Using (*) to compute limits:

ex: compute the sequences' limits, or show they diverge:

1) $a_n = \cos(1/n)$
2) $a_n = \tan\left(\pi n / (2n+1)\right)$
3) $a_n = \sin(\pi n / 2)$

note: $\cos(x)$, $\sin(x)$, $\tan(x)$ are cts on their domains.

also: $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, $\lim\limits_{n\to\infty} \frac{\pi n}{2n+1} = \frac{\pi}{2}$,

$\lim\limits_{n\to\infty} \pi n = \infty$

sol'n by (*):

1) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \cos(1/n)$

$\overset{(*)}{=} \cos\left(\lim\limits_{n\to\infty} 1/n\right)$

$= \cos(0) = 1$

2) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \tan\left(\pi n / (2n+1)\right)$

$\to \pi/2$, but tan discontinuous @ $\pi/2$, so (*) doesn't apply can still evaluate by inspection:

$\lim_{n\to\infty} \tan\left(\pi n/2n+1\right)$

approaches $\pi/2$ from the left

approaches ∞

3) (*) doesn't apply since $\lim_{n\to\infty} \pi n = \infty$

evaluate by inspection:

$$\begin{aligned}
a_1 &= \sin(\pi/2) = 1 \\
a_2 &= \sin(2\pi/2) = 0 \\
a_3 &= \sin(3\pi/2) = -1 \\
a_4 &= \sin(4\pi/2) = 0 \\
a_5 &= 1 \\
a_6 &= 0 \\
&\vdots
\end{aligned}$$

Sequence oscillates $\Rightarrow$ limit DNE.

Theorem (Squeeze thm)

if a_n, b_n, c_n are sequences s.t. for every n,

$$a_n \le b_n \le c_n$$

then:

if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$

then $\lim\limits_{n\to\infty} b_n = L$ as well.

ex: Find $\lim\limits_{n\to\infty} \dfrac{\cos(e^n)}{n}$

sol'n: Observe: $-1 \le \cos(e^n) \le 1$ for every n

hence $\dfrac{-1}{n} \le \dfrac{\cos(e^n)}{n} \le \dfrac{1}{n}$ " "

and since $\lim\limits_{n\to\infty} -\dfrac{1}{n} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$

by squeeze theorem:

$$\lim_{n\to\infty} \frac{\cos(e^n)}{n} = 0, \text{ as well.}$$

The sequence $a_n = r^n$

if $r=2$: $a_n = 2^n$: $2, 4, 8, 16, \ldots$

$\lim_{n\to\infty} 2^n = \infty$, diverges.

if $r=1$: $a_n = 1^n = 1$: $1, 1, 1, \ldots$

$\lim_{n\to\infty} 1^n = 1$, converges

if $r=\frac{1}{2}$: $a_n = (\frac{1}{2})^n$: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$\lim_{n\to\infty} (\frac{1}{2})^n = 0$, converges

if $r=-1$: $a_n = (-1)^n$: $-1, 1, -1, 1, \ldots$

$\lim_{n\to\infty} (-1)^n$ DNE, diverges.

In general:

- if $-1<r<1$, $\lim_{n\to\infty} r^n = 0$
- if $r=1$, $\lim_{n\to\infty} r^n = 1$
- diverges otherwise.

Monotone Convergence:

Def'n: – a sequence is *increasing* if $a_{n+1} > a_n$ for every n.

(ex: $1/2, 2/3, 3/4, \ldots, n/n+1, \ldots$)

– " " *decreasing* if $a_{n+1} < a_n$ " "

(ex: $1/2, 1/3, 1/4, \ldots, 1/n, \ldots$)

– a sequence is *monotone* if it is either increasing or decreasing.

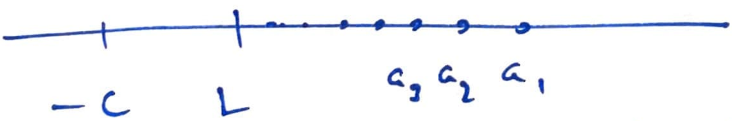
– " " is *bounded* if there is a constant C s.t. $|a_n| \leq C$ for every n.

(ex: if $a_n = (-1/2)^n$: $-1/2, 1/4, -1/8, 1/16, \ldots$ then $|a_n| \leq 1$ for every n; say: sequence is bounded by 1)

Theorem (Monotone convergence)
Suppose that the sequence a_n is monotone and bounded. Then the sequence converges (i.e. $\lim_{n\to\infty} a_n$ exists).

Picture:

must have a limit L
(might be $< C$)



must have limit L
(might be $> -C$)

Theorem is useful: allows us to apply lim laws to recursively defined sequences.

Recursively Defined Sequences

ex: Define a sequence by

$$a_1 = 1 \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

so:

$$a_2 = 3 - \frac{1}{a_1} = 3 - \frac{1}{1} = 2$$

$$a_3 = 3 - \frac{1}{a_2} = 3 - \frac{1}{2} = 2.5$$

$$a_4 = 3 - \frac{1}{2.5} = 13/5 = 2.6$$

$$a_5 = 3 - \frac{1}{2.6} = 34/13 = 2.615$$

$$\vdots$$

- It can be shown (by induction - we won't cover) that a_n is increasing.
- and observe: $|a_n| \le 3$ for every n.
- so: by monotone convergence, a_n has a limit.

But what is it?

why? See below

Observe: $\displaystyle \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$

defn

$\displaystyle = \lim_{n\to\infty} 3 - \frac{1}{a_n}$

lim laws

since we know $\displaystyle \lim_{n\to\infty} a_n$ exists

$\displaystyle = 3 - \frac{1}{\lim_{n\to\infty} a_n}$