Mass in Kähler Geometry

Hans-Joachim Hein* and Claude LeBrun†

Abstract

We prove a simple, explicit formula for the mass of any asymptotically locally Euclidean (ALE) Kähler manifold, assuming only the sort of weak fall-off conditions required for the mass to actually be well-defined. For ALE scalar-flat Kähler manifolds, the mass turns out to be a topological invariant, depending only on the underlying smooth manifold, the first Chern class of the complex structure, and the Kähler class of the metric. When the metric is actually AE (asymptotically Euclidean), our formula not only implies a positive mass theorem for Kähler metrics, but also yields a Penrose-type inequality for the mass.

A complete connected non-compact Riemannian manifold (M, g) of dimension $n \geq 3$ is said to be *asymptotically Euclidean* (or *AE*) if there is a compact subset $\mathbf{K} \subset M$ such that $M - \mathbf{K}$ consists of finitely many components, each of which is diffeomorphic to the complement of a closed ball $\mathbf{D}^n \subset \mathbb{R}^n$, in a manner such that g becomes the standard Euclidean metric plus terms that fall off sufficiently rapidly at infinity. More generally, a Riemannian n-manifold (M, g) is said to be *asymptotically locally Euclidean* (or *ALE*) if the complement of a compact set $\mathbf{K}$ consists of finitely many components, each of which is diffeomorphic to a quotient $(\mathbb{R}^n - \mathbf{D}^n)/\Gamma_j$, where $\Gamma_j \subset \mathbf{O}(n)$ is a finite subgroup which acts freely on the unit sphere, in such a way that g again becomes the Euclidean metric plus error terms that fall off sufficiently rapidly at infinity. The components of $M - \mathbf{K}$ are called the *ends* of M; their fundamental groups are the afore-mentioned groups Γ_j, which may in principle be different for different ends of the manifold.

*Research funded in part by NSF grant DMS-1514709.
†Research funded in part by NSF grant DMS-1510094.

The *mass* of an ALE Riemannian n-manifold is an invariant which assigns a real number to each end. This concept originated in general relativity, where an asymptotically flat 3-manifold could be interpreted as representing a time-symmetric slice of some 4-dimensional space-time, in which case this invariant becomes the so-called ADM mass [4], which reads off the apparent mass of an isolated gravitational source from the asymptotics of its gravitational field. Our conventions are chosen so that, at a given end, the mass of an ALE manifold is given by

$$\mathfrak{m}(M,g) := \lim_{\varrho\to\infty} \frac{\mathbf{\Gamma}(\frac{n}{2})}{4(n-1)\pi^{n/2}} \int_{S_\varrho/\Gamma_j} [g_{k\ell,k} - g_{kk,\ell}]\, \mathbf{n}^\ell d\mathfrak{a}_E$$

where commas represent derivatives in the given asymptotic coordinates, summation over repeated indices is implicit, S_ϱ is the Euclidean coordinate sphere of radius ϱ, $d\mathfrak{a}_E$ is the $(n-1)$-dimensional volume form induced on this sphere by the Euclidean metric, and $\vec{\mathbf{n}}$ is the outward-pointing Euclidean unit normal vector. While our choice here of normalization factor is of course primarily a matter of convention, an explanation of this choice is provided in the Appendix. Perhaps the most controversial feature of our definition is that we have specified that the integral is to be taken over S_ϱ/Γ_j rather than over S_ϱ, so that the mass, by our conventions, is $1/|\Gamma_j|$ times the value one might otherwise expect.

Needless to say, this peculiar definition of the mass seems to depend on the choice of asymptotic coordinates. Indeed, without additional assumptions, the relevant limit might not even exist, or might be coordinate dependent. However, Bartnik [7] and Chruściel [15] independently discovered that the mass is finite and independent of the choice of asymptotic coordinates provided we impose weak fall-off conditions of the following type:

(i) the scalar curvature s of the C^2 metric g belongs to L^1; and

(ii) in some asymptotic chart at each end of M^n, the components of the metric satisfy $g_{jk}-\delta_{jk} \in C^{1,\alpha}_{-\tau}$ for some $\tau > (n-2)/2$ and some $\alpha \in (0,1)$.

Here the weighted Hölder spaces $C^{k,\alpha}_{-\tau}$ consist of $C^{k,\alpha}$ functions such that

$$\left(\sum_{j=0}^{k} |x|^j |\nabla^j f(x)|\right) + |x|^{k+\alpha}[\nabla^k f]_{C^{0,\alpha}(B_{|x|/10}(x))} = O(|x|^{-\tau}).$$

This definition can naturally be extended to tensor fields, and the resulting $C^{k,\alpha}_{-\tau}$ spaces then become Banach spaces when equipped with the obvious weighted analogs of the usual Hölder norms. While Bartnik actually does mention these weighted Hölder spaces in passing [7, Theorem 1.2 (v)], the state of the literature at the time led him to instead impose a slightly stronger condition in lieu of (ii), by instead requiring $g-\delta$ to belong to the weighted Sobolev spaces $W^{2,q}_{-\tau}$ for some $q>n$ and some $\tau>(n-2)/2$. Bartnik's condition implies (ii), and condition (ii) in turn implies that, for some $\varepsilon>0$, the metric g satisfies the Chruściel-type fall-off condition

$$g_{jk}=\delta_{jk}+O(|x|^{1-\frac{n}{2}-\varepsilon}), \qquad g_{jk,\ell}=O(|x|^{-\frac{n}{2}-\varepsilon})$$

in suitable coordinates; and this Chruściel-type fall-off is actually all that is needed for many of our key results. The central issue is really the range of fall-off rates τ that are to be allowed; as emphasized by both Bartnik and Chruściel, allowing slower rates of fall-off than indicated above would make the mass coordinate-dependent, and so essentially ill-defined. Our definition of an ALE manifold will therefore *by default* include conditions (i) and (ii), except where we clearly specify that a weaker assumption suffices for a given result. When $n=4$, some our proofs will also require analogous control of an extra derivative of the the metric, and so *by default* we will strengthen assumption (ii) in this special dimension to instead require that $g_{jk}-\delta_{jk}\in C^{2,\alpha}_{-\tau}$ for some $\tau>(n-2)/2=1$, although we will also sometimes explicitly weaken this assumption when it is not needed for a given result.

The coordinate-based definition of the mass makes it seem both enigmatic and chimerical. In this article, however, we will show that the mass has a completely transparent meaning when the ALE space in question is a *Kähler* manifold. Along the way, we will incidentally learn that an ALE Kähler manifold only has one end; thus, in the Kähler setting, a choice of end is not required in order to be able to discuss the mass in the first place!

Rather than beginning with general ALE Kähler manifolds, let us first highlight the setting that originally motivated our investigation: the so-called scalar-flat case, where the scalar curvature is assumed to vanish identically. In this context, we will demonstrate the following result:

Theorem A. *The mass of an ALE scalar-flat Kähler manifold (M,g,J) is a topological invariant, determined entirely by the smooth manifold M, together with the first Chern class $c_1=c_1(M,J)\in H^2(M)$ of the complex structure and the Kähler class $[\omega]\in H^2(M)$ of the metric.*

In fact, our proof actually provides an *explicit* formula for the mass in terms of these data. Revisiting familiar examples, this in particular gives a pure-thought explanation of the second author's observation [33] that there are ALE scalar-flat Kähler surfaces[1] of negative mass. Rather more interestingly, though, a quick glance at other known examples immediately now gives a negative answer[2] to a question posed by Arezzo [3] that naturally arose in connection with gluing constructions for cscK metrics:

Theorem B. *There are infinitely many topological types of ALE scalar-flat Kähler surfaces that have zero mass, but are not Ricci-flat.*

By contrast, Corollary 4.8 below, which was pointed out to us by Cristiano Spotti, gives a systematic explanation of why the mass actually turns out to be negative for so many other concrete examples.

We now come to the actual formula for the mass. Because M is a smooth manifold, one can define the compactly supported de Rham cohomology $H^k_c(M)$, as well as the usual de Rham cohomology. If M is a complex manifold, it is in particular oriented, and Poincaré duality therefore gives us an isomorphism $H^2_c(M) \cong [H^{2m-2}(M)]^*$. On the other hand, there is a natural map $H^2_c(M) \to H^2(M)$ induced by the inclusion of compactly supported forms into all differential forms, and in the ALE setting, this map is actually an *isomorphism*. We may therefore define

$$\clubsuit : H^2(M) \to H^2_c(M)$$

to be its inverse. Using this notation, we may now state our explicit formula for the mass:

Theorem C. *Any ALE Kähler manifold (M, g, J) of complex dimension m has mass given by*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s_g d\mu_g$$

where s_g and $d\mu_g$ are respectively the scalar curvature and volume form of g, while $c_1 = c_1(M, J) \in H^2(M)$ is the first Chern class of the complex structure, $[\omega] \in H^2(M)$ is the Kähler class of g, and $\langle \ , \ \rangle$ is the duality pairing between $H^2_c(M)$ and $H^{2m-2}(M)$.

[1] Throughout the article, we use the term *complex surface* to indicate a complex manifold of *complex* dimension 2, and thus of real dimension 4.

[2] We would like to thank Ioana Şuvaina for pointing out to us that this answer was already implicit in results of Rollin-Singer [44, §6.7] regarding the toric case.

Here we remind the reader that the Bartnik-Chruściel fall-off condition (i) requires the scalar curvature s to be integrable. It is remarkable that this feature[3] plays a direct role in our setting, by ensuring that the second term on the right-hand side is well-defined. It is worth noting that our discussion will not simply rely on the Bartnik-Chruściel theorem on the coordinate-invariance of the mass, but rather will actually provide an independent verification of it in the Kähler setting.

The reader may find it illuminating to compare Theorem C with the more familiar compact case. If (M^{2m}, g, J) is a *compact* Kähler manifold of complex dimension m, then its total scalar curvature is well known to be topologically determined [8, 11] by the first Chern class of the complex structure and the Kähler class of the metric via the Gauss-Bonnet-type formula

$$\int_M s \, d\mu = \frac{4\pi}{(m-1)!} \langle c_1, [\omega]^{m-1} \rangle .$$

The gist of Theorem C is that the mass measures the degree to which this formula fails in the ALE case:

$$\frac{4\pi^m(2m-1)}{(m-1)!} \mathfrak{m}(M, g) = \int_M s \, d\mu - \frac{4\pi}{(m-1)!} \langle ♣c_1, [\omega]^{m-1} \rangle .$$

In other words, the mass may be understood as an *anomaly* in the formula for the total scalar curvature, encapsulating an essential difference between the ALE and compact cases.

Of course, the formula in Theorem C simplifies when g is scalar-flat; the integral on the right drops out, and the mass is then expressed purely in terms of topological data. Theorem A is thus an immediate corollary. Theorem B is then proved by applying this formula to some ALE scalar-flat Kähler surfaces constructed by the second author in [34].

As we've already noted, there are ALE manifolds of non-negative scalar curvature which nonetheless have negative mass. However, one expects this to never happen for AE (asymptotically Euclidean) manifolds. Indeed, this is actually a theorem [38, 46, 51] if one is willing to further assume that the manifold is either low-dimensional or spin. Here we can add something new to the discussion, by demonstrating that the conjecture also always holds in the Kähler case, even if the manifold is high-dimensional and non-spin:

[3] As explained to us by Gustav Holzegel, the intimate relationship between mass and scalar curvature apparently first came to light in the work of Brill [10] on stationary axisymmmetric space-times.

Theorem D (Positive Mass Theorem for Kähler Manifolds)**.** *Any asymptotically Euclidean (AE) Kähler manifold with non-negative scalar curvature has non-negative mass:*

$$s \geq 0 \quad \Longrightarrow \quad \mathfrak{m}(M,g) \geq 0.$$

Moreover, $\mathfrak{m}(M,g) = 0$ *in this context iff* (M,g) *is Euclidean space.*

Our proof of this version of the positive mass theorem uses nothing but our mass formula and some complex manifold theory. Indeed, the argument actually tells us a great deal more; it in fact shows that the mass can be bounded from below by the $(2m-2)$-volume of a subvariety. This is reminiscent of the Penrose inequality [9, 28, 43], which gives a sharp lower bound for the mass of an AE 3-manifold in terms of the area of a minimal surface. Our Kähler analog goes as follows:

Theorem E (Penrose Inequality for Kähler Manifolds)**.** *Let* (M^{2m}, g, J) *be an AE Kähler manifold with scalar curvature* $s \geq 0$*. Then* (M,J) *carries a canonical divisor* D *that is expressed as a sum* $\sum n_j D_j$ *of compact complex hypersurfaces with positive integer coefficients, with the property that* $\bigcup_j D_j \neq \varnothing$ *whenever* $(M,J) \neq \mathbb{C}^m$*. In terms of this divisor, we then have*

$$\mathfrak{m}(M,g) \geq \frac{(m-1)!}{(2m-1)\pi^{m-1}} \sum_j n_j \, Vol(D_j)$$

and equality holds if and only if (M,g,J) *is scalar-flat Kähler.*

Much of the intrinsic interest of our subject arises from the case of real dimension 4, where a plethora of known examples leads to a wealth of applications, including Theorem B. However, the complex-surface case entails technical subtleties that simply disappear in higher dimensions. Our presentation therefore begins with proofs of Theorems A and C in complex dimension $m \geq 3$. Using this high-dimensional case as our guide, but now emphasizing the coordinate-invariant nature of the mass, we then develop a second, more robust proof of the asymptotic form of our mass formula, in a manner that also shows that this formula remains valid in complex dimension 2. We then prove some global results regarding ALE Kähler surfaces, culminating in a proof of the $m = 2$ case of Theorem C, along with various applications, including Theorem B. We then conclude by showing that Theorems D and E are straightforward corollaries of our other results.

1 The High-Dimensional Case

We begin by proving Theorems A and C when the complex dimension is $m \geq 3$. Our high-dimensional proofs will prefigure many of the ideas needed for the complex-surface ($m = 2$) case, but manage to avoid a number of difficult technical complications. Our journey begins with the following step:

Lemma 1.1. *Let M_∞ be an end of an ALE Kähler manifold (M^{2m}, g, J), $m \geq 3$, and let $(x^1, \ldots, x^{2m})$ be a real asymptotic coordinate system on the universal cover $\widetilde{M}_\infty$ of M_∞ in which g satisfies the weak fall-off hypothesis*

$$g_{jk} = \delta_{jk} + O(|x|^{1-m-\varepsilon}), \qquad g_{jk,\ell} = O(|x|^{-m-\varepsilon})$$

for some $\varepsilon > 0$. Then there is a (non-compact) complex m-manifold $\mathscr{X}$ containing an embedded complex hypersurface $\Sigma \cong \mathbb{CP}_{m-1}$ with normal bundle of degree $+1$, such that $\widetilde{M}_\infty$ is biholomorphic to $\mathscr{X} - \Sigma$.

Proof. We first identify the range $\mathbb{R}^{2m} - \mathbf{D}^{2m}$ of our asymptotic coordinate system with $\mathbb{C}^m - \mathbf{D}^{2m}$ in a reasonably intelligent manner, by choosing a constant-coefficient almost-complex structure J_0 on $\mathbb{R}^{2m}$ such that $J \to J_0$ at infinity. We can do this by identifying all the tangent spaces of $\mathbb{R}^{2m}$ in the usual way, using the flat Euclidean connection $\triangledown$. Since $\nabla J = 0$, where ∇ is the Levi-Civita connection of g, and since $\nabla = \triangledown + \mathbf{\Gamma}$, where $\triangledown$ is the coordinate Euclidean connection and $\mathbf{\Gamma} = O(\varrho^{-m-\varepsilon})$, the value of J will approach a well-defined limit J_0 along some chosen radial ray, and we then extend this limit as a constant-coefficient tensor field on our asymptotic coordinate domain. Along the chosen ray, we then have $J - J_0 = O(\varrho^{1-m-\varepsilon})$, and integrating along great circles in spheres of constant radius then shows that $J - J_0$ has $O(\varrho^{1-m-\varepsilon})$ fall-off everywhere. The same argument similarly shows that the derivative of J falls off at the same rate as the derivative of the metric g.

Now think of $(\mathbb{C}^m, J_0)$ as an affine chart on $\mathbb{CP}_m$, whose complex structure we will also denote by J_0. Let $\Sigma \subset \mathbb{CP}_m$ be the hyperplane at infinity, and notice that our asymptotic coordinates give us a diffeomorphism between $\widetilde{M}_\infty$ and $\mathscr{X} - \Sigma$, where $\mathscr{X} \subset \mathbb{CP}_m$ is some neighborhood of this hyperplane. We may then define a "rough" almost complex structure J on $\mathscr{X}$ by taking it to be the given J on $\mathscr{X} - \Sigma$, and J_0 along Σ. This J is then at least C^1 on $\mathscr{X}$. Indeed, if $(z^1, z^2, \ldots, z^m)$ are the standard affine coordinates on $\mathbb{C}^m$, then, in the cone $|z^1| \geq \max\{|z^j| \mid j > 1\}$, we may inspect the behavior of J

near infinity by observing that there is a unique $(m,0)$-form with respect to J given by

$$\varphi = (dz^1 + \varphi_1^{\bar{j}} d\bar{z}^j) \wedge (dz^2 + \varphi_2^{\bar{j}} d\bar{z}^j) \wedge \cdots \wedge (dz^m + \varphi_m^{\bar{j}} d\bar{z}^j),$$

and that the functions $\varphi_k^{\bar{j}}$ then have the same fall-off as J. Setting

$$(w_1, w_2, \ldots, w_m) = \left(\frac{1}{z^1}, \frac{z^2}{z^1}, \ldots, \frac{z^m}{z^1} \right),$$

one can then analogously determine the components of J from those of

$$\psi := -w_1^{m+1}\varphi = (dw_1 + \psi_1^{\bar{j}} d\bar{w}_j) \wedge (dw_2 + \psi_2^{\bar{j}} d\bar{w}_j) \wedge \cdots \wedge (dw_m + \psi_m^{\bar{j}} d\bar{w}_j).$$

Reading off the coefficients $\varphi_k^{\bar{j}}$ and $\psi_k^{\bar{j}}$ by inspecting the type $(m-1,1)$ parts of φ and ψ with respect to the background complex structure J_0, one then sees that the coefficients $\{\psi_k^{\bar{j}}\}$ behave like the $\{\varphi_k^{\bar{j}}\}$ times, at worst, $O(|w_1|^{-1})$, while their first derivatives behave like those of the $\{\varphi_k^{\bar{j}}\}$ times, at worst, $O(|w_1|^{-3})$. Since $\varrho^{-1} = O(|w_1|)$ in the region in question, our fall-off conditions therefore guarantee that the almost-complex structure is at least C^1. In particular, the Nijenhuis tensor of J is continuous, and since it vanishes on the dense set $\mathscr{X} - \Sigma$, it vanishes identically. The Hill-Taylor version [24] of Newlander-Nirenberg therefore guarantees the existence of complex coordinates on $(\mathscr{X}, J)$. These will at least have Hölder regularity $C^{1,\alpha}$ with respect to the original atlas, for any $\alpha \in (0,1)$. □

In fact, our fall-off conditions are noticeably stronger than what is actually needed for the proof of this lemma. In any case, whenever we can add such a hypersurface at infinity, then, provided the complex dimension is $m \geq 3$, the following result will force the complex structure J to become completely *standard* at infinity:

Lemma 1.2. *Let $(\mathscr{X}, J)$ be a (possibly non-compact) complex m-manifold, $m \geq 3$, that contains an embedded hypersurface $\Sigma \subset \mathscr{X}$ which is biholomorphic to $\mathbb{CP}_{m-1}$ and has normal bundle of degree $+1$. Then $\Sigma \subset \mathscr{X}$ has an open neighborhood $\mathscr{U}$ which is biholomorphic to an open neighborhood of a hyperplane $\mathbb{CP}_{m-1} \subset \mathbb{CP}_m$.*

Proof. Since the normal bundle of $\Sigma \cong \mathbb{CP}_{m-1}$ is isomorphic to $\mathcal{O}(1)$, and since $H^1(\mathbb{CP}_{m-1}, \mathcal{O}(1)) = 0$, a theorem of Kodaira [31] implies that there is a complete analytic family of compact complex submanifolds of dimension $h^0(\mathbb{CP}_{m-1}, \mathcal{O}(1)) = m$ which represents all small deformations of $\Sigma \subset \mathscr{X}$ through compact complex submanifolds. Since $\mathbb{CP}_{m-1}$ is rigid, and since $h^{0,1}(\mathbb{CP}_{m-1}) = 0$, we may assume, by shrinking the size of the family if necessary, that every submanifold in the family is biholomorphic to $\mathbb{CP}_{m-1}$, and has normal bundle $\mathcal{O}(1)$. Let us use $\mathscr{Y}$ to denote the complex m-manifold which parameterizes these hypersurfaces; and for any $y \in \mathscr{Y}$, let $\Sigma_y \subset \mathscr{X}$ be the corresponding complex hypersurface. Note that, by construction, $\Sigma = \Sigma_o$ for some base-point $o \in \mathscr{Y}$.

Now Kodaira's theorem also gives us a natural identification of the tangent space $T^{1,0}_y\mathscr{Y}$ with the holomorphic sections of the normal bundle of $\Sigma_y \subset \mathscr{X}$. Since $H^0(\mathbb{CP}_{m-1}, \mathcal{O}(1))$ consists of linear functions on $\mathbb{C}^m$, the space of complex directions $\mathbb{P}(T^{1,0}_y\mathscr{Y})$ can thus be naturally identified with the space of hyperplanes $\mathbb{CP}_{m-2} \subset \Sigma_y \cong \mathbb{CP}_{m-1}$; in other words, each Σ_y is exactly the dual projective space $\mathbb{P}(\Lambda^{1,0}_y\mathscr{Y})$ of the projectivized tangent space $\mathbb{P}(T^{1,0}_y\mathscr{Y})$.

This leads us to consider the space $\mathscr{Z}$ of those embedded $\mathbb{CP}_{m-2}$'s in $\mathscr{X}$ that arise as hyperplanes in the various Σ_y. Thus, by definition, each $z \in \mathscr{Z}$ corresponds to a submanifold $\Pi_z \cong \mathbb{CP}_{m-2}$ of $\mathscr{X}$. But since $\Sigma_y = \mathbb{P}(\Lambda^{1,0}_y\mathscr{Y})$, any point of the projectivized tangent bundle

$$p : \mathbb{P}(T^{1,0}\mathscr{Y}) \to \mathscr{Y}$$

also gives rise to some such submanifold Π_z. Since any $\Pi_z \cong \mathbb{CP}_{m-2}$ has normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$, the family Π_z, $z \in \mathscr{Z}$, is therefore complete in the sense of Kodaira, because every section of the normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ can be realized by some variation in $\mathbb{P}(T^{1,0}\mathscr{Y})$. In fact, this observation actually tells us a great deal more; not only is $\mathscr{Z}$ a complex manifold of complex dimension $2m-2$, but there is a natural surjective holomorphic submersion $q : \mathbb{P}(T^{1,0}\mathscr{Y}) \to \mathscr{Z}$. We thus obtain a double fibration

$$\begin{tikzcd} & \mathbb{P}(T^{1,0}\mathscr{Y}) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathscr{Z} & & \mathscr{Y} \end{tikzcd}$$

which embeds $\mathbb{P}(T^{1,0}\mathscr{Y})$ into the product $\mathscr{Z}\times\mathscr{Y}$, and thereby realizes it as

$$\mathbb{P}(T^{1,0}\mathscr{Y}) = \{(z,y)\in\mathscr{Z}\times\mathscr{Y} \mid \Pi_z\subset\Sigma_y\}.$$

In particular, for any $z\in\mathscr{Z}$, the curve $\gamma_z\subset\mathscr{Y}$ given by $\mathfrak{p}[\mathfrak{q}^{-1}(z)]$ exactly consists of those $y\in\mathscr{Y}$ for which $\Sigma_y\supset\Pi_z$. But this also shows that γ_z is an immersed complex curve, with tangent space at y exactly consisting of sections of the normal bundle $\mathcal{O}(1)$ of $\Sigma_y\cong\mathbb{CP}_{m-1}$ which vanish at the hyperplane $\Pi_z\cong\mathbb{CP}_{m-2}$. Hence the lift $\tilde{\gamma}_z\to\mathbb{P}(T^{1,0}\mathscr{Y})$ of γ_z defined by $\tilde{\gamma}_z := T^{1,0}\gamma_z$ coincides with $\mathfrak{q}^{-1}(z)$. In particular, the holomorphic system of complex curves γ_z, $z\in\mathscr{Z}$, has the property that there is exactly one such curve tangent to each direction in $\mathscr{Y}$. By [32, Proposition 1.2.I], the curves γ_z are therefore the unparameterized geodesics of a unique holomorphic projective connection on $\mathscr{Y}$; moreover, by replacing $\mathscr{Y}$ with a smaller neighborhood of o if necessary, we may arrange that this projective connection is globally represented by some torsion-free holomorphic affine connection ∇, with respect to which $\mathscr{Y}$ is geodesically convex. This allows us to identify $\mathscr{Z}$ with the space of unparameterized complex geodesics of ∇.

Let $\mathscr{U}\subset\mathscr{X}$ be the open set defined by

$$\mathscr{U} = \bigcup_{y\in\mathscr{Y}}\Sigma_y.$$

The fact that this is open follows from the fact that the normal bundle $\mathcal{O}(1)$ of every Σ_y is everywhere generated by its global sections. But now, by construction, every $x\in\mathscr{U}$ belongs to Σ_y for some $y\in\mathscr{Y}$. For each $x\in\mathscr{U}$, we can therefore define a non-empty hypersurface $\mathscr{S}_x\subset\mathscr{Y}$ by

$$\mathscr{S}_x := \{y\in\mathscr{Y} \mid x\in\Sigma_y\}.$$

This is a non-singular hypersurface, because the normal bundle of each $y\in\mathscr{S}_x$ has a global holomorphic section which is non-zero at $x\in\Sigma_y$; the set of normal sections vanishing at x thus has complex codimension 1, and exactly corresponds to $T^{1,0}_y\mathscr{S}_x\subset T^{1,0}_y\mathscr{Y}$. Moreover, since $\Sigma_y=\mathbb{P}(\Lambda^{1,0}_y\mathscr{Y})$, the tangent space $T^{1,0}_y\mathscr{S}_x$, for any $x\in\Sigma_y$, is exactly the hyperplane in $T^{1,0}_y\mathscr{Y}$ annihilated by the 1-dimensional subspace $x\subset\Lambda^{1,0}_y\mathscr{Y}$. It follows that there is a hypersurface $\mathscr{S}_x$ tangent to any given hyperplane in $T^{1,0}\mathscr{Y}$.

However, with respect to ∇, the hypersurfaces $\mathscr{S}_x$ are all *totally geodesic!* Indeed, if $y\in\mathscr{S}_x$ and $\xi\in T^{1,0}_y\mathscr{S}_x-0$, the section of the normal bundle of Σ_y

which represents ξ must vanish at x, and must do so at some $\Pi_z \cong \mathbb{CP}_{m-2}$ containing x. The geodesic γ_z through y in the direction ξ therefore precisely consists of those $y' \in \gamma_z \subset \mathscr{Y}$ for which $\Pi_z \subset \Sigma_{y'}$. But since $x \in \Pi_z$, we therefore have $x \in \Sigma_{y'}$ for every $y' \in \gamma_z$, and it therefore follows that $\gamma_z \subset \mathscr{S}_x$. This shows that $\mathscr{S}_x$ is totally geodesic, as claimed.

However, a classical theorem of Schouten and Struik [47, p. 182] asserts that a projective connection in dimension $m \geq 3$ is projectively flat iff every hyperplane element is tangent to a totally geodesic hypersurface; cf. [48, p. 290]. Thus ∇ is projectively flat, and $o \in \mathscr{Y}$ therefore has a neighborhood which can be identified with a ball in $\mathbb{C}^m$, in such a manner that the unparameterized geodesics of ∇ are just the intersections of complex lines in $\mathbb{C}^m$ with the ball. Let us again shrink $\mathscr{Y}$ by replacing it with this ball about o. The hypersurfaces $\mathscr{S}_x$ are now just the intersections of hyperplanes in $\mathbb{C}^m$ with the ball $\mathscr{Y}$; in other words, thinking of $\mathbb{C}^m$ as an affine chart on $\mathbb{CP}_m$, they are just the intersections of projective hyperplanes with a fixed ball about o. For the smaller $\mathscr{U}$ that corresponds to this smaller $\mathscr{Y}$, we therefore get an injective holomorphic map to the dual projective space $\mathbb{CP}_m^*$ by sending $x \in \mathscr{U}$ to the hyperplane which intersects $\mathscr{Y}$ in $\mathscr{S}_x$. This provides the promised biholomorphism between $\mathscr{U} \supset \Sigma_o = \Sigma$ and a neighborhood of a hyperplane in projective m-space. $\square$

Remark The above-cited result of Schouten and Struik is proved by showing that the Weyl projective curvature of the projective connection vanishes, and then using the fact, due to Weyl [50, p. 105], that, when $m \geq 3$, this curvature condition is equivalent to the projective connection being projectively flat. The fact that this fails when $m = 2$ gives the complex surface case an entirely different flavor, as we will see in Lemma 3.4 below.

There are certainly many other ways of proving the above result. One alternative strategy would proceed by first using [18] to show the infinitesimal neighborhoods of $\Sigma \subset \mathscr{X}$ are all standard, and then invoking [16] or [25] to conclude that a neighborhood of $\Sigma \subset \mathscr{X}$ is therefore biholomorphic to a neighborhood of $\mathbb{CP}_{m-1} \subset \mathbb{CP}_m$. $\diamondsuit$

Perhaps the single most important consequence of Lemma 1.2 is that J must always be standard at infinity when $m \geq 3$. For us, it is vital that the asymptotic coordinates which put J in standard form can moreover be chosen to be consistent with the hypothesized fall-off of the metric:

Lemma 1.3. *Let (M^{2m}, g, J) be an ALE Kähler manifold of complex dimension $m \geq 3$ which, in some real coordinate system on each end, merely satisfies condition* (ii)*, as set out on page 2 above. Then there are asymptotic complex coordinates $(z^1, \dots, z^m)$ on the universal cover of $\widetilde{M}_\infty$ of any end, in which the complex structure J becomes the standard one on $\mathbb{C}^m$, and in which the metric has fall-off*

$$g_{jk} = \delta_{jk} + O(|z|^{1-m-\varepsilon}), \quad g_{jk,\ell} = O(|z|^{-m-\varepsilon})$$

for some $\varepsilon > 0$.

Proof. Let $(\tilde{x}^1, \dots, \tilde{x}^{2m})$ be some given asymptotic coordinate system in which $g_{jk} - \delta_{jk} \in C^{1,\alpha}_{-\tau}$ for some $\tau > m-1$ and some $\alpha \in (0,1)$, and let us once again set $\varepsilon = \min(\tau - (m-1), \alpha)$. We now think of $\mathbb{R}^{2m}$, with real coordinates $(\tilde{x}^1, \dots, \tilde{x}^{2m})$ and the constant-coefficient almost-complex structure J_0 of the proof of Lemma 1.1, as an affine chart on $\mathbb{CP}_m$. Lemma 1.2, in conjunction with the proof of Lemma 1.1, then gives us a C^1 diffeomorphism Ψ between neighborhoods of $\mathbb{CP}_{m-1} \subset \mathbb{CP}_m$ that, by [42, 40], restricts as a $C^{2,\varepsilon}$ diffeomorphism between the complement of a compact set in $\mathbb{R}^{2m}$, with coordinates $\tilde{x}$, and the complement of a compact set in $\mathbb{C}^m$, equipped with standard complex coordinates $(z^1, \dots, z^m)$; and let $(x^1, \dots, x^{2m})$ be the real and imaginary parts of $(z^1, \dots, z^m)$, so that $z^j = x^{2j-1} + ix^{2j}$. Now notice that, for some large constant C, we in particular have

$$C^{-1}|x| < |\tilde{x}| < C|x|$$

outside a large ball, simply because Ψ is uniformly Lipschitz near $\mathbb{CP}_{m-1} \subset \mathbb{CP}_m$. Because Ψ is by construction holomorphic with respect to the complex structure J, the functions $z^\mu := \Psi^* z^\mu$ are holomorphic with respect to the complex structure associated with our Kähler metric, so their real and imaginary parts $x^j := \Psi^* x^j$ are harmonic functions with respect to the Kähler metric g. We now use a partition of unity to construct a $C^{1,\varepsilon}$ Riemannian metric $\bar{g}$ on $\mathbb{R}^{2m}$ which coincides with g outside some large ball, and use a smooth cut-off function to construct $C^{2,\varepsilon}$ functions f^j on $\mathbb{R}^{2m}$ which coincide with the $x^j := \Psi^* x^j$ outside this same ball. The Laplacians $\Delta_{\bar{g}} f^j$ of these functions are then compactly supported $C^{0,\varepsilon}$ functions on $\mathbb{R}^{2m}$.

The fall-off of the first derivative of g in $\tilde{x}$-coordinates implies that

$$\Delta_{\bar{g}} \tilde{x}^j = g^{k\ell} \mathbf{\Gamma}^j_{k\ell} \in C^{0,\varepsilon}_{-m-\varepsilon}(\mathbb{R}^{2m}).$$

On the other hand, since $-m-\varepsilon \in (-2m, -2)$, the Laplacian $\Delta_{\bar{g}}$ induces an isomorphism [30, Theorem 8.3.6] between $C^{2,\varepsilon}_{2-m-\varepsilon}(\mathbb{R}^{2m})$ and $C^{0,\varepsilon}_{-m-\varepsilon}(\mathbb{R}^{2m})$. Thus, for each j, there is a unique $u^j \in C^{2,\varepsilon}_{2-m-\varepsilon}(\mathbb{R}^{2m})$ with $\Delta_{\bar{g}} u^j = \Delta_{\bar{g}} \tilde{x}^j$. The functions $\tilde{y}^j := \tilde{x}^j - u^j$ are then $\bar{g}$-harmonic functions on $\mathbb{R}^{2m}$, and provide coordinates at infinity that are asymptotic to the $\tilde{x}^j$. But, since $\Delta_{\bar{g}} f^j \in C^{0,\varepsilon}_{\beta}$ for $\beta < -2m$, [30, Theorem 8.3.6] also asserts that there is, for each j, a unique function $v^j \in C^{2,\varepsilon}_{2-2m}$ with $\Delta_{\bar{g}} v^j = \Delta_{\bar{g}} f^j$. The functions $y^j = f^j - v^j$ are then yet another set of $\bar{g}$-harmonic functions which give us coordinates at infinity, this time instead asymptotic to the x^j.

Now choose some $\eta \in (1,2)$ and some $q > 2m$. Since the $\tilde{y}^j$ and the y^j are $O(|x|) = O(|\tilde{x}|)$ at infinity, they therefore belong to the weighted space L^q_η used by Bartnik [7]. On the other hand, our $C^{1,\alpha}_{-\tau}$ fall-off condition on the metric guarantees that $\bar{g}_{jk} - \delta_{jk} \in W^{1,q}_{1-m-\varepsilon/2}$ in $\tilde{x}$ coordinates, so one of Bartnik's key results [7, Theorem 3.1] now asserts that

$$\mathcal{H}_{q,\eta} := \{ f \in L^q_\eta \mid \Delta_{\bar{g}} f = 0 \}$$

has dimension $n + 1 = 2m + 1$, and hence that

$$\mathrm{span}\,\{1, \tilde{y}^1, \ldots \tilde{y}^{2m}\} = \mathcal{H}_{q,\eta} = \mathrm{span}\,\{1, y^1, \ldots, y^{2m}\}.$$

It follows that the y^j are affine-linear combinations of the $\tilde{y}^k$; in other words,

$$y^j = a^j + A^j_k \tilde{y}^k$$

for an appropriate translation $\vec{a} \in \mathbb{R}^{2m}$ and an appropriate invertible linear transformation $A \in \mathbf{GL}(2m, \mathbb{R})$. Consequently,

$$x^j = a^j + A^j_k \tilde{x}^k + w^j$$

outside a large ball, where

$$w^j = v^j - A^j_k u^k \in C^{2,\varepsilon}_{2-m-\varepsilon}.$$

In particular, $\frac{\partial x^j}{\partial \tilde{x}^k} - A^j_k \in C^{1,\varepsilon}_{1-m-\varepsilon}$, and inverting the Jacobian matrix then tells us that, as functions of $\tilde{x}$, $\frac{\partial \tilde{x}^k}{\partial x^j} - (A^{-1})^k_j \in C^{1,\varepsilon}_{1-m-\varepsilon}$. We thus have

$$\frac{\partial}{\partial x^j} = \left(A^k_j + U^k_j \right) \frac{\partial}{\partial \tilde{x}^k}$$

where $U^k_j = \frac{\partial w^k}{\partial \tilde{x}^\ell}\frac{\partial \tilde{x}^\ell}{\partial x^j} \in C^{1,\varepsilon}_{1-m-\varepsilon}$. In $\tilde{x}$ coordinates, we therefore see that

$$g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) - A^\ell_j A^\ell_k \in C^{1,\varepsilon}_{1-m-\varepsilon}$$

and that

$$\frac{\partial}{\partial x^\ell}\left[g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})\right] \in C^{0,\varepsilon}_{-m-\varepsilon}.$$

Since $C^{-1}|x| < |\tilde{x}| < C|x|$, this now immediately implies that

$$g_{jk} = (A^t A)_{jk} + O(|x|^{1-m-\varepsilon}), \quad g_{jk,\ell} = O(|x|^{-m-\varepsilon})$$

in x coordinates. However, since the Kähler metric g is Hermitian in the complex coordinate system defined by the $z^j = x^{2j-1} + ix^{2j}$, the matrix $A^t A$ must represent a Hermitian inner product on $\mathbb{C}^m$, and so can be written as B^*B for some $B \in \mathbf{GL}(m, \mathbb{C})$. Thus, after a complex-linear change of coordinates $\vec{z} \mapsto B^{-1}\vec{z}$, we will then have

$$g_{jk} = \delta_{jk} + O(|z|^{1-m-\varepsilon}), \quad g_{jk,\ell} = O(|z|^{-m-\varepsilon}),$$

as desired. □

Remark The above proof dovetails with Bartnik's weighted-Sobolev results in a way that lets us avoid having to reinvent the wheel. However, we certainly could have avoided passing to a complete manifold or citing Bartnik's count of harmonic functions of sub-quadratic growth. Indeed, the results in [41, Chapter 6] allow one to argue directly that any harmonic function on $(\widetilde{M}_\infty, g)$ of polynomial growth is asymptotic to a harmonic polynomial on Euclidean space $(\mathbb{R}^{2m}, \delta)$. ♢

While Lemma 1.3 is still phrased in terms of *any* end, we will soon see that there can actually only be *one* end. Indeed, Lemma 1.2 opens up a thoroughfare to this and other global results, via the following remarkable consequence:

Lemma 1.4. *Let (M^{2m}, g, J) be an ALE Kähler manifold of complex dimension $m \geq 3$. Then we may compactify (M, J) as a complex orbifold (X, J_X) by adding a copy of $\mathbb{CP}_{m-1}/\Gamma_j$ to each end. Moreover, the resulting complex orbifold* **admits Kähler metrics**.

Proof. Lemma 1.1 already told us that we could smoothly cap off the universal cover of each end by adding a $\mathbb{CP}_{m-1}$, and Lemma 1.2 then showed that each such capped-off space is biholomorphic to a neighborhood $\mathscr{U}$ of $\mathbb{CP}_{m-1} \subset \mathbb{CP}_m$. Since the action of each Γ_j extends continuously to $\mathscr{U}$, and since it is holomorphic on the complement of $\mathbb{CP}_{m-1}$, the induced action is actually holomorphic; and since Hartogs' theorem also tells us that this action extends holomorphically to all of $\mathbb{CP}_m$, Γ_j therefore acts on $\mathscr{U}$ by projective linear transformations. This allows us to compactify (M, J) as a complex orbifold (X, J_X) by adding a copy of the appropriate $\mathbb{CP}_{m-1}/\Gamma_j$ to each end. Here, of course, Γ_j is identified with a finite subgroup of $\mathbf{U}(m) \subset \mathbf{SU}(m+1)$, and so acts on a neighborhood of $\mathbb{CP}_{m-1} \subset \mathbb{CP}_m$ in a manner that preserves not only the complex structure, but also the standard Fubini-Study metric.

Using this last observation, we will now construct a Kähler metric $\hat{g}$ on (X, J_X). To do this, we first use our asymptotic coordinates on the complement of a suitable $\mathbf{K} \Subset M$ to identify the universal cover of each end with the complement of a large closed ball $\mathbf{D}^{2m} \subset \mathbb{C}^m$ of radius ϱ_0 in a Γ_j-invariant manner. Since $\mathbb{C}^m - \mathbf{D}^{2m}$ is 2-connected, we can then write the Kähler form ω of our given ALE Kähler metric g as

$$\omega = d(\beta + \bar{\beta}) = \partial\beta + \bar{\partial}\bar{\beta}$$

for some $\bar{\partial}$-closed $(0,1)$-form β on $\mathbb{C}^m - \mathbf{D}^{2m}$. However, since $m \geq 3$, a result of Andreotti-Grauert [2, p. 225] tells us that $H^{0,1}_{\bar{\partial}}(\mathbb{C}^m - \mathbf{D}^{2m}) = 0$. Thus $\beta = \bar{\partial} h$ for some function h, and we therefore have

$$\omega = i\partial\bar{\partial} f$$

where $f = 2\,\Im m\, h$. By averaging over the action of Γ_j, we then improve our choice of the potential f so as to make it Γ_j-invariant on each end.

We now introduce the function $u = \varrho^2 = \sum |z^j|^2$ on each end. If $F(u)$ is any smooth function, then along the z^1-axis we have

$$\partial\bar{\partial} F(u) = (uF')'(u) dz^1 \wedge d\bar{z}^1 + F'(u) \sum_{j=2}^{m} dz^j \wedge d\bar{z}^j \ ,$$

so that $\mathbf{U}(m)$-invariance implies that $i\partial\bar{\partial} F$ is positive semi-definite iff $uF'(u)$ is a non-negative, non-decreasing function. Now choose some radius $\varrho_1 > \varrho_0$, and let $\psi(u)$ be a non-decreasing cut-off function which is $\equiv 0$ near $u = \varrho_0^2$

and $\equiv 1$ for $u \geq \varrho_1^2$. Let $F : [0, \infty) \to [0, \infty)$ be the smooth function defined by

$$F(u) = \int_0^u \frac{\psi(t)\, dt}{1+t}, \tag{1.1}$$

so that $uF'(u) = \psi(u)\, [1 - (1+u)^{-1}]$ is non-negative and non-decreasing. Since this ensures that $i\partial\bar{\partial}F$ is positive semi-definite, it follows that, for any constant $\mathbf{N} > 0$, M admits a Kähler metric $g_{\mathbf{N}}$ that equals $g/\mathbf{N}$ on the compact set $\mathbf{K} \subset M$, and which has Kähler form given by

$$\omega_{\mathbf{N}} = \frac{\omega}{\mathbf{N}} + i\partial\bar{\partial}F = i\partial\bar{\partial}\left(F + \frac{f}{\mathbf{N}}\right)$$

on the ends. In particular, since $i\partial\bar{\partial}F$ coincides with the Fubini-Study Kähler form $\omega_{FS} = i\partial\bar{\partial}\log(1+u)$ when $u > \varrho_1^2$, we actually have

$$\omega_{\mathbf{N}} = \omega_{FS} + i\mathbf{N}^{-1}\partial\bar{\partial}f$$

when $\varrho > \varrho_1$. Now choose some $\varrho_2 > \varrho_1$, and let $\phi(u) \geq 0$ be a second smooth cut-off function which is $\equiv 0$ for $u \leq \varrho_1^2$ and $\equiv 1$ for $u \geq \varrho_2^2$. We can then consider the $(1,1)$-form on M which is defined by

$$\hat{\omega}_{\mathbf{N}} = \omega_{\mathbf{N}} - i\mathbf{N}^{-1}\partial\bar{\partial}[\phi(u)f]$$

in the asymptotic regions, and given by $\omega_{\mathbf{N}}$ on the compact set $\mathbf{K}$; the fact that f has been taken to be Γ_j-invariant guarantees that this coordinate expression is Γ_j-invariant, and so descends to a well-defined form on each end. However, we then have

$$\hat{\omega}_{\mathbf{N}} = \omega_{FS} + i\mathbf{N}^{-1}\partial\bar{\partial}[(1-\phi)f]$$

in the asymptotic regions $\varrho \geq \varrho_1$. Since the Hessian of $(1-\phi)f$ is uniformly bounded with respect to the Fubini-Study metric on the compact union of the transition annuli $\varrho_1 \leq \varrho \leq \varrho_2$, it follows that $\hat{\omega}_{\mathbf{N}}$ will be positive-definite on these annuli for $\mathbf{N} \gg 0$. On the other hand, since $\hat{\omega}_{\mathbf{N}}$ agrees with either ω_{FS} or $\omega_{\mathbf{N}}$ everywhere else, it follows that, provided we take $\mathbf{N}$ to be sufficiently large, $\hat{\omega}_{\mathbf{N}}$ will be a Kähler form on all of M. But the Kähler metric $\hat{g}$ corresponding to $\hat{\omega} := \hat{\omega}_{\mathbf{N}}$ for some such suitably large $\mathbf{N}$ then exactly coincides with the standard Fubini-Study metric of each $\mathbb{CP}_m/\Gamma_j$ in the asymptotic region $\varrho > \varrho_2$ of each end, and so naturally extends to all of (X, J_X) as a Kähler metric. This shows that the complex orbifold X is indeed of Kähler type, as claimed. □

Remark As was pointed out to us by Ronan Conlon, the above result can be generalized to asymptotically conical Kähler manifolds, even when the end is not rationally 2-connected. For details, see [17, Theorem A (iv)]. $\diamondsuit$

This now implies a previously promised result:

Proposition 1.5. *If $m \geq 3$, an ALE Kähler m-manifold has only one end.*

Proof. Let (M^{2m}, g, J) be an ALE Kähler manifold, where $m \geq 3$. Consider the orbifold compactification (X, J_X) of (M, J) given by Lemma 1.4, and let $\hat{g}$ be an orbifold Kähler metric on X, with Kähler form $\hat{\omega}$. We may then consider the intersection pairing

$$\begin{array}{ccc} H^{1,1}(X, \mathbb{R}) \times H^{1,1}(X, \mathbb{R}) & \xrightarrow{Q} & \mathbb{R} \\ (\ [\alpha] \ , \ [\beta] \ ) & \longmapsto & \displaystyle\int_X \alpha \wedge \beta \wedge \hat{\omega}^{m-2} \ . \end{array}$$

on $H^{1,1}(X) := H^{1,1}_{orb}(X)$. However, because Hodge theory is valid in the orbifold setting, there is a Lefschetz decomposition

$$H^{1,1}(X, \mathbb{R}) = \mathbb{R}[\hat{\omega}] \oplus P^{1,1}(X, \mathbb{R}),$$

where the primitive harmonic $(1,1)$-forms $P^{1,1}$ are pointwise orthogonal to the Kähler form $\hat{\omega}$. This implies a generalization of the Hodge index theorem: the intersection form Q is of *Lorentz type*. Indeed, the Hodge-Riemann bilinear relations [21] tell us that Q is positive-definite on $\mathbb{R}[\hat{\omega}]$, and negative-definite on $P^{1,1}(X, \mathbb{R})$.

Let us now define a closed non-negative $(1,1)$-form α_j on X supported in the closure of the j^{th} end of M by setting $\alpha_j = i\partial\bar{\partial}F$ in the j^{th} asymptotic region, where F is the function defined by (1.1), and then extending α_j across the hyperplane at infinity as the Fubini-Study form ω_{FS}, while setting $\alpha_j \equiv 0$ outside the closure of the j^{th} end. The semi-positivity of α_j then guarantees that $Q([\alpha_j], [\alpha_j]) > 0$ for each j. However, $Q([\alpha_j], [\alpha_k]) = 0$ if $j \neq k$, since the supports of α_j and α_k are then disjoint. If M had two or more ends, Q would thus admit two orthogonal positive directions. But since the generalized Hodge index theorem says that Q is of Lorentz type, this is impossible. To avoid this contradiction, we are thus forced to conclude that M can only have one end. $\square$

Remark The classes $[\alpha_j] \in H^{1,1}(X)$ in the above proof are proportional to the Poincaré duals of the hypersurfaces $\Sigma_j = \mathbb{CP}_{m-1}/\Gamma_j$ arising as the hyperplanes at infinity of the various ends. The fact that $Q([\alpha_j], [\alpha_k]) = 0$ for $j \neq k$ reflects the fact that $\Sigma_j \cap \Sigma_k = \varnothing$, while the fact that $Q([\alpha_j], [\alpha_j]) > 0$ reflects the fact that the homological self-intersection of Σ_j is represented by a positive multiple of the complex sub-orbifold $\mathbb{CP}_{m-2}/\Gamma_j$. This geometrical idea is the link between the above argument and our proof of Proposition 3.2 in the complex surface case.

Various other means for proving Proposition 1.5 are also available. For example, a proof directly based on the pseudo-convexity of the boundary can be found in [49]. Alternatively, uniqueness of the end can be deduced by applying [45, Theorem 6.3] to the Remmert reduction [20] of (M, J). $\diamondsuit$

With Lemma 1.3 and Proposition 1.5 in hand, Theorem C becomes comparatively easy to prove in complex dimension $m \geq 3$. Here is the key step:

Proposition 1.6. *Let (M^{2m}, g, J) be an ALE Kähler manifold, $m \geq 3$, satisfying both conditions (i) and (ii), as set forth on page 2. Then, in* **any** *asymptotic coordinate system, its mass is given by*

$$\mathfrak{m}(M, g) = \lim_{\varrho \to \infty} \frac{1}{2(2m-1)\pi^m} \int_{S_\varrho/\Gamma} \theta \wedge \omega^{m-1}$$

for any 1-form θ with $d\theta = \rho$ on the end M_∞, where ρ is the Ricci form of g.

Proof. Taking the Bartnik-Chruściel coordinate invariance of the mass [7, 15] as given, we will begin by first checking that the assertion is true in a particular asymptotic coordinate system and for a particular choice of θ.

Because g is Kähler, the asymptotic complex coordinates $(z^1, \ldots, z^m)$ of Lemma 1.3 are all harmonic, and the same therefore applies to the real coordinates $(x^1, \ldots, x^n)$ obtained by taking their real and imaginary parts. Thus

$$\mathbf{\Gamma}^\ell := g^{jk} \mathbf{\Gamma}^\ell_{jk} = \Delta x^\ell = 0,$$

so that

$$g^{jk} \left(g_{ji,k} - \frac{1}{2} g_{jk,i} \right) = 0$$

and

$$g^{jk} \left(g_{j\ell,k} - g_{jk,\ell} \right) = -\frac{1}{2} g^{jk} g_{jk,\ell} = -\left(\log \sqrt{\det g} \right)_{,\ell}$$

in this asymptotic coordinate system. On the other hand, our fall-off conditions guarantee that

$$g^{jk}\left(g_{j\ell,k}-g_{jk,\ell}\right)=[\delta^{jk}+O(\varrho^{1-m-\varepsilon})]\left(g_{j\ell,k}-g_{jk,\ell}\right)=g_{i\ell,i}-g_{ii,\ell}+O(\varrho^{1-2m-2\varepsilon}),$$

and that the Hodge star operators associated with g and δ differ by $O(\varrho^{1-m-\varepsilon})$. Thus

$$\int_{S_\varrho/\Gamma}[g_{ij,i}-g_{ii,j}]\,\mathbf{n}^j\,d\mathfrak{a}_E=-\int_{S_\varrho/\Gamma}\star\, d\log\sqrt{\det g}\quad+O(\varrho^{-2\epsilon})$$

in these coordinates, and the mass is therefore given by

$$\mathfrak{m}(M,g)=-\lim_{\varrho\to\infty}\frac{(m-1)!}{4(2m-1)\pi^m}\int_{S_\varrho/\Gamma}\star\, d\log\sqrt{\det g}\ .$$

However, the Kähler condition allows us to rewrite the integrand as

$$\star\, d\log\sqrt{\det g}=\left[-i(\partial-\bar\partial)\log\frac{\omega^m}{|dz|^{2m}}\right]\wedge\frac{\omega^{m-1}}{(m-1)!}$$

and since

$$d\left[\frac{i}{2}(\partial-\bar\partial)\log\frac{\omega^m}{|dz|^{2m}}\right]=-i\partial\bar\partial\log\frac{\omega^m}{|dz|^{2m}}=\rho$$

on our Kähler manifold, we therefore have

$$\mathfrak{m}(M,g)=\lim_{\varrho\to\infty}\frac{1}{2(2m-1)\pi^m}\int_{S_\varrho/\Gamma}\theta\wedge\omega^{m-1}$$

for a *particular* 1-form

$$\theta=\frac{i}{2}(\partial-\bar\partial)\log\frac{\omega^m}{|dz|^{2m}}$$

with $d\theta=\rho$ on the end M_∞.

On the other hand, since $b_1(M_\infty)=0$, the most general 1-form $\tilde\theta$ on M_∞ with $d\tilde\theta=\rho$ is given by $\tilde\theta=\theta+df$ for a function f. Choosing a different θ would thus change the integrand by an exact form, and so leave the integral on each S_ϱ/Γ completely unchanged.

Finally, the limit is independent of the asymptotic coordinate system. Indeed, notice that

$$d\left[\theta\wedge\omega^{m-1}\right]=\rho\wedge\omega^{m-1}=\frac{s}{2m}\omega^m=\tfrac{(m-1)!}{2}s\ d\mu.$$

Consequently, if $\mathscr{S}$ is a real hypersuface in the region E_ϱ exterior to S_ϱ/Γ such that S_ϱ/Γ and $\mathscr{S}$ are the boundary components of a bounded region $\mathscr{V} \subset E_\varrho$, then

$$\tfrac{2}{(m-1)!}\left|\int_{\mathscr{S}} \theta\wedge\omega^{m-1} - \int_{S_\varrho/\Gamma} \theta\wedge\omega^{m-1}\right| = \left|\int_{\mathscr{V}} s\ d\mu\right| \leq \int_{\mathscr{V}} |s|\ d\mu \leq \int_{E_\varrho} |s|\ d\mu,$$

and the expression at the far right tends to zero as $\varrho \to \infty$, since, by hypothesis, the scalar curvature s belongs to L^1. $\square$

Remark If the metric g is *scalar-flat* Kähler, the form $\theta\wedge\omega^{m-1}$ is actually closed, so the integral becomes independent of the radius ϱ, and the mass can be calculated without the need for taking a limit.

When $m = 2$, the above argument still works if one simply *assumes* that there is an asymptotic chart in which J is standard and g falls off as in Lemma 1.3. While this assumption does in fact hold for many interesting examples, it unfortunately fails for the general ALE Kähler surface. This complication will force us to develop a more flexible approach in order to be able to definitively treat the complex-surface case. $\diamondsuit$

We now provide some key conceptual underpinning for our mass formula.

Lemma 1.7. *Let (M, g) be any ALE manifold of real dimension $n \geq 4$. Then the natural map $H^2_c(M) \to H^2_{dR}(M)$ from compactly supported cohomology to ordinary de Rham cohomology is an isomorphism. Consequently, every element of $H^2(M)$ is represented by a unique L^2 harmonic 2-form.*

Proof. We can compactify M as a manifold-with-boundary $\overline{M}$ by adding a copy of S^{n-1}/Γ_j to each end. The natural map in question therefore fits into an exact sequence

$$\cdots \to H^1_{dR}(\cup_i[S^{n-1}/\Gamma_i]) \to H^2_c(M) \to H^2_{dR}(M) \to H^2_{dR}(\cup_i[S^{n-1}/\Gamma_i]) \to \cdots$$

corresponding to the exact cohomology sequence of the pair $(\overline{M}, \partial M)$. On the other hand, since de Rham cohomology injects upon passing to a finite cover, we have $H^k_{dR}(S^{n-1}/\Gamma_i) \subset H^k_{dR}(S^{n-1}) = 0$ when $0 < k < n-1$. It therefore follows that $H^2_c(M) \to H^2_{dR}(M)$ is an isomorphism. Moreover, since g is asymptotically conical, this in turn implies [13, 23] that $H^2(M)$ can be identified with the space $\mathcal{H}^2_2(M, g)$ of L^2 harmonic 2-forms on (M, g). $\square$

This entitles us to make the following definition:

Definition 1.8. *If (M, g, J) is any ALE Kähler manifold, we will use*

$$\clubsuit : H^2_{dR}(M) \to H^2_c(M)$$

to denote the **inverse** *of the natural map $H^2_c(M) \to H^2_{dR}(M)$.*

We are now ready to state and prove our mass formula.

Theorem 1.9. *The mass of any ALE Kähler manifold (M, g, J) of complex dimension $m \geq 3$ is given by the formula*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s \, d\mu$$

where $\langle \ , \ \rangle$ is the duality pairing between $H^2_c(M)$ and $H^{2m-2}(M)$.

Proof. Choose some 1-form θ on M_∞ such that $d\theta = \rho$, where ρ is the Ricci form of (M, g, J). Next, choose some asymptotic coordinate system on M_∞, and temporarily let $\mathfrak{r}$ denote the corresponding coordinate radius on M_∞. Finally, choose a smooth cut-off function $f : M \to [0, 1]$ which is $\equiv 0$ on $M - M_\infty$ and $\equiv 1$ for $\mathfrak{r} \geq$ я, where я is some fixed large real number. We then set $\psi := \rho - d(f\theta)$. Since M only has one end M_∞ by Proposition 1.5, this ψ is then a compactly supported closed 2-form on M. Since ψ is moreover cohomologous to ρ, it therefore represents $\clubsuit([\rho]) = 2\pi\clubsuit(c_1)$ in compactly supported cohomology.

For any $\varrho >$ я, we now let $M_\varrho \subset M$ be the compact manifold-with-boundary obtained by removing $\mathfrak{r} > \varrho$ from M, so that $\partial M_\varrho = S_\varrho/\Gamma$. Since

$$\rho \wedge \omega^{m-1} = \frac{s}{2m}\omega^m = \frac{(m-1)!}{2} s \, d\mu_g,$$

we have

$$\frac{(m-1)!}{2} \int_{M_\varrho} s \, d\mu = \int_{M_\varrho} \rho \wedge \omega^{m-1} = \int_{M_\varrho} [\psi + d(f\theta)] \wedge \omega^{m-1} \ .$$

It follows that

$$\begin{aligned}
2\pi\langle \clubsuit(c_1), [\omega]^{m-1}\rangle &= \int_M \psi\wedge\omega^{m-1} = \int_{M_\varrho} \psi\wedge\omega^{m-1} \\
&= -\int_{M_\varrho} d(f\theta\wedge\omega^{m-1}) + \frac{(m-1)!}{2}\int_{M_\varrho} s\, d\mu \\
&= -\int_{\partial M_\varrho} f\theta\wedge\omega^{m-1} + \frac{(m-1)!}{2}\int_{M_\varrho} s\, d\mu \\
&= -\int_{S_\varrho/\Gamma} \theta\wedge\omega^{m-1} + \frac{(m-1)!}{2}\int_{M_\varrho} s\, d\mu\ .
\end{aligned}$$

In other words,

$$\frac{1}{2(2m-1)\pi^m}\int_{S_\varrho/\Gamma}\theta\wedge\omega^{m-1} = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_{M_\varrho} s\, d\mu.$$

Taking the limit of both sides as $\varrho\to\infty$ therefore yields

$$\mathfrak{m}(M,g) = -\frac{\langle \clubsuit(c_1), [\omega]\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s\, d\mu$$

by Proposition 1.6. This proves the desired mass formula. □

Specializing to the scalar-flat case, we now obtain the high-dimensional version of Theorem A:

Theorem 1.10. *If (M^{2m}, g, J) is any ALE* scalar-flat *Kähler m-manifold, $m\geq 3$, its mass is given by*

$$\mathfrak{m}(M,g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}}.$$

In particular, the mass is a topological invariant in this context, entirely determined by the smooth manifold M, the first Chern class of the complex structure and the Kähler class of the metric.

We now conclude our discussion of the high-dimensional case by pointing out some other useful consequences of Lemma 1.4.

Lemma 1.11. *The orbifold (X, J_X) of Lemma 1.4 satisfies $H^1(X,\mathcal{O})=0$.*

Proof. By Lemma 1.2, the orbifold (X, J) contains an open set of the form $\mathscr{U}/\Gamma$, where $\mathscr{U} \subset \mathbb{CP}_m$ is a tubular neighborhood of a hyperplane $\mathbb{CP}_{m-1}$, and this tubular neighborhood then contains a (perhaps smaller) neighborhood $\check{\mathscr{U}}$ of $\mathbb{CP}_{m-1}$ which is the union of all the projective lines $\mathbb{CP}_1 \subset \mathscr{U}$. If $\alpha \in H^0(X, \Omega^1)$ is a global holomorphic 1-form on the orbifold X, we can restrict it to $\mathscr{U}/\Gamma$ and then pull it back to obtain a holomorphic 1-form $\hat{\alpha} \in H^0(\mathscr{U}, \Omega^1)$. However, the cotangent bundle of $\mathbb{CP}_m$ restricted to a projective line is isomorphic to $\mathcal{O}(-2)\oplus\mathcal{O}(-1)\oplus\cdots\oplus\mathcal{O}(-1)$, so any holomorphic 1-form on $\mathscr{U}$ must vanish identically along any projective line $\mathbb{CP}_1 \subset \mathscr{U}$. It follows that $\hat{\alpha}$ vanishes identically on $\check{\mathscr{U}}$. Hence $\alpha \equiv 0$ on a non-empty open set, and hence $\alpha \equiv 0$ on X by the uniqueness of analytic continuation. Thus $H^0(X, \Omega^1) = 0$. However, $H^0(X, \Omega^1) = H^{1,0}(X)$ and $H^1(X, \mathcal{O}) = H^{0,1}(X)$ are conjugate by Hodge symmetry, since X admits Kähler metrics. This shows that $H^1(X, \mathcal{O}) = 0$, as claimed. □

In the asymptotically Euclidean case, this now allows us to prove a result that will play a leading role in §5 below:

Proposition 1.12. *When $m \geq 3$, any AE Kähler m-manifold (M^{2m}, g, J) admits a proper holomorphic degree-one map $M \to \mathbb{C}^m$.*

Proof. Since $\Gamma = \{1\}$ by assumption, the compactification (X, J_X) of (M, J) is a manifold, and by Lemma 1.2 it contains an open set biholomorphic to some tubular neighborhood $\mathscr{U}$ of $\mathbb{CP}_{m-1} \subset \mathbb{CP}_m$. In particular, X contains a complete, m-complex-dimensional family of hypersurfaces arising as hyperplanes in $\mathscr{U} \subset \mathbb{CP}_m$. Since $H^1(X, \mathcal{O}) = 0$, holomorphic line bundles on X are classified by their Chern classes, and it therefore follows that all of these hypersurfaces determine the same divisor line bundle $L \to X$; that is, they all belong the same m-dimensional linear system $|H^0(X, \mathcal{O}(L))|$. Since no point belongs to all of these hypersurfaces, this linear system has no base locus, and it therefore gives rise to a globally defined holomorphic map

$$\Phi : X \to \mathbb{P}[H^0(X, \mathcal{O}(L))^*] \cong \mathbb{CP}_m.$$

Since the hyperplanes we initially considered lie entirely within $\mathscr{U}$ and give projective coordinates on some smaller tubular neighborhood $\check{\mathscr{U}}$ of $\mathbb{CP}_{m-1}$, this map takes $\check{\mathscr{U}}$ biholomorphically to its image, and no point of $\Phi(\check{\mathscr{U}})$ has any other pre-image in X. Thus Φ has degree 1, and the hyperplane $\Sigma \cong \mathbb{CP}_{m-1}$ used to compactify M is taken biholomorphically to a hyperplane

$\mathbb{CP}_{m-1} \subset \mathbb{CP}_m$. The restriction of Φ to $M = X - \Sigma$ therefore defines a proper, degree-one holomorphic map $M \to \mathbb{C}^m$, as desired. $\square$

Remark While the last result roughly says that any AE Kähler manifold is a generalized blow-up of $\mathbb{C}^m$, it should be emphasized that some rather complicated scenarios are in principle allowed when $m \geq 3$. For example, one could blow up $\mathbb{C}^m$ at a point, then choose a smooth sub-variety V in the resulting exceptional $\mathbb{CP}_{m-1}$, then modify the blow-up $\tilde{\mathbb{C}}^m$ by replacing V with its projectivized normal bundle, and then repeat this procedure. While Proposition 3.3 below will provide an analogous result when $m = 2$, the low-dimensional picture is simpler, as the most general AE Kähler surface will turn out to just be an iterated blow-up of $\mathbb{C}^2$ at isolated points. $\diamondsuit$

More generally, the Kodaira-Baily embedding theorem [5] and Lemma 1.11 together imply that the Kähler orbifold compactification X of M given by Lemma 1.4 is always a complex projective variety; and Lemma 3.1 below leads to a similar result when $m = 2$. After resolution of singularities, it is therefore easy to show that any ALE Kähler manifold is biholomorphic to the complement of a rationally connected hypersurface in a rationally connected smooth projective variety.

2 Coordinate Invariance of the Mass

Proposition 1.6 shows that the mass of an ALE Kähler m-manifold of complex dimension $m \geq 3$ can be calculated by integrating a coordinate-independent differential form over a family of hypersurfaces that tends to infinity. This perhaps sounds like it should imply the Kähler case of the coordinate-invariance of the mass, in the sense of the celebrated results of Bartnik [7, Theorem 4.2] and Chruściel [15, Theorem 2]. However, our proof actually proceeded by checking our asymptotic mass formula in a *special* coordinate system, and then noticing that this formula actually has an interpretation that is essentially coordinate-free; to know that our expression also coincides with the standard expression for the mass in other charts, we still had to rely on Bartnik-Chruściel. In this section, we will remedy this by proving a more robust version of Proposition 1.6 that *directly* relates our integral to the standard mass expression in *any* asymptotic chart in which the metric satisfies a weak fall-off hypothesis. One remarkable consequence of this argument will be that *Proposition 1.6 still holds when* $m = 2$, even though Lemma 1.3 cannot be generalized to this setting. The following technical result is the linchpin of our argument:

Proposition 2.1. *Let g be a C^2 Kähler metric on $(\mathbb{R}^{2m} - \mathbf{D}^{2m})/\Gamma$, $m \geq 2$, where $\Gamma \subset \mathbf{SO}(2m)$ is some finite group that acts without fixed-points on S^{2m-1}. In the given coordinate system $(x^1, \ldots, x^{2m})$ on $\mathbb{R}^{2m} - \mathbf{D}^{2m}$, assume that g satisfies the weak fall-off hypothesis*

$$g_{jk} = \delta_{jk} + O(\varrho^{-\tau}), \qquad g_{jk,\ell} = O(\varrho^{-\tau-1})$$

where $\varrho = |x|$ and where $\tau = m - 1 + \varepsilon$ for some $\varepsilon > 0$. Then there is a continuously differentiable 1-form θ on $(\mathbb{R}^{2m} - \mathbf{D}^{2m})/\Gamma$ such that

$$\int_{S_\varrho/\Gamma} [g_{kj,k} - g_{kk,j}]\, \mathbf{n}^j d\mathfrak{a}_E - \frac{2}{(m-1)!} \int_{S_\varrho/\Gamma} \theta \wedge \omega^{m-1} = O(\varrho^{-2\varepsilon})$$

and such that $d\theta = \rho$, where ρ is the Ricci form of g with respect to a given compatible integrable almost-complex structure J.

Proof. Let J be a given almost-complex structure which is parallel with respect to g, and recall that this implies that J is integrable. By the argument used in the proof of Lemma 1.1, we may then find a unique constant-coefficient almost-complex structure J_0 on $\mathbb{R}^{2m}$ such that

$$J = J_0 + O(\varrho^{-\tau}), \qquad \triangledown J = O(\varrho^{-\tau-1}),$$

where $\triangledown$ denotes the Euclidean connection associated with the coordinate system and where $\varrho = |x|$. By rotating our coordinates if necessary, we may then assume that J_0 is the usual complex-structure tensor of $\mathbb{C}^m$. Since Γ preserves J and acts by linear transformations, it automatically preserves J_0, too, so we actually have $\Gamma \subset \mathbf{U}(m)$. We will now systematically work in the complex coordinates $(z^1, \ldots, z^m)$ associated with this picture of J_0.

Per standard conventions [8], we let J act on 1-forms ϕ by $J\phi = -\phi \circ J$, thereby making it consistent with index-raising. With this understood, then, at least at large radius,

$$J\, dz^\mu = -i(dz^\mu + \mathscr{K}^\mu_{\bar{\nu}} d\bar{z}^{\bar{\nu}} + \mathscr{L}^\mu_\nu dz^\nu)$$

for a uniquely determined collection of coefficients $\mathscr{K}^\mu_{\bar{\nu}}$ and $\mathscr{L}^\mu_\nu$ with the same $C^1_{-\tau}$ fall-off as $J - J_0$. Since we consequently also have

$$J\, d\bar{z}^{\bar{\mu}} = +i(d\bar{z}^{\bar{\mu}} + \overline{\mathscr{K}^\mu_{\bar{\nu}}} dz^\nu + \overline{\mathscr{L}^\mu_\nu} d\bar{z}^{\bar{\nu}}),$$

applying J again therefore gives us

$$J^2 dz^\mu \equiv -\left(dz^\mu + 2\mathscr{L}^\mu_\nu dz^\nu\right) \mod C^1_{-2\tau}.$$

The fact that $J^2 = -I$ therefore implies that $\mathscr{L}^\mu_\nu \in C^1_{-2\tau}$, and hence that

$$J\, dz^\mu \equiv -i(dz^\mu + \mathscr{K}^\mu_{\bar\nu} d\bar z^{\bar\nu}) \mod C^1_{-2\tau},$$

thus allowing us to sweep the $\mathscr{L}^\mu_\nu$ into the error term in our calculations.

Now consider the collection of 1-forms defined by

$$\zeta^\mu := \tfrac{1}{2}\left(dz^\mu + iJdz^\mu\right) \equiv dz^\mu + \tfrac{1}{2}\mathscr{K}^\mu_{\bar\nu} d\bar z^{\bar\nu} \mod C^1_{-2\tau}.$$

These are all $(1,0)$-forms with respect to J, so the m-form

$$\varphi = \zeta^1 \wedge \cdots \wedge \zeta^m$$

is consequently of type $(m,0)$ with respect to J. If we now let

$$\varphi_0 = dz^1 \wedge \cdots \wedge dz^m$$

denote the standard coordinate $(m,0)$-form with respect to J_0, then

$$\varphi \;\equiv\; \varphi_0 \;-\; \tfrac{1}{2}\sum_{\mu,\bar\nu=1}^{m} (-1)^\mu \mathscr{K}^\mu_{\bar\nu}\, d\bar z^{\bar\nu} \wedge dz^1 \wedge \cdots \wedge \widehat{dz^\mu} \wedge \cdots \wedge dz^m \mod C^1_{-2\tau}.$$

Consequently,

$$\varphi \wedge \bar\varphi \;\equiv\; \varphi_0 \wedge \bar\varphi_0 \qquad \mod C^1_{-2\tau}, \tag{2.1}$$

even though we merely have $\varphi - \varphi_0 \in C^1_{-\tau}$.

Because g is Kähler with respect to J, we of course have $\nabla J = 0$, where ∇ denotes the Levi-Civita connection of g. In our real asymptotic coordinate system $(x^1, \ldots, x^{2m})$ this statement takes the explicit form

$$\triangledown_j J^\ell_k + \mathbf{\Gamma}^\ell_{je} J^e_k - \mathbf{\Gamma}^e_{jk} J^\ell_e = 0$$

where $\triangledown$ denotes the flat Euclidean coordinate connection, and where $\mathbf{\Gamma}$ are once again the Christoffel symbols of g. By thinking of $\mathbf{\Gamma}^\ell_{jk}$ as a matrix-valued 1-form $[\mathbf{\Gamma}_j]$, we can now usefully rewrite this as

$$\triangledown_j J = \Big[\; J \,,\, [\mathbf{\Gamma}_j] \;\Big]$$

or, equivalently, as

$$\nabla_j(J - J_0) = \Big[\, J_0\,,\, [\mathbf{\Gamma}_j] \Big] + \Big[\, (J - J_0)\,,\, [\mathbf{\Gamma}_j] \Big].$$

But since our fall-off conditions tells us that $\mathbf{\Gamma} = O(\varrho^{-\tau-1})$ and that $J - J_0 = O(\varrho^{-\tau})$, we therefore have

$$\nabla_j(J - J_0) = \Big[\, J_0\,,\, [\mathbf{\Gamma}_j] \Big] + O(\varrho^{-2\tau-1}). \tag{2.2}$$

Now set

$$\mathscr{K} := \mathscr{K}^{\mu}_{\bar{\nu}} \frac{\partial}{\partial z^{\mu}} \otimes dz^{\bar{\nu}},$$

where the Einstein summation convention is understood, and notice that

$$J \equiv J_0 + i\mathscr{K} - i\overline{\mathscr{K}} \qquad \bmod C^1_{-2\tau}.$$

Expressing the endomorphism $[\mathbf{\Gamma}_j]$ as

$$[\mathbf{\Gamma}_j] = \mathbf{\Gamma}^{\mu}_{j\nu} \frac{\partial}{\partial z^{\mu}} \otimes dz^{\nu} + \mathbf{\Gamma}^{\mu}_{j\bar{\nu}} \frac{\partial}{\partial z^{\mu}} \otimes dz^{\bar{\nu}} + \mathbf{\Gamma}^{\bar{\mu}}_{j\nu} \frac{\partial}{\partial z^{\bar{\mu}}} \otimes dz^{\nu} + \mathbf{\Gamma}^{\bar{\mu}}_{j\bar{\nu}} \frac{\partial}{\partial z^{\bar{\mu}}} \otimes dz^{\bar{\nu}}$$

we can thus rewrite (2.2) as

$$i\mathscr{K}^{\mu}_{\bar{\nu},j} = 2i\mathbf{\Gamma}^{\mu}_{j\bar{\nu}} + O(\varrho^{-2\tau-1})$$

and so deduce that

$$\mathbf{\Gamma}^{\mu}_{j\bar{\nu}} = \tfrac{1}{2}\mathscr{K}^{\mu}_{\bar{\nu},j} + O(\varrho^{-2\tau-1}).$$

In particular, after decomposing the index j into parts of type $(1,0)$ and $(0,1)$ with respect to J_0, we consequently have

$$\mathscr{K}^{\mu}_{\bar{\nu},\lambda} = 2\mathbf{\Gamma}^{\mu}_{\lambda\bar{\nu}} + O(\varrho^{-2\tau-1}) \tag{2.3}$$

and

$$\mathscr{K}^{\mu}_{\bar{\nu},\bar{\lambda}} = 2\mathbf{\Gamma}^{\mu}_{\bar{\lambda}\bar{\nu}} + O(\varrho^{-2\tau-1}). \tag{2.4}$$

Since the Levi-Civita connection ∇ is torsion-free, equation (2.4) then implies that

$$\mathscr{K}^{\mu}_{\bar{\nu},\bar{\lambda}} - \mathscr{K}^{\mu}_{\bar{\lambda},\bar{\nu}} = O(\varrho^{-2\tau-1}).$$

Thus, with the Einstein summation convention understood,

$$\mathscr{K}^{\mu}_{\bar{\nu},\bar{\lambda}} d\bar{z}^{\bar{\nu}} \wedge d\bar{z}^{\bar{\lambda}} = O(\varrho^{-2\tau-1}), \tag{2.5}$$

in contrast to the $O(\varrho^{-\tau-1})$ fall-off we might have naïvely expected.

The same sort of decomposition also allows us to express the metric as

$$g = g_{\mu\nu}dz^\mu \otimes dz^\nu + g_{\mu\bar\nu}dz^\mu \otimes d\bar z^{\bar\nu} + g_{\bar\mu\nu}d\bar z^{\bar\mu} \otimes dz^\nu + g_{\bar\mu\bar\nu}d\bar z^{\bar\mu} \otimes d\bar z^{\bar\nu}$$

where symmetry and reality imply that

$$g_{\mu\nu} = g_{\nu\mu} = \overline{g_{\bar\mu\bar\nu}} = \overline{g_{\bar\nu\bar\mu}}, \qquad g_{\mu\bar\nu} = g_{\bar\nu\mu} = \overline{g_{\bar\mu\nu}} = \overline{g_{\nu\bar\mu}}.$$

Our fall-off hypothesis now becomes

$$g_{\mu\nu} \in C^1_{-\tau}, \qquad g_{\mu\bar\nu} - \delta_{\mu\bar\nu} \in C^1_{-\tau},$$

with the understanding that δ denotes the standard Euclidean metric, so that $[\delta_{\mu\bar\nu}]$ is *one-half* times the identity matrix. The Kähler form is thus given by

$$\begin{aligned}
\omega &= g(J\cdot,\cdot) \\
&\equiv i\, g_{\mu\nu}(dz^\mu + \mathscr{K}^\mu_{\bar\lambda} dz^{\bar\lambda}) \otimes dz^\nu + i\, g_{\mu\bar\nu}(dz^\mu + \mathscr{K}^\mu_{\bar\lambda} dz^{\bar\lambda}) \otimes d\bar z^{\bar\nu} \\
&\quad -i\, g_{\bar\mu\nu}(d\bar z^{\bar\mu} + \overline{\mathscr{K}^\mu_{\bar\lambda}} dz^\lambda) \otimes dz^\nu - i\, g_{\bar\mu\bar\nu}(d\bar z^{\bar\mu} + \overline{\mathscr{K}^\mu_{\bar\lambda}} dz^\lambda) \otimes d\bar z^{\bar\nu} \quad \mod C^1_{-2\tau} \\
&\equiv i g_{\mu\bar\nu} dz^\mu \wedge d\bar z^{\bar\nu} - \tfrac{i}{2}\delta_{\lambda[\bar\mu}\mathscr{K}^\lambda_{\bar\nu]} d\bar z^{\bar\mu} \wedge d\bar z^{\bar\nu} + \tfrac{i}{2}\delta_{\bar\lambda[\mu}\overline{\mathscr{K}}^{\bar\lambda}_{\nu]} dz^\mu \wedge dz^\nu \quad \mod C^1_{-2\tau},
\end{aligned}$$

where we have used the fact that ω is anti-symmetric, and hence equal to its own skew part; here the square brackets, denoting skew-symmetrization, have simply been added for clarity. Taking the $(0,2)$ component of ω with respect to J_0 thus gives us the interesting complex 2-form

$$\omega^{0,2} \equiv -\tfrac{i}{2}\delta_{\lambda[\bar\mu}\mathscr{K}^\lambda_{\bar\nu]} d\bar z^{\bar\mu} \wedge d\bar z^{\bar\nu} \quad \mod C^1_{-2\tau}.$$

Our fall-off conditions now imply that

$$[d^*(\omega^{0,2})]_\ell = -g^{jk}\nabla_j(\omega^{0,2})_{k\ell} = -\delta^{jk}(\omega^{0,2})_{k\ell,j} + O(\varrho^{-2\tau-1}),$$

so that

$$[d^*(\omega^{0,2})]_\lambda = O(\varrho^{-2\tau-1}),$$

while (2.3) implies that

$$\begin{aligned}
[d^*(\omega^{0,2})]_{\bar\kappa} &= -\delta^{\mu\bar\nu}(\omega^{0,2})_{\bar\nu\bar\kappa,\mu} + O(\varrho^{-2\tau-1}) \\
&= \tfrac{i}{2}\delta^{\mu\bar\nu}\delta_{\lambda[\bar\nu}\mathscr{K}^\lambda_{\bar\kappa],\mu} + O(\varrho^{-2\tau-1}) \\
&= i\delta^{\mu\bar\nu}\delta_{\lambda[\bar\nu}\mathbf{\Gamma}^\lambda_{\bar\kappa]\mu} + O(\varrho^{-2\tau-1}) \\
&= \tfrac{i}{2}\delta^{\mu\bar\nu}\left(g_{[\bar\kappa\bar\nu],\mu} + g_{\mu[\bar\nu,\bar\kappa]} - g_{\mu[\bar\kappa,\bar\nu]}\right) + O(\varrho^{-2\tau-1}) \\
&= -i\delta^{\mu\bar\nu}g_{\mu[\bar\kappa,\bar\nu]} + O(\varrho^{-2\tau-1}).
\end{aligned}$$

Thus

$$d^*(\omega^{0,2}) = -i\delta^{\mu\bar{\nu}} g_{\mu[\bar{\lambda},\bar{\nu}]} d\bar{z}^{\bar{\lambda}} + O(\varrho^{-2\tau-1}),$$

and complex conjugation then gives us

$$d^*(\omega^{2,0}) = i\delta^{\nu\bar{\mu}} g_{\bar{\mu}[\lambda,\nu]} dz^{\lambda} + O(\varrho^{-2\tau-1}).$$

Setting

$$\varOmega := i\omega^{2,0} - i\omega^{0,2},$$

we therefore obtain a real co-exact 1-form

$$\gamma = d^* \varOmega = - \star d \star \varOmega$$

that is explicitly given by

$$\gamma = -2\,\Re e\,(\delta^{\mu\bar{\nu}} g_{\mu[\bar{\lambda},\bar{\nu}]} d\bar{z}^{\bar{\lambda}}) + O(\varrho^{-2\tau-1}).$$

Next, we consider the 1-form

$$\gimel = g_{jk} \mathbf{\Gamma}^j dx^k$$

obtained from the vector field $\mathbf{\Gamma}^j \frac{\partial}{\partial x^j}$ by index-lowering, where

$$\mathbf{\Gamma}^j := g^{k\ell} \mathbf{\Gamma}^j_{k\ell} = \Delta x^j.$$

The usual formula for the Christoffel symbol then tells us that

$$\begin{aligned}
\gimel_{\bar{\nu}} &= g_{\kappa\bar{\nu}} \mathbf{\Gamma}^{\kappa} = \delta_{\kappa\bar{\nu}} \mathbf{\Gamma}^{\kappa} + O(\varrho^{-2\tau-1}) \\
&= \delta_{\kappa\bar{\nu}} g^{jk} \mathbf{\Gamma}^{\kappa}_{jk} + O(\varrho^{-2\tau-1}) \\
&= 2\delta_{\kappa\bar{\nu}} \delta^{\mu\bar{\lambda}} \mathbf{\Gamma}^{\kappa}_{\mu\bar{\lambda}} + O(\varrho^{-2\tau-1}) \\
&= \delta^{\mu\bar{\lambda}} (g_{\mu\bar{\nu},\bar{\lambda}} + g_{\bar{\lambda}\bar{\nu},\mu} - g_{\mu\bar{\lambda},\bar{\nu}}) + O(\varrho^{-2\tau-1}) \\
&= \delta^{\mu\bar{\lambda}} g_{\bar{\lambda}\bar{\nu},\mu} + 2\delta^{\mu\bar{\lambda}} g_{\mu[\bar{\nu},\bar{\lambda}]} + O(\varrho^{-2\tau-1}).
\end{aligned}$$

On the other hand, the trace of equation (2.3) tells us that

$$\begin{aligned}
\mathscr{K}^{\mu}_{\bar{\nu},\mu} &= 2\mathbf{\Gamma}^{\mu}_{\bar{\nu}\mu} + O(\varrho^{-2\tau-1}) \\
&= g^{\mu\bar{\lambda}} (g_{\mu\bar{\lambda},\bar{\nu}} + g_{\bar{\lambda}\bar{\nu},\mu} - g_{\mu\bar{\nu},\bar{\lambda}}) + O(\varrho^{-2\tau-1}) \\
&= \delta^{\mu\bar{\lambda}} (g_{\mu\bar{\lambda},\bar{\nu}} + g_{\bar{\lambda}\bar{\nu},\mu} - g_{\mu\bar{\nu},\bar{\lambda}}) + O(\varrho^{-2\tau-1}) \\
&= \delta^{\mu\bar{\lambda}} g_{\bar{\lambda}\bar{\nu},\mu} - 2\delta^{\mu\bar{\lambda}} g_{\mu[\bar{\nu},\bar{\lambda}]} + O(\varrho^{-2\tau-1}).
\end{aligned}$$

This gives us an identity

$$\Re e\,(\mathscr{K}^{\mu}_{\bar{\nu},\mu} d\bar{z}^{\bar{\nu}}) = \tfrac{1}{2}\gimel + 2\gamma \tag{2.6}$$

which will eventually prove to be invaluable.

Now, because g is Kähler, the Levi-Civita connection ∇ of g induces a connection on $K := \Lambda^{m,0}$ by restriction, and we will simply denote this connection by ∇, too. Thus, in the asymptotic region where $\varphi \neq 0$ has been defined,

$$\nabla\varphi = \vartheta \otimes \varphi$$

for some complex-valued connection 1-form ϑ. Setting

$$\alpha := \vartheta^{0,1}, \qquad \beta := \vartheta^{1,0},$$

we will now use (2.5) and the fact [8, 29] that the Chern and Levi-Civita connections on $K = \Lambda^{m,0}$ coincide to compute α and β modulo harmless error terms. Now since ∇ is actually the Chern connection, $\nabla^{0,1} = \bar{\partial} := \bar{\partial}_J$, and hence

$$\begin{aligned}
\alpha \wedge \varphi &= \nabla^{0,1}\varphi = \bar{\partial}\varphi = d\varphi = d(\varphi - \varphi_0) \\
&= -d\left[\tfrac{1}{2}\sum_{\mu,\bar{\nu}=1}^{m} (-1)^{\mu} \mathscr{K}^{\mu}_{\bar{\nu}}\, d\bar{z}^{\bar{\nu}} \wedge dz^1 \wedge \cdots \wedge \widehat{dz^{\mu}} \wedge \cdots \wedge dz^m\right] + O(\varrho^{-2\tau-1}) \\
&= -\tfrac{1}{2}\sum_{\mu,\bar{\nu}=1}^{m} (-1)^{\mu} d\mathscr{K}^{\mu}_{\bar{\nu}} \wedge d\bar{z}^{\bar{\nu}} \wedge dz^1 \wedge \cdots \wedge \widehat{dz^{\mu}} \wedge \cdots \wedge dz^m + O(\varrho^{-2\tau-1}) \\
&= \tfrac{1}{2}\sum_{\mu,\bar{\nu},\bar{\kappa}=1}^{m} (-1)^{\mu} \mathscr{K}^{\mu}_{\bar{\nu},\bar{\kappa}} d\bar{z}^{\bar{\nu}} \wedge d\bar{z}^{\bar{\kappa}} \wedge dz^1 \wedge \cdots \wedge \widehat{dz^{\mu}} \wedge \cdots \wedge dz^m \\
&\quad -\tfrac{1}{2}\sum_{\mu,\bar{\nu}=1}^{m} \mathscr{K}^{\mu}_{\bar{\nu},\mu} d\bar{z}^{\bar{\nu}} \wedge dz^1 \wedge \cdots \wedge dz^m + O(\varrho^{-2\tau-1}) \\
&= \left(-\tfrac{1}{2}\sum_{\mu,\bar{\nu}=1}^{m} \mathscr{K}^{\mu}_{\bar{\nu},\mu} d\bar{z}^{\bar{\nu}}\right) \wedge \varphi_0 + O(\varrho^{-2\tau-1}) \\
&= \left(-\tfrac{1}{2}\sum_{\mu,\bar{\nu}=1}^{m} \mathscr{K}^{\mu}_{\bar{\nu},\mu} \bar{\zeta}^{\bar{\nu}}\right) \wedge \varphi + O(\varrho^{-2\tau-1}),
\end{aligned}$$

where we have used (2.5) to sweep $\mathscr{K}^{\mu}_{\bar\nu,\bar\kappa} d\bar z^{\bar\nu}\wedge d\bar z^{\bar\kappa}$ into the error term. Since the $\bar\zeta^{\bar\nu}$ are a basis for $\Lambda^{0,1}_J$, this shows that

$$\begin{aligned}\alpha &= -\tfrac{1}{2}\mathscr{K}^{\mu}_{\bar\nu,\mu}\bar\zeta^{\bar\nu} + O(\varrho^{-2\tau-1})\\ &= -\tfrac{1}{2}\mathscr{K}^{\mu}_{\bar\nu,\mu} d\bar z^{\bar\nu} + O(\varrho^{-2\tau-1}),\end{aligned}$$

where the Einstein summation convention is of course understood.

On the other hand, the Chern connection is also compatible with the Hermitian inner product $\langle\ ,\ \rangle$ induced on the canonical line bundle K by g. Thus, if $h := \|\varphi\|^2 = \langle\varphi,\varphi\rangle$, we have

$$\begin{aligned}\partial h &= \nabla^{1,0}\langle\varphi,\varphi\rangle\\ &= \langle\nabla^{1,0}\varphi,\varphi\rangle + \langle\varphi,\nabla^{0,1}\varphi\rangle\\ &= \langle\beta\otimes\varphi,\varphi\rangle + \langle\varphi,\alpha\otimes\varphi\rangle\\ &= \langle\varphi,\varphi\rangle\beta + \langle\varphi,\varphi\rangle\bar\alpha\\ &= h(\beta+\bar\alpha)\end{aligned}$$

and hence

$$\beta = -\bar\alpha + \partial\log h.$$

On the other hand,

$$h = \|\varphi\|^2 = i^{m^2} m!\ \frac{\varphi\wedge\bar\varphi}{\omega^m},$$

and equation (2.1) therefore tells us that

$$\begin{aligned}\partial\log h &= \partial\log\frac{\varphi\wedge\bar\varphi}{\omega^m}\\ &= -\partial\log\frac{\omega^m}{\varphi_0\wedge\bar\varphi_0} + \partial\log\frac{\varphi\wedge\bar\varphi}{\varphi_0\wedge\bar\varphi_0}\\ &= -\partial\log\sqrt{\det g} + O(\varrho^{-2\tau-1}),\end{aligned}$$

since $\varphi_0\wedge\bar\varphi_0$ is just a constant times the coordinate volume element $|dz|^{2m}$, while ω^m is just a constant times the metric volume element of g. Thus, relative to the trivialization given by φ, the connection form ϑ of the Chern connection on K is given by

$$\vartheta = \alpha+\beta = \alpha - \bar\alpha - \partial\log\sqrt{\det g} + O(\varrho^{-2\tau-1})$$

where $\partial := \partial_J$, and where

$$\alpha = -\tfrac{1}{2}\mathscr{K}^{\mu}_{\bar\nu,\mu} d\bar z^{\bar\nu} + O(\varrho^{-2\tau-1}).$$

However, the curvature of the Chern connection on K is given by $i\rho$, where ρ once again denotes the Ricci form of (g, J). Thus

$$d\vartheta = i\rho,$$

and the real 1-form θ defined by

$$\theta = \Im m\, \vartheta = -\tfrac{i}{2}(\vartheta - \bar{\vartheta})$$

therefore satisfies $d\theta = \rho$. In conjunction with (2.6), the above calculation thus shows that

$$\begin{aligned}
\theta &= i(\bar{\alpha} - \alpha) + \tfrac{i}{2}(\partial - \bar{\partial}) \log \sqrt{\det g} \qquad + O(\varrho^{-2\tau-1}) \\
&= -J(\alpha + \bar{\alpha}) + \tfrac{i}{2}(\partial - \bar{\partial}) \log \sqrt{\det g} \qquad + O(\varrho^{-2\tau-1}) \\
&= J\, \Re e\, (\mathscr{K}^{\mu}_{\bar{\nu},\mu} d\bar{z}^{\bar{\nu}}) - \tfrac{1}{2} J\, d\log \sqrt{\det g} \quad + O(\varrho^{-2\tau-1}) \\
&= \tfrac{1}{2} J\, (\gimel + 4\gamma - \, d\log \sqrt{\det g}) \quad + O(\varrho^{-2\tau-1})
\end{aligned}$$

where the 1-form $\gimel$ corresponds to $\mathbf{\Gamma}^j := g^{k\ell}\mathbf{\Gamma}^j_{k\ell}$ by index lowering, and where the 1-form γ is co-exact. On the other hand, the Kähler condition also implies that any 1-form ϕ satisfies

$$\star\phi = J\phi \wedge \frac{\omega^{m-1}}{(m-1)!},$$

so it follows that

$$\star(\gimel + 4\gamma - d\log\sqrt{\det g}) = \frac{2}{(m-1)!}\, \theta \wedge \omega^{m-1} + O(\varrho^{-2\tau-1}).$$

Notice, moreover, that both sides are invariant under the isometric linear action of $\Gamma \subset \mathbf{U}(m)$. Setting $\tau = m - 1 + \varepsilon$, where $\varepsilon > 0$, we therefore have

$$\int_{S_\varrho/\Gamma} \star(\gimel - d\log\sqrt{\det g}) + 4\int_{S_\varrho/\Gamma} \star\gamma = \frac{2}{(m-1)!}\int_{S_\varrho/\Gamma} \theta \wedge \omega^{m-1} + O(\varrho^{-2\varepsilon}).$$

However, since $\star\gamma = d\star\Omega$, the second integral on the left vanishes by Stokes' theorem, and we therefore deduce that

$$\int_{S_\varrho/\Gamma} \star(\gimel - d\log\sqrt{\det g}) = \frac{2}{(m-1)!}\int_{S_\varrho/\Gamma} \theta \wedge \omega^{m-1} + O(\varrho^{-2\varepsilon}).$$

But since $\gimel - d\log\sqrt{\det g} = O(\varrho^{-\tau-1})$, and since $\star$ differs from the Euclidean Hodge star by $O(\varrho^{-\tau})$, this implies that

$$\int_{S_\varrho/\Gamma}\left[\gimel_j - (\log\sqrt{\det g})_{,j}\right]\mathbf{n}^j d\mathfrak{a}_E = \frac{2}{(m-1)!}\int_{S_\varrho/\Gamma}\theta\wedge\omega^{m-1} + O(\varrho^{-2\varepsilon}).$$

However,

$$\gimel_j = \tfrac{1}{2}g^{k\ell}(g_{jk,\ell}+g_{j\ell,k}-g_{k\ell,j}) = g_{jk,k} - \tfrac{1}{2}g_{kk,j} + O(\varrho^{-2\tau-1})$$

and

$$(\log\sqrt{\det g})_{,j} = \tfrac{1}{2}g_{kk,j} + O(\varrho^{-2\tau-1}),$$

and we therefore have

$$\int_{S_\varrho/\Gamma}[g_{kj,k}-g_{kk,j}]\,\mathbf{n}^j d\mathfrak{a}_E - \frac{2}{(m-1)!}\int_{S_\varrho/\Gamma}\theta\wedge\omega^{m-1} = O(\varrho^{-2\varepsilon}),$$

as claimed. $\square$

This now implies our coordinate-invariant reformulation of the mass:

Theorem 2.2. *Let (M^{2m}, g, J) be a ALE Kähler manifold of any complex dimension $m \geq 2$. Suppose only that g is a C^2 metric whose scalar curvature s belongs to L^1, and that, in* some *real asymptotic coordinate system $(x^1, \ldots, x^{2m})$ on a given end M_∞, the metric g has fall-off*

$$g_{jk} = \delta_{jk} + O(|x|^{1-m-\varepsilon}), \qquad g_{jk,\ell} = O(|x|^{-m-\varepsilon})$$

for some $\varepsilon > 0$. Then the mass at the given end, expressed as the limit of an integral computed in these coordinates, is well-defined, and satisfies

$$\mathfrak{m}(M,g) = \lim_{\varrho\to\infty}\frac{1}{2(2m-1)\pi^m}\int_{S_\varrho/\Gamma}\theta\wedge\omega^{m-1}$$

for any 1-form θ with $d\theta = \rho$ on the end M_∞, where ρ is the Ricci form of g. Moreover, the mass, determined in this manner, is coordinate independent; computing it in any other asymptotic coordinate system in which the metric satisfies this weak fall-off hypothesis will produce exactly the same answer.

Proof. By Proposition 2.1, there is a *particular* 1-form θ with $d\theta = \rho$ such that

$$\lim_{\varrho\to\infty} \tfrac{(m-1)!}{4\pi^m(2m-1)} \int_{S_\varrho/\Gamma} [g_{kj,k} - g_{kk,j}]\, \mathbf{n}^j d\mathfrak{a}_E = \lim_{\varrho\to\infty} \int_{S_\varrho/\Gamma} \frac{\theta\wedge\omega^{m-1}}{2\pi^m(2m-1)} \qquad (2.7)$$

provided either limit exists. The left-hand side of equation (2.7) is of course the coordinate definition of the mass associated with the given asymptotic chart. On the other hand, the last paragraph of the proof of Proposition 1.6 shows that, when $s \in L^1$, the limit on the right-hand side of (2.7) exists and actually coincides with the limit obtained by instead performing the relevant integrals on the level sets of an *arbitrary* exhaustion function for $\widetilde{M_\infty}$. But since Γ is finite, $H^1_{dR}(M_\infty) = \mathrm{Hom}(\Gamma, \mathbb{R}) = 0$, and any other primitive $\tilde{\theta}$ for the Ricci form can be expressed as $\tilde{\theta} = \theta + df$; thus, choosing a different primitive $\tilde{\theta}$ would just change the integrand by an exact form, and so leave the right-hand-side of (2.7) unchanged. This shows that the right-hand limit is coordinate-independent. Consequently, the limit on the left-hand side of (2.7) exists and is also independent of the choice of coordinates, provided we restrict ourselves to asymptotic charts in which g satisfies the above weak fall-off hypothesis. □

Note that we obtain something stronger if (M, g, J) is a *scalar-flat* Kähler manifold. Indeed, when $s \equiv 0$, the differential form $\theta\wedge\omega^{m-1}$ is *closed*, and the integral $\int_{\mathscr{S}} \theta\wedge\omega^{m-1}$ then only depends on the homology class of the compact hypersurface $\mathscr{S} \subset M_\infty$. One can thus replace the limit on the right-hand of (2.7) with the integral on a single hypersurface! This remarkable fact played a central role in the process of discovery that led to the present results.

In order to extend our proof of Theorem C to the $m = 2$ case, we now lack only one last ingredient: the fact that an ALE Kähler surface can only have one end. In the next section, we will show that this is indeed true. In the process, we will also discover other interesting and useful results governing the complex-analytic behavior of ALE Kähler surfaces.

3 Complex Asymptotics: The Surface Case

As we saw in Lemma 1.3, the complex structure of any ALE Kähler manifold of complex dimension $m \geq 3$ is *standard at infinity*, in the sense that the complement of a suitable compact set is biholomorphic to $(\mathbb{C}^m - \mathbf{D}^{2m})/\Gamma$, where $\mathbf{D}^{2m} \subset \mathbb{C}^m$ is the closed unit ball. However, concrete examples show [19, 27] that this is not generally true when $m = 2$. Nonetheless, many of our high-dimensional results still have workable analogs in the complex surface case. For example, here is an $m = 2$ version of Lemma 1.1:

Lemma 3.1. *Let M_∞ be an end of an ALE Kähler surface (M^4, g, J), where we just assume that, in some asymptotic chart, the metric either has fall-off*

$$g_{jk} - \delta_{jk} \in C^1_{-\tau}$$

for some $\tau > 3/2$, or else that

$$g_{jk} - \delta_{jk} \in C^{2,\alpha}_{-\tau}$$

for some $\tau > 1$. Then there is a (non-compact) complex surface $\mathscr{X}$ containing an embedded holomorphic curve $\Sigma \cong \mathbb{CP}_1$ with self-intersection $+1$, such that the universal cover $\widetilde{M}_\infty$ of M_∞ is biholomorphic to $\mathscr{X} - \Sigma$.

Proof. If $\tau > 3/2$, the proof of Lemma 1.1 goes through with only minor improvements. Indeed, suppose $\tau \geq 1 + \varepsilon$ for some $\varepsilon \in (1/2, 1)$. Then the almost-complex structure J constructed by our previous method will still be of Hölder class $C^{0,\varepsilon}$. Since we have assumed that $\varepsilon > 1/2$, the Hill-Taylor version [24] of Newlander-Nirenberg thus says that J is integrable, in the sense of the existence of complex coordinate charts, iff its Nijenhuis tensor vanishes in the distributional sense. However, our J belongs to $W^{1,p} \cap C^{0,\varepsilon}$ for any $p \in (4, 2/(1 - \varepsilon))$, and its Nijenhuis tensor thus has components of class L^p. But since the Nijenhuis tensor of J vanishes in the classical sense on $M_\infty = \mathscr{X} - \Sigma$, it therefore vanishes almost everywhere; and since its components belong to L^p, this means they also vanish as distributions. The Hill-Taylor theorem then tells us that $(\mathscr{X}, J)$ can be covered with local complex coordinate charts, and that these will be at least $C^{1,\varepsilon}$ with respect to the original atlas.

However, when $\tau \in (1, 3/2]$, this argument breaks down, and we instead need to assume that $g_{jk} - \delta_{jk} \in C^{2,\alpha}_{-\tau}$ in order to obtain the desired conclusion. We proceed by an argument exactly parallel to that given in [22, Section 3.2]. The key idea is to first change coordinates on $\mathbb{C}^2 - \mathbf{D}^4$ in such a manner that all the complex lines through the origin in $\mathbb{C}^2$ become J-holomorphic curves. The reason for doing this is that, when passing from $\mathbb{C}^m$ to $\mathbb{CP}_m$ by inverting a coordinate, the worst loss of regularity in our previous construction occurred in the radial directions. Improving the radial behavior of J by imposing this gauge choice will allow us to overcome this difficulty. Indeed, assuming that $g - \delta \in C^{2,\alpha}_{-\tau}$, Picard iteration [22, Section 3.2, Step 1] allows one to construct such a change of coordinates $\Phi : \mathbb{C}^2 - \mathbf{B} \to \mathbb{C}^2 - \mathbf{D}^4$, outside a sufficiently large ball $\mathbf{B}$, such that the components of $\Phi - \mathrm{id}$ belong to $C^{2,\alpha}_{1-\tau}$. (One does gain control of an extra derivative only along radial complex lines, where the problem we are solving is elliptic, but, due to the lack of ellipticity in the transverse directions, this is all that we can expect.) Thus $(\Phi^* J) - J_0$ is now of class $C^{1,\alpha}_{-\tau}$, and moreover vanishes in radial complex directions. Thus, our previous fall-off analysis shows that J induces a complex structure on a neighborhood of a projective line $\mathbb{CP}_1 \subset \mathbb{CP}_2$ that is actually of Hölder class $C^{1,\varepsilon}$, where $\varepsilon = \min(\alpha, \tau - 1)$. In particular, the resulting almost-complex structure has vanishing Nijenhuis tensor by continuity, and so, by standard versions [42, 40] of Newlander-Nirenberg, becomes standard in complex charts that are at least $C^{2,\varepsilon}$ with respect to our original atlas. □

We remark that this Lemma was first discovered in the special case of *scalar-flat* Kähler metrics [14, 35, 36, 37], where, at the outset, one can arrange for g to have much better fall-off, and where the relevant complex surface $\mathscr{X}$ actually arises as a hypersurface in a twistor space. Lemma 3.1 thus allows us to generalize various proofs from the narrow world of scalar-flat Kähler surfaces to the present, broader context. In particular, an argument used in [36] now yields an analog of Proposition 1.5:

Proposition 3.2. *Any ALE Kähler surface (M^4, g, J) has only one end.*

Proof. Lemma 3.1 allows us to construct an orbifold compactification X of M by adding a quotient of $\mathbb{CP}_1$ to each end. After blowing up, this produces a smooth compactification X of M which is a non-singular complex surface. Moreover, the closure of each end of M contains smoothly immersed rational curves of positive normal bundle, and each such curve has positive self-intersection. Grauert's criterion therefore guarantees [6] that X is projective, and so in particular is of Kähler type. The Hodge index theorem therefore tells us that the intersection form on $H^{1,1}(X, \mathbb{R})$ must be of Lorentz type. However, the curves arising from two different ends of M would necessarily be disjoint, and therefore would be orthogonal with respect to the intersection form. Since this would contradict the Hodge index theorem if there were two or more ends, we are therefore forced to conclude that M can only have one end. □

Similarly, an argument from [35, 37] proves the following:

Proposition 3.3. *Any AE Kähler surface is biholomorphic to an iterated blow-up of* $\mathbb{C}^2$.

Proof. In the asymptotically Euclidean case, the compactification X is actually a complex manifold, obtained by adding a $\mathbb{CP}_1$ of self-intersection $+1$ to M. Grauert's criterion [6] thus implies that X is projective, and in particular is Kähler. However, we must have $H^{1,0}(X) = 0$, since the Kodaira deformations [31] of this $\mathbb{CP}_1$ sweep out an open subset of X, and since $\Lambda^{1,0}X$ must be isomorphic to $\mathcal{O}(-2) \oplus \mathcal{O}(-1)$ on any one of these smoothly embedded copies of $\mathbb{CP}_1$. Hodge symmetry therefore tells us that $H^{0,1}(X) = 0$, and it therefore follows that all of these rational curves actually belong to the same linear system. Since the intersection of all these rational curves is empty, this linear system has no base locus. Since Kodaira's theorem [31] moreover tells us that the dimension of this family is 2, this linear system defines a non-singular holomorphic map $X \to \mathbb{CP}_2$ which sends a neighborhood of $\Sigma \subset X$ biholomorphically to a neighborhood of a projective line $\mathbb{CP}_1 \subset \mathbb{CP}_2$. Since bimeromorphic maps between compact surfaces always factor into blow-ups and blow-downs, it follows that X is obtained from $\mathbb{CP}_2$ by blowing up points away from this $\mathbb{CP}_1$. Deleting the line at infinity, we thus see that M is simply a blow-up of $\mathbb{C}^2$. □

In particular, Proposition 3.3 tells us that the complex structure of an AE Kähler surface is always standard at infinity, just as it was in higher dimensions. We emphasize, however, that the corresponding statement is generally *false* for ALE Kähler surfaces. Here it is perhaps worth emphasizing that the proof of Proposition 3.3 is *global* in nature. This should be contrasted with the *local* type of rigidity displayed by Lemma 1.2, the proof of which fails in a crucial respect when $m = 2$.

Indeed, if $\Sigma \subset \mathscr{X}$ is an embedded $\mathbb{CP}_1$ of self-intersection $+1$ in a non-compact complex surface, the Kodaira family $\mathscr{Y}$ of its deformations still carries a holomorphic projective structure, but now *any* holomorphic projective structure on a complex surface locally arises in this fashion [26, 32]. While the Weyl curvature always vanishes for a 2-dimensional projective structure, most such structures are certainly not flat. Indeed, the obstruction to projective flatness in dimension $m = 2$ is actually measured by the *projective Cotton tensor*, which can locally be expressed as $C_{\lambda\mu\nu} = \nabla_{[\mu} r_{\nu]\lambda}$, where ∇ is any torsion-free holomorphic connection that both represents the projective structure and induces a flat connection on the canonical line bundle of $\mathscr{Y}$, and where r denotes the Ricci tensor of ∇. The Cotton tensor of $\mathscr{Y}$ at the base-point o is actually the obstruction to the triviality of the *fourth* infinitesimal neighborhood $\Sigma^{(4)}$. Nonetheless, one can still prove the following:

Lemma 3.4. *Let $\mathscr{X}$ be a (possibly non-compact) complex surface, and let $\Sigma \subset \mathscr{X}$ be an embedded $\mathbb{CP}_1$ of self-intersection $+1$. Then the third infinitesimal neighborhood $\Sigma^{(3)}$ of $\Sigma \subset \mathscr{X}$ is isomorphic to the third infinitesimal neighborhood of a projective line $\mathbb{CP}_1 \subset \mathbb{CP}_2$.*

Combining this with the proof of Proposition 1.3 then yields

Proposition 3.5. *Let (M^4, g, J) be an ALE Kähler surface. Then there is an asymptotic coordinate system $(x^1, \ldots, x^4)$ on the universal cover of the end M_∞ of M in which*

$$g = \delta + O(|x|^{-1-\varepsilon}), \qquad \nabla g = O(|x|^{-2-\varepsilon})$$

and

$$J = J_0 + O(|x|^{-3}), \qquad \nabla J = O(|x|^{-4})$$

where ∇ is the coordinate (Euclidean) derivative, and J_0 is the familiar constant-coefficient almost complex structure tensor on $\mathbb{C}^2 = \mathbb{R}^4$.

Thus, while one cannot always arrange for J to be standard at infinity, it is at least asymptotic to the standard complex structure to a higher order than the fall-off of the metric would naïvely lead one to expect. In the asymptotic coordinates provided by Proposition 3.5, the proof of Proposition 2.1 then simplifies dramatically, because the 1-forms $\daleth$ and γ become negligible error terms. Assuming the Bartnik-Chruściel coordinate-invariance of the mass, a variant of the demonstration of Proposition 1.6 thus suffices to prove the $m = 2$ case of the result. This was how we first obtained the asymptotic mass formula in the complex-surface case.

While Lemma 3.4 cannot be improved in general, one can still do systematically better in many cases of interest. Indeed, notice the action of Γ on $\widetilde{M}_\infty$ always extends to a holomorphic action on $\mathscr{X}$, and that this then induces an action on $\mathscr{Y}$ preserving both the holomorphic projective structure and the base-point $o \in \mathscr{Y}$. Moreover, the induced action of Γ on $T^{1,0}\mathscr{Y} \cong \mathbb{C}^2$ is just given by the tautological 2-dimensional representation of $\Gamma \subset \mathbf{U}(2)$. Since the Cotton tensor at o must be invariant under the action of Γ, it either vanishes, or else the action of Γ on $[\mathbb{C}^2 \otimes \Lambda^2(\mathbb{C}^2)]^*$ must have a trivial

1-dimensional sub-representation. In our context, this will force $\Sigma^{(4)}$ to be standard, allowing one to osculate J by J_0 to higher order at infinity, unless Γ is a cyclic group $\mathbb{Z}_\ell$, where ℓ is odd, acting on $\mathbb{C}^2$ with generator

$$\begin{bmatrix} e^{2\pi i/\ell} & 0 \\ 0 & e^{-4\pi i/\ell} \end{bmatrix}.$$

However, the $\ell = 3$ examples of Honda [27] show that Proposition 3.5 is actually optimal for certain ALE scalar-flat Kähler surfaces.

4 The Mass Formula for Complex Surfaces

All the pieces needed to finish the proofs of Theorem A and C are now in place. Of course, the remaining step is to demonstrate the $m = 2$ case of the mass formula. Once this is done, we will then obtain Theorem B by simply re-examining some off-the-shelf examples using these new instruments.

Theorem 4.1. *The mass of any ALE Kähler surface* (M^4, g, J) *is given by*

$$m(M, g) = -\frac{1}{3\pi}\langle \clubsuit(c_1), [\omega]\rangle + \frac{1}{12\pi^2}\int_M s_g \, d\mu_g \ .$$

Proof. Theorem 2.2 shows that the asymptotic mass formula of Proposition 1.6 also holds in the $m = 2$ case. Meanwhile, Proposition 3.2 shows that the $m = 2$ version of Proposition 1.5 also holds. With these minor substitutions, the proof of Theorem 1.9, with m set now equal to 2, then proves the desired cohomological mass formula. □

In conjunction with Theorem 1.9, Theorem 4.1 now implies Theorem C. We also obtain the following corollary:

Theorem 4.2. *The mass of any ALE* scalar-flat *Kähler surface* (M^4, g, J) *is given by*

$$m(M, g) = -\frac{1}{3\pi}\langle \clubsuit(c_1), [\omega]\rangle.$$

In particular, the mass is a topological invariant in this setting, and depends only on the underlying manifold M*, together with the cohomology classes* $c_1(M, J)$ *and* $[\omega]$*.*

Theorem A is now an immediate consequence of Theorems 1.10 and 4.2.

Let us now recast Theorem 4.2 in a more concrete form by identifying $H^2_c(M,\mathbb{R})$ with the homology group $H_2(M,\mathbb{R})$ via Poincaré duality. In this setting, the intersection pairing on $H^2_c(M)$ becomes the geometric pairing on $H_2(M)$ obtained by counting intersection numbers of compact (real) surfaces in M. Note that Lemma 1.7 implies that this pairing

$$H_2(M,\mathbb{R}) \times H_2(M,\mathbb{R}) \to \mathbb{R}$$

is non-degenerate on any ALE 4-manifold M.

Theorem 4.3. *Let (M, g, J) be an ALE scalar-flat Kähler surface. Let $E_1, \ldots E_{\mathfrak{b}}$ be a basis for $H_2(M,\mathbb{R})$, and let $Q = [Q_{jk}] = [E_j \cdot E_k]$ be the corresponding intersection matrix. If we define $a_1, \ldots, a_{\mathfrak{b}}$ by*

$$\begin{bmatrix} a_1 \\ \vdots \\ a_{\mathfrak{b}} \end{bmatrix} = Q^{-1} \begin{bmatrix} \int_{E_1} c_1 \\ \vdots \\ \int_{E_{\mathfrak{b}}} c_1 \end{bmatrix} \tag{4.1}$$

then the mass of (M, g) is given by

$$\mathfrak{m} = -\frac{1}{3\pi} \sum_{j=1}^{\mathfrak{b}} a_j \int_{E_j} [\omega] \tag{4.2}$$

where $[\omega]$ denotes the Kähler class of (M, g, J).

Proof. The cycle $\sum a_j E_j$ is exactly determined by the requirement that

$$\left(\sum a_j E_j\right) \cdot D = \int_D c_1$$

for any $D \in H_2(M)$. However, since $H_2(M,\mathbb{R}) = H^2_c(M)$ by Poincaré duality, this is equivalent to saying that

$$\sum a_j \int_{E_j} \mho = \langle c_1, \mho\rangle$$

for any $\mho \in H^2_c(M)$. However, for any $\Omega \in H^2(M)$, we have

$$\langle \clubsuit(c_1), \Omega\rangle = \langle \clubsuit(c_1), \clubsuit(\Omega)\rangle = \langle c_1, \clubsuit(\Omega)\rangle$$

so that, setting $\mho = \clubsuit(\Omega)$, we have

$$\langle \clubsuit(c_1), \Omega\rangle = \sum a_j \int_{E_j} \clubsuit(\Omega) = \sum a_j \int_{E_j} \Omega \ .$$

Setting $\Omega = [\omega]$, we therefore have

$$\mathfrak{m}(M, g, J) = -\frac{1}{3\pi}\langle \clubsuit(c_1), [\omega]\rangle = -\frac{1}{3\pi} \sum a_j \int_{E_j} [\omega]$$

by Theorem 4.1. $\square$

Recalling Proposition 3.3, we thus obtain the following:

Corollary 4.4. *Let (M, g, J) be an AE scalar-flat Kähler surface. We may then choose a homology basis $E_1, \ldots, E_{\mathfrak{b}} \in H_2(M, \mathbb{Z})$ with intersection matrix $Q = -I$ in which $c_1(M)$ is Poincaré dual to $-\sum E_j$. Consequently,*

$$\mathfrak{m}(M, g) = \frac{1}{3\pi} \sum_{j=1}^{\mathfrak{b}} \int_{E_j} [\omega]$$

where $[\omega]$ is the Kähler class of (M, g, J).

Proof. By Proposition 3.3, (M, J) is an iterated blow-up of $\mathbb{C}^2$ at $\mathfrak{b}$ points, and so has a small deformation which is a blow-up of $\mathbb{C}^2$ at distinct points. One can then take the E_j to be the homology classes of the exceptional divisors of these distinct points. $\square$

When $\mathbb{C}^2$ is blown up at distinct points, the expression for the mass provided by Corollary 4.4 is obviously a sum of areas of holomorphic curves, and so is certainly positive if $\mathfrak{b} > 0$. However, $\sum_{j=1}^{\mathfrak{b}} E_j$ is *always* homologous to a sum of holomorphic curves with positive integer coefficients, even in the degenerate cases, so this expression for the mass will actually always be positive whenever $M \neq \mathbb{C}^2$. We will return to this point in Theorem 5.1 below.

Corollary 4.5. *Let (M, g, J) be an ALE scalar-flat Kähler surface, where (M, J) is obtained from the total space of the $\mathcal{O}(-\ell)$ line bundle over $\mathbb{CP}_1$ by blowing up $\mathfrak{b} - 1$ distinct points that do not lie on the zero section. Let F be* *the homology class of the zero section, and let $E_1, \ldots, E_{\mathfrak{b}-1}$ be the homology classes of the exceptional divisors of the blown up points. Then*

$$\mathfrak{m}(M,g) = \frac{1}{3\pi}\left[\frac{2-\ell}{\ell}\int_F \omega + \sum_{j=1}^{\mathfrak{b}-1}\int_{E_j}\omega\right] .$$

Proof. In the homology basis $F, E_1, \ldots, E_{\mathfrak{b}-1}$, the intersection form is represented by the matrix

$$\mathcal{Q} = \begin{bmatrix} -\ell & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}$$

while

$$\begin{bmatrix} \int_F c_1 \\ \int_{E_1} c_1 \\ \vdots \\ \int_{E_{\mathfrak{b}-1}} c_1 \end{bmatrix} = \begin{bmatrix} 2-\ell \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and the result therefore follows from Theorem 4.3. □

In particular, one sees that the mass is negative when $\ell \geq 3$ and no points are blown up. This was laboriously discovered by hand for specific explicit examples in [33], but now we see that this phenomenon occurs as a matter of general principle.

Of course, the mass formula we have discovered is purely topological, and thus insensitive to deformations of complex structure. As an application, we immediately now see the following:

Corollary 4.6. *Let (M, g, J) be an ALE scalar-flat Kähler surface, where (M, J) is obtained from the total space of the $\mathcal{O}(-\ell)$ line bundle over $\mathbb{CP}_1$ by blowing up $\mathfrak{b}-1$ distinct points that lie on the zero section. Let $\tilde{F}$ be the homology class of the proper transform of the zero section, and let $E_1, \ldots, E_{\mathfrak{b}-1}$ be the homology classes of the exceptional divisors of the blown up points. Then*

$$\mathfrak{m}(M,g) = \frac{1}{3\pi\ell}\left[(2-\ell)\int_{\tilde{F}} \omega + 2\sum_{j=1}^{\mathfrak{b}-1}\int_{E_j}\omega\right] .$$

Proof. This example is diffeomorphic to the previous one, in a manner that preserves the first Chern class. The mass formula therefore follows from Corollary 4.5, together with the observation that $\tilde{F} + E_1 + \cdots + E_{\mathfrak{b}-1}$ is homologous to F. $\square$

Applying Corollary 4.6 to some examples constructed in [34], we now immediately obtain Theorem B:

Theorem 4.7. *There are infinitely many topological types of ALE scalar-flat Kähler surfaces that have zero mass, but are not Ricci-flat. Indeed, for any $\ell \geq 3$, the blow-up of the $\mathcal{O}(-\ell)$ line bundle on $\mathbb{CP}_1$ at any non-empty collection of distinct points on the zero section admits such metrics.*

Proof. Let $p_0, p_1, \ldots, p_{\mathfrak{b}-1}$ be distinct points in hyperbolic 3-space $\mathcal{H}^3$, chosen so that that the geodesic rays $\overrightarrow{p_0p_1}, \ldots, \overrightarrow{p_0p_{\mathfrak{b}-1}}$ all have distinct initial tangent directions at p_0. Let $\mathfrak{r}_j$, $j = 0, \ldots, \mathfrak{b}-1$, denote the hyperbolic distance from p_j, considered as a function on $\mathcal{H}^3$. Let $X = \mathcal{H}^3 - \{p_0, p_1, \ldots, p_{\mathfrak{b}-1}\}$, and let $P \to X$ be the principal $\mathbf{U}(1)$-bundle with $c_1 = -\ell$ on a small 2-sphere around p_0 and $c_1 = -1$ on a small 2-sphere around any other p_j. Set

$$V := 1 + \frac{\ell}{e^{2\mathfrak{r}_0} - 1} + \sum_{j=1}^{\mathfrak{b}-1} \frac{1}{e^{2\mathfrak{r}_j} - 1},$$

on X, and let ϑ be a connection 1-form on $P \to X$ with curvature

$$d\vartheta = \star dV,$$

where the Hodge star is computed with respect to the hyperbolic metric $\mathscr{h}$ on $X \subset \mathcal{H}^3$ and standard orientation. Finally, let

$$\mathscr{g} = \frac{1}{4\sinh^2 \mathfrak{r}_0}\left[V\mathscr{h} + V^{-1}\vartheta^2\right]$$

on P, and let (M, g) be the metric completion of $(P, \mathscr{g})$. Then (M, g) is an ALE scalar-flat Kähler surface, and (M, J) is obtained [34, p. 244] from the $\mathcal{O}(-\ell)$ line bundle on $\mathbb{CP}_1$ by blowing up $\mathfrak{b} - 1$ distinct points on the zero section. The proper transform $\tilde{F}$ of the zero section is represented in this picture by the sphere at infinity of $\mathcal{H}^3$, and the restriction of g to $\tilde{F}$ is just the standard Fubini-Study metric, with total area π. On the other hand, the exceptional curve E_j is the closure in M of the inverse image in P of the geodesic ray in $\mathcal{H}^3$ which starts at p_j and points diametrically away from p_0; its total area is given by $2\pi/(e^{2\text{я}_j}-1)$, where $\text{я}_j = \mathfrak{r}_0(p_j)$ is the hyperbolic distance from p_0 to p_j. By Corollary 4.6, the mass of the resulting metric is therefore given by

$$\mathfrak{m}(M,g) = \frac{1}{3\ell}\left[2-\ell+4\sum_{j=1}^{\mathfrak{b}-1}\frac{1}{e^{2\text{я}_j}-1}\right]$$

and so, if $\ell \geq 3$ and $\mathfrak{b}-1 \geq 1$, this obviously changes sign as we let the я_j range over all of $\mathbb{R}^+$. To be more concrete and specific, we in particular obtain a non-Ricci-flat example with zero mass if we take $\mathfrak{b}-1 = \ell-2 \geq 1$ and $\text{я}_j = \log\sqrt{5}$ for every $j = 1,\ldots,\mathfrak{b}-1$. $\square$

Interestingly, though, the above construction depends in practice on a choice of (M,J) which is *non-minimal*, in the sense of being the blow-up of another complex surface. This appears to be essential. Indeed, the following consequence of Theorem 4.3, which was graciously pointed out to us by Cristiano Spotti, offers a systematic result along these lines:

Corollary 4.8. *Let (M^4,g,J) be an ALE scalar-flat Kähler surface, and suppose that (M,J) is the* **minimal** *resolution of a surface singularity. Then $\mathfrak{m}(M,g)\leq 0$, with equality iff g is Ricci-flat.*

Proof. Choose a basis for H_2 that is represented by a collection of smooth rational curves E_j. Because the resolution is assumed to be minimal, each E_j has self-intersection ≤ -2, and adjunction therefore tells us that $\int_{E_j} c_1 \leq 0$ for every j. However, it is also known [1, Remark 3.1.2] that every entry in the inverse Q^{-1} of the intersection matrix of such a minimal resolution is *non-positive*. Thus, the coefficients defined by equation (4.1) all satisfy $a_j \geq 0$, and the mass formula (4.2) therefore produces a non-positive answer. Moreover, if the mass is zero, then $a_j = 0$ for all j, so that $Q\vec{a}$ vanishes and $c_1 = 0$. But Lemma 1.7 tells us that c_1 is represented by a unique L^2 harmonic 2-form, and, since g is scalar-flat Kähler, one such representative is $\rho/2\pi$, where ρ is the Ricci-form of (M,g,J). The mass therefore vanishes for such a manifold if and only if the metric is Ricci-flat. $\square$

Lock and Viaclovsky [39] have recently given a systematic construction of ALE scalar-flat Kähler metrics on minimal resolutions of surface singularities, thereby putting the earlier examples of Calderbank and Singer [12] into a broader context. The above Corollary now shows that all of these examples actually have negative mass.

5 The Positive Mass Theorem

We conclude this article by proving the positive mass theorem for Kähler manifolds, along with our related Penrose-type inequality.

Suppose that (M^{2m}, g, J) is an AE Kähler manifold. Then, as we saw in Proposition 1.12, there is a proper holomorphic map $F : M \to \mathbb{C}^m$ which has degree 1, and which is a biholomorphism outside a compact set. We now consider the holomorphic m-form

$$\Upsilon = F^* dz^1 \wedge \cdots \wedge dz^m,$$

which is a holomorphic section of the canonical line bundle of M, and which exactly vanishes at the set of critical points of F. Because this zero set is locally the zero set of a non-trivial holomorphic function, it is purely of complex codimension 1, and we moreover know this locus is compact because F is a biholomorphism outside of a compact set. Breaking up the locus $\Upsilon = 0$ as a finite union of its irreducible components D_j, and assigning each of these an integer multiplicity n_j given by the order of vanishing of Υ along D_j, we can thus express the divisor D as

$$D = \sum_j n_j D_j \ .$$

Since Υ is a holomorphic section of the canonical line bundle K_M, the homology class $[D] = \sum n_j [D_j]$ is then Poincaré dual to $\clubsuit(c_1(K_M)) = -\clubsuit(c_1(M, J))$. The mass formula of Theorem C therefore can be rewritten as

$$\mathfrak{m}(M, g) = \frac{1}{(2m-1)\pi^{m-1}} [\omega]^{m-1}(D) + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s_g d\mu_g,$$

and we therefore obtain the Penrose inequality promised by Theorem E:

Theorem 5.1. *Suppose that (M^{2m}, g, J) is an AE Kähler manifold with scalar curvature $s \geq 0$. Then, in terms of the complex hypersurfaces D_j and positive integer multiplicities n_j described above,*

$$\mathfrak{m}(M, g) \geq \frac{(m-1)!}{(2m-1)\pi^{m-1}} \sum_j n_j \, Vol(D_j) \ ,$$

with equality iff (M, g, J) is scalar-flat Kähler. Moreover, $\bigcup_j D_j \neq \varnothing$ if $(M, J) \neq \mathbb{C}^m$.

Proof. Since we have assumed that $s \geq 0$, the scalar curvature integral in the mass formula is non-negative, and equals zero only if g is scalar-flat. Since the volume form induced by g on the regular locus of D_j is just $\omega^{m-1}/(m-1)!$, we can therefore transform $[\omega]^{m-1}(D)$ into $(m-1)!$ times a sum of volumes, weighted by multiplicities, and the stated inequality now follows from the mass formula.

Finally, $\bigcup_j D_j$ can only be empty if $\Upsilon = F^* dz^1 \wedge \cdots \wedge dz^m$ is everywhere non-zero. But this happens iff F has no critical points, or in other words iff F is a local diffeomorphism. However, F is a degree 1 proper holomorphic map. Thus the fact that f is a local diffeomorphism implies that it is actually a global biholomorphism. □

With this in hand, we can now easily read off our Positive Mass Theorem, announced in the introduction as Theorem D:

Theorem 5.2. *Suppose that (M^{2m}, g, J) is an AE Kähler manifold with scalar curvature $s \geq 0$. Then its mass $\mathfrak{m}(M, g)$ is non-negative, and equals zero only if (M, g, J) is flat.*

Proof. By Theorem 5.1, the mass is positive unless g is scalar-flat Kähler, $\bigcup_j D_j = \varnothing$, and $(M, J) = \mathbb{C}^m$. However, the Ricci form ρ of an ALE scalar-flat Kähler metric is an L^2 harmonic form, and so, by Lemma 1.7, must vanish if the cohomology class $2\pi c_1$ it represents vanishes. Thus, g would necessarily be a Ricci-flat AE metric on $\mathbb{C}^m$. But the AE condition implies that a metric's volume growth is asymptotically exactly Euclidean, and the Bishop-Gromov inequality thus implies that a complete Ricci-flat metric with this property is necessarily flat. □

Appendix A: Normalization of the Mass

In this appendix, we provide a "physical" explanation of our normalization of the mass integral. We work throughout in units where $G = c = 1$.

In the absence of matter, tidal forces in Newtonian gravitation distort the shape of a cloud of test particles without changing its volume, to lowest order in time. Thus, the acceleration vector field due to gravitation should be divergence-free in empty space. If we assume that an isolated object generates an acceleration field that points towards the object, with magnitude only depending on the distance ϱ from the source, the acceleration field in dimension n must therefore take the form

$$\vec{a} = \nabla\left(\frac{\mathscr{M}}{\varrho^{n-2}}\right)$$

for some constant $\mathscr{M}$, which we now declare to be the *mass* of the source. In the classical case of $n = 3$, this of course reproduces Newton's law of gravitation. Since a test particle following a circular orbit of radius ϱ and angular frequency ω about the origin exhibits an in-pointing radial acceleration of magnitude $\varrho\omega^2$, this acceleration can be ascribed to our gravitational field iff

$$\omega^2 = (n-2)\frac{\mathscr{M}}{\varrho^n}.$$

This is a crude generalization of Kepler's third law of planetary motion.

Einstein's vacuum equations state that the Ricci curvature of the space-time metric should vanish in the absence of matter; inspection of Jacobi's equation reveals that this is again equivalent to requiring that tidal forces distort the shape of a cloud of test particles without changing its volume, to lowest order in time. In space-time dimension $n+1$, the general spherically symmetric solution of these equations is the generalized Schwarzschild metric

$$\mathscr{g} = -\left(1 - \frac{A}{\varrho^{n-2}}\right)dt^2 + \left(1 - \frac{A}{\varrho^{n-2}}\right)^{-1} d\varrho^2 + \varrho^2\mathfrak{h}$$

where $\mathfrak{h}$ denotes the standard unit-radius metric on S^{n-1} and A is a real constant. Notice that

$$\xi = \frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\theta}$$

is a Killing field for this metric, where $\partial/\partial\theta$ is the usual generator of rotation of $S^{n-1} \subset \mathbb{R}^n$ around $\mathbb{R}^{n-2} \subset \mathbb{R}^n$. Now a flow line of a Killing field is a geodesic iff it passes through a critical point of $\mathscr{g}(\xi,\xi)$; indeed, Killing's equation $\nabla_{(a}\xi_{b)} = 0$ tells us that

$$(\nabla_\xi\xi)_b = \xi^a\nabla_a\xi_b = -\xi^a\nabla_b\xi_a = -\frac{1}{2}\nabla_b\xi^a\xi_a = -\frac{1}{2}\nabla_b\mathscr{g}(\xi,\xi),$$

and the claim therefore follows from the fact that $\mathscr{g}(\xi,\xi)$ is constant along the flow. However, restricting to the great circle in S^{n-1} where $\partial/\partial\theta$ has maximal length,

$$\mathscr{g}(\xi,\xi) = -\left(1 - \frac{A}{\varrho^{n-2}}\right) + \omega^2\varrho^2\ ,$$

and the critical-point condition then reduces to

$$0 = \frac{d}{d\varrho}\left[-\left(1-\frac{A}{\varrho^{n-2}}\right)+\omega^2\varrho^2\right] = -\frac{(n-2)A}{\varrho^{n-1}} + 2\omega^2\varrho\ .$$

Thus, when

$$\omega^2 = \frac{(n-2)}{2}\frac{A}{\varrho^n}$$

a flow-line is a space-time geodesic, and represents a test particle moving in a circular orbit at constant angular velocity ω. Comparison with circular orbits in the Newtonian model discussed above therefore leads us to interpret the Schwarzschild metric as representing the gravitational field of an object of mass

$$\mathcal{M} = \frac{A}{2}.$$

The spatial slice $t = 0$ of the Schwarzschild metric

$$g = \left(1-\frac{A}{\varrho^{n-2}}\right)^{-1} d\varrho^2 + \varrho^2\mathfrak{h}$$

is totally geodesic, and provides the prototype for defining the mass of an ALE manifold. If we interpret ϱ as the Euclidean radius in $\mathbb{R}^n$, this metric takes the form

$$g_{jk} = \delta_{jk} + \frac{A}{\varrho^n}x_jx_k + O(\frac{1}{\varrho^{n-1}})$$

so that

$$g_{jk,\ell} = \frac{A}{\varrho^n}(\delta_{j\ell}x_k + \delta_{k\ell}x_j - n\frac{x_jx_kx_\ell}{\varrho^2}) + O(\frac{1}{\varrho^n})$$

Thus

$$g_{ij,i} - g_{ii,j} = \frac{A}{\varrho^n}(\delta_{ji}x_i + \delta_{ii}x_j - \delta_{ij}x_i - \delta_{ij}x_i) + O(\frac{1}{\varrho^n}) = (n-1)\frac{A}{\varrho^{n-1}}\nu_j + O(\frac{1}{\varrho^n})$$

and

$$\begin{aligned}\lim_{\varrho\to\infty}\int_{S_\rho}[g_{ij,i} - g_{ii,j}]\,\mathbf{n}^j\,d\mathfrak{a}_E &= (n-1)A\,\mathrm{Vol}(S^{n-1})\\ &= 2(n-1)\mathcal{M}\,\mathrm{Vol}(S^{n-1})\\ &= \frac{4(n-1)\pi^{n/2}}{\mathbf{\Gamma}(\frac{n}{2})}\mathcal{M}\end{aligned}$$

Thus, defining the mass of an n-dimensional ALE manifold (at a given end) to be

$$m(M, g) := \lim_{\varrho \to \infty} \frac{\mathbf{\Gamma}(\frac{n}{2})}{4(n-1)\pi^{n/2}} \int_{S_\rho/\Gamma_j} [g_{ij,i} - g_{ii,j}]\, \mathbf{n}^j d\mathfrak{a}_E$$

will result in a mass of $m = \mathcal{M}$ for the $t = 0$ spatial slice of the Schwarzschild metric. In particular, when $n = 3$, the normalizing constant simplifies to $1/16\pi$, which is the well-established value found throughout the literature.

Acknowledgments: The authors would like to thank the Isaac Newton Institute, Cambridge, for its hospitality during the writing of this article. They would also like to thank Claudio Arezzo, Ronan Conlan, Michael Eastwood, Gary Gibbons, Denny Hill, Gustav Holzegel, John Lott, Rafe Mazzeo, Bianca Santoro, Cristiano Spotti, Ioana Şuvaina, and the anonymous referees for many useful comments and suggestions.

References

[1] V. Alexeev, *Classification of log canonical surface singularities: arithmetical proof*, in Flips and Abundance for Algebraic Threefolds, Société Mathématique de France, Paris, 1992, pp. 47–58. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[2] A. Andreotti and H. Grauert, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France, 90 (1962), pp. 193–259.

[3] C. Arezzo, R. Lena, and L. Mazzieri, *On the resolution of extremal and constant scalar curvature Kaehler orbifolds*, Int. Math. Res. Not. IMRN, (2016). First published online December 19, 2015, doi:10.1093/imrn/mv346.

[4] R. Arnowitt, S. Deser, and C. W. Misner, *Coordinate invariance and energy expressions in general relativity.*, Phys. Rev. (2), 122 (1961), pp. 997–1006.

[5] W. L. Baily, *On the imbedding of V-manifolds in projective space*, Amer. J. Math., 79 (1957), pp. 403–430.

[6] W. Barth, C. Peters, and A. Van de Ven, *Compact Complex Surfaces*, vol. 4 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1984.

[7] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math., 39 (1986), pp. 661–693.

[8] A. L. Besse, *Einstein Manifolds*, vol. 10 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1987.

[9] H. L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom., 59 (2001), pp. 177–267.

[10] D. R. Brill, *On the positive definite mass of the Bondi-Weber-Wheeler time-symmetric gravitational waves*, Ann. Physics, 7 (1959), pp. 466–483.

[11] E. Calabi, *Extremal Kähler metrics*, in Seminar on Differential Geometry, vol. 102 of Ann. Math. Studies, Princeton Univ. Press, Princeton, N.J., 1982, pp. 259–290.

[12] D. M. J. Calderbank and M. A. Singer, *Einstein metrics and complex singularities*, Invent. Math., 156 (2004), pp. 405–443.

[13] G. Carron, *Cohomologie L^2 et parabolicité*, J. Geom. Anal., 15 (2005), pp. 391–404.

[14] X. X. Chen, C. LeBrun, and B. Weber, *On conformally Kähler, Einstein manifolds*, J. Amer. Math. Soc., 21 (2008), pp. 1137–1168.

[15] P. Chruściel, *Boundary conditions at spatial infinity from a Hamiltonian point of view*, in Topological Properties and Global Structure of Space-Time (Erice, 1985), vol. 138 of NATO Adv. Sci. Inst. Ser. B Phys., Plenum, New York, 1986, pp. 49–59. Digitized version available at `http://homepage.univie.ac.at/piotr.chrusciel/scans/index.html`.

[16] M. Commichau and H. Grauert, *Das formale Prinzip für kompakte komplexe Untermannigfaltigkeiten mit 1-positivem Normalenbündel*, in Recent Developments in Several Complex Variables, vol. 100 of Ann. of Math. Stud., Princeton Univ. Press, Princeton, N.J., 1981, pp. 101–126.

[17] R. J. Conlon and H.-J. Hein, *Asymptotically conical Calabi-Yau manifolds, III.* e-print arXiv:1405.7140 [math.DG], 2014.

[18] M. Eastwood and C. LeBrun, *Thickening and supersymmetric extensions of complex manifolds*, Amer. J. Math., 108 (1986), pp. 1177–1192.

[19] G. W. Gibbons and S. W. Hawking, *Classification of gravitational instanton symmetries*, Comm. Math. Phys., 66 (1979), pp. 291–310.

[20] H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann., 146 (1962), pp. 331–368.

[21] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[22] M. Haskins, H.-J. Hein, and J. Nordström, *Asymptotically cylindrical Calabi-Yau manifolds*, J. Differential Geom., 101 (2015), pp. 213–265.

[23] T. Hausel, E. Hunsicker, and R. Mazzeo, *Hodge cohomology of gravitational instantons*, Duke Math. J., 122 (2004), pp. 485–548.

[24] C. D. Hill and M. Taylor, *Integrability of rough almost complex structures*, J. Geom. Anal., 13 (2003), pp. 163–172.

[25] A. Hirschowitz, *On the convergence of formal equivalence between embeddings*, Ann. of Math. (2), 113 (1981), pp. 501–514.

[26] N. J. Hitchin, *Complex manifolds and Einstein's equations*, in Twistor Geometry and Nonlinear Systems (Primorsko, 1980), Springer, 1982, pp. 73–99.

[27] N. Honda, *Scalar flat Kähler metrics on affine bundles over* $\mathbb{CP}^1$, SIGMA. Symmetry Integrability Geom. Methods Appl., 10 (2014), pp. 1–25, Paper 046.

[28] G. Huisken and T. Ilmanen, *The Riemannian Penrose inequality*, Internat. Math. Res. Notices, (1997), pp. 1045–1058.

[29] D. Huybrechts, *Complex Geometry. An Introduction*, Universitext, Springer-Verlag, Berlin, 2005.

[30] D. D. Joyce, *Compact Manifolds with Special Holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[31] K. Kodaira, *A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds*, Ann. of Math. (2), 75 (1962), pp. 146–162.

[32] C. LeBrun, *Spaces of Complex Geodesics and Related Structures*, PhD thesis, Oxford University, 1980. Digitized version available at `http://ora.ox.ac.uk/objects/uuid:e29dd99c-0437-4956-8280-89dda76fa3f8`.

[33] ——, *Counter-examples to the generalized positive action conjecture*, Comm. Math. Phys., 118 (1988), pp. 591–596.

[34] ——, *Explicit self-dual metrics on* $\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2$, J. Differential Geom., 34 (1991), pp. 223–253.

[35] ——, *Twistors, Kähler manifolds, and bimeromorphic geometry. I*, J. Amer. Math. Soc., 5 (1992), pp. 289–316.

[36] C. LeBrun and B. Maskit, *On optimal 4-dimensional metrics*, J. Geom. Anal., 18 (2008), pp. 537–564.

[37] C. LeBrun and Y. S. Poon, *Self-dual manifolds with symmetry*, in Differential geometry: geometry in mathematical physics and related topics (Los Angeles, CA, 1990), vol. 54 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 1993, pp. 365–377.

[38] J. Lee and T. Parker, *The Yamabe problem*, Bull. Am. Math. Soc., 17 (1987), pp. 37–91.

[39] M. Lock and J. Viaclovsky, *A smörgåsbord of scalar-flat Kähler ALE surfaces*. e-print arXiv:1410.6461 [math.DG], 2014.

[40] B. Malgrange, *Sur l'intégrabilité des structures presque-complexes*, in Symposia Mathematica, Vol. II (INDAM, Rome, 1968), Academic Press, London, 1969, pp. 289–296.

[41] S. Marshall, *Deformations of Special Lagrangian Submanifolds*, PhD thesis, Oxford University, 2002. Digitized version available at `http://people.maths.ox.ac.uk/joyce/theses/theses.html`.

[42] A. Nijenhuis and W. B. Woolf, *Some integration problems in almost-complex and complex manifolds.*, Ann. of Math. (2), 77 (1963), pp. 424–489.

[43] R. Penrose, *Naked singularities*, Ann. New York Acad. Sci., 224 (1973), pp. 125–134. Sixth Texas Symposium on Relativistic Astrophysics.

[44] Y. Rollin and M. Singer, *Constant scalar curvature Kähler surfaces and parabolic polystability*, J. Geom. Anal., 19 (2009), pp. 107–136.

[45] H. Rossi, *Vector fields on analytic spaces*, Ann. of Math. (2), 78 (1963), pp. 455–467.

[46] R. M. Schoen and S. T. Yau, *Complete manifolds with nonnegative scalar curvature and the positive action conjecture in general relativity*, Proc. Nat. Acad. Sci. U.S.A., 76 (1979), pp. 1024–1025.

[47] J. Schouten and D. Struik, *Einführung in die neueren Methoden der Differentialgeometrie, vol. 2*, P. Noordhoff, Groningen, 1938.

[48] J. A. Schouten, *Ricci-Calculus. An Introduction to Tensor Analysis and its Geometrical Applications*, Springer-Verlag, Berlin, 1954. 2nd ed.

[49] B. Weber, *First Betti numbers of Kähler manifolds with weakly pseudoconvex boundary.* e-print arXiv:1110.4571 [math.DG], 2011.

[50] H. Weyl, *Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung*, Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl., 1921 (1921), pp. 99–112.

[51] E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys., 80 (1981), pp. 381–402.

Department of Mathematics, University of Maryland, College Park, MD 20742-4015, USA

Department of Mathematics, State University of New York, Stony Brook, NY 11794-3651, USA