# Identities and properties for associated Legendre functions

DBW

This note is a personal note with a personal history; it arose out off my incapacity to find references on the internet that prove relations that exist between the associated Legendre functions. The goal is to put notes on the internet that at least contain the 14 recurrence relations and some other identites found on Wikipedia[1], without resorting to advanced methods, such as generating functions; in these notes merely basic analysis and algebraic manipulations are used. Wikipedia cites many references, such as Hilbert and Courant (1953), or Abramowitz and Stegun (1983). But most of these were not at my disposal, and some are very useful mathematical tables, but with very little proofs.

I start with the definition and some basic properties of Legendre polynomials $P_n$, then introduce associated Legendre functions $P_l^m$. Then follows the main text, in which I give proofs of a number of relations among the $P_l^m$. I then consider the number of zeroes of the $P_n$ and $P_l^m$, the values at the endpoints, expansions of $P_l^m$ in terms of $P_l$ and also shortly consider two sets of orthogonal functions for $m = 1$. After all that, I show proofs of some integral relations, which are used on occasion in the main text. On notation: I often omit the argument $x$ of the functions and write d for differentiation with respect to $x$ on many occasions. Finally I give (part of) a screen shot of the Wikipedia-website showing the 14 recurrence relations mentioned above.

I have little hope this text will be free of typos and more serious errors; please write an email if you find one to westradennis at gmail dot com!

## 1 Legendre Polynomials

We define the Legendre polynomials $P_l$ for $l = 0, 1, 2, \dots$ by Rodriguez formula

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l (x^2 - 1)^l\,. \tag{1}$$

By inspection, $P_l$ has degree $l$, and we have $P_l(-x) = (-1)^l P_l(x)$ as the $l$th derivative of an even function has parity $(-1)^l$. The $P_l$ form an orthogonal set of polynomials on $[-1; 1]$

[1] See the section recurrence relations from `https://en.wikipedia.org/wiki/Associated_Legendre_polynomials` .

and proofs of the relation

$$\int_{-1}^{1} P_l(x)P_k(x)\mathrm{d}x = \frac{\delta_{k,l}}{2l+1}\,, \tag{2}$$

can be found in many (online) resources, and we present one in section 7. The symbol $\delta_{k,l}$ is the Kronecker delta, which equals 1 if $k = l$ and zero otherwise.

Our first relation needs the orthogonality relation and some hard work, although a proof using the generating function is faster (but then, one first has to show the generating function has the right properties).

> *The Legendre polynomials satisfy the following recurrence relation*
>
> $$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{3}$$

*Proof:* Consider the polynomial $xP_n(x)$. It has degree $n+1$ and is thus in the linear span of $P_0, \ldots, P_{n+1}$. We can hence write $xP_n(x)$ as a linear combination of the first $n+2$ Legendre polynomials and the $k$th Legendre polynomial appears with coefficient

$$a_k = \frac{2k+1}{2}\int_{-1}^{1} xP_n(x)P_k(x)\mathrm{d}x\,.$$

We are interested in integrals of $xP_r(x)P_s(x)$ for general $r$ and $s$. In section 7, we show that these integrals vanish unless $r = s \pm 1$ and for this case, we can use

$$\int_{-1}^{1} xP_r(x)P_{r-1}(x)\mathrm{d}x = \frac{2r}{(2r-1)(2r+1)}\,.$$

Writing $xP_n(x) = \alpha P_{n+1}(x) + \beta P_{n-1}(x)$, and first integrating the product with $P_{n+1}$ we find $\alpha = \frac{n+1}{2n+1}$ and similarly $\beta = \frac{n}{2n+1}$. Hence

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)\,.$$

This is what we wanted to prove.

> *The difference $P_{l+1} - P_{l-1}$ satsifies the following differential relation:*
>
> $$(2l+1)P_l(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(P_{l+1}(x) - P_{l-1}(x)\Big)\,. \tag{4}$$

*Proof:* The second derivative of $(x^2-1)^{l+1}$ equals

$$\begin{aligned}\mathrm{d}^2(x^2-1)^{l+1} &= 2(l+1)(x^2-1)^l + 4l(l+1)x^2(x^2-1)^{l-1} \\ &= 2(2l+1)(l+1)(x^2-1)^l + 4l(l+1)(x^2-1)^{l-1}\,.\end{aligned} \tag{5}$$

Acting on this result with $\frac{1}{2^{l+1}(l+1)!}\mathrm{d}^l$ gives the required result.

| $l$ | $P_l(x)$ |
|---|---|
| 0 | $1$ |
| 1 | $x$ |
| 2 | $\frac{1}{2}(3x^2-1)$ |
| 3 | $\frac{1}{2}(5x^3-3x)$ |
| 4 | $\frac{1}{8}(35x^4-30x^2+3)$ |
| 5 | $\frac{1}{8}(63x^5-70x^3+15x)$ |

## 2 Associated Legendre Functions

We define the associated Legendre functions $P_l^m$ for $-l \le m \le l$ by

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^{l+m}(x^2-1)^l\,. \tag{6}$$

One immediately sees that $P_l^0 = P_l$ and that for $m \ge 0$ we have

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^m P_l(x)\,. \tag{7}$$

The functions $P_l^m$ are polynomials of degree $l$ for $m$ even and if $m$ is odd, then $P_l^m$ is $\sqrt{1-x^2}$ times a polynomial of degre $l-1$. The parity of $P_l^m$ is $(-1)^{l+m}$. If one tries to apply the defining formula (6) for $m > l$ one finds zero.

*We have:*

$$\sqrt{1-x^2}P_l^m(x) = \frac{1}{2l+1}\Big(P_{l-1}^{m+1} - P_{l+1}^{m+1}\Big)\,, \qquad m \ge 0\,. \tag{8}$$

*Proof:* We act on identity (4) with $(-1)^m(1-x^2)^{(m+1)/2}\mathrm{d}^m$ and the result follows. Later, when we have shown (21) we will see that the condition $m \ge 0$ can be relaxed.

The relation (8) is the sixth on the list of recurrence relations of Wikipedia.

*For negative $m$ we can use the following relation between $P_l^m$ and $P_l^{-m}$:*

$$P_l^{-m} = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m\,. \tag{9}$$

*Proof:* By induction one easily shows that for $k = 0,1,2,3,\ldots$ and some differentiable functions $f$ and $g$

$$d^k(fg) = \sum_{r=0}^{k}\binom{k}{r}d^r(f)\cdot d^{k-r}(g)\,.$$

We consider $f = x - 1$ and $g = x + 1$, so that and for $p = 0, 1, 2, 3, \ldots$

$$d(f^p) = pf^{p-1}\,, \qquad d^k(f^r) = (r)_k f^{r-k}$$

where $(r)_k$ is defined to be 0 if $r < k$ and $(r)_k = \frac{r!}{(r-k)!}$ for $r \geq k$.

We now show that

$$d^{l-m}(f^l g^l) = (fg)^m \frac{(l-m)!}{(l+m)!} d^{l+m}(f^l g^l)\,.$$

This will prove the claim about the associated Legendre polynomials.

To show the intermediate statement, we first rewrite

$$d^{l+m}(f^l g^l) = \sum_{r=0}^{l+m} \binom{l+m}{r} d^r(f^l) \cdot d^{l+m-r}(g^l) = \sum_{r=m}^{l} \binom{l+m}{r} d^r(f^l) \cdot d^{l+m-r}(g^l)$$

since the omitted terms in the sum vanish; $d^r(f^s) = 0$ if $r > s$. Then we can write put in the expressions for $d^r(f^l)$ and for $d^{l+m-r}(g^l)$, and change the summation variable to $s = r - m$ to obtain

$$d^{l+m}(f^l g^l) = \sum_{s=0}^{l-m} \frac{l!^2 (l+m)!}{(l-s-m)! s! (s+m)! (l-s)!} f^{l-s-m} g^s\,. \qquad (10)$$

Second, we rewrite the other term $d^{l-m}(f^l g^l)$ out as

$$d^{l-m}(f^l g^l) = \sum_{r=0}^{l-m} \frac{(l-m)!}{r!(l-m-r)!} \frac{l!}{(l-r)!} f^{l-r} \frac{l!}{(l-(l-m-r))!} g^{l-(l-m-r)}$$

which by inspection is proportional to $(fg)^m$ times (10), and the constant of proportionality can be read off to be $(-1)^m \frac{(l-m)!}{(l+m)!}$. This proves that what we set out to show.

Equation (9) makes it clear why $P_l^m = 0$ if $|m| > l$.

*The following relation is the first of the 14 recurrence relations listed by Wikipedia:*

$$(2l+1)xP_l^m = (l-m+1)P_{l+1}^m + (l+m)P_{l-1}^m\,. \qquad (11)$$

*Proof:* First we remark that it suffices to proof (11) for nonnegative $m$: If (11) holds for positive $m$, then one can insert the relation $P_l^m = \frac{(l+m)!}{(l-m)!}(-1)^m P_l^{-m}$ and one shows that it follows that $(2l+1)xP_l^{-m} = (l+m+1)P_{l+1}^{-m} + (l-m)P_{l-1}^{-m}$, which is precisely (11) subject to the substitution $m \mapsto -m$.

Differentiating relation (3) $m$ times and multiplying with $(-1)^m(1-x^2)^{m/2}$ we find for nonnegative $m$ the relation

$$(l+1)P_{l+1}^m(x) = (2l+1)xP_l^m(x) - m(2l+1)(1-x^2)^{1/2}P_l^{m-1}(x) - lP_{l-1}^m(x)\,. \qquad (12)$$

Combining eqns.(12) and (8) one obtains the result.

The following two results are rather cheap.

*If we differentiate the defining relation (6) we obtain:*

$$\mathrm{d}P_l^m(x) = -\frac{mx}{1-x^2}P_l^m(x) - \frac{P_l^{m+1}(x)}{\sqrt{1-x^2}}\,. \qquad (13)$$

Now it is a simple step for the following.

*Multiplying (13) with $-(1-x^2)$ one obtains nr. 13 of the list of recurrence relations from Wikipedia:*

$$(x^2-1)\mathrm{d}P_l^m(x) = mxP_l^m(x) + \sqrt{1-x^2}P_l^{m+1}\,. \qquad (14)$$

*The following relation gives an easy way to let $l$ go one up:*

$$P_{l+1}^m(x) = xP_l^m(x) - (l+m)\sqrt{1-x^2}P_l^{m-1}(x)\,. \qquad (15)$$

*Proof:* We consider

$$\mathrm{d}^{l+m+1}(x^2-1)^{l+1} = \mathrm{d}^{l+m}\Big(2(l+1)x(x^2-1)^l\Big)\,,$$

and use the binomial relation for derivatives $\mathrm{d}^r(fg) = \sum\binom{r}{k}\mathrm{d}^k f \mathrm{d}^{r-k}g$ to rewrite the right-hand side as

$$2(l+1)(l+m)\mathrm{d}^{l+m-1}(x^2-1)^l + 2(l+1)x\mathrm{d}^{l+m}(x^2-1)^l\,.$$

Multiplying with $\frac{(-1)^m}{2^{l+1}(l+1)!}(1-x^2)^{m/2}$ one obtains the result.

*The next is the second in the list on Wikipedia:*

$$2mxP_l^m(x) = -\sqrt{1-x^2}\Big(P_l^{m+1}(x) + (l-m+1)(l+m)P_l^{m-1}(x)\Big)\,. \qquad (16)$$

*Proof:* We consider the following identity

$$\mathrm{d}^{l+m+1}(x^2-1)^{l+1} = \mathrm{d}^{l+m+1}\Big((x^2-1)^l(x^2-1)\Big)$$

and use the binomial identity for derivatives rewrite the right-hand side as

$$(x^2-1)\mathrm{d}^{l+m+1}(x^2-1)^l + (l+m+1)2x\mathrm{d}^{l+m}(x^2-1)^l + (l+m)(l+m+1)\mathrm{d}^{l+m-1}(x^2-1)^l\,.$$

Multiplying through by $\frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}$ and regrouping some terms we find

$$2(l+1)P_{l+1}^m - 2x(l+m+1)P_l^m = \sqrt{1-x^2}\Big(P_l^{m+1} - (l+m)(l+m+1)P_l^{m-1}\Big)\,. \quad (17)$$

If we insert relation (15) to eliminate $P_{l+1}^m$ we obtain the (16).

*This identity is the third on the list of Wikipedia:*

$$\frac{1}{\sqrt{1-x^2}}P_l^m(x) = -\frac{1}{2m}\Big(P_{l-1}^{m+1}(x) + (l+m-1)(l+m)P_{l-1}^{m-1}(x)\Big)\,. \quad (18)$$

*Proof:* We start with (16) and substitute $2mxP_l^m = 2mP_{l+1}^m + 2m(l+m)\sqrt{1-x^2}P_l^{m-1}$, which follows from (15). Then it follows that

$$2mP_{l+1}^m = -\sqrt{1-x^2}\Big(P_l^{m+1} + (l+m)(l+m+1)P_l^{m-1}\Big)\,,$$

from which (18) follows immediately if we change $l$ to $l-1$.

The following identity is not on Wikipedia's list, but is definitely usefull to relate some identities. Its proof is rather cumbersome and tedious.

*The following is rather surprising:*

$$P_{l-1}^{m+1} + (l+m)(l+m-1)P_{l-1}^{m-1} = P_{l+1}^{m+1} + (l-m+1)(l-m+2)P_{l+1}^{m-1}\,. \quad (19)$$

Remark: Although at first glance the right-hand side seems to be of degree $l+1$, the highest-order terms cancel, and so is also of degree $l-1$.

*Proof:* If we plug in the definitions of the $P_l^m$, we see that we are to prove that the two terms

$$L = 4l(l+1)\Big((1-x^2)\mathrm{d}^{l+m} + (l+m)(l+m-1)\mathrm{d}^{l+m-2}\Big)(x^2-1)^{l-1}$$

and

$$R = \Big((1-x^2)\mathrm{d}^{l+m+2} + (l-m+1)(l-m+2)\mathrm{d}^{l+m}\Big)(x^2-1)^{l+1}$$

are equal. We distinguish between two cases: (I) $l+m=2n$, (II) $l+m=2n+1$. We first consider case (I), and eliminate all occurrences of $m$.

We expand $(x^2-1)^{l\pm1}$ by the binomial theorem, apply $\mathrm{d}^A x^B = \frac{B!}{(B-A)!}x^{B-A}$ and find that

$$L = \sum_{k=0}^{l-n} 4l(l+1)L_k x^{2k}\,,$$

where

$$L_k = 2(-1)^{l+n+k+1}\binom{l-1}{n+k-1}\frac{(2k+2n-2)!}{(2k)!}\Big((2n{+}2l{-}1)(l{-}n{-}k){+}k(2k{-}1){-}n(2n{-}1)\Big)\,.$$

We also expand $R = \sum_{k=0}^{l-n+1} R_k x^{2k}$ and find

$$R_k = 2(-1)^{l+n+k}\binom{l+1}{n+k1}\frac{(2k+2n)!}{(2k)!}\Big((2n{+}2k{+}1)(l{+}1{-}n{-}k){+}k(2k{-}1){-}(2l{-}2n{+}1)(l{-}n{+}1)\Big)\,.$$

In particular, $R_{l-n+1}=0$. We can write

$$4l(l+1)L_k = \frac{4(l+1)!(2k+2n-2)!}{(n+k)!(l+1-n-k)!(2k)!}(-1)^{l+n+l+1}\cdot 2(n+1)(l+1-n-k)a_k$$

with

$$a_k = (2n+2l-1)(l-n-k) + k(2k-1) - n(2n-1)$$

and similarly

$$R_k = \frac{4(l+1)!(2k+2n-2)!}{(n+k)!(l+1-n-k)!(2k)!}(-1)^{l+n+l+1}(n+k)(2k+2n-1)b_k$$

with

$$b_k = (2l-2n+1)(l-n+1) - (2n+2k+1)(l+1-n-k) - k(2k-1)\,.$$

By inspection, it suffices to show $2a_k(l+1-n-k) = (2k+2n-1)b_k$, which is seen rather easily if we rewrite $a_k$ and $b_k$ as

$$a_k = (l-2n)(2n+2k-1)\,,\quad b_k = 2(l-2n)(l+1-n-k)\,.$$

This shows identity (19) for $l+m$ is even.

For the case, where $l+m$ is odd, we take $l+m=2n+1$, and it boils down to proving that the terms

$$L = \Big((1-x^2)\mathrm{d}^{2n+1} + 2n(2n+1)\mathrm{d}^{2n-1}\Big)(x^2-1)^{l-1}$$

and

$$R = \Big((1-x^2)\mathrm{d}^{2n+3} + (2l-2n)(2l-2n+1)\mathrm{d}^{2n+1}\Big)(x^2-1)^{l+1}$$

are related by $4l(l{+}1)L = R$. Again we write $L = \sum_{k=1}^{l-n} x^{2k-1} L_k$ and $R = \sum_{k=1}^{l-n+1} x^{2k-1} R_k$. We find with some algebra that

$$L_k = 2(-1)^{l+n+k+1} \frac{(l-1)!(2n+2k-2)!}{(n+k-1)!(l-n-k)!(2k-1)!} a_k$$

where $a_k = (l-2n-1)(2n+2k-1)$. This leads to

$$4l(l+1)L_k = 4(-1)^{l+n+k+1} \frac{(l+1)!(2n+2k)!(2n+2k-1)!}{(n+k)!(l-n-k)!(2k-1)!}(l-2n-1)\,.$$

Similar and similarly tedious algebra leads to

$$R_k = 2(-1)^{l+n+k+1} \frac{(l+1)!(2n+2k)!}{(n+k)!(l+1-n-k)!(2k-1)!} b_k$$

with

$$b_k = (l-n)(2l-2n+1) - (2k-1)(k-1) - (2n+2k+1)(l+1-n-k)\,,$$

which can be brought into

$$b_k = 2(l-2n-1)(l+1-n-k)\,.$$

Substituting this expression into $R_k$ shows that $4l(l+1)L_k = R_k$ and the proof is done – indeed, also $R_{l-n+1} = 0$.

*Nr. 4 on Wikipedia's list follows directly from (18) and (19)*

$$\frac{1}{\sqrt{1-x^2}} P_l^m = \frac{-1}{2m}\Big(P_{l+1}^{m+1} + (l-m+1)(l-m+2)P_{l+1}^{m-1}\Big)\,. \tag{20}$$

*Nr. 5 on Wikipedia's with $m \geq 0$ list follows directly from (8) and (19)*

$$\sqrt{1-x^2} P_l^m = \frac{1}{(2l+1)}\Big((l-m+1)(l-m+2)P_{l+1}^{m-1} - (l+m-1)(l+m)P_{l-1}^{m-1}\Big)\,. \tag{21}$$

Let us now show: *Equations (8) and (21) also hold for $m \leq 0$.* By virtue of (9) we substitute $P_l^m = (-1)^m \frac{(l+m)!}{(l-m)!} P_l^{-m}$, $P_{l+1}^{m-1} = (-1)^{m+1} \frac{(l+m)!}{(l-m+2)!} P_{l+1}^{-m+1}$ and $P_{l-1}^{m-1} = (-1)^m \frac{(l+m)!}{(l-m)!} P_{l-1}^{-m+1}$ in (21), which leads directly to $\sqrt{1-x^2} P_l^{-m} = \frac{1}{2l+1}(P_{l-1}^{-m+1} - P_{l+1}^{-m+1})$, which is precisely (8) for negative $m$. The same substitutions in (8) show that (21) also holds for negative $m$.

As nr. 6 was already proven, we now proceed to number 7 on Wikipedia's list.

*The following identity holds*

$$\sqrt{1-x^2}P_l^{m+1} = x(l-m)P_l^m - (l+m)P_{l-1}^m\,. \tag{22}$$

*Proof:* We use (16) to write

$$\sqrt{1-x^2}P_l^{m+1} = -(l+m)(l-m+1)\sqrt{1-x^2}P_l^{m-1} - 2mxP_l^m\,.$$

Then we insert (8) to eliminate $\sqrt{1-x^2}P_l^{m-1}$ and find

$$\sqrt{1-x^2}P_l^{m+1} = -\frac{(l+m)(l-m+1)}{2l+1}P_{l-1}^m + \frac{(l+m)(l-m+1)}{2l+1}P_{l+1}^m - 2mxP_l^m\,.$$

Inserting now $(l-m+1)P_{l+1}^m = (2l+1)xP_l^m - (l+m)P_{l-1}^m$, a.k.a. (11), and doing the algebra then finishes the proof.

*The following identity is nr. 8 on Wikipedia's list:*

$$\sqrt{1-x^2}P_l^{m+1} = (l-m+1)P_{l+1}^m - (l+m+1)xP_l^m\,. \tag{23}$$

*Proof:* We rewrite the right-hand side of (22) as

$$x(l-m)P_l^m - (l+m)P_{l-1}^m = (2l+1)xP_l^m - (l+m+1)xP_l^m - (l+m)P_{l-1}^m\,.$$

The proof is finished by applying (11) to eliminate the term $(2l+1)xP_l^m$.

We now turn to some differential properties of the associated Legendre functions.

*We present nr. 9:*

$$\sqrt{1-x^2}\mathrm{d}P_l^m(x) = -\frac{1}{2}P_l^{m+1}(x) + \frac{1}{2}(l+m)(l+1-m)P_l^{m-1}(x)\,. \tag{24}$$

We can rewrite equation (13) as

$$\sqrt{1-x^2}\mathrm{d}P_l^m(x) = -\frac{mx}{\sqrt{1-x^2}}P_l^m(x) - P_l^{m+1}(x)\,.$$

Using equation (16) and eliminating $mxP_l^m(x)$ leads to the result.

The remaining equations from the list of identities on Wikipedia are then mainly just simple algebra and combining all the other results:

*Number 10:*

$$(1-x^2)\mathrm{d}P_l^m = \frac{1}{2l+1}\Big((l+1)(l+m)P_{l-1}^m - l(l-m+1)P_{l+1}^m\Big) \tag{25}$$

*Proof:* Using eqn.(24) we write the left-hand side of equation to be proven as

$$\frac{1}{2}\Big((l+m)(l-m+1)\sqrt{1-x^2}P_l^{m-1} - \sqrt{1-x^2}P_l^{m+1}\Big)\,.$$

Then we use equation (8) for $\sqrt{1-x^2}P_l^{m-1}$ and equation (21) for $\sqrt{1-x^2}P_l^{m+1}$. This finishes the proof.

*Number 11:*

$$(x^2-1)\mathrm{d}P_l^m = lxP_l^m - (l+m)P_{l-1}^m\,. \tag{26}$$

*Proof:* If combine (11) and (25) this identity follows immediately.

*Number 12:*

$$(x^2-1)\mathrm{d}P_l^m = -(l+1)xP_l^m + (l+1-m)P_{l+1}^m\,. \tag{27}$$

*Proof:* The proof follows if we use $l = (2l+1) - (l+1)$ and write by virtue of (26)

$$(x^2-1)\mathrm{d}P_l^m = (2l+1)xP_l^m - (l+1)xP_l^m - (l+m)P_{l-1}^m$$

and then use relation (11).

As we already had number 13 before, we go on to the last on the list:

*Number 14:*

$$(x^2-1)\mathrm{d}P_l^m = -(l+m)(l-m+1)\sqrt{1-x^2}P_l^{m-1} - mxP_l^m\,. \tag{28}$$

First we write $mxP_l^m = 2mxP_l^m - mxP_l^m$, then we insert (16) and subsequently we add $\sqrt{1-x^2}P_l^{m+1}$, which shows

$$mxP_l^m + \sqrt{1-x^2}P_l^{m+1} = -\sqrt{1-x^2}(l+m)(l-m+1)P_l^{m-1} - mxP_l^m\,,$$

which is the right-hand side of the equation to be proven by virtue of (14).

## 3 On the zeros of (associated) Legendre polynomials

In this section we prove the following claims:

(1) The Legendre polynomials $P_l(x)$ have $l$ distinct simple zeros on the interval $(-1;+1)$;

(2) The associated Legendre functions $P_l^m(x)$ have $l-m$ distinct simple zeros on the interval $(-1;+1)$.

The case $l = 0$ is trivial, since $P_0(x) = 1$. Claim (1) is claim (2) restricted to $m = 0$. We first consider this case. To make the notation easier, we introduce the short-hand

$$\langle f, g\rangle = \int_{-1}^{+1} f(x)g(x)\mathrm{d}x\,.$$

Then the integral of $P_n$ over the interval $[-1; +1]$ equals zero as this integral equals $\langle P_l, 1\rangle = \langle P_l, P_0\rangle = 0$. Thus $P_l$, being continuous, must change sign somewhere on $(-1; +1)$ and thus has at least one zero of odd multiplicity. Let $\xi_1, \ldots, \xi_k$ be all the different zeros of odd multiplicity of $P_l$, so that $1 \leq k \leq l$ and the continuous function $ZP_l$ with $Z = (x - \xi_1)\cdot(x - \xi_2)\cdots(x - \xi_k)$ does not change on $(-1; +1)$. Thus $\langle Z, P_l\rangle \neq 0$.

If $k < l$ then $Z$ is a polynomial of degree smaller than $l$ and thus can be written as a linear combination of $P_0$, $P_1$, $\ldots$, $P_{l-1}$. But then $\langle Z, P_l\rangle$ vanishes, which cannot be, and thus $k = l$. Thus $P_l$ has $l$ distinct zeros.

For the second claim, we need a preliminary result, which is

$$\langle P_l^m, P_{l'}^m\rangle = \int_{-1}^{+1} P_l^m(x)P_{l'}^m\mathrm{d}x = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l,l'} \qquad (29)$$

which is proved in section 7.

Since we may assume $l > 0$ and $m > 0$, we first consider $l = m$. Then $P_m^m$ is proportional to $(1 - x^2)^{m/2}$ and thus has zero zeros on $(-1; +1)$. Thus in this case the claim is true. For the remaining we fix $l > m > 0$. For each $k \geq 0$ the functions $P_k^m$ is of the form $(1 - x^2)^m \cdot p_{k-m}(x)$, where $p_{k-m}$ is a polynomial of degree $k - m$. Thus for given $N > m$ the polynomials $(1 - x^2)^{-m/2} \cdot P_m^m(x)$, $(1 - x^2)^{-m/2} \cdot P_{m+1}^m(x)$, $(1 - x^2)^{-m/2} \cdot P_N^m(x)$ span the space of polynomials with degree at most $N - m$.

Since $\langle P_l^m, P_m^m\rangle = 0$ as $l > m$ and $P_m^m$ does not change sign, so does $P_l^m$ have to change sign. Thus there are $k \geq 1$ zeros $\xi_1$, $\ldots$, $\xi_k$ of $P_l^m$ of odd multiplicity on $(-1; +1)$. We introduce $Z = (1 - x^2)^{m/2} \cdot (x - \xi_1)\cdots(x - \xi_k)$, so that $ZP_l^m$ does not change sign on $(-1; +1)$ and therefore $\langle Z, P_l^m\rangle \neq 0$.

If $l < k$ then $Z$ can be written as a linear sum of less then $l$ associated Legendre functions $P_m^m$, $\ldots P_k^m$, so that in this case we should have $\langle Z, P_l^m\rangle = 0$, which we already argued cannot be. Hence $l = k$ and the claim is proved.

## 4 On the values at the endpoints $x = \pm 1$

From the first view Legendre polynomials, one sees $P_l(1) = 1$. To prove this for all $l$, we use the relation

$$(l + 1)P_{l+1}(x) = (2l + 1)xP_l(x) - lP_{l-1}(x)$$

and find that

$$(l+1)P_{l+1}(1) = (2l+1)P_l(1) - lP_{l-1}(1)\,.$$

Thus by induction, if $P_0(1) = P_1(1) = 1$, then for all $l$ we have $P_l(1) = 1$. Using the parity property of the Legendre polynomials, we find $P_l(-1) = (-1)^l$.

We now turn to the derivatives. From $(2l+1)P_l(x) = \mathrm{d}P_{l+1}(x) - \mathrm{d}P_{l-1}(x)$ we see

$$\mathrm{d}P_{l+1} = (2l+1)P_l + (2l-3)P_{l-2} + (2l-7)P_{l-4} + \dots \tag{30}$$

We now claim that

$$P_l'(\pm 1) = (\pm 1)^{l-1} l(l+1)/2\,.$$

From eqn.(30) we see that $\mathrm{d}P_{l+1} = (2l+1)P_l + \mathrm{d}P_{l-1}$. Also, we have $P_0'(\pm 1) = 0$, $P_1'(\pm 1) = 1$, $P_2'(\pm 1) = 3$, $P_3'(\pm 1) = \pm 6$, so the claim is true for $l = 0, 1, 2, 3$. By parity, we only need to prove the claim $P_l'(1) = l(l+1)/2$. If the claim holds for lower values of $l$, then $P_{l+1}'(1) = (2l+1) + l(l-1)/2 = (l^2+3l+1)/2 = (l+1)(l+2)/2$, which proves the claim.

## 5 Expanding $P_l^m$ in terms of $P_l$

If $m$ is even, the associated Legendre polynomials $P_l^m$ are polynomials of degree $l$. Hence they can be expanded in terms of the Legendre polynomials $P_k$ for $0 \le k \le l$, that is, $P_l^m = \sum_{k \le l} c_k P_k$. By parity arguments, only if $k + l = 0 \bmod 2$ the coefficient $c_k$ can be nonzero. We will only treat the case $m = 2$ as an example.

We write

$$P_l^2 = \sum_{k=0}^{l} c_k P_k$$

so that by multiplying with $P_k$ and integrating this relation we have

$$c_k = \frac{2k+1}{2} \int_{-1}^{1} P_l^2(x) P_k(x) \mathrm{d}x\,.$$

We have $P_l^2(x) = (1-x^2)\mathrm{d}^2 P_l(x)$, $P_k(x) = a_k \mathrm{d}^k (x^2-1)^k$, $a_k = \frac{1}{2^k l!}$ and we define $G_k(x) = (1-x^2)P_k(x)$. Hence $G(\pm 1) = 0$ and $G_k'(x) = -2xP_k(x) + (1-x^2)P_k'(x)$ and $G_k'(\pm 1) = -2(\pm 1)^{k+1}$. By partially integrating twice, the expression

$$\int_{-1}^{1} P_l^2(x) P_k(x) \mathrm{d}x$$

can be shown to be equal to

$$2(1-(-1)^{k+l+1}) + \int_{-1}^{1} P_l(x) \mathrm{d}^2 G_k(x) \mathrm{d}x\,.$$

The first term vanishes, unless $k+l=0 \bmod 2$. The second term vanishes unless $k=l$; from the start, we have $k \leq l$ and $\mathrm{d}^2 G_k(x)$ is of degree $k$. But since $P_l(x) = a_l \mathrm{d}^l (x^2-1)^l$ we can partially integrate $l$ times to obtain

$$\int_{-1}^{1} P_l(x) \mathrm{d}^2 G_k(x) \mathrm{d}x = (-1)^l a_l \int_{-1}^{1} (x^2-1)^l \mathrm{d}^{l+2} G_k(x)\,.$$

Now since the degree of $G_k$ is $k+2$, which does not exceed $l+2$, the term $\mathrm{d}^{l+2}G_k(x)$ vanishes unless $k=l$. In fact, we have

$$\mathrm{d}^{l+2} G_k(x) = -2a_k \binom{l+2}{2} \mathrm{d}^{k+l}(x^2-1)^k = -\delta_{k,l}(l+1)(l+2)a_l(2l)!\,.$$

Performing the integration explicitly, using $\int_{-1}^{1}(1-x^2)^l \mathrm{d}x = 2^{2l+1}\frac{l!l!}{(2l)!}$ and putting all ends together, we have

$$c_k = (2k+1)(1+(-1)^{k+l} - \delta_{k,l}\frac{(l+1)(l+2)}{2l+1} \tag{31}$$

Using a little algebra, one shows that $c_l = -l(l-1)$ and $c_k = (2k+1)(1+(-1)^{k+l})$ if $k<l$.

For example $P_4^2 = -12P_4 + 10P_2 + 2P_0$.

Since $P_l^2$ vanishes at the endpoints and $P_k(1)=1$, the coefficients $c_k$ must add up to 0. By inspection, we see this holds if and only if

$$\sum_{0 \leq k \leq l,\, k+l=0 \mathrm{mod} 2} (2k+1) = \frac{1}{2}(l+1)(l+2)\,.$$

Let us first show this for $l=2t$. In this case the sum consists of $l+1$ summands and equals

$$(4t+1) + (2(t-2)+1) + \ldots + 1 = (4t+1) + (4t-3) + \ldots + 1\,.$$

First adding the 1 of each summand we find the above expression equals $t+1+4(1+2+\ldots+t)$. Hence

$$\sum_{0 \leq k \leq l,\, k+l=0 \mathrm{mod} 2} (2k+1) = t+1+2t^2+2t = \frac{1}{2}(2t+1)(2t+2)\,,$$

which proves the case $l=2t$. If $l=2t-1$ we have $t$ summands and

$$\sum_{0 \leq k \leq l,\, k+l=0 \mathrm{mod} 2} (2k+1) = (4t-1) + (4t-5) + \ldots + 3$$

and adding 1 to each summand we see that the last expression equals $2t^2+2t-t = 2t^2+t = (2t+1)t$, which equals $\frac{1}{2}(l+1)(l+2)$. Hence we have shown that in the expression $P_l^2 = \sum_{k=0}^{l} c_k P_k$ the coefficients $c_k$ given by eqn(31) indeed add up to zero.

One further interesting identity can be obtained using the expansion of $P_l^2$. Since the coefficients $c_k$ for $k < l$ in the expansion $P_l^2 = \sum_k c_k P_k$ do not depend on $l$, the mostly cancel in the expression $P_{l+1}^2 - P_{l-1}^2$. A little algebra shows

$$P_{l+1}^2 - P_{l-1}^2 = -l(l+1)\Big(P_{l+1} - P_{l-1}\Big)\,, \tag{32}$$

which is just an occasion of eqn.(19) choosing $m = 1$.

## 6 Orthogonal functions for $m = 1$

We define the polynomials

$$\Pi_l^{\pm}(x) = \frac{P_l^1(x)}{\sqrt{1-x^2}} \pm \sqrt{1-x^2}\,\frac{\mathrm{d}}{\mathrm{d}x}P_l^1(x)\,. \tag{33}$$

We claim that the set $\{\Pi_l^+;\ l = 1, 2, 3, \ldots\}$ is a set of orthogonal functions on $[-1; 1]$, and that the same holds for the set $\{\Pi_l^-;\ l = 1, 2, 3, \ldots\}$.

We write out $\int \Pi_l^{\pm}\Pi_{l'}^{\pm}\mathrm{d}x$ as a sum $A + B \pm C$, where

$$A = \int_{-1}^{1} \frac{P_l^1 P_{l'}^1}{1-x^2}\mathrm{d}x$$

$$B = \int_{-1}^{1} (1-x^2)\mathrm{d}P_l^1\mathrm{d}P_{l'}^1\mathrm{d}x$$

$$C = \int_{-1}^{1} \Big(P_l^1\mathrm{d}P_{l'}^1 + P_{l'}^1\mathrm{d}P_l^1\, l.$$

The integrand in $C$ is the derivative of $P_l^1 P_{l'}^1$, which vanishes at $x = \pm 1$ and hence $C = 0$. A partial integration shows

$$B = -\int_{-1}^{1} P_l^1\mathrm{d}\Big((1-x^2)\mathrm{d}P_{l'}^1\Big)\mathrm{d}x\,.$$

Since $P_{l'}^1$ satisfies the differential equation $((1-x^2)y')' + (l'(l'+1) - \frac{1}{1-x^2})y = 0$ we see that

$$B = -A + l'(l'+1)\int_{-1}^{1} P_l^1 P_{l'}^1\mathrm{d}x$$

Thus $A+B+C = l'(l'+1)\int_{-1}^{1} P_l^1 P_{l'}^1 \mathrm{d}x$, which vanishes unless $l = l'$. Indeed, we can perform a partial integration and see that

$$\begin{aligned}\int_{-1}^{1} P_l^1 P_{l'}^1 \mathrm{d}x &= \int_{-1}^{1} (1-x^2)\mathrm{d}P_l \mathrm{d}P_{l'} \mathrm{d}x \\ &= -\int_{-1}^{1} P_{l'} \mathrm{d}\Big((1-x^2)\mathrm{d}P_l\Big)\mathrm{d}x \\ &= l(l+1)\int_{-1}^{1} P_{l'} P_l \mathrm{d}x = \frac{2l(l+1)}{2l+1}\,.\end{aligned} \tag{34}$$

We thus obtain the final answer

$$\int_{-1}^{1} \Pi_l^{\pm}(x)\Pi_{l'}^{\pm}(x)\mathrm{d}x = \frac{2l^2(l+1)^2}{2l+1}\delta_{l,l'}\,. \tag{35}$$

In the following section it is shown that the $P_l^m$ for fixed $m$ also constitute a set of orthogonal functions – see eqn.(40).

# 7 Proof of integral relations

We consider the function $B(a,b)$ defined by

$$B(a,b) = \int_0^1 t^a (1-t)^b \mathrm{d}t\,.$$

A partial integration shows

$$\int_0^1 t^a(1-t)^b \mathrm{d}t = \frac{b}{a+1}\int_0^1 t^{a+1}(1-t)^{b-1}\mathrm{d}t = \frac{b}{a+1}B(a+1,b-1)\,.$$

Since $B(a,0) = \frac{1}{a+1}$, it then follows rather easily

$$B(a,b) = \frac{a!b!}{(a+b+1)!}\,. \tag{36}$$

Using the substitution $x = 2t-1$ we deduce

$$\int_{-1}^{1} (1-x^2)^l \mathrm{d}x = 2^{2l+1}B(l,l) = 2^{2l+1}\frac{l!l!}{(2l+1)!}\,. \tag{37}$$

Let us now show that

$$\int_{-1}^{1} P_k(x)P_l(x)\mathrm{d}x = \frac{2}{2l+1}\,. \tag{38}$$

Plugging in the definition, we have

$$\int_{-1}^{1} P_k(x)P_l(x)\mathrm{d}x = \frac{1}{2^{k+l}l!k!}\int_{-1}^{1} \mathrm{d}^k(x^2-1)^k \cdot \mathrm{d}^l(x^2-1)^l \mathrm{d}x\,.$$

Since $\mathrm{d}^a(x^2-1)^b$ vanishes at $x=\pm 1$ if $a<b$, partial integration leads to

$$\int_{-1}^{1} P_k(x)P_l(x)\mathrm{d}x = \frac{(-1)^l}{2^{k+l}l!k!}\int_{-1}^{1} \mathrm{d}^{k+l}(x^2-1)^k \cdot (x^2-1)^l \mathrm{d}x\,.$$

If $k \neq l$ we may assume $l > k$ and then $\mathrm{d}^{k+l}(x^2-1)^k = 0$. Hence the integral vanishes unless $k=l$. In this case we use $\mathrm{d}^{2l}(x^2-1)^l = \mathrm{d}^{2l}x^{2l} = (2l)!$ and $(x^2-1)^l = (-1)^l(1-x^2)^l$ to find

$$\int_{-1}^{1} P_k(x)P_l(x)\mathrm{d}x = \delta_{k,l}\frac{(2l)!}{2^{2l}l!l!}\int_{-1}^{1}(x^2-1)^l\mathrm{d}x\,,$$

which directly leads to the result using (37).

Let us now show that

$$\int_{-1}^{1} xP_s(x)P_r(x)\mathrm{d}x = \delta_{r,s+1}\frac{2r}{(2r-1)(2r+1)} + \delta_{s,r+1}\frac{2s}{(2s+1)(2s-1)}\,. \qquad (39)$$

First we consider the case that $r=s$. In this case the integral is easily found to be zero – $P_r$ has parity $(-1)^r$, and thus $x(P_r)^2$ has parity $-1$. Therefore the integral over the symmetric interval $[-1;1]$ vanishes.

Now we consider the case $r>s$. We find, using Rodriguez formula (1), partial integration and $(xf)^{(n)} = xf^{(n)} + nf^{(n-1)}$, that $\int_{-1}^{1} xP_r(x)P_s(x)\mathrm{d}x$ equals up to some numerical factors the expression

$$\int_{-1}^{1}(x^2-1)^r\left[x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r+s}(x^2-1)^s + r\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r+s-1}(x^2-1)^s\right]\mathrm{d}x\,.$$

We see that the first term vanishes since $r>s$; the second term only survives if $r \leq s+1$. Hence we need only consider the case $s = r-1$.

Writing $c_r = \frac{1}{2^r r!}$ we find, again using Rodriguez formula, partial integration and $(xf)^{(n)} = xf^{(n)} + nf^{(n-1)}$ but also that the $(2r-2)$th derivative of $(x^2-1)^{r-1}$ equals $(2r-2)!$, that

$$\int_{-1}^{1} xP_r(x)P_{r-1}(x)\mathrm{d}x = (-1)^r c_r c_{r-1}(2r-2)!\int_{-1}^{1}(x^2-1)^r\mathrm{d}x\,.$$

Putting in the numbers, we find $\int_{-1}^{1} xP_r(x)P_{r-1}(x)\mathrm{d}x = \frac{2r}{(2r-1)(2r+1)}$. The case $s>r$ leads to the same expression, but $s$ and $r$ interchanged.

Finally, we show

$$\int_{-1}^{+1} P_k^m(x) P_l^m(x) \mathrm{d}x = \delta_{k,l} \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \,. \tag{40}$$

Plugging in the definition, we have

$$\int_{-1}^{+1} P_k^m(x) P_l^m(x) \mathrm{d}x = \frac{1}{2^{k+l} k! l!} \int_{-1}^{1} (1-x^2)^m \mathrm{d}^{k+m} (x^2-1)^k \cdot \mathrm{d}^{l+m} (x^2-1)^l \mathrm{d}x \,.$$

The degree of $g(x) = (1-x^2)^m \mathrm{d}^{k+m}(x^2-1)^k$ is $k+m$. The degree of $h(x) = \mathrm{d}^{l+m}(x^2-1)^l$ is $l-m$. If we first assume $k \neq l$, we may as well take $l > k$. The first $m-1$ derivatives vanish at $x = \pm 1$ and the first $l-1$ derivatives of $(x^2-1)^l$ vanish at well at $x = \pm$. Therefire, if we perform partial integrations, moving the derivatives from $h$ to $g$, all boundary terms vanish. Since $\mathrm{d}^{l+m} g(x) = 0$ if $l > k$, the integral (40) vanishes if $k \neq l$.

If $k = l$, we find

$$\int_{-1}^{+1} P_k^m(x) P_l^m(x) \mathrm{d}x = \frac{(-1)^{l+m}}{2^{k+l} l! l!} \int_{-1}^{1} (x^2-1)^l \mathrm{d}^{l+m} \Big( (1-x^2)^m \mathrm{d}^{l+m} (x^2-1)^l \Big) \mathrm{d}x \,.$$

Since the term in brackets has degree $l+m$, we only need to consider the highest order term;

$$\mathrm{d}^{l+m} \Big( (1-x^2)^m \mathrm{d}^{l+m} (x^2-1)^l \Big) = (-1)^m \mathrm{d}^{l+m} (x^{2m} \mathrm{d}^{l+m} x^{2l}) = (-1)^m \frac{(2l)!(l+m)!}{(l-m)!} \,.$$

Combining this result and the previous integrals one easily finds the stated result.

# 8 The list of recurrence relations from Wikipedia

## Recurrence formula [ edit ]

These functions have a number of recurrence properties:

$$(\ell - m + 1)P_{\ell+1}^{m}(x) = (2\ell + 1)xP_{\ell}^{m}(x) - (\ell + m)P_{\ell-1}^{m}(x)$$

$$2mxP_{\ell}^{m}(x) = -\sqrt{1-x^2}\left[P_{\ell}^{m+1}(x) + (\ell + m)(\ell - m + 1)P_{\ell}^{m-1}(x)\right]$$

$$\frac{1}{\sqrt{1-x^2}}P_{\ell}^{m}(x) = \frac{-1}{2m}\left[P_{\ell-1}^{m+1}(x) + (\ell + m - 1)(\ell + m)P_{\ell-1}^{m-1}(x)\right]$$

$$\frac{1}{\sqrt{1-x^2}}P_{\ell}^{m}(x) = \frac{-1}{2m}\left[P_{\ell+1}^{m+1}(x) + (\ell - m + 1)(\ell - m + 2)P_{\ell+1}^{m-1}(x)\right]$$

$$\sqrt{1-x^2}P_{\ell}^{m}(x) = \frac{1}{2\ell+1}\left[(\ell - m + 1)(\ell - m + 2)P_{\ell+1}^{m-1}(x) - (\ell + m - 1)(\ell + m)P_{\ell-1}^{m-1}(x)\right]$$

$$\sqrt{1-x^2}P_{\ell}^{m}(x) = \frac{1}{2\ell+1}\left[-P_{\ell+1}^{m+1}(x) + P_{\ell-1}^{m+1}(x)\right]$$

$$\sqrt{1-x^2}P_{\ell}^{m+1}(x) = (\ell - m)xP_{\ell}^{m}(x) - (\ell + m)P_{\ell-1}^{m}(x)$$

$$\sqrt{1-x^2}P_{\ell}^{m+1}(x) = (\ell - m + 1)P_{\ell+1}^{m}(x) - (\ell + m + 1)xP_{\ell}^{m}(x)$$

$$\sqrt{1-x^2}\frac{d}{dx}P_{\ell}^{m}(x) = \frac{1}{2}\left[(\ell + m)(\ell - m + 1)P_{\ell}^{m-1}(x) - P_{\ell}^{m+1}(x)\right]$$

$$(1-x^2)\frac{d}{dx}P_{\ell}^{m}(x) = \frac{1}{2\ell+1}\left[(\ell + 1)(\ell + m)P_{\ell-1}^{m}(x) - \ell(\ell - m + 1)P_{\ell+1}^{m}(x)\right]$$

$$(x^2-1)\frac{d}{dx}P_{\ell}^{m}(x) = \ell xP_{\ell}^{m}(x) - (\ell + m)P_{\ell-1}^{m}(x)$$

$$(x^2-1)\frac{d}{dx}P_{\ell}^{m}(x) = -(\ell + 1)xP_{\ell}^{m}(x) + (\ell - m + 1)P_{\ell+1}^{m}(x)$$