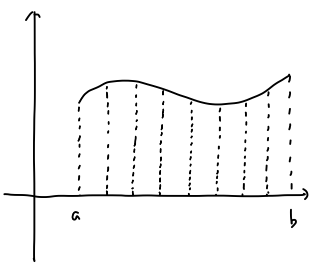
Tu 5/23 9:00 – 10:30, 10:30 – 10:40 break, 10:40 – 12:10

Double integral over rectangles & general regions.

- Review of definite integrals.

Given a continuous function $f: [a, b] \to \mathbb{R}$,



we divide the interval $[a, b]$ into n-parts

Δ_i = the i-th subinterval, has length $\frac{b-a}{n}$

choose an arbitrary $x_i^* \in \Delta_i$, the form the "Riemann Sum"

$$\sum_{i=1}^{n} f(x_i^*) \cdot |\Delta_i|$$

when $n \to +\infty$, the following limit exists regardless of the choice of x_i^*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot |\Delta_i|$$

we denote this by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot |\Delta_i|$$

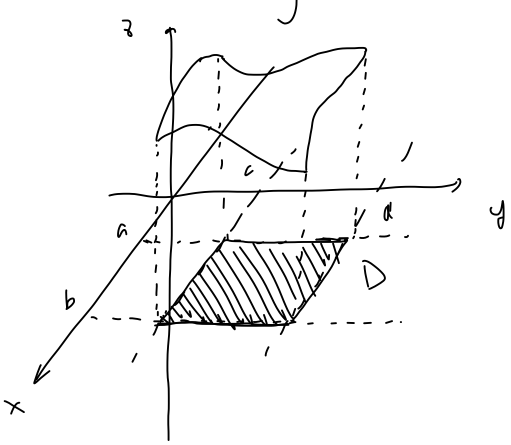
the integration of f over the interval $[a, b]$ 1-dim'l

which is also called the "area" of f

- Double integrals and Volumes

Now we want to generalize this to higher dimensional, we first consider the case 1-dim'l higher, i.e., we want to integrate over a rectangle 2-dim'l

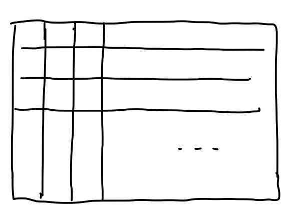
Given a rectangle $D = [a, b] \times [c, d] \subset \mathbb{R}^2$, and a function f on D, i.e., $f: D \to \mathbb{R}$



we divide D into smaller rectangles:

divide $[a, b]$ into m parts,

divide $[c, d]$ into n parts.



Δ_{ij} : the subrectangle formed by the i-th & j-th subinterval

Choose an arbitrary point $(x_i^*, y_j^*) \in \Delta_{ij}$, then we form the following "Riemann Sum"

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \, |\Delta_{ij}|$$

We say f is integrable if when $n, m \to +\infty$, the following limit exists regardless of the choice of (x_i^*, y_j^*)

$$\lim_{m,n\to+\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \, |\Delta_{ij}|$$

We denote it by

$$\iint_D f(x,y)\, dA = \lim_{m,n\to+\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \, |\Delta_{ij}|$$

Here we use two integral symbols to emphasize that we are integrating over a <u>2-dim'l region</u>
we call it the double integral of f over D

Example:

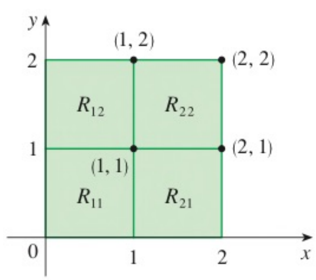
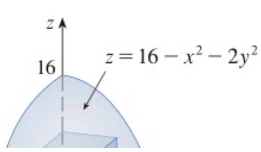


FIGURE 6



EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

SOLUTION The squares are shown in Figure 6. The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$. Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$\begin{aligned} V &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A \\ &= f(1, 1)\,\Delta A + f(1, 2)\,\Delta A + f(2, 1)\,\Delta A + f(2, 2)\,\Delta A \\ &= 13(1) + 7(1) + 10(1) + 4(1) = 34 \end{aligned}$$

This is the volume of the approximating rectangular boxes shown in Figure 7. ■

If we choose $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$

then $V \approx f(0,0)\,\Delta A + f(0,1)\,\Delta A + f(1,0)\,\Delta A + f(1,1)\,\Delta A$

$= 16 + 14 + 15 + 13 = 58$

If we choose $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{2})$, $(\frac{3}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{3}{2})$,

then $V \approx f(\frac{1}{2}, \frac{1}{2})\,\Delta A + f(\frac{1}{2}, \frac{3}{2})\,\Delta A + f(\frac{3}{2}, \frac{1}{2})\,\Delta A + f(\frac{3}{2}, \frac{3}{2})\,\Delta A$

$= 64 - 15 = 49$

· Iterated integrals

Goal: Find a computable way to calculate double integrals

It is difficult to calculate integral by Riemann sums, because taking limits means that you have to do calculations infinitely many times

Recall our situation: $R = [a,b] \times [c,d]$, $f: D \to \mathbb{R}$ is a continuous function
we want to know

$$\iint_R f(x,y)\, dA$$

First, fix x, consider the following integral

$$\int_c^d f(x,y)\, dy$$

this is a function in $x \in [a,b]$, we denote it by $A(x) = \int_c^d f(x,y)\, dy$
then we consider

$$\int_a^b A(x)\, dx = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

or we can first fix y, and consider the following integral

$$\int_a^b f(x,y)\, dx$$

this is a function in $y \in [c,d]$, we denote it by $B(y) = \int_a^b f(x,y)\, dx$
then we consider

$$\int_c^d B(y)\, dy = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

Both are called iterated integral

Natural question: does these two values agree? what's their relation to $\iint_R f(x,y)\, dA$?

10 Fubini's Theorem If f is continuous on the rectangle

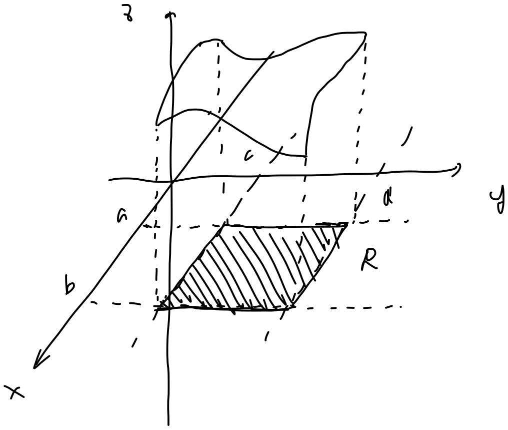
$$R = \{(x,y) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$$

then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Geometric Intuition



Volumes are "sum" of all the areas of x-section/y-section

Examples:

- $$\int_0^1 \int_0^2 x^2 y^3 \, dx \, dy$$

$$= \int_0^1 y^3 \frac{x^3}{3}\Big|_0^2 \, dy = \frac{8}{3}\int_0^1 y^3 \, dy = \frac{8}{3} \times \frac{y^4}{4}\Big|_0^1 = \frac{2}{3}$$

$$\int_0^2 \int_0^1 x^2 y^3 \, dy \, dx$$

$$= \int_0^2 x^2 \frac{1}{4} \, dx = \frac{1}{4} \times \frac{x^3}{3}\Big|_0^2 = \frac{2}{3}$$

- $$\int_a^b \int_c^d xy \, dx \, dy \qquad \int_c^d \int_a^b xy \, dy \, dx$$

$$= \int_a^b \frac{d^2-c^2}{2} y \, dy = \frac{(b^2-a^2)(d^2-c^2)}{4} = \int_c^d \frac{b^2-a^2}{2} x \, dx$$

- **EXAMPLE 5** Evaluate the double integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$. (Compare with Example 3.)

SOLUTION 1 Fubini's Theorem gives

$$\iint_R (x - 3y^2)\, dA = \int_0^2 \int_1^2 (x - 3y^2)\, dy\, dx = \int_0^2 \Big[xy - y^3\Big]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\, dx = \frac{x^2}{2} - 7x \Big]_0^2 = -12$$

SOLUTION 2 Again applying Fubini's Theorem, but this time integrating with respect to x first, we have

$$\iint_R (x - 3y^2)\, dA = \int_1^2 \int_0^2 (x - 3y^2)\, dx\, dy = \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2 - 6y^2)\, dy = 2y - 2y^3\Big]_1^2 = -12$$ ■

Integration order is important !!

It can happen that for some double integral, it's easier to integrate x or y first but turns out to be very difficult if you start from the other one

EXAMPLE 6 Evaluate $\iint_R y \sin(xy)\, dA$, where $R = [1, 2] \times [0, \pi]$.

SOLUTION If we first integrate with respect to x, we get

$$\begin{aligned}
\iint_R y \sin(xy)\, dA &= \int_0^\pi \int_1^2 y \sin(xy)\, dx\, dy \\
&= \int_0^\pi y \left[-\frac{1}{y} \cos(xy) \right]_{x=1}^{x=2} dy \\
&= \int_0^\pi (-\cos 2y + \cos y)\, dy \\
&= -\tfrac{1}{2} \sin 2y + \sin y \Big]_0^\pi = 0
\end{aligned}$$

NOTE In Example 6, if we reverse the order of integration and first integrate with respect to y, we get

$$\iint_R y \sin(xy)\, dA = \int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx$$

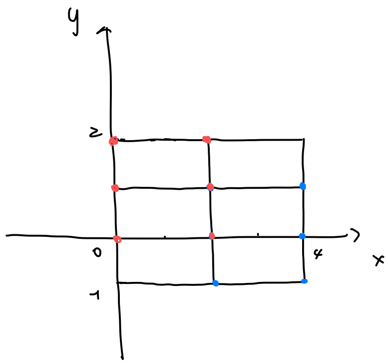
but this order of integration is much more difficult than the method given in the example because it involves integration by parts twice. Therefore, when we evaluate double integrals it is wise to choose the order of integration that gives simpler integrals.

special case: $f(x,y) = g(x)h(y)$ factorizable, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d g(x)h(y)\, dx\, dy = \left(\int_a^b g(x)\, dx\right) \cdot \left(\int_c^d h(y)\, dy\right)$$

Exercise

2.



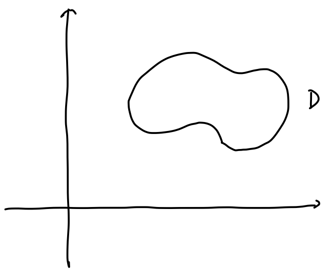
16. $\int_0^1 \int_0^1 (x+y)^2 \, dx \, dy$

$$= \int_0^1 \frac{(x+y)^3}{3} \Big|_0^1 dy = \int_0^1 \left(\frac{(y+1)^3}{3} - \frac{y^3}{3} \right) dy = \frac{(y+1)^4}{12} - \frac{y^4}{12} \Big|_0^1 = \frac{5}{4}$$

32. $\iint_R \frac{x}{1+xy} \, dA \qquad [0,1] \times [0,1]$

· Double integral over general regions

In general, a 2-dim'l region may not be a rectangle, such as the following



if we have a function $f: D \to \mathbb{R}$

how do we integrate it ?

Geometric intuition: Volume of a solid over D

When D is bounded, we can find a rectangle $R = [a,b] \times [c,d]$ containing it, then consider the following function

$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D; \\ 0, & (x,y) \notin D. \end{cases}$$

then define

$$\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA,$$ the latter is the integral we have studied before

Question: Does this definition depend on the choice of R containing D?

Ans: When f is continuous, No.

· Some specific types of the region

Type I:

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x,y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a,b]$. Some examples of type I regions are shown in Figure 5.

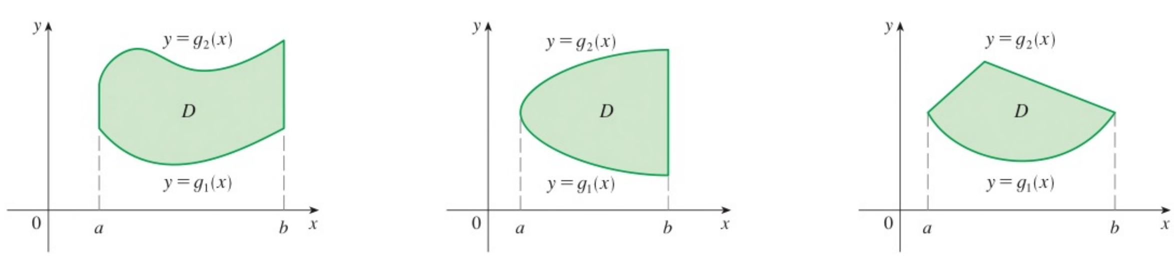


FIGURE 5
Some type I regions

NOTE For a type I region, the functions g_1 and g_2 must be continuous but they do not need to be defined by a single formula. For instance, in the third region of Figure 5, g_2 is a continuous piecewise defined function.

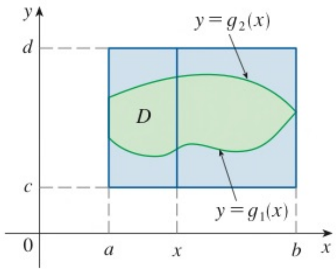


FIGURE 6

In order to evaluate $\iint_D f(x, y)\, dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as in Figure 6, and we let F be the function given by Equation 1; that is, F agrees with f on D and F is 0 outside D. Then, by Fubini's Theorem,

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Observe that $F(x, y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x, y) then lies outside D. Therefore

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \leqslant y \leqslant g_2(x)$. Thus we have the following formula that enables us to evaluate the double integral as an iterated integral.

3 If f is continuous on a type I region D described by

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then $$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

The integral on the right side of (3) is an iterated integral that is similar to the ones we considered in Section 15.1, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

We also consider plane regions of **type II**, which can be expressed as

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Three such regions are illustrated in Figure 7.

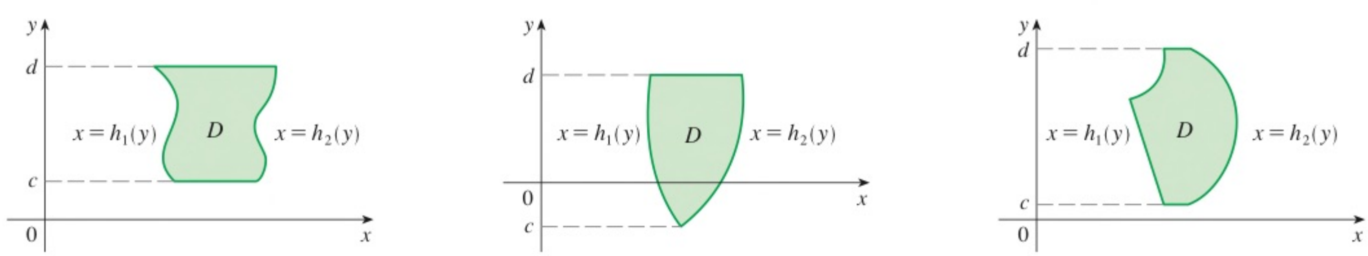


FIGURE 7
Some type II regions

Using the same methods that were used in establishing (3), we can show that the following result holds.

4 If f is continuous on a type II region D described by

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

then $$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Examples for Type I

EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

SOLUTION The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region D, sketched in Figure 8, is a type I region but not a type II region and we can write

$$D = \{(x, y) \mid -1 \leqslant x \leqslant 1,\ 2x^2 \leqslant y \leqslant 1 + x^2\}$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \Big[xy + y^2\Big]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Bigg]_{-1}^{1} = \frac{32}{15}$$

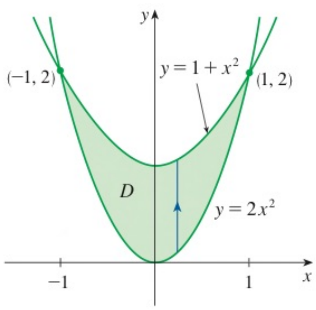


FIGURE 8

Step 1: find intersection points & sketch the graph

$2x^2 = 1 + x^2 \Rightarrow x = \pm 1$

Step 2: Evaluate the integral by the graph

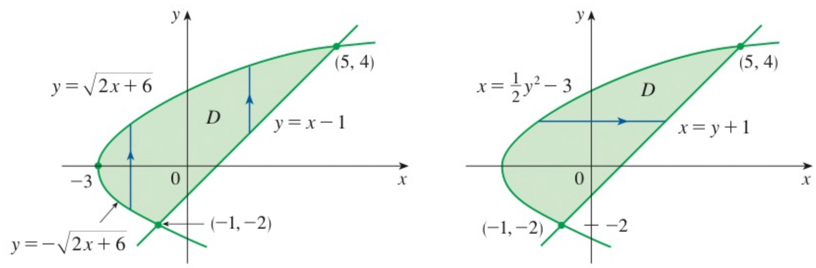
For a fixed x-point, you got two points
For a fixed y-point, you got two points } compare

Example for type II

EXAMPLE 3 Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION The region D is shown in Figure 12. Again D is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

$$D = \left\{(x, y) \mid -2 \leqslant y \leqslant 4, \tfrac{1}{2}y^2 - 3 \leqslant x \leqslant y + 1\right\}$$



Sometimes it is both type I & II, but you can chose the simpler one

FIGURE 12

(a) D as a type I region

(b) D as a type II region

Then (4) gives

$$\iint_D xy\, dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy\, dx\, dy = \int_{-2}^{4} \left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

$$= \tfrac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - \left(\tfrac{1}{2}y^2 - 3\right)^2\right] dy$$

$$= \tfrac{1}{2}\int_{-2}^{4} \left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy$$

$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36$$

In Example 3, if we had expressed D as a type I region using Figure 12(a), then the lower boundary curve would be

$$g_1(x) = \begin{cases} -\sqrt{2x+6} & \text{if } -3 \leqslant x \leqslant -1 \\ x - 1 & \text{if } -1 < x \leqslant 5 \end{cases}$$

and we would have obtained

$$\iint_D xy\, dA = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\, dy\, dx + \int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} xy\, dy\, dx$$

$x + 2y + z = 2$

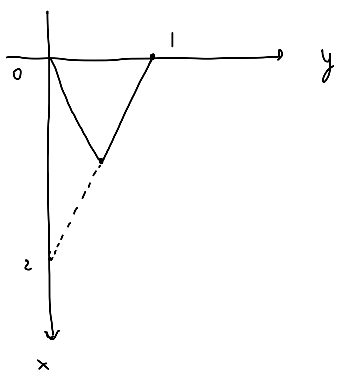
which would have involved more work than the other method.

Examples where both type I & II work

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

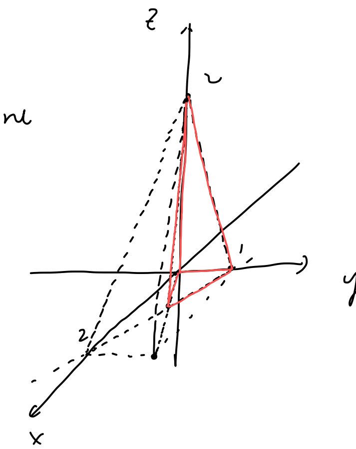
- Sketh the graph:
  - Sketch the area on the xy-plane, yz-plane, xz-plane

in the xy-plane



height

$z = 2 - x - 2y$



xy: $x + 2y = 2$

xz: $x + z = 2$

yz: $2y + z = 2$

type I: upper bound: $y = 1 - \frac{x}{2}$, lower bound: $y = \frac{x}{2}$

then

$$\text{Vol} = \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} (2 - x - 2y)\, dy\, dx$$

$$= \int_0^1 \left(2y - xy - y^2\right)\Big|_{\frac{x}{2}}^{1-\frac{x}{2}} dx = \int_0^1 \left(1 - 2x + x^2\right) dx = \frac{1}{3}$$

type II: upper bound $x = \begin{cases} 2y, & 0 \le y \le \frac{1}{2} \\ 2 - 2y, & \frac{1}{2} \le y \le 1 \end{cases}$. lower bound $x = 0$

$$\text{Vol} = \int_0^{\frac{1}{2}} \int_0^{2y} (2 - x - 2y)\, dx\, dy + \int_{\frac{1}{2}}^1 \int_0^{2-2y} (2 - x - 2y)\, dx\, dy$$

$$= \int_0^{\frac{1}{2}} \left(2x - \frac{1}{2}x^2 - 2xy\right)\Big|_0^{2y} dy + \int_{\frac{1}{2}}^1 \left(2x - \frac{1}{2}x^2 - 2xy\right)\Big|_0^{2-2y} dy$$

$$= \int_0^{\frac{1}{2}} 4y - 6y^2\, dy + \int_{\frac{1}{2}}^1 2 - 4y + 2y^2\, dy$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{12} = \frac{1}{3}$$

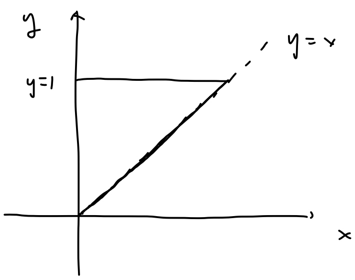
· Changing the order of integration

EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

first try: $\int \sin(y^2)\, dy$ ⇝ we don't know how to integrate it

that means we have to change the order of integration and give another try

Now we sketch the integration area



for a fixed x, y ranges from x to 1

If we integrate from x first, then

for a fixed y, x ranges from 0 to y, then

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \int_0^1 \int_0^y \sin(y^2)\, dx\, dy$$

$$= \int_0^1 y \sin(y^2)\, dy = -\frac{1}{2}\cos(y^2)\Big|_0^1 = \frac{1}{2} - \frac{1}{2}\cos(1)$$

· Properties of Double integrals

We assume that all of the following integrals exist. For rectangular regions D the first three properties can be proved in the same manner as in Section 5.2. And then for general regions the properties follow from Definition 2.

5 $$\iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$$

6 $$\iint_D cf(x, y)\, dA = c \iint_D f(x, y)\, dA \qquad \text{where } c \text{ is a constant}$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in D, then

7 $$\iint_D f(x, y)\, dA \geqslant \iint_D g(x, y)\, dA$$

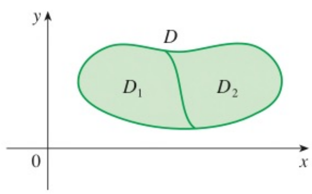


FIGURE 17

The next property of double integrals is similar to the property of single integrals given by the equation $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ (Property 5 in Section 5.2).

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

8 $$\boxed{\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA}$$

Property 8 can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure. (See Exercises 67 and 68.)

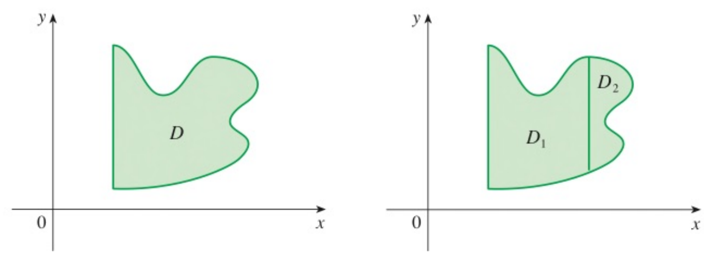


FIGURE 18

(a) D is neither type I nor type II.

(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

9 $$\boxed{\iint_D 1\, dA = A(D)}$$

Figure 19 illustrates why Equation 9 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.

Finally, we can combine Properties 6, 7, and 9 to prove the following property. (See Exercise 73.)

10 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$m \cdot A(D) \leqslant \iint_D f(x, y)\, dA \leqslant M \cdot A(D)$$

EXAMPLE 6 Use Property 10 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

SOLUTION Since $-1 \leqslant \sin x \leqslant 1$ and $-1 \leqslant \cos y \leqslant 1$, we have $-1 \leqslant \sin x \cos y \leqslant 1$ and, because the natural exponential function is increasing, we have

$$e^{-1} \leqslant e^{\sin x \cos y} \leqslant e^{1} = e$$

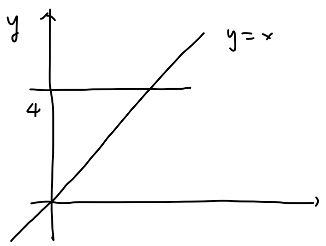
Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$ in Property 10, we obtain

$$\frac{4\pi}{e} \leqslant \iint_D e^{\sin x \cos y}\, dA \leqslant 4\pi e \qquad \blacksquare$$

Exercise.

20. $\displaystyle\iint_D y^2 e^{xy}\, dA$, $\quad D$ is bounded by $y = x$, $y = 4$, $x = 0$

Step 1: Sketch the region



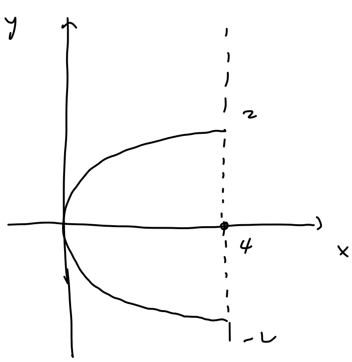
Step 2: use type II. for fixed y, x range .. 0 to y

$$\iint_D y^2 e^{xy}\, dA = \int_0^4 \int_0^y y^2 e^{xy}\, dx\, dy$$

$$= \int_0^4 y e^{xy} \Big|_0^y \, dy = \int_0^4 \left(y e^{y^2} - y\right) dy$$

32. Under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$

Step 1: Sketch the region on xy-plane:



type II: fixed y, x from y^2 to 4

$$V = \int_{-2}^{2} \int_{y^2}^{4} (1 + x^2y^2)\, dx\, dy$$