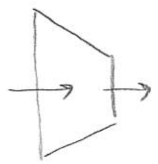
Lecture 35 - supplemental

A. Proof of $Ma = 1$ limit in converging nozzle.



The mass flow rate is a constant.

$$w = \rho A v$$

In incompressible flow ρ is a constant too, so when A decreases v must increase. In compressible flow ρ can change. This puts a "speed limit" on v.

$\boxed{\text{Proof}}$

$$w = \rho A v$$

$$\ln w = \ln \rho + \ln A + \ln v$$

$$d(\ln w) = 0 = \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dv}{v}$$

$$-\frac{dv}{v} = \frac{d\rho}{\rho} + \frac{dA}{A} \qquad (1)$$

* Bernoulli's equation (isentropic = no friction)

$$\frac{dP}{\rho} + \frac{1}{2} d(v^2) = 0$$

$$d(v^2) = 2v\,dv$$

$$\frac{dP}{\rho} + v\,dv = 0 \quad \Rightarrow \quad -v\,dv = \frac{dP}{\rho} \qquad (2)$$

* Equation of state for ρ: $\rho = \rho(P, S)$

$$d\rho = \left(\frac{\partial \rho}{\partial P}\right)_S dP + \underbrace{\left(\frac{\partial \rho}{\partial S}\right)_P dS}_{dS = 0 \text{ , isentropic}}$$

$$d\rho = \underbrace{\left(\frac{d\rho}{dP}\right)_S}_{\frac{1}{c^2} \Rightarrow \text{definition of speed of sound}} dP \quad \Rightarrow \quad dP = c^2\,d\rho \qquad (3)$$

* combine (2) & (3)

$$\frac{c^2\,d\rho}{\rho} = -v\,dv$$

$$\frac{d\rho}{\rho} = -\frac{v^2}{c^2}\frac{dv}{v} \qquad (4)$$

* combine (1) & (4)

$$-\frac{dv}{v} = \underbrace{\frac{-v^2}{c^2}}_{Ma^2}\frac{dv}{v} + \frac{dA}{A}$$

$$\frac{dv}{v}\left(-1 + Ma^2\right) = \frac{dA}{A}$$

$$\boxed{\frac{dv}{v} = \frac{1}{Ma^2 - 1}\frac{dA}{A}}$$

* when $Ma = 1$, $dv = 0$

* when $Ma < 1$:

$dv > 0$ when $dA < 0$
speeds up when converging

* when $Ma > 1$

$dv > 0$ when $dA > 0$
speeds up when diverging

B. Proof of P, T relations for compressible, isentropic flow

(1) Mass Balance

$$\boxed{w = \rho_1 A_1 v_1} \qquad (1)$$

(2) Mechanical Energy Balance

$$\frac{\Delta P}{\rho} + \frac{\Delta v^2}{2} + \underbrace{\cancel{g\Delta h}^{\,0}}_{\text{no height change}} = \frac{1}{w}\left(\underbrace{\cancel{W_m}^{\,0}}_{\text{no pumps}} - \underbrace{\cancel{E_v}^{\,0}}_{\text{very fast isentropic/inviscid}} - E_c\right)$$

$$\frac{\Delta P}{\rho} + \frac{\Delta v^2}{2} = -\frac{E_c}{w}$$ ← what is E_c?

$$\frac{E_c}{w} = -\frac{\Delta P}{\rho} + \int_{P_0}^{P_1} \frac{dP}{\rho}$$ ← ρ is a function of P

See Dean p. 336 for derivation of E_c

$$\cancel{\frac{\Delta P}{\rho}} + \frac{\Delta v^2}{2} = \cancel{\frac{\Delta P}{\rho}} - \int_{P_0}^{P_1} \frac{dP}{\rho} \Rightarrow \boxed{\int_{P_0}^{P_1} \frac{dP}{\rho} + \frac{\Delta v^2}{2} = 0} \qquad (2)$$

(3) Equation of State

Assume an ideal gas: $\boxed{\rho = \frac{MP}{RT}}$ (3a)

(see Thermodynamics textbook for more.)

$$c = \sqrt{\frac{\gamma RT}{M}} \qquad (3b)$$

$$\gamma = \frac{\hat{C}_p}{\hat{C}_v} = 1.4 \qquad (3c)$$

$$\hat{C}_p - \hat{C}_v = \frac{R}{M} \quad \text{or} \quad \hat{C}_p = \frac{R}{M}\frac{\gamma}{\gamma - 1}$$

(3d)

(4) Total Energy Equation

$$\underbrace{d\Big(u}_{\text{internal energy}} + \underbrace{\frac{v^2}{2} + \frac{P}{\rho} + gz}_{\text{mech. energy}}\Big) = 0$$

$\Leftarrow$ $d(gz) = 0$: no height change

$H \equiv u + P/\rho$: enthalpy (definition)

$$d\left(H + \frac{v^2}{2}\right) = 0 \Rightarrow dH = -\frac{1}{2}\,d(v^2) \quad (*)$$

$$\hat{C}_p \equiv \left.\frac{\partial H}{\partial T}\right|_P \quad : \text{definition of heat capacity}$$

$(**)$ $dH = \hat{C}_p\, dT$ So: $(*)$ & $(**)$ give

$$\boxed{\hat{C}_p\, dT = -\frac{1}{2}\,d(v^2)} \quad (4)$$

Relationship for T

Equation (4): $\hat{C}_p\, dT = -\frac{1}{2}\,d(v^2)$

$$\hat{C}_p\, \Delta T = -\frac{1}{2}\,\Delta(v^2)$$

$$\hat{C}_p\,(T_1 - T_0) = -\frac{1}{2}\left(v_1^2 - v_0^2\right)$$

$\leftarrow$ let $v_0 = 0$, the "stagnation point" (no velocity in the tank).

$$\hat{C}_p\,(T_1 - T_0) = -\frac{v_1^2}{2}$$

divide by T_1, rearrange.

$$\hat{C}_p\left(\frac{T_1}{T_1} - \frac{T_0}{T_1}\right) = \frac{-v_1^2}{2T_1}$$

$$\frac{T_0}{T_1} = \frac{v_1^2}{2\hat{C}_p T_1} + 1$$

$\leftarrow$ recall that $c = \sqrt{\frac{\gamma R T}{M}}$ or $c^2 = \frac{\gamma R T}{M}$ for an ideal gas.

$$\left(T = \frac{Mc^2}{\gamma R}\right)$$

$$\frac{T_0}{T_1} = \frac{v_1^2}{2}\,\frac{\gamma R}{M c_1^2}\,\frac{1}{\hat{C}_p} + 1$$

$$\frac{T_0}{T_1} = \frac{1}{2}\underbrace{\frac{v_1^2}{c_1^2}}_{Ma^2}\cdot\frac{\gamma R}{M\hat{C}_p} + 1$$

recall that

$$Ma = \frac{v_1}{c_1}$$

For an ideal gas: $\hat{C}_p = \frac{R}{M}\frac{\gamma}{\gamma-1}$, $\gamma = \frac{\hat{C}_p}{\hat{C}_v} = 1.4$

(Equation 3d)

$$\frac{T_0}{T_1} = \frac{1}{2}Ma^2\cdot\frac{\gamma \cancel{R}}{\cancel{M}}\cdot\frac{\cancel{M}}{\cancel{R}}\frac{\gamma-1}{\cancel{\gamma}}$$

$$\boxed{\frac{T_0}{T_1} = 1 + \frac{\gamma-1}{2}Ma^2}$$

Relationship for P

Equation (2): $\int_{P_0}^{P_1}\frac{dP}{\rho} + \frac{\Delta v^2}{2} = 0$ $\quad\frac{dP}{\rho} + \frac{1}{2}d(v^2) = 0$ (on a streamline)

• differential version

• Bernoulli's Equation!

Equation (4): $-\frac{1}{2}d(v^2) = \hat{C}_p\, dT$

* Combine: $\frac{dP}{\rho} = \hat{C}_p\, dT$

* Equation (3a): $\rho = \frac{MP}{RT} \longrightarrow \frac{dP}{(MP/RT)} = \hat{C}_p\, dT$

$$\frac{dP}{P} = \frac{M}{RT}\hat{C}_p\, dT$$

Equation (3d): $\hat{C}_p = \frac{R}{M}\frac{\gamma}{\gamma-1}$

$$\frac{dP}{P} = \frac{\cancel{M}}{\cancel{R}T}\cdot\frac{\cancel{R}}{\cancel{M}}\frac{\gamma}{\gamma-1}dT$$

$$\frac{dP}{P} = \frac{\gamma}{\gamma-1}\frac{dT}{T} \Rightarrow \text{integrate} \Rightarrow \int_{P_0}^{P_1}\frac{dP}{P} = \frac{\gamma}{\gamma-1}\int_{T_0}^{T_1}\frac{dT}{T}$$

$$\ln P_1 - \ln P_0 = \frac{\gamma}{\gamma-1}\left(\ln T_1 - \ln T_0\right)$$

$$\ln\left(\frac{P_1}{P_0}\right) = \frac{\gamma}{\gamma-1}\ln\left(\frac{T_1}{T_0}\right)$$ multiply both sides by (-1)

$$\ln\left(\frac{P_0}{P_1}\right) = \frac{\gamma}{\gamma-1}\ln\left(\frac{T_0}{T_1}\right) = \ln\left(T_0/T_1\right)^{\gamma/\gamma-1}$$ exponentiate both sides

$$\boxed{\frac{P_0}{P_1} = \left(\frac{T_0}{T_1}\right)^{\gamma/\gamma-1}} \quad \text{or} \quad \boxed{\frac{T_0}{T_1} = \left(\frac{P_0}{P_1}\right)^{\frac{\gamma-1}{\gamma}}}$$

$$\boxed{\frac{P_0}{P_1} = \left[1 + \frac{\gamma-1}{2}Ma^2\right]^{\gamma/\gamma-1}}$$

Relationship for ρ.

Equation (3a): $\rho = \frac{MP}{RT}$, $P = \frac{\rho RT}{M}$

$$\frac{P_0}{P_1} = \left(\frac{T_0}{T_1}\right)^{\gamma/\gamma-1} \Rightarrow \frac{\rho_0 RT_0/M}{\rho_1 RT_1/M} = \left(\frac{T_0}{T_1}\right)^{\gamma/\gamma-1}$$

$$\frac{\rho_0}{\rho_1}\cdot\frac{T_0}{T_1} = \left(\frac{T_0}{T_1}\right)^{\gamma/\gamma-1} \Rightarrow \frac{\rho_0}{\rho_1} = \left(\frac{T_0}{T_1}\right)^{\frac{\gamma}{\gamma-1}-1}$$

$$\frac{\gamma}{\gamma-1} - \frac{\gamma-1}{\gamma-1} = \frac{1}{\gamma-1}$$

$$\boxed{\frac{\rho_0}{\rho_1} = \left(\frac{T_0}{T_1}\right)^{\frac{1}{\gamma-1}}} \quad \boxed{\frac{T_0}{T_1} = \left(\frac{\rho_0}{\rho_1}\right)^{\gamma-1}}$$

$$\boxed{\frac{\rho_0}{\rho_1} = \left[1 + \frac{\gamma-1}{2}Ma^2\right]^{\frac{1}{\gamma-1}}}$$