Chapter 1

Problem-1 Determine whether or not each of the following signal is periodic and if yes then determine its fundamental period.

(a) $x(t) = [\sin(4t-1)]^2$

(b) $x[n] = \cos(4n + \pi/4)$

(c) $x[n] = (-1)^n \cos(2\pi n/7)$

Answer:

(a) In order to figure out if a CT signal $x(t)$ is periodic, we need to find a finite, non-zero value of T such that $x(t) = x(t+T)$ for all t. The smallest T that satisfies this is the fundamental period.

This function is quite straightforward. We know that the function $\sin(4t-1)$ is periodic with period $\frac{\pi}{2}$. Since the positive and negative cycles of sinusoids have the same shape, the square of this function, i.e. $x(t) = [\sin(4t-1)]^2$ is periodic with fundamental period $\frac{\pi}{4}$.

Also, we can use the relation

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\sin 2x,$$

which is periodic with period $\frac{\pi}{4}$.

(b) For a DT function $x[n]$, we need to find a finite, non-zero *integer* N such that $x[n] = x[n+N]$ for all n. The smallest integer N for which this holds is the fundamental period. If we cannot find such an N, then the function is not periodic.

We need

$$\cos\left[4(n+N) + \frac{\pi}{4}\right] = \cos\left[4n + \frac{\pi}{4}\right]$$

For the above to hold, the following has to be true for some integer(s) k.

$$\begin{aligned} 4n + 4N + \frac{\pi}{4} &= 4n + \frac{\pi}{4} + 2\pi k \\ N &= \frac{\pi}{2}k \end{aligned}$$

Since π is not a rational number, we cannot find an integer N that satisfies this. Thus, the function is not periodic.

(c) We can use the same steps as we did above but we can start with finding the fundamental period of the simpler function $y[n] = \cos\left(\frac{2\pi n}{7}\right)$.

We need the following to hold

$$\cos\left(\frac{2\pi n}{7}\right) = \cos\left(\frac{2\pi(n+N)}{7}\right)$$

So we need the following to hold for at least one integer value of k.

$$\begin{aligned} \frac{2\pi n}{7} + 2\pi k &= \frac{2\pi(n+N)}{7} \\ k &= \frac{N}{7} \end{aligned}$$

So, $N = 7, 14, 21, \cdots$ satisfy this.

Now, for $x[n] = (-1)^n \cos\left(\frac{2\pi(n+N)}{7}\right)$ we immediately see that its period has to be an even number because $(-1)^n$ takes on the value of 1 for even n and -1 for odd n. So, the fundamental period is 14.

Problem 2: Determine whether the following systems are memoryless, linear, causal, TI, and stable?

(a) $y(t) = x(t+3) - x(1-t)$

(b) $y[n] = \begin{cases} (-1)^n x[n], & x[n] \geq 0 \\ 2x[n], & x[n] < 0 \end{cases}$

(c) $y[n] = \sum_{k=n}^{\infty} x[k]$

For this problem, assume that $y_1(t), y_2(t), y_3(t), y_4(t)$ are the outputs of the CT systems when the inputs are $x_1(t), x_2(t), x_3(t), x_4(t)$, respectively. Also, a and b are any (possibly complex) numbers, t_0 is any real number and n_0 is any integer.

(a) (1) **Memoryless - NO:** Clearly, this is not memoryless because $y(t)$ depends on $x(t+3)$ which is a future value.

(2) **Time-invariant - NO:** Consider the output $y_1(t)$ and a time-shifted version of it $y_1(t+t_0)$ as follows:

$$\begin{aligned} y_1(t) &= x_1(t+3) - x_1(1-t) \\ y_1(t+t_0) &= x_1(t+t_0+3) - x_1(1-t-t_0) \end{aligned}$$

If $x_2(t) = x_1(t+t_0)$ is the input, then the output is given by:

$$\begin{aligned} y_2(t) &= x_2(t+3) - x_2(1-t) \\ &= x_1(t+3+t_0) - x_2(1-t+t_0) \\ &\neq y_1(t+t_0) \end{aligned}$$

Therefore it is not time-invariant

(3) **Linear - YES:**
If $x_3(t) = ax_1(t) + bx_2(t)$. Then,

$$\begin{aligned} y_3(t) &= x_3(t+3) + x_3(1-t) \\ &= ax_1(t+3) + bx_2(t+3) + ax_1(1-t) + bx_2(1-t) \\ &= ax_1(t+3) + ax_1(1-t) + bx_2(t+3) + bx_2(1-t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

So, this system is linear.

(4) **Causal - NO:** Clearly, this system is not causal because $y(t)$ depends on $x(t+3)$ which is a future value of the input.

(5) **Stable - YES:** Since $y(t)$ is a finite sum of the input $x(t)$ at different time lags, if $x(t)$ is bounded, so is $y(t)$.

(b) (1) **Memoryless - YES:** It is memoryless since $y[n]$ depends only on $x[n]$.

(2) **Time-invariant - NO:**

$$y_1[n] = \begin{cases} (-1)^n x_1[n], & x_1[n] \geq 0 \\ 2x_1[n], & x_1[n] > 0 \end{cases}$$

So, if $x_1[n] = x[n + n_0]$,

$$y_1[n] = \begin{cases} (-1)^n x[n + n_0], & x[n + n_0] \geq 0 \\ 2x[n + n_0], & x[n + n_0] > 0 \end{cases}$$

but,

$$y[n + n_0] = \begin{cases} (-1)^{n+n_0} x[n + n_0], & x[n + n_0] \geq 0 \\ 2x[n + n_0], & x[n + n_0] > 0 \end{cases}$$

So, if n_0 is odd, $y[n + n_0] \neq y_1[n]$. Therefore, it is not time-invariant.

(3) **Linear - NO:** Say that $x[0] = 1$, then $y[0] = 1$. Now, if $x_1[0] = -1 \cdot x[0] = -1$, then

$$\begin{aligned} y_1[0] &= -2 \\ &\neq -y[0] \end{aligned}$$

Therefore, it is not linear.

(4) **Causal - YES:** Since the system is memoryless, it is also causal

(5) **Stable - YES** :Any value of $y[n]$ is just a scaled version of the input. So, if $x[n]$ is bounded, so is $y[n]$.

(c) (1) **Memoryless - NO:** It is not memoryless because $y[n]$ depends on the input signal from the time index n to ∞.

(2) **Time-Invariant - YES:**
If $x_1[n] = x[n + n_0]$, then

$$\begin{aligned} y_1[n] &= \sum_{k=n}^{\infty} x[k + n_0] \\ &= \sum_{k=n+n_0}^{\infty} x[k] \\ &= y[n + n_0] \end{aligned}$$

Thus, it is time invariant.

(3) **Linear - YES:** Let $x_3[n] = ax_1[n] + bx_2[n]$. So,

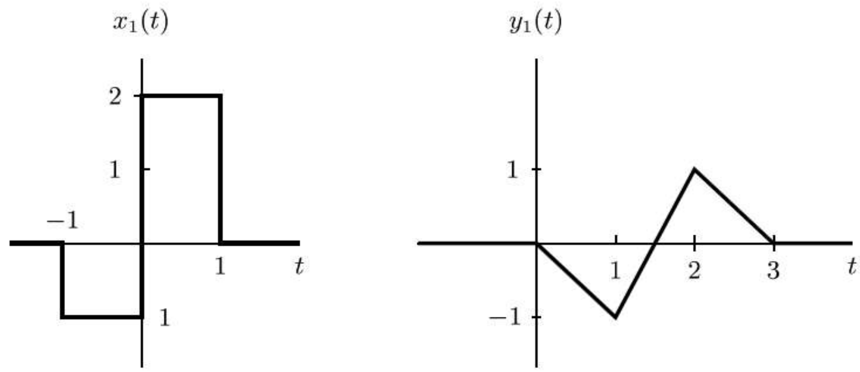
$$\begin{aligned} y_3[n] &= \sum_{k=n}^{\infty} x_3[n] \\ &= \sum_{k=n}^{\infty} ax_1[n] + \sum_{k=n}^{\infty} bx_2[n] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Thus, the system is linear. Also, since the output is just a sum of the input at different time lags, we can conclude that the system is linear.

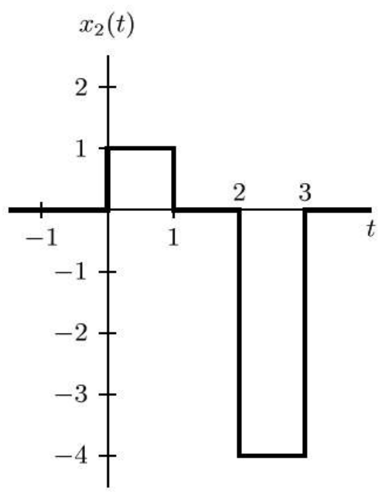
(4) **Causal - NO:**
It is not causal because $y[n]$ depends on $x[n], x[n+1] \cdots x[\infty]$.

(5) **Stable - NO:** The output is an infinite sum of the input sequence at time lags of $n \to \infty$. Thus, if the input signal is bounded (e.g. $x[n] = 1$), the output could be unbounded.

Problem 3 Consider an LTI system whose response the to signal $x_1(t)$ is the signal $y_1(t)$, where these signals are depicted below:

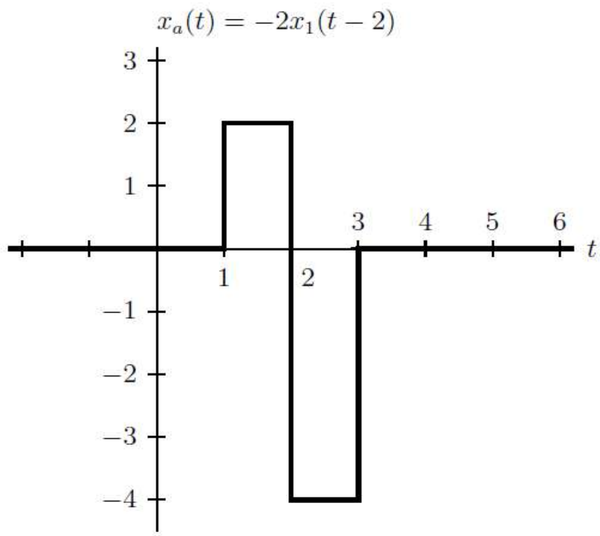


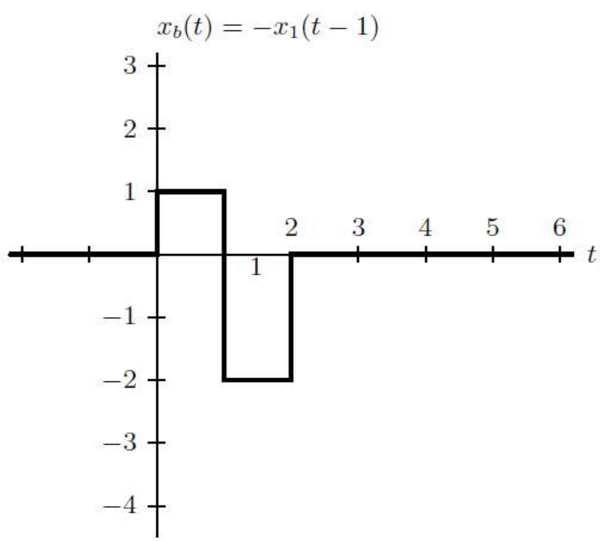
Determine and provide a labeled sketch of the response to the input $x_2(t)$ depicted below.



Answer 3

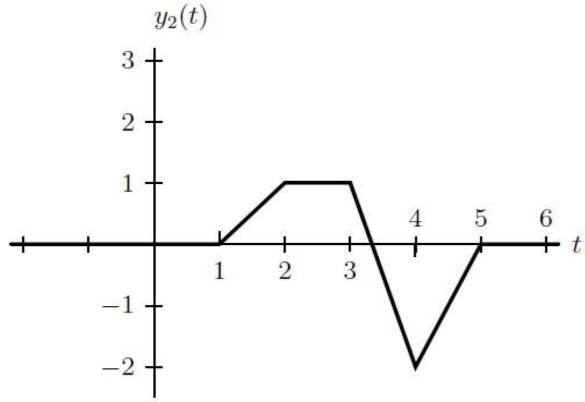
Since we are dealing with an LTI system, we need to express the input signal $x_2(t)$ in terms of a linear, time-shifted combination of the input whose output is known, i.e. $x_1(t)$. $x_2(t)$ can be expressed as the sum of $x_a(t) = -2x_1(t-2)$ and $x_b(t) = -x_1(t-1)$ as depicted below:

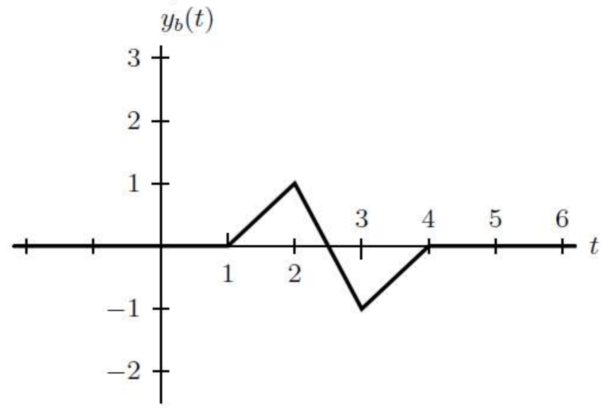




Let $y_a(t)$ and $y_b(t)$ be the outputs of the system if the inputs are $x_a(t)$ and $x_b(t)$, respectively. From the given input-output pair and using the LTI property, we have the following signals for $y_a(t)$ and $y_b(t)$

The sum of these two signals results in the desired output $y_2(t)$:





Problem 4: O&W 1.54

(a) For the $r = 1$ case, we have:

$$\begin{aligned}\sum_{n=0}^{n=N-1} 1 &= 1 + 1^1 + 1^2 + \cdots + 1^{N-1} \\ &= N\end{aligned}$$

For the $r \neq 1$ case, by carrying out the long division, we can see that

$$\begin{aligned}\frac{1}{1-r} &= 1 + r + r^1 + r^2 + \cdots + r^{N-1} + \frac{r^N}{1-r} \\ &= \sum_{n=0}^{N-1} r^n + \frac{r^N}{1-r} \\ \sum_{n=0}^{N-1} r^n &= \frac{1-r^N}{1-r}\end{aligned}$$

(b) Using the formula we just derived for the $r \neq 1$ case, we have

$$\begin{aligned}\sum_{n=0}^{\infty} r^n &= \lim_{N\to\infty} \sum_{n=0}^{N-1} r^n \\ &= \lim_{N\to\infty} \frac{1-r^N}{1-r} \\ &= \frac{1}{1-r} - \lim_{N\to\infty} \frac{r^N}{1-r}\end{aligned}$$

If $|r| < 1$

$$\lim_{N\to\infty} \frac{r^N}{1-r} = 0$$

So,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

(c)

$$\begin{aligned}\sum_{n=0}^{\infty} n\alpha^n &= \alpha + 2\alpha^2 + 3\alpha^3 + \cdots \\ &= \alpha\left(1 + 2\alpha + 3\alpha^2 + \cdots\right)\end{aligned}$$

Now we can seperate the contents of the paranthesis on the right-hand-side (RHS) of the equation above as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} n\alpha^n &= \alpha\left(1 + \alpha + \alpha^2 + \cdots + \alpha + 2\alpha^2 + 3\alpha^3 + \cdots\right) \\ &= \alpha\left(1 + \alpha + \alpha^2 + \cdots\right) + \alpha\left(\alpha + 2\alpha^2 + 3\alpha^3 + \cdots\right)\end{aligned}$$

Note that the contents of the second paranthesis on the RHS is the very expression we are trying to evaluate:

$$\begin{aligned}\sum_{n=0}^{\infty} n\alpha^n &= \alpha\left(1+\alpha+\alpha^2+\cdots+\sum_{n=0}^{\infty} n\alpha^n\right) \\ (1-\alpha)\sum_{n=0}^{\infty} n\alpha^n &= \alpha\left(1+\alpha+\alpha^2+\cdots\right) \\ &= \alpha\sum_{n=0}^{\infty}\alpha^n\end{aligned}$$

Using the result from part (b) for $|\alpha| < 1$,

$$\begin{aligned}(1-\alpha)\sum_{n=0}^{\infty} n\alpha^n &= \frac{\alpha}{1-\alpha} \\ \sum_{n=0}^{\infty} n\alpha^n &= \frac{\alpha}{(1-\alpha)^2}\end{aligned}$$

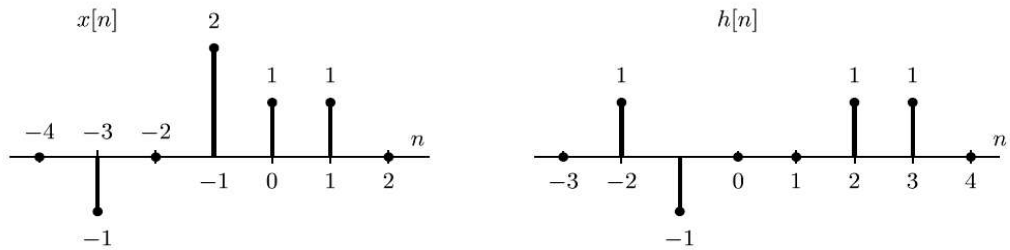
(d)

$$\begin{aligned}\sum_{n=k}^{\infty}\alpha^n &= \sum_{n=0}^{\infty}\alpha^n - \sum_{n=0}^{k-1}\alpha^n \\ &= \frac{1}{1-\alpha} - \frac{1-\alpha^k}{1-\alpha} \\ &= \frac{\alpha^k}{1-\alpha}.\end{aligned}$$

Chapter 2

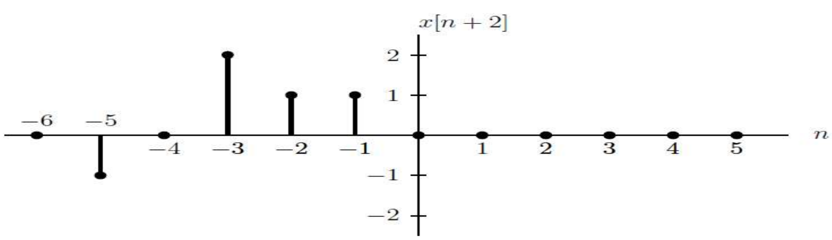
Problem 1 Compute the convolution $y[n] = x[n] * h[n]$ of each of the two following pairs of signals:

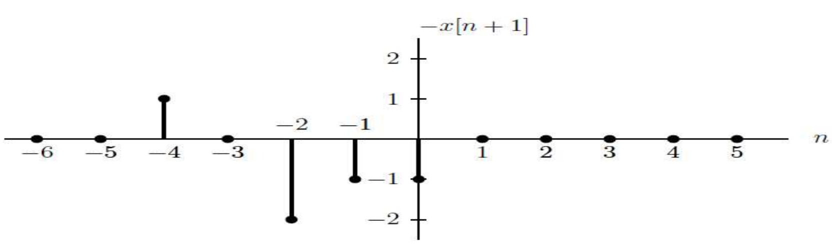
(a). $x[n]$ and $h[n]$ are depicted below

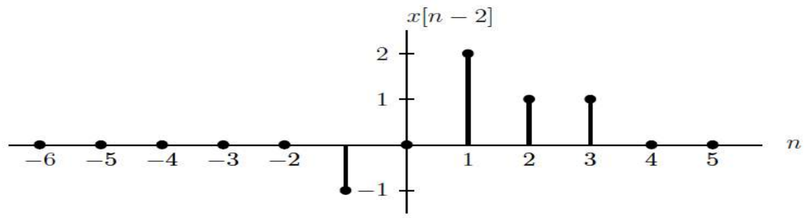


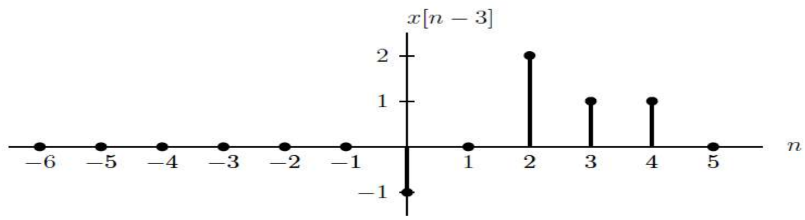
(b). $x[n] = u[n+4] - u[n-1]$, $h[n] = 2^n u[2-n]$.

(a) With short duration DT sequences, it is often simplest to find their convolution by centering copies of one of the signals about each of the non-zero samples of the other signal and scaled by the value of the sample at that location. The result is the sum of all the shifted and scaled signals. Thus, $y[n]$ is given by the sum of the following signals.

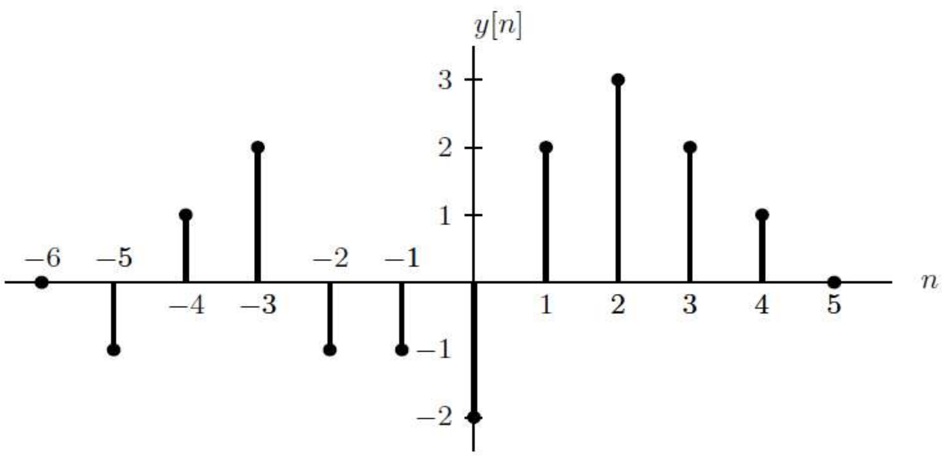




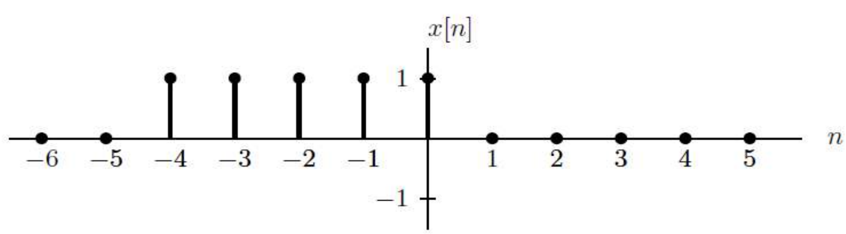




The sum of these yields the following sequence for $y[n]$:



(b) For this part, we can again use the shift and scale method since the sequence $x[n]$ is of a short duration as given below:



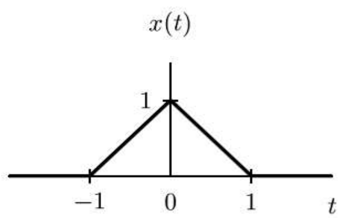
Thus, we can write the output as a sum of scaled shifted inputs as follows:

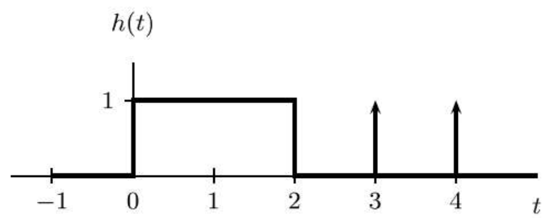
$$\begin{aligned} y[n] &= 2^n u[2-n] + 2^{n+1}u[1-n] + 2^{n+2}u[-n] + 2^{n+3}u[-n-1] + 2^{n+4}u[-n-2] \\ &= \sum_{k=0}^{4} 2^{n+k} u[2-n-k] \end{aligned}$$

Problem 2 Compute the convolution $y(t) = x(t) * h(t)$ for each of the following pairs of signals:

(a). $x(t) = e^{-t}u(t+1)$, $h(t) = e^{2t}u(-t)$

(b). $x(t)$ and $h(t)$ are depicted below:

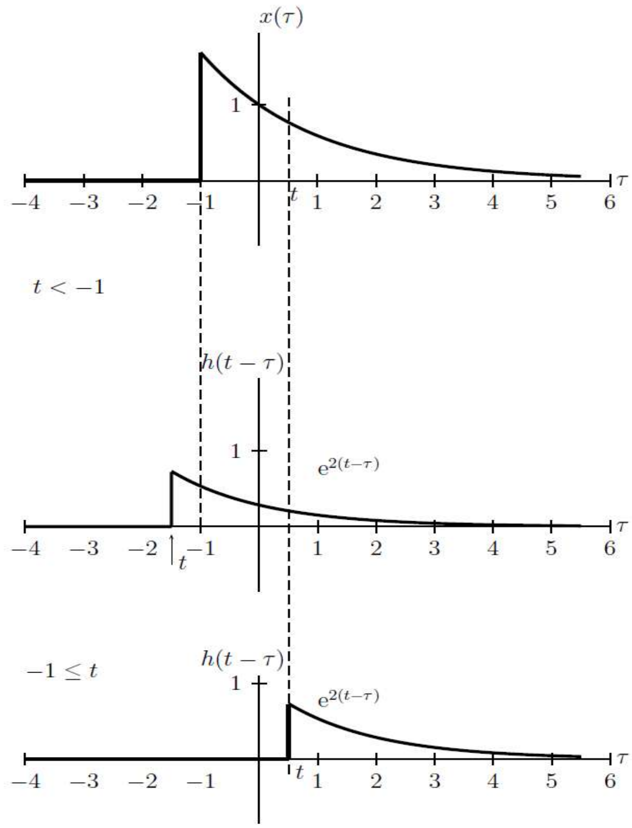




(a) From the definition of the convolution, we have the following expression for the output $y(t)$:

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

Based on the given $x(t)$ and $h(t)$, we can break the integration up into 2 regions as illustrated in the diagram. The ranges are $t < -1$ and $t \geq -1$.



For the range $t < -1$, the region where $x(\tau)h(t-\tau)$ is non-zero is from $-1 \to \infty$. So, the expression for $y(t)$ is given by:

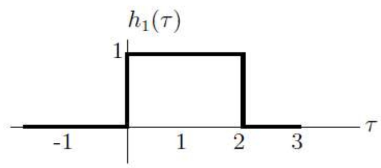
$$
\begin{aligned}
y(t) &= \int_{-1}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{-1}^{\infty} \mathrm{e}^{2(t-\tau)}\mathrm{e}^{-\tau}d\tau \\
&= \mathrm{e}^{2t}\int_{-1}^{\infty} \mathrm{e}^{-3\tau}d\tau = \mathrm{e}^{2t}\left[-\frac{1}{3}\mathrm{e}^{-3\tau}\right]_{-1}^{\infty} \\
&= \frac{1}{3}\mathrm{e}^{2t+3}
\end{aligned}
$$

For the range $t \geq -1$, the $x(\tau)h(t-\tau)$ is non-zero for $\tau > t$. So the expression for $y(t)$ is given by:

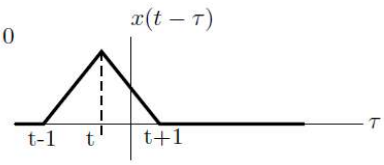
$$
\begin{aligned}
y(t) &= \int_{t}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{t}^{\infty} \mathrm{e}^{2(t-\tau)}\mathrm{e}^{-\tau}d\tau \\
&= \mathrm{e}^{2t}\int_{t}^{\infty} \mathrm{e}^{-3\tau}d\tau = \mathrm{e}^{2t}\left[-\frac{1}{3}\mathrm{e}^{-3\tau}\right]_{t}^{\infty} \\
&= \mathrm{e}^{2t}\left[-\frac{1}{3}\mathrm{e}^{-3t}\right] \\
&= \frac{1}{3}\mathrm{e}^{-t}
\end{aligned}
$$

(b) Here, we can break $h(t)$ up into $h(t) = h_1(t) + h_2(t)$ where $h_1(t)$ is the "box" part of $h(t)$ and $h_2(t)$ are the two impulses. Let $y_1(t)$ and $y_2(t)$ denote the result of convolving $x(t)$ with $h_1(t)$ and $h_2(t)$ respectively.

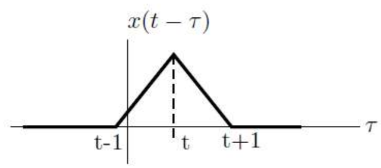
First let us compute $y_1(t)$. To do this, we fix $h_1(t)$ and flip and slide $x(t)$. The following figure illustrates the different regions of overlap.



$-1 < t < 0$



$0 \le t < 1$



For the range $-1 < t < 0$, the result of the convolution is the area under the product of the two signals which is given by:

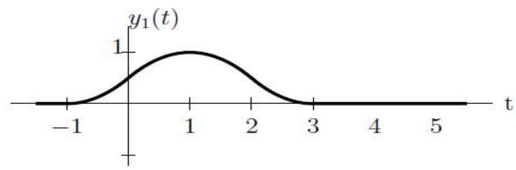
$$
\begin{aligned}
y_1(t) &= \frac{1}{2}(t+1)(t+1) \\
&= \frac{1}{2}(t^2 + 2t + 1)
\end{aligned}
$$

For the range $0 \le t < 1$, the area under the product is given by:

$$
\begin{aligned}
y_1(t) &= t(1-t) + \frac{1}{2}t(1-(1-t)) + \frac{1}{2} \\
&= t - t^2 + \frac{1}{2}t^2 + \frac{1}{2} \\
&= \frac{1}{2}(1 + 2t - t^2)
\end{aligned}
$$

Now both $x(t)$ and $h_1(t)$ are symmetric signals which are symmetric about $t = 0$ and $t = 1$ respectively. Therefore, the convolution of the two is symmetric about $t = 1$.

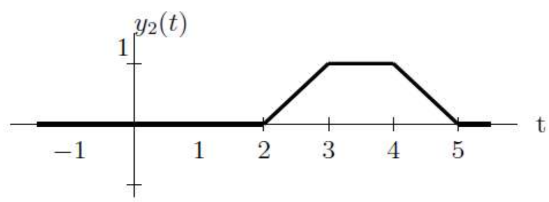
The plot for $y_1(t)$ looks like the following:



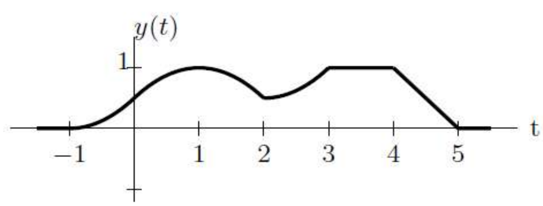
With the different regions of the curve as follows:

$$
\begin{aligned}
-1 < t < 0, \qquad & y_1(t) = \frac{1}{2}(t^2 + 2t + 1) \\
0 \leq t < 1, \qquad & y_1(t) = \frac{1}{2}(1 + 2t - t^2) \\
1 \leq t < 2, \qquad & y_1(t) = \frac{1}{2}(1 + 2t - t^2) \\
2 \leq t < 3, \qquad & y_1(t) = \frac{1}{2}(t^2 - 6t + 9)
\end{aligned}
$$

The convolution with $h_2(t)$ is straightforward because it is a convolution with impulses. To do this, all we need is to center the triangle around both impulses and scale by the area under each impulse which in this case is 1. This gives the following plot for $y_2(t)$.



The final result is the sum of the two as follows:



The curved parts of the plot are given by the following expressions:

$$
\begin{aligned}
-1 < t < 0, \qquad & y(t) = \frac{1}{2}(t^2 + 2t + 1) \\
0 \leq t < 1, \qquad & y(t) = \frac{1}{2}(1 + 2t - t^2) \\
1 \leq t < 2, \qquad & y(t) = \frac{1}{2}(1 + 2t - t^2) \\
2 \leq t < 3, \qquad & y(t) = \frac{1}{2}(t^2 - 4t + 5)
\end{aligned}
$$

Problem 3 The following are impulse responses of either discrete-time or continuous-time LTI systems. Determine whether each system is causal and/or stable. Justify your answer:

(a). $h[n] = 2^n u[3-n]$

(b). $h(t) = u(1-t) - \frac{1}{2}e^{-t}u(t)$

(c). $h[n] = [1-(0.99)^n]u[n]$

(d). $h(t) = e^{15t}\left[u(t-1) - u(t-100)\right]$

(a) Since the unit sample response is non-zero for $n < 0$, the system is not causal. For stability, we need to ensure that the impulse response is absolutely summable.

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |h[n]| &= \sum_{k=-\infty}^{3} 2^k \\ &= \sum_{k=\infty}^{-3} \left(\frac{1}{2}\right)^k \end{aligned}$$

which is finite. Thus, the system is stable

(b) Since $h(t)$ is 1 for $t < 0$, the system is not causal. For stability, the impulse response has to be absolutely integrable:

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)|dt &= \int_{-\infty}^{\infty} h(t)dt \qquad \text{since } h(t) \text{ is never negative} \\ &= \int_{-\infty}^{\infty} \left(u(1-t) - \frac{1}{2}\mathrm{e}^{-t}u(t)\right) dt \\ &= \int_{-\infty}^{0} u(1-t)dt + \int_{0}^{\infty} \left(u(1-t) - \frac{1}{2}\mathrm{e}^{-t}u(t)\right) dt \\ &= \int_{-\infty}^{0} 1dt + \int_{0}^{\infty} \left(u(1-t) - \frac{1}{2}\mathrm{e}^{-t}u(t)\right) dt \end{aligned}$$

The first term on the r.h.s. of the equation integrates to ∞ but the second term is finite, which means the sum of the two terms is infinite. So, the system is not stable.

(c) This system is causal because the impulse response is zero for $n < 0$. For stability, the impulse response has to be absolutely summable.

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |h[k]| &= \sum_{k=0}^{\infty} h[k] + \sum_{k=-\infty}^{-1} -h[k] \\ &= \sum_{k=0}^{\infty} [1-(0.99)^k]u[k] + \sum_{k=-\infty}^{-1} [1-(0.99)^k]u[k] \\ &= \sum_{k=0}^{\infty} [1-(0.99)^k] \\ &= \sum_{k=0}^{\infty} 1 - \sum_{k=0}^{\infty} (0.99)^k \end{aligned}$$

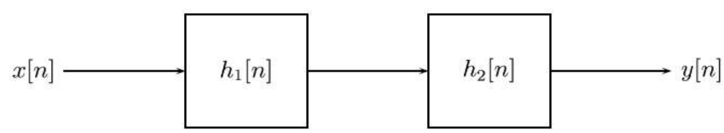
The second term on the r.h.s. is finite, as we know from power series, and the formulae we derived in problem set 1. The first term on the r.h.s. of the equation is infinite. So, the r.h.s. is infinite, which means the system is not stable.

(d) Since $h(t) = 0$ for all $t < 0$, this system is causal. Now, let's check for stability by taking the integral of the absolute value.

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)|dt &= \int_{-\infty}^{\infty} h(t)dt \qquad \text{since } h(t) \text{ is always positive} \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{15t}[u(t-1) - u(t-100)]dt \\ &= \int_{1}^{100} \mathrm{e}^{15t}dt \\ &= \frac{1}{15}\mathrm{e}^{15t}\Big|_{1}^{100} \\ &= \frac{1}{15}(\mathrm{e}^{1500} - \mathrm{e}^{15}) \end{aligned}$$

Which is finite. So, the system is stable.

Problem 4 Consider the cascade of LTI systems with unit sample responses $h_1[n]$ and $h_2[n]$ depicted below:

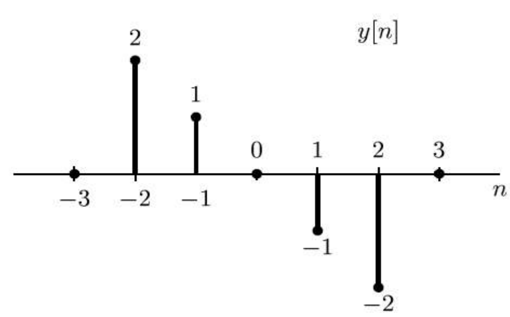


Suppose we are given the following information:

- $h_2[n] = \delta[n] - \delta[n-1]$
- If the input is

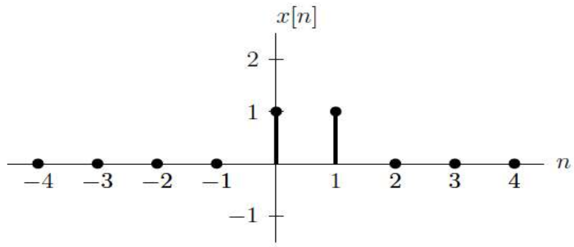
  $$x[n] = u[n] - u[n-2]$$

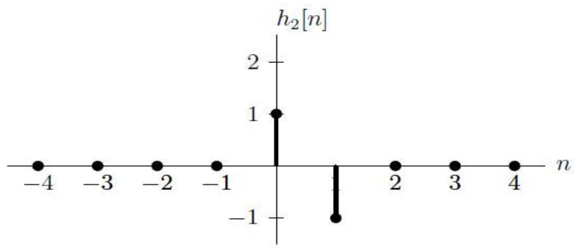
  then the output is as depicted below



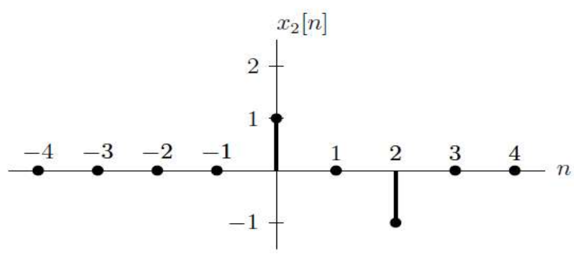
Find $h_1[n]$.

Since convolution is commutative, we can convolve $x[n]$ with $h_2[n]$ first followed by a convolution with $h_1[n]$ to get $y[n]$. Let's start by convolving $x[n]$ with $h_2[n]$ and denote the result of this as $x_2[n]$. $x[n]$and $h[n]$ are given by the following:

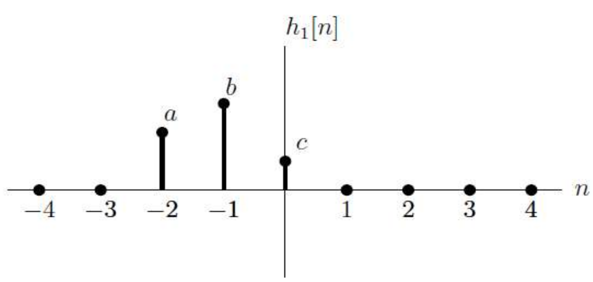




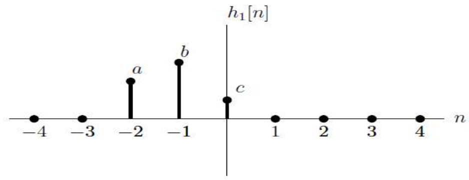
So, $x_2[n]$ is given by the following:

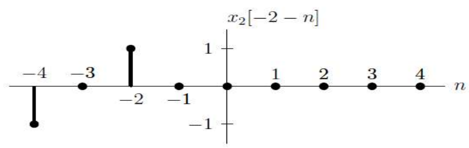


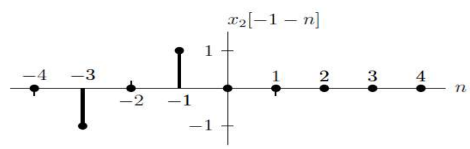
Now, we need to figure out the sequence $h_1[n]$ that when convolved with $x_2[n]$ produces $y[n]$. First, based on the starting and ending points of $y[n]$, we can determine the first and last non-zero points of $h_1[n]$. This is because the first non-zero point of $y[n]$ is at the time index that is the sum of the first non-zero indices of $x_2[n]$ and $h_1[n]$. Which means that the first non-zero point of $h_1[n]$ is at $n = -2$. Similarly, we know that the ending point is at the index $n = 0$ We can use flip and slide mechanics to determine the values of the samples between $n = -2$ and $n = 0$. Let the following be a general stem plot of $h_1[n]$:

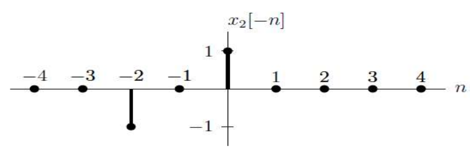


If we flip and slide $x_2[n]$ against $h_2[n]$ and compare against the given $y[n]$, we have the following:









From the plot of $x[-2-n]$ we find that:

$$\begin{aligned} y[-2] &= a+0 \\ a &= 2 \end{aligned}$$

From the plot of $x[-1-n]$ we find that:

$$\begin{aligned} y[-1] &= b+0 \\ b &= 1 \end{aligned}$$

From the plot of $x[-n]$ we find that:

$$\begin{aligned} y[0] &= c-a \\ 0 &= c-2 \\ c &= 2 \end{aligned}$$

Thus, we have found the entire sequence $h_1[n]$ which is given as follows: