- Lagrange multiplier (with multiple constraints)

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(a) = c \end{cases} \rightsquigarrow \begin{cases} \nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_k \nabla g_k \\ g_i(a) = c_i \end{cases}.$$

- Implicit function theorem

$$x^2+y^2+z^2=2 \rightarrow z = z(x,y), \quad y = y(x,z), \; x = x(y,z)?$$

eg Given $\begin{cases} F_1(x,y,z)=c_1 \\ F_2(x,y,z)=c_2 \end{cases}$. Suppose (a,b,c)
$F_i(a,b,c)=c_i$ for $i=1,2$

Do there exist differentiable functions $y=y(x)$, $z=z(x)$ near (a,b,c) s.t. $\begin{cases} F_1(x,y(x),z(x))=c_1 \\ F_2(x,y(x),z(x))=c_2 \end{cases}$?

If such functions exist, by taking implicit differentiation,

$$F_1: \quad \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial F_1}{\partial z}\cdot\frac{dz}{dx} = 0$$

$$F_2: \quad \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial F_2}{\partial z}\cdot\frac{dz}{dx} = 0.$$

$$\Rightarrow \begin{pmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{pmatrix} = \begin{pmatrix} -\frac{\partial F_1}{\partial x} \\ -\frac{\partial F_2}{\partial x} \end{pmatrix}$$

If this linear system not solvable, $y=y(x), z=z(x)$ ONE.

If it has a solution (e.g. $\begin{pmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{pmatrix}$ is invertible) $\quad$ $y=y(x)$, $z=z(x)$ $\quad$ map exist.

Generally, given $n+k$ variables $(x_1 \cdots x_n, y_1 \cdots y_k)$

k equations $(F_1 \cdots F_k)$

$$\begin{cases} F_1(x_1 \cdots x_n, y_1 \cdots y_k) = c_1 \\ \quad \vdots \\ F_k(x_1 \cdots x_n, y_1 \cdots y_k) = c_k \end{cases}$$

when $y_1 \cdots y_k$ can be expressed as functions of $x_1 \cdots x_n$ locally?

Thm (Implicit function theorem)

Let $\Omega \subseteq \mathbb{R}^{n+k}$ open. $F: \Omega \to \mathbb{R}^k$ be C^1

Denote $x = (x_1 \cdots x_n)$ $y = (y_1 \cdots y_k)$

$$F(x,y) = \begin{pmatrix} F_1(x,y) \\ \vdots \\ F_k(x,y) \end{pmatrix}.$$

Suppose $(a,b) \in \Omega$ $\left(\begin{matrix} a \in \mathbb{R}^n \\ b \in \mathbb{R}^k \end{matrix} \right)$

s.t. $F(a,b) = c \in \mathbb{R}^k$ and the matrix

$$\left[\frac{\partial F_i}{\partial y_j}(a,b) \right]_{\substack{1 \le i \le k \\ 1 \le i \le k}} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(a,b) & \cdots & \frac{\partial F_1}{\partial y_k}(a,b) \\ \vdots & & \\ \frac{\partial F_k}{\partial y_1}(a,b) & \cdots & \frac{\partial F_k}{\partial y_k}(a,b) \end{pmatrix}_{k \times k}$$

is invertible.

Then there are open set $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^k$ containing a, b

s.t. there exists a unique function $\varphi: U \to V$

with $\varphi(a) = b$ and $F(x, \varphi(x)) = c$ for all $x \in U$.

Moreover, φ is C^1.

By implicit differentiation, $\varphi = (\varphi_1 \cdots \varphi_k)$

$$\frac{\partial}{\partial x_i} : \quad F(x, \varphi(x)) = c$$

$$\underset{k\times k}{\left[\frac{\partial F_i}{\partial y_j}(x, \varphi(x))\right]_{\substack{1\le i\le k\\1\le j\le k}}} \underset{k\times n}{\left(\frac{\partial \varphi_j}{\partial x_l}(x,\varphi(x))\right)_{\substack{1\le j\le k\\1\le l\le n}}} = -\underset{k\times n}{\left(\frac{\partial F_i}{\partial x_l}(x,\varphi(x))\right)_{\substack{1\le i\le k\\1\le l\le n}}}$$

$$\underset{k\times n}{\left(\frac{\partial \varphi_j}{\partial x_l}(x,\varphi(x))\right)} = \underset{k\times k}{\left[\frac{\partial F_i}{\partial y_j}(x,\varphi(x))\right]^{-1}} \underset{k\times n}{\left(-\frac{\partial F_i}{\partial x_l}(x,\varphi(x))\right)}$$

Special case when $k=1$.

<u>eg</u> (last class) $x^2+y^2+z^2=2$. Can we solve $z = z(x,y)$ near $(0,1,1)$?

$$F: \mathbb{R}^{2+1} \longrightarrow \mathbb{R}^1 \qquad F(x,y,z) = x^2+y^2+z^2$$

$$a = (0,1) \in \mathbb{R}^2 \quad b = 1 \in \mathbb{R}^1 \quad c = F(a,b) = 2$$

$$\frac{\partial F}{\partial z}(0,1,1) = 2z\Big|_{(0,1,1)} = 2 \neq 0$$

By IFT, $\exists U \subseteq \mathbb{R}^2$ containing $a = (0,1)$
$\exists V \subseteq \mathbb{R}$ " $b = 1$.

s.t. $\exists$ a unique function $z : U \to V$
$(x,y) \mapsto z(x,y)$

with $z(0,1) = 1$ and $F(x,y,z(x,y)) = 2$
for all $x \in U$. Also $z(x,y)$ C^1.
(We know it is $z = \sqrt{2-x^2-y^2}$)

More generally, $F : \Omega \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^1$

$$F(x,y) = F(x_1 \cdots x_n, y)$$

Suppose $a = (a_1 \cdots a_n) \in \mathbb{R}^n$, $b \in \mathbb{R}^1$, $F(a,b) = c$

By IFT, if $\frac{\partial F}{\partial y}(a,b) \neq 0$, then

$\exists$ a function $y = y(x_1 \cdots x_n)$ near $(a_1 \cdots a_n)$

solving $\begin{cases} F(x_1 \cdots x_n, y) = c \\ y(a_1 \cdots a_n) = b \end{cases}$ locally.

Rmk $\frac{\partial y}{\partial x_1} \cdots \frac{\partial y}{\partial x_n}$ can be computed by implicit differentiation

Special case $k=2$, $n=1$.

eg (last class) $\begin{cases} x^2+y^2+z^2=2 \\ x+z=1 \end{cases}$ can we solve $y=y(x)$, $z=z(x)$? near $(0.1.1)$

$F: \mathbb{R}^{\overset{n}{1}+\overset{k}{2}} \longrightarrow \mathbb{R}^{\overset{k}{2}}$ $\qquad F(x.y.z)=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$F(x.y.z)=\begin{pmatrix} x^2+y^2+z^2 \\ x+z \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{pmatrix}_{(0.1.1)} = \begin{pmatrix} 2y & 2z \\ 0 & 1 \end{pmatrix}\Bigg|_{(0.1.1)} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

$\therefore$ invertible.

By IFT, $\exists\, y=y(x)$, $z=z(x)$ near $(0.1.1)$

s.t. $\begin{cases} F_1(x, y(x), z(x)) = 2 \\ F_2(\quad \text{''} \quad) = 1 \\ y(0)=1 \\ z(0)=1 \end{cases}$

More generally, $F: \Omega \subseteq \mathbb{R}^{1+2} \to \mathbb{R}^2$

$$F(x, y_1, y_2) = \begin{pmatrix} F_1(x, y_1, y_2) \\ F_2(x, y_1, y_2) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Suppose $(a, b_1, b_2) \in \Omega$ satisfie $F(a, b_1, b_2) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

By IFT, if $\left.\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix}\right|_{(a, b_1, b_2)}$ is invertible,

then $\exists\, y_1 = y_1(x),\ y_2 = y_2(x)$ near a

solving the constraints. $\begin{cases} F_i(x, y_1(x), y_2(x)) = c_i \\ y_i(a) = b_i \end{cases}$ $i = 1, 2$

locally.

<u>Rmk</u> We write $F(x, y)$ and solve y as a function of x, but the ordering of the variables is not important.

<u>eg</u> Consider the constraints $\begin{cases} F_1 = xz + \sin(yz - x^2) = 8 \\ F_2 = x + 4y + 3z = 18 \end{cases}$

$\{ x = x(y)$? $\{ y = y(x)$? $\{ x = x(z)$ ...

$\left\{ z = z(y) \right.$, $\left\{ z = z(x) \right.$, $\left\{ y = y(z) \right.$

$\begin{pmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \end{pmatrix}$ $\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial t} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}$

Near $(2.1.4)$, can we solve 2 of the variables as functions of the remaining variable

(sol) If we want to check $\begin{cases} x = x(y) \\ z = z(y) \end{cases}$ possible,

$$\left.\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial z} \end{pmatrix}\right|_{(2.1.4)}$$

$$= \left.\begin{pmatrix} z - 2x\cos(yz - x^2) & x + y\cos(yz - x^2) \\ 1 & 3 \end{pmatrix}\right|_{(2.1.4)}$$

$$= \begin{pmatrix} 4 - 2\cdot 2\cos 0 & 2 + \cos 0 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} \quad \text{Invertible}$$

$\therefore$ $x = x(y)$, $z = z(y)$ locally near $(2.1.4)$

Similar computation; IFT says x, y can be expressed as functions on z.

Is $y=y(x)$, $z=z(x)$ possible?

$\det\begin{vmatrix} 4 & 3 \\ 4 & 3 \end{vmatrix} = 0$. In this case, we can't determine if y, z can be expressed as functions on x near $(2,1,4)$ from IFT.

<u>Rmk</u> If $y=y(x)$, $z=z(x)$ near $(2,1,4)$ exist and differentiable, by taking implicit differentiation

$$\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} dy/dx \\ dz/dx \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$ This linear system has no solution. $\Rightarrow$ contradiction.

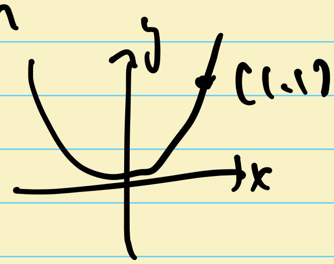
<u>Rmk</u> Implicit function theorem has many important application, such as rigorous proof of

① Implicit differentiation
② Tangent plane of surface $F(x,y,z)=c$
③ Lagrange multiplier

Inverse function theorem

eg $y = f(x) = x^2$

$f : \mathbb{R} \to \mathbb{R}$



Near $(1,1)$ x is a function of y $(x = \sqrt{y})$

$$\left.\frac{dx}{dy}\right|_{(1,1)} = \left.\frac{1}{2\sqrt{y}}\right|_{(1,1)} = \frac{1}{2}$$

$$= \frac{1}{\left.\left(\frac{dy}{dx}\right)\right|_{(1,1)}}$$

$f : \mathbb{R}^n \to \mathbb{R}^n$

Thm (Inverse function theorem)

$\Omega \subseteq \mathbb{R}^n$ a open subset. $f : \Omega \to \mathbb{R}^n$ be C^1.

$f(a) = b$.

If $Df(a)$ is an invertible matrix,

then there are open sets $U \subseteq \mathbb{R}^n$ containing a
$V \subseteq \mathbb{R}^n$ " b

s.t. there exist a unique function $g : V \to U$

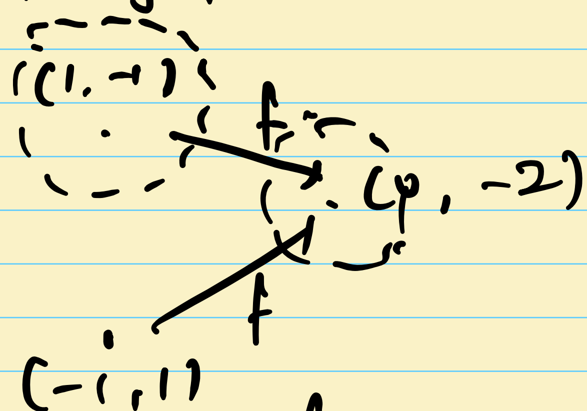
with $g(b) = a$, $\begin{cases} g(f(x)) = x & \forall x \in U \\ f(g(y)) = y & \forall y \in V. \end{cases}$

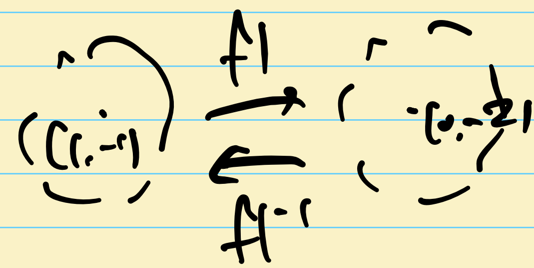
Moreover, g is C^1 and $Dg(y) = Df(g(y))^{-1}$ $\forall y \in V$.

Rmk In fact, Inverse function theorem is equivalent to Implicit function theorem.

eg $f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(x,y) = (x^2-y^2, 2xy)$

Since $f(x,y) = f(-x,-y)$ hence f is not injective and it has no global inverse.





$$\mathbb{R}^2 \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{R}^2$$

$$Df(x,y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$ and $\det Df(x,y) = 4x^2+4y^2 > 0$ for $(x,y) \neq (0,0)$.

By Inverse function theorem, f is locally invertible with differentiable local inverse e.g. let $g(u,v)$ be a local inverse of $f(x,y)$ near $(1,-1)$. Then $g(0,-2) = (1,-1)$

$$Dg(0,-2) = Df(1,-1)^{-1} = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Rmk In fact, we can find $g(u,v)$ explicitly.

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$ Near $(1,-1)$, $x \neq 0$

hence $y = \frac{v}{2x}$

$$\Rightarrow u = x^2 - \left(\frac{v}{2x}\right)^2$$

$$\Rightarrow 4x^4 - 4ux^2 - v^2 = 0$$

$$\Rightarrow x^2 = \frac{4u \pm \sqrt{(-4u)^2 - 4\cdot 4\cdot(-v^2)}}{8}$$

$$= \frac{u \pm \sqrt{u^2+v^2}}{2}$$

Put $(x,y) = (1,-1)$ and $(u,v) = (0,-2)$

$$\Rightarrow 1^2 = \frac{0 \pm \sqrt{0^2+2^2}}{2}$$ + correct sign $\Rightarrow x^2 = \frac{u + \sqrt{u^2+v^2}}{2}$

$$\Rightarrow x = \sqrt{\frac{u+\sqrt{u^2+v^2}}{2}}, \quad y = \frac{v}{2x} = \frac{2\sqrt{2}\,v}{\sqrt{u+\sqrt{u^2+v^2}}}$$

$$\therefore g(u,v) = \left(\sqrt{\frac{u+\sqrt{u^2+v^2}}{2}}, \frac{2\sqrt{2}\,v}{\sqrt{u+\sqrt{u^2+v^2}}} \right)$$

Rmk In both Implicit & Inverse function theorems, we assume a Jacobian matrix to be invertible. Without this assumption, the theorems are inconclusive.

Exercise Consider a set of points S $(x,y,u,v) \in \mathbb{R}^4$ satisfying

$$\begin{cases} xu + yvu^2 = 2 \\ xu^3 + y^2v^4 = 2 \end{cases}$$

(a) Show that near $(1,1,1,1)$ there exist a differentiable function $(u,v) = f(x,y)$ s.t. S is a graph of f.

(b) Compute $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ at $(1,1,1,1)$.

(sol) (a) $F(x,y,u,v) = (F_1, F_2) = (xu + yvu^2,\ xu^3 + y^2v^4)$

$F(1,1,1,1) = (2,2)$

$$\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix}\Bigg|_{(1,1,1,1)} = \begin{pmatrix} x + 2yvu & yu^2 \\ 3xu^2 & 4y^2v^3 \end{pmatrix}\Bigg|_{(1,1,1,1)}$$

$$= \begin{bmatrix} 3 & 1 \\ 3 & 4 \end{bmatrix}$$

$\det\begin{pmatrix} 3 & 1 \\ 3 & 4 \end{pmatrix} \neq 0$. Hence by Implicit function theorem, there exist a differentiable $f(x,y) = (u,v)$ near $(1,1;1,1)$ s.t. $f(1,1) = (1,1)$ and $F(x,y,f(x,y)) = (2,2)$

(b) Taking implicit differentiation $\frac{\partial}{\partial x}$

$$u + x\frac{\partial u}{\partial x} + yu^2\frac{\partial v}{\partial x} + 2yvu\frac{\partial u}{\partial x} = 0$$

$$u^3 + 3xu^2\frac{\partial u}{\partial x} + y^2(4v^3)\frac{\partial v}{\partial x} = 0$$

at $(1.1.1.1)$

$$\Rightarrow \begin{cases} 1+3\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial x}=0 \\ 1+3\dfrac{\partial u}{\partial x}+4\dfrac{\partial v}{\partial x}=0 \end{cases}$$

$$\begin{pmatrix} 3 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} \qquad \square$$

$$xu+yvu^2=2 \quad \overset{\frac{\partial}{\partial x}}{\Rightarrow} \quad \frac{\partial (xu)}{\partial x} = u + x\frac{\partial u}{\partial x}$$

$$\frac{\partial (yvu^2)}{\partial x} = y\cdot\frac{\partial v}{\partial x}\cdot u^2 + yv\cdot\frac{\partial u^2}{\partial x}$$

$$= yu^2\frac{\partial v}{\partial x} + yv\cdot 2u\cdot\frac{\partial u}{\partial x}$$