Radial vector fields

Definition 1.

We set

$$\vec{r} \qquad \mathbf{r} = \langle x, y \rangle$$

Then

General definition: A radial vector field is of the form

$$\mathbf{F} = f(x,y)\,\mathbf{r}, \quad \text{with} \quad f(x,y) \in \mathbb{R}$$

Fields of special interest:

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p}$$

Normal and tangent vectors (1) $C: x^2+y^2=a^2$

Situation: We consider

- Function $g(x,y)=x^2+y^2$
- Circle $C:\{(x,y);\, g(x,y)=a^2\}$ → radius a
- Field $\mathbf{F}=\frac{\mathbf{r}}{|\mathbf{r}|}$

Problem: For $(x,y)\in C$, prove that

$$\mathbf{F}(x,y)\perp \text{tangent line to } C \text{ at } (x,y)$$

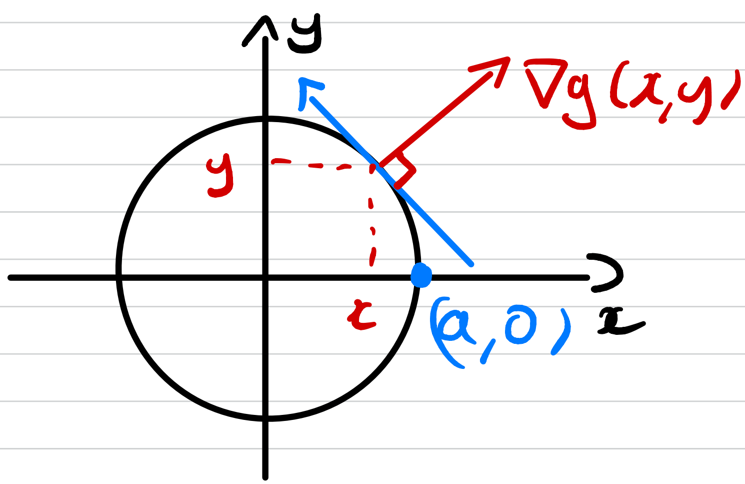
$\Leftrightarrow$ $\vec{F}(x,y)$ parallel to $\nabla g(x,y)$
for each $(x,y)\in C$

$\nabla g = \langle g_x, g_y \rangle$

Function $\quad g(x,y) = x^2+y^2 = z$

Curve : Level curve for g $\quad (z = a^2)$

we get the circle $C: \ x^2+y^2 = a^2$



Claim : $\vec{F}(x,y) \perp$ tgt line

Since $\nabla g \perp$ tgt line,
this will happen iff
$\vec{F}(x,y) \parallel \nabla g(x,y)$
to be checked

Gradient

$$\nabla g(x,y) = \langle g_x, g_y \rangle = \langle 2x, 2y \rangle = 2\overbrace{\langle x,y \rangle}^{=\vec{r}}$$

$$\nabla g(x,y) = 2\vec{r}$$

Comparison with $\vec{F}$

We have found $\nabla g(x,y) = 2\vec{r}$

We have $\vec{F}(x,y) = \frac{1}{|\vec{r}|}\vec{r}$

Therefore

$$\vec{F}(x,y) = \frac{1}{2|\vec{r}|} \times 2\vec{r}$$

$$\vec{F}(x,y) = \underbrace{\frac{1}{2|\vec{r}|}}_{b \in \mathbb{R}} \nabla g(x,y)$$

Rmk If $(x,y) \in C$, $|\vec{r}| = a$

$\Rightarrow b = \frac{1}{2a}$

Since $\vec{F}(x,y) = b\,\nabla g(x,y)$ with $b \in \mathbb{R}$, we have $\vec{F}(x,y) \parallel \nabla g$

Thus $\boxed{\vec{F}(x,y) \perp \text{tgt line at } (x,y) \in C}$

Normal and tangent vectors (2)

Recall: From level curves considerations, we have

$$\nabla g(x, y) \perp \text{tangent line to } C \text{ at } (x, y)$$

Computing the gradient: We get

$$\langle 2x, 2y \rangle \perp \text{tangent line to } C \text{ at } (x, y)$$

Conclusion: Since $\langle 2x, 2y \rangle = 2\mathbf{r}$, we end up with

$$\mathbf{r} \perp \text{tangent line to } C \text{ at } (x, y)$$

Vector field in $\mathbb{R}^3$

Definition of vector fields in $\mathbb{R}^3$:

- Of the form $\mathbf{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z)\rangle$
- For each (x, y, z), $\mathbf{F} \in \mathbb{R}^3$, namely $\mathbf{F}$ is a vector

Radial vector fields: Of the form

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x, y, z\rangle}{|\mathbf{r}|^p}$$

Example of vector field in $\mathbb{R}^3$ (1)

Definition of the vector field:

$$\mathbf{F}(x, y, z) = \left\langle x, y, e^{-z} \right\rangle$$

Problem:
Give a representation of $\mathbf{F}$

Example of vector field in $\mathbb{R}^3$ (2)

Hyp $z \geq 0$

Vector field $\vec{n} \in \mathbb{R}^2$

if $z=0$, then

decreases exp. fast

Recall:

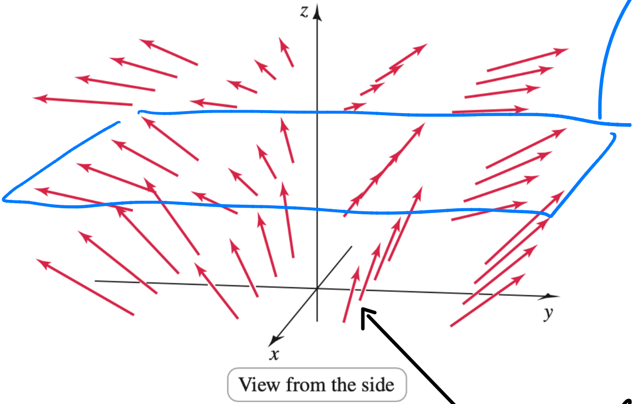
$$\mathbf{F}(x,y) = \langle \boxed{x, y}, \boxed{e^{-z}} \rangle$$

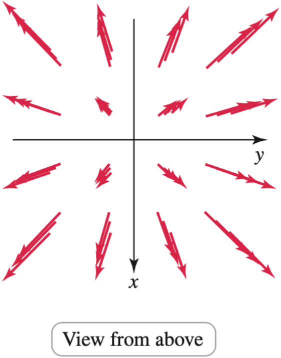
Information about the vector field:

1. xy-trace: $\mathbf{F} = \langle x, y, 1 \rangle$
   $\hookrightarrow$ Radial in the plane, with component 1 in vertical direction
2. In horizontal plane $z = z_0$: $\mathbf{F} = \langle x, y, e^{-z_0} \rangle$
   $\hookrightarrow$ Radial in the plane, with smaller component in vert. direction
3. As $z \to \infty$: $\mathbf{F} \to \langle x, y, 0 \rangle$
   $\hookrightarrow$ Radial in the plane, with 0 component in vertical direction
4. Magnitude increases as we move away from vertical axis

Example of vector field in $\mathbb{R}^3$ (3)

In $z = z_0$





In xy-plane
$\langle x, y, 1\rangle$

Outline

Motivation

Work of a force

$W = \vec{F} \cdot \text{displacement} \quad (\text{in } \mathbb{R}^3)$

This formula works if the displacement is linear.
What happens if we move along a curve?

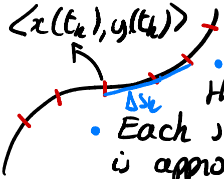
Physical situation: Assume we want to compute

- Work of gravitational field **F**
- Along the (curved) path C of a satellite

Needed quantity: integral of **F** along C
$\hookrightarrow$ How to compute that?

Rmk We start with $F = f$ (scalar function)
Then we will look at $\vec{F}$ vector field

Approximation procedure



Idea:

- Discretize the curve C.
- Each small piece is approximated by the tgt line.

Notation: We consider

- Curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$
- Partition $a = t_0 < \cdots < t_n = b$ of time interval $[a, b]$
- Arc length s of $\mathbf{r}$
- Function f defined on $\mathbb{R}^2$

- On each small piece, compute work as usual

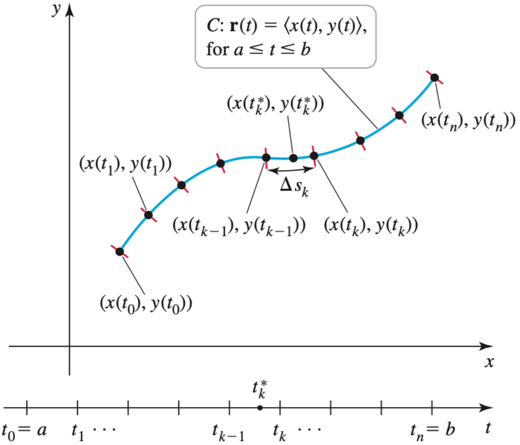
Approximation:

$$S_n = \sum_{k=1}^{n} f\left(x(t_k), y(t_k)\right) \Delta s_k$$

Approximation procedure: illustration

Recall:

$$S_n = \sum_{k=1}^{n} f\left(x(t_k), y(t_k)\right) \Delta s_k$$



Computation of line integrals in $\mathbb{R}^2$

Theorem 2.

We consider

- Curve C defined by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$
- Time interval $[a, b]$
- Arc length s of $\mathbf{r}$
- Function f defined on $\mathbb{R}^2$

Then we have

$$\int_C f \, \mathrm{d}s = \int_a^b f\left(x(t), y(t)\right) |\mathbf{r}'(t)| \mathrm{d}t$$

Computation of line integrals

$$s = \int_0^t |\vec{r}'(u)|\, du$$
$$ds = |\vec{r}'(t)|\, dt$$

Recipe:

1. Find parametric description of C
   $\hookrightarrow \mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $t \in [a, b]$
2. Compute $|\mathbf{r}'(t)| = \sqrt{x^2(t) + y^2(t)}$
3. Make substitutions for x and y and evaluate ordinary integral

$$\int_a^b f\left(x(t), y(t)\right) |\mathbf{r}'(t)| \mathrm{d}t$$

Average temperature (1)

Situation:

- Circular plate

$$R = \left\{x^2 + y^2 = 1\right\}$$

- Temperature distribution in the plane:

$$T(x, y) = 100\left(x^2 + 2y^2\right)$$

Problem:

Compute the average temperature on the edge of the plate

Curve   Circle : $\langle \cos(t), \sin(t) \rangle = \vec{r}(t)$

$0 \leq t \leq 2\pi$

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

Average temp :

$|\vec{r}\,'(t)| = \sin^2(t) + \cos^2(t) = 1$

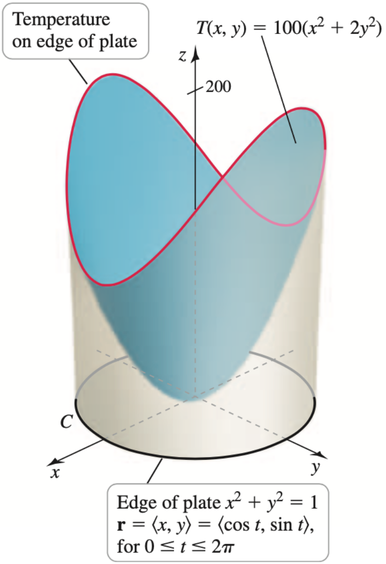
$$\bar{T} = \underbrace{\frac{1}{2\pi}}_{\text{length of } C} \int_C f(x,y)\, ds$$

$$\overset{\text{Thm 2}}{=} \frac{1}{2\pi} \int_0^{2\pi} 100 \left(\cos^2(t) + 2\sin^2(t) \right) \times 1 \, dt$$

$$= \frac{50}{\pi} \int_0^{2\pi} \left(1 + \underbrace{\sin^2(t)}_{\frac{1}{2}(1-\cos(2t))} \right) dt$$

$$\boxed{\bar{T} = 150}$$

Average temperature (2)



Average temperature (3)

Parametric description of C: $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$

Arc length: $|\mathbf{r}'(t)| = 1$

Line integral:

$$\begin{aligned}
\int_C T(x,y)\,\mathrm{d}s &= 100 \int_0^{2\pi} \left(x(t)^2 + 2y(t)^2 \right) |\mathbf{r}'(t)|\,\mathrm{d}t \\
&= 100 \int_0^{2\pi} \left(\cos^2(t) + 2\sin^2(t) \right)\,\mathrm{d}t \\
&= 100 \int_0^{2\pi} \left(1 + \sin^2(t) \right)\,\mathrm{d}t
\end{aligned}$$

Thus

$$\int_C T(x,y)\,\mathrm{d}s = 300\pi$$

Average temperature (4)

Recall:

$$\int_C T(x,y)\,\mathrm{d}s = 300\pi$$

Average temperature: Given by

$$\overline{T} = \frac{\int_C T(x,y)\,\mathrm{d}s}{\text{Length}(C)}$$

We get

$$\overline{T} = \frac{300\pi}{2\pi} = 150$$