^D_0 = D$ and $R^D_{<0} = R^D_0 = \{t^p \mid p \in D\}$.
- For $\alpha > 0$, let $H^D_{<\alpha} = \bigcup_{\beta<\alpha} H^D_\beta$ and $R^D_{<\alpha} = \bigcup_{\beta<\alpha} R^D_\beta$.
  - Let H^D_α be the set of $p \in \mathbb{P}$ such that $t^{(p)_\delta} \in R^D_{<\alpha}$ for every $\delta \in \mathrm{succ}_{T^p}(p^{\mathrm{mc}})$.
  - Let R^D_α be the set of t^p for $p \in \mathbb{P}$ such that H^D_α is $(\leqslant, \leqslant^*)$-dense below p, i.e. for every $q \leqslant p$, there is a $r \leqslant^* q$ in H^D_α.

The following properties follow immediately from the definition.

Proposition 5.1. *The H^D and R^D-hierarchies have the following properties:*

(1) *$\alpha \leqslant \beta$ implies that $H^D_\alpha \subseteq H^D_\beta$ and $R^D_\alpha \subseteq R^D_\beta$.*
(2) *$R^D_{<\infty} = R^D_{<|\mathbb{P}|^+}$ and $H^D_{<\infty} = H^D_{<|\mathbb{P}|^+}$.*
(3) *R^D_α is open with respect to $(\mathbb{P}, \leqslant)$, i.e. if $q \leqslant p$ and $t^p \in R^D_\alpha$ then $t^q \in R^D_\alpha$.*
(4) *H^D_α is $\leqslant^*$-open, i.e. if $p \in H^D_\alpha$ and $q \leqslant^* p$, then $q \in H^D_\alpha$.*
(5) *H^D_α "$\subseteq$" R^D_α, i.e. $\{t^p \mid p \in H^D_\alpha\} \subseteq R^D_\alpha$.*

Proof. 1. First, as D is open, $H^D_0 \subseteq H^D_1$ and $R^D_0 \subseteq R^D_1$. Note that $R^D_{<\alpha} \subseteq R^D_\alpha$ implies that $H^D_\alpha \subseteq H^D_{\alpha+1}$, and $H^D_\alpha \subseteq H^D_{\alpha+1}$ implies that $R^D_\alpha \subseteq R^D_{\alpha+1}$. Therefore 1 follows by induction.

2. This follows immediately from 1.

3. Suppose $p \in R^D_\alpha$ and $q \leqslant p$. If H^D_α is $(\leqslant, \leqslant^*)$-dense below p, it is also $(\leqslant, \leqslant^*)$-dense below q. So $q \in R^D_\alpha$.

4. The case H_0^D is trivial. Suppose $p \in H_\alpha^D$ and $q \leqslant^* p$. For every $\zeta \in \mathrm{succ}_{T^q}(q^{\mathrm{mc}})$, $(q)_\zeta \leqslant^* (p)_{\pi_{q,p}(\zeta)}$. Since $R_{<\alpha}^D$ is open with respect to $(\mathbb{P}, \leqslant^*)$, $t^{(q)_\zeta} \in R_{<\alpha}^D$. Therefore $q \in H_\alpha^D$.

5. Suppose $p \in H_\alpha^D$ and $q \leqslant p$. Let $r = q$ and $\zeta \in \mathrm{succ}_{T^r}(r^{\mathrm{mc}})$. Then $(r)_\zeta \leqslant^* (p)_s$ for some $s \in T_{p^{\mathrm{mc}}}^p \backslash \{\varnothing\}$. As $p \in H_\alpha^D$, $t^{(p)_{\min(s)}} \in R_{<\alpha}^D$. By 3, $t^{(p)_s} \in R_{<\alpha}^D$ and $t^{(r)_\zeta} \in R_{<\alpha}^D$. Therefore, $r \in H_\alpha^D$. So H_α^D is $(\leqslant, \leqslant^*)$-dense below p, hence $t^p \in R_\alpha^D$. □

Definition 10. For $p \in \mathbb{P}$, $\mathrm{rank}_D(t^p)$, the *D-rank of t^p*, is the least ordinal α such that $t^p \in R_\alpha^D$, if it exists; otherwise $\mathrm{rank}_D(t^p) = \infty$.[7] We often write the relativized notation $\mathrm{rank}_{p,D}(s)$, in which case called (p, D)-rank of s, to abbreviate for $\mathrm{rank}_D(t^{(p)_s})$, for $s \in T_{p^{\mathrm{mc}}}^p$, although its value only depends on t^p.

Here are some quick facts about ranks.

Proposition 5.2. *Suppose $D \subseteq \mathbb{P}$ is open and $p, q \in \mathbb{P}$.*

(1) If $\mathrm{rank}_D(t^p) < \infty$, then $\mathrm{rank}_D(t^p) < |\mathbb{P}|^+$.
(2) If $\mathrm{rank}_D(t^p) < \infty$ and $q \leqslant p$, then $\mathrm{rank}_D(t^q) \leqslant \mathrm{rank}_D(t^p)$.

Proof. 1. This follows immediately from Proposition 5.1-1.

2. If $q \leqslant p$ and $\mathrm{rank}_D(t^p) < \infty$, then by Proposition 5.1-3,

$$\varnothing \neq \{\alpha \in \mathrm{Ord} \mid t^p \in R_\alpha^D\} \subseteq \{\alpha \in \mathrm{Ord} \mid t^q \in R_\alpha^D\}.$$

Thus $\mathrm{rank}_D(t^q) \leqslant \mathrm{rank}_D(t^p)$. □

Definition 11. Suppose $D \subseteq \mathbb{P}$ is open and $p \in \mathbb{P}$. We say that p is *D-good* if $p \in H_\alpha^D$ and for every $s \in T_{p^{\mathrm{mc}}}^p$ and for $\beta \leqslant \alpha$,

$$(p)_s \in H_\beta^D \quad \Rightarrow \quad (p)_{s^\frown\langle\delta\rangle} \in H_{<\beta}^D, \text{ for all } \delta \in \mathrm{succ}_{T_{p^{\mathrm{mc}}}^p}(s).$$

Clearly if p is D-good, then so is $(p)_s$ for every $s \in T_{p^{\mathrm{mc}}}^p$.

Proposition 5.3. *Suppose $D \subseteq \mathbb{P}$ is open. Let $E_D =_{\mathrm{def}} \{p \in \mathbb{P} \mid p$ is D-good$\}$. Then E_D is $\leqslant^*$-dense below any p with $\mathrm{rank}_D(t^p) < \infty$; or equivalently, for every p such that $\mathrm{rank}_D(t^p) < \infty$, there is a $q \leqslant^* p$ in E_D.*

Proof. Take an $N \prec V_\mu$ for a sufficiently large μ and such that $|N| = \lambda^+$, $N^\lambda \subseteq N$. Let $\kappa < \eta$ be an ordinal such that $\kappa \geqslant_E \zeta$ for all $\zeta \in N \cap [\lambda, \eta)$. We write $R_\alpha^{D,N}$ and H_α^D for the corresponding notions defined in N, and write $\mathrm{rank}_D^N(t^p)$ and $\mathrm{rank}_{p,D}^N(s)$[8], $s \in T_{p^{\mathrm{mc}}}^p$, for the corresponding notions computed in N. By the elementarity of N, these notions are absolute between N and V, more precisely, $R_\alpha^{D,N} = R_\alpha^D \cap N$, $H_\alpha^{D,N} = H_\alpha^D \cap N$ for $\alpha \in \mathrm{Ord} \cap N$, and $\mathrm{rank}_D^N(t^p) = \mathrm{rank}_D(t^p)$ for $p \in \mathbb{P} \cap N$. Proposition 5.3 follows from the following lemma.

Lemma 5.1. *Suppose $p \in \mathbb{P} \cap N$ and T is a U_κ-tree with trunk s_κ and such that $t^p \cup \{\langle\kappa, s_\kappa, T\rangle\} \leqslant^* p$. Suppose $\mathrm{rank}_D^N(t^p) < \infty$. Then there are a $q \in N$ and a U_κ-subtree $T^r \subseteq T$ such that $r = q \cup \{\langle\kappa, s_\kappa, T^r\rangle\}$ is a D-good direct extension of p.*

Grant Lemma 5.1. Suppose $p \in N$ and $\mathrm{rank}_D^N(t^p) < \infty$. By Lemma 5.1, there is a $q \in V$ that is D-good and directly extends p° and hence p. Since $\mathrm{rank}_D(\cdot)$ is absolute between N and V, for every $p \in \mathbb{P} \cap N$ with $\mathrm{rank}_D(t^p) < \infty$, there is

[7] We demand that $\infty > \alpha$ for all $\alpha \in \mathrm{Ord}$.

[8] More precisely, should be $\mathrm{rank}_{D \cap N}^N(t^p)$ and $\mathrm{rank}_{p,D \cap N}^N(s)$.

a D-good direct extension of p in V. By elementarity, for every $p \in \mathbb{P} \cap N$ with $\operatorname{rank}^N_D(t^p) < \infty$, there is a D-good direct extension of p in N. Using elementarity again, every $p \in \mathbb{P}$ in V with $\operatorname{rank}_D(t^p) < \infty$ has a D-good direct extension. Thus the set E_D is dense below p°. □

Now we prove Lemma 5.1.

Proof of Lemma 5.1. The proof proceeds by induction on $\alpha = \operatorname{rank}^N_D(t^p)$ in N. For $\alpha = 0$, it is trivial. We follow the idea in Gitik's proof of his Lemma 3.12 in [7] (page 1387). Assume that for all $\beta \in \alpha \cap N$, the claim holds.

Assume $p \in \mathbb{P}$ and $t^p \in R^D_\alpha \cap N$. By definition, we may replace p with a $p^\circ \leqslant^* p$ in N with least $\alpha \leqslant \operatorname{rank}^N_D(t^p)$ in N such that $p^\circ \in H^D_\alpha \cap N$. So we may assume in addition that $\operatorname{rank}^N_D(t^q) = \operatorname{rank}^N_D(t^p) = \alpha$ for any $q \leqslant^* p$ in $H^D_\alpha \cap N$. By elementarity, for any $q \in H^D_\alpha$, $\operatorname{rank}_D(t^q) = \operatorname{rank}_D(t^p) = \alpha$. Let $A = \operatorname{succ}_T(s_\kappa)$. We shall construct inductively $\langle (p_\xi, T^\xi) : \xi \in A \rangle$. To simplify the presentation, we may assume that $p^- = \varnothing$ and $s_\kappa = \varnothing$.

Suppose we already have $\langle (p_\xi, T^\xi) : \delta \in A \cap \zeta \rangle$. Now we construct p_ζ and T^ζ. Let $p'_\zeta = p \cup (\bigcup\{p_\xi \mid \xi \in A \cap \zeta\})$ and

$$r'_\zeta = p'_\zeta \cup \{\langle \kappa, \varnothing, \bigcup\{T_{\langle \xi \rangle} \mid \xi \in A \backslash \zeta\}\rangle\}.$$

Then $(r'_\zeta)_\zeta \leqslant^* (p)_{\pi_{\kappa,p}(\zeta)}$. As $t^{(p)_{\pi_{\kappa,p}(\zeta)}} \in R^{D,N}_\beta$ for some $\beta \in \alpha \cap N$, $\operatorname{rank}^N_D(t^{(r'_\zeta)_\zeta}) \leqslant \beta$, by the inductive hypothesis, there are a $q \in N$ and a U_κ-subtree $T_\zeta \subseteq T_{\langle \zeta \rangle}$ such that $q \cup \{\langle \kappa, \langle \zeta \rangle, T_\zeta \rangle\}$ is a D-good direct extension of $(r'_\zeta)_\zeta$. Let $p_\zeta = p'_\zeta \cup \{\langle \iota, q^\iota \rangle \mid \iota \in \operatorname{supp}(q) \backslash \operatorname{supp}(r'_\zeta)\}$. This completes the inductive construction.

At the end, let $q = \bigcup_{\xi < \lambda} p_\xi$. For $i < \lambda$, let

$$C_i = \{\bigcap \operatorname{succ}_{T^\xi}(\langle \xi \rangle) \mid \xi \in A \wedge \xi^* = i\}.$$

Since the set of $\xi \in A$ such that $\xi^* = i$ is bounded, $C_i \in U_\kappa$ for each $i < \lambda$. Set $A^* = A \cap (\Delta^*_{i<\lambda} C_i)$. By the weak normality for U_κ, $A^* \in U_\kappa$. Let T^r be the tree obtained from $\bigcup\{T_{\langle \xi \rangle} \mid \xi \in A^*\}$ by intersecting all levels of it with A^*. Then by Claim 3.12.1 in Gitik's [7] (page 1388), $r = q \cup \{\langle \kappa, \varnothing, T^r \rangle\}$ is in $\mathbb{P}$ and directly extends p.

By our construction, for every $\zeta \in \operatorname{succ}_{T^r}(\varnothing)$, $(r)_\zeta$ is D-good and directly extends $(p)_{\pi_{\kappa,q}(\zeta)}$. Since $(r)_\zeta \leqslant^* (p)_{\pi_{\kappa,q}(\zeta)}$, $\operatorname{rank}_D(t^{(r)_\zeta}) \leqslant \operatorname{rank}_D(t^{(p)_{\pi_{\kappa,q}(\zeta)}}) < \alpha$. So $r \in H^D_\alpha$. By our additional assumption on p, $\operatorname{rank}_D(t^r) = \operatorname{rank}_D(t^p) = \alpha$. So r is D-good. □

The Prikry condition for $\mathbb{P}$ (see Lemma 3.12, [7]) can be stated in terms of our rank notion as follows.

Proposition 5.4 (Gitik)**.** *Suppose $D \subseteq \mathbb{P}$ is dense and open. Let $\mathbb{1}_\mathbb{P}$ denote the largest element of $\mathbb{P}$. Then $\operatorname{rank}_D(t^{\mathbb{1}_\mathbb{P}}) < \infty$; or equivalently, for every $p \in \mathbb{P}$, there is a $q \leqslant^* p$ in $H^D_{<\infty}$.*

Proof. Rerun Gitik's proof but with "p decides σ" replaced by "$p \in H^D_{<\infty}$". □

Next we define a notion of rank on members of E_D to isolate a set of "D-better" conditions. For every $p \in E_D$, we define a rank function $\rho_{p,D}(\cdot)$ on $T^p_{p^{\mathrm{mc}}}$ inductively as follows:

- if $(p)_s \in D$, then $\rho_{p,D}(s) = 0$;

- if $(p)_s \notin D$, then $\rho_{p,D}(s)$ is the least α such that there is a U_p-measure one $A \subseteq \text{succ}_{T^p_{p^{\text{mc}}}}(s)$ such that $\alpha \geqslant \rho_{p,D}(s^\frown\langle\delta\rangle) + 1$ for all $\delta \in A$.

By the definition of D-goodness, if $p \in E_D$, then the set $\{s \in T^p_{p^{\text{mc}}} \mid \text{rank}_{p,D}(s) > 0\}$ is a wellfounded subtree of $T^p_{p^{\text{mc}}}$. Thus $\rho_{p,D}(s)$ is defined for all $s \in T^p_{p^{\text{mc}}}$ if $p \in E_D$.

Below is a simple observation to be used in our proof of λ-goodness for $\mathbb{P}$.

Proposition 5.5. *Suppose $D \subseteq \mathbb{P}$ is open and $p \in E_D$. If $q \leqslant p$, then for every $s \in T^q_{q^{\text{mc}}}$, $\text{rank}_D(t^{(q)_s}) \leqslant \text{rank}_D(t^{(q)_{\pi_{q,p}(s)}})$ and $\rho_{q,D}(s) \leqslant \rho_{p,D}(\pi_{q,p}(s))$.*

Proof. It suffices to consider only the case $q \leqslant^* p$. The proof proceeds by induction on $\rho_{q,D}(s)$. We leave the details to the readers. □

Lemma 5.2. *Suppose $D \subseteq \mathbb{P}$ is open and $p \in E_D$. Then $\rho_{p,D}(\varnothing) < \omega$. More precisely, there is a U_p-subtree $S_p \subseteq T^p_{p^{\text{mc}}}$ such that for every $s \in S_p$, $\rho_{p,D}(s) = \max(\rho_{p,D}(\varnothing) - |s|, 0)$.*

Proof. Clearly, the range of $\rho_{p,D}(\cdot)$ is an ordinal. The lemma follows from the fact that U_p is countably complete. Assume towards a contradiction that $\rho_{p,D}(\varnothing) \geqslant \omega$, then there is an $s \in T^p_{p^{\text{mc}}}$ such that $\rho_{p,D}(s) = \omega$. But due to the countably completeness of U_p, there is a finite number k such that $\rho_{p,D}(s^\frown\langle\delta\rangle) < k$ for a U_p-measure one set of $\delta \in \text{succ}_{T^p_{p^{\text{mc}}}}(s)$. Therefore $\rho_{p,D}(s) \leqslant k < \omega$. Contradiction!

Using the idea in §3.4 of [14], by trimming off nodes s in $T^p_{p^{\text{mc}}} \backslash \{\varnothing\}$ such that $\rho_{p,D}(s) \geqslant \rho_{p,D}(s \restriction (|s| - 1)) > 0$, one obtain a U_p-subtree $S_p \subseteq T^p_{p^{\text{mc}}}$ such that for every $s \in S_p$, either $\rho_{p,D}(s) = 0$ or $\rho_{p,D}(s) = \sup\{\rho_{p,D}(s^\frown\langle\delta\rangle) + 1 \mid \delta \in \text{succ}_{S_p}(s)\}$. It is easy to see that this S_p works as desired. □

Definition 12. Suppose $D \subseteq \mathbb{P}$ is open. For a $p \in E_D$, we say p is *D-better* if $T^p_{p^{\text{mc}}}$ satisfies the condition that for every $s \in T^p_{p^{\text{mc}}}$, $\rho_{p,D}(s) = \max\{\rho_{p,D}(\varnothing) - |s|, 0\}$.

From Lemma 5.2, we have

Corollary 4. *B_D is $\leqslant^*$-dense in E_D, therefore $\leqslant^*$-dense in $\mathbb{P}$.*

Now we are ready to prove the main result of this section.

Lemma 5.3. *$\mathbb{P}$ is λ-good.*

Proof. Fix a $p \in \mathbb{P}$ and $\mathscr{D}$, a collection of dense open subsets of $\mathbb{P}$ with $|\mathscr{D}| < \lambda$. Enumerate $\mathscr{D}$ as $\{D_\iota \mid \iota < |\mathscr{D}|\}$. Start with p, we inductively construct a $\leqslant^*$-decreasing sequence $\langle p_\iota : \iota < |\mathscr{D}|\rangle$ and a sequence of integers $\langle k_\iota : \iota < |\mathscr{D}|\rangle$ as follows:

First, let p_0 be a D_0-better direct extension of p and $k_0 = \rho_{p_0,D}(\varnothing)$. Suppose we have constructed the two sequences up to some $\iota > 0$, i.e. $\langle p_\zeta : \zeta < \iota\rangle$ and $\langle k_\zeta : \zeta < \iota\rangle$. Since $(\mathbb{P}, \leqslant^*)$ is λ-closed, there is a $q_\iota \in \mathbb{P}$ such that $p_\zeta \leqslant^* q_\iota$ for all $\zeta < \iota$. Let p_ι be a D_ι-better direct extension of q_ι and $k_\iota = \rho_{q_\iota,D}(\varnothing)$.

At the end, pick a $p^\circ \in \mathbb{P}$ such that $p^\circ \leqslant^* p_\iota$ for all $\iota < |\mathscr{D}|$. For each $k < \omega$, let $\mathscr{D}_{p,k} = \{D_\iota \mid k_\iota \leqslant k\}$. We may assume that $\mathscr{D}_{p,k} \neq \varnothing$ for all $i < \omega$. We claim that $\bigcap \mathscr{D}_{p,k}$ is dense below p° for all $k < \omega$.

Fix a $k < \omega$. Suppose $r \leqslant p^\circ$. By replacing r with a $q \leqslant^* p^\circ$ such that $r = q^\frown s$ for some $s \in T^q_{q^{\text{mc}}}$, we may assume that $r \leqslant^* p^\circ$. Note that by Proposition 5.5 for any $s \in T^r_{r^{\text{mc}}}$ and any $\xi < |\mathscr{D}|$ such that $D_\xi \in \mathscr{D}_{p,k}$, $\text{rank}_D(t^{(r)_s}) \leqslant \text{rank}_D(t^{(q_\xi)_{\pi_{r,q_\xi}(s)}})$ and $\rho_{r,D_\xi}(\pi_{r,q_\xi}(s)) \leqslant \rho_{p_\xi,D_\xi}(s)$. Pick an $s \in T^r_{r^{\text{mc}}}$ such that $|s| \geqslant k$, then $\rho_{r,D}(s) = 0$

for every $D \in \mathscr{D}_{p,k}$. Hence, $(r)_s \in D$ for every $D \in \mathscr{D}_{p,k}$. This shows that $\bigcap \mathscr{D}_{p,k}$ is dense below p°. □

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