Sydney Girls High School
2020
Trial Higher School Certificate
Examination

Mathematics Extension 1

General Instructions	• Reading time – 10 minutes • Working time – 2 hours • Write using a black pen • Calculators approved by NESA may be used • A reference sheet is provided • In Questions 11-14, show relevant mathematical reasoning and/or calculations
Total marks: 70	**Section I – 10 marks (pages 2-5)** • Attempt Questions 1-10 • Allow about 15 minutes for this section **Section II – 60 marks (pages 6-13)** • Attempt Questions 11-14 • Allow about 1 hour and 45 minutes for this section

Name: …………………………………………………… Teacher: ……………………………………………………	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2020 HSC Examination Paper in this subject.

Section 1

10 marks

Attempt Questions 1-10

Use the Multiple-choice answer sheet for questions 1-10

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1) An object moves with velocity $18\ \text{m/s}$ at an angle of $150°$ in the anticlockwise direction to the vector $\underset{\sim}{i}$. The vector representation of this velocity is :

A. $-9\sqrt{3}\,\underset{\sim}{i} - 9\,\underset{\sim}{j}$

B. $-9\,\underset{\sim}{i} - 9\sqrt{3}\,\underset{\sim}{j}$

C. $-9\sqrt{3}\,\underset{\sim}{i} + 9\,\underset{\sim}{j}$

D. $-9\,\underset{\sim}{i} + 9\sqrt{3}\,\underset{\sim}{j}$

2) What is the value of $\sin 2\theta$ given that $\sin\theta = \frac{\sqrt{3}}{4}$ and θ is obtuse ?

A. $-\frac{\sqrt{39}}{8}$

B. $-\frac{\sqrt{3}}{2}$

C. $\frac{\sqrt{3}}{2}$

D. $\frac{\sqrt{39}}{8}$

3) The equation $y = e^{ax}$ satisfies the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0.$

What are the possible values of a ?

A. $a = -2$ or $a = 3$

B. $a = -1$ or $a = 6$

C. $a = 2$ or $a = -3$

D. $a = 1$ or $a = -6$

4) Given that $4\sin\theta + 3\cos\theta = -5$, what is the value of $\tan\frac{\theta}{2}$?

A. -2

B. -1

C. 1

D. 2

5) Ten people are to be seated around a circular table. The ten people consists of five students and five parents. In how many ways can the people be arranged around the table if at least two of the students must sit next to each other ?

A. 3 600 000

B. 3 614 400

C. 360 000

D. 361 440

6) Given $0 < x < 1$, which of the following represents the derivative of $\sin^{-1}\sqrt{1-x^2}$?

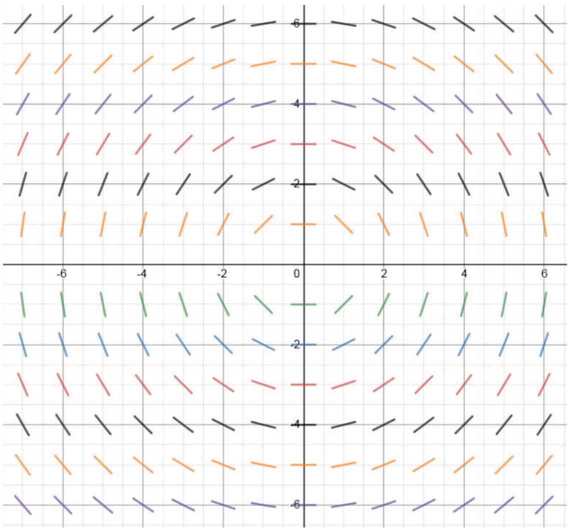
A. $\frac{-1}{x\sqrt{1-x^2}}$

B. $\frac{1}{\sqrt{1-x^2}}$

C. $\frac{1}{x\sqrt{1-x^2}}$

D. $\frac{-1}{\sqrt{1-x^2}}$

7) The slope field for a first order differential equation is shown below.



Which of the following could be the differential equation represented?

A. $\frac{dy}{dx} = \frac{x}{y}$

B. $\frac{dy}{dx} = -\frac{x}{y}$

C. $\frac{dy}{dx} = xy$

D. $\frac{dy}{dx} = -xy$

8) Given that $\sin\left(x - \frac{\pi}{3}\right) = 3\cos\left(x - \frac{\pi}{4}\right)$, find the exact value of $\tan x$.

A. $\frac{\sqrt{6}+6}{6-\sqrt{2}}$

B. $\frac{\sqrt{6}+6}{\sqrt{2}-6}$

C. $\frac{\sqrt{2}-6}{\sqrt{6}+6}$

D. $\frac{6-\sqrt{2}}{\sqrt{6}+6}$

9) Beryl made an error proving that $2^n + (-1)^{n+1}$ is divisible by 3 for all integers $n \geq 1$ using mathematical induction. The proof is shown below.

Step 1: Prove $2^n + (-1)^{n+1}$ is divisible by 3 (n is an integer) for $n = 1$.

$2^1 + (-1)^{1+1} = 2 + 1$

$= 3 \times 1$ <u>Line 1</u>

Result is true for $n = 1$.

Step 2: Assume true for $n = k$.

$2^k + (-1)^{k+1} = 3m$ (where m is an integer) <u>Line 2</u>

Step 3: Prove true for $n = k + 1$.

$2^{k+1} + (-1)^{k+1+1} = 2(2^k) + (-1)^{k+2}$ <u>Line 3</u>

$= 2[3m + (-1)^{k+1}] + (-1)^{k+2}$ <u>Line 4</u>

$= 2 \times 3m + 2 \times (-1)^{k+2} + (-1)^{k+2}$

$= 3[2m + (-1)^{k+2}]$

Which is a multiple of 3 since m and k are integers.

Step 4: Hence, proven true by mathematical induction for all integers $n \geq 1$.

In which line did Beryl make an error ?

A. Line 1

B. Line 2

C. Line 3

D. Line 4

10) The polynomial $x^2 + x + 2$ is a factor of the polynomial $x^4 + 4x^2 + x + a$.

Determine the value of the constant a.

A. $a = -6$

B. $a = -3$

C. $a = 3$

D. $a = 6$

Section II

60 marks

Attempt questions 11-14

Start each question on a NEW piece of paper

Question 11 (15 marks)

Use a new sheet of paper.

(a) Find $\frac{dy}{dx}$ given $y = x^3 \tan^{-1}(2x)$. 2

(b) Find : 2

$$\int \frac{dx}{\sqrt{12 - 4x^2}} \ .$$

(c) Use the substitution $x = \cos\theta$ (where $0 < \theta < \frac{\pi}{2}$) to determine : 3

$$\int \sqrt{\frac{x^2}{1-x^2}}\, dx \ .$$

(d) The movements of a drone flying straight and level at a constant speed are being monitored via a GPS screen. One unit on the screen represents 100 m in the air. The initial position vector of the drone on the screen is $5\underset{\sim}{i} + 12\underset{\sim}{j}$. An hour later its position vector on the screen is $15\underset{\sim}{i} + 30\underset{\sim}{j}$.

Find the distance travelled by the drone in km, correct to two decimal places. 2

(e)

(i) Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x + \alpha)$. 2

(ii) Hence, state the greatest value of the function : 1

$$f(x) = \cos x - \sqrt{3}\sin x + 10\,.$$

(f) Use the substitution $u = e^x$ to evaluate (correct to three decimal places) : 3

$$\int_{\ln 3}^{\ln 5} \frac{2e^x}{2 + e^{2x}}\, dx\ .$$

Question 12 (15 marks)

Use a new sheet of paper

(a) Consider the vectors $\underset{\sim}{a} = \begin{bmatrix} 4 \\ -\sqrt{5} \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 2 \\ \sqrt{5} \end{bmatrix}$.

Find the vector projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$. Express your answer in component form. 2

(b) Use the relationship $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$, or otherwise, to determine where the curve $y = \sin 5x + \sin x$ cuts the x −axis for $0 \le x \le \frac{\pi}{2}$. 2

(c) The variables x and θ satisfy the differential equation :

$$x \cos^2\theta \, \frac{dx}{d\theta} = 1 + 2\tan\theta \, ,$$

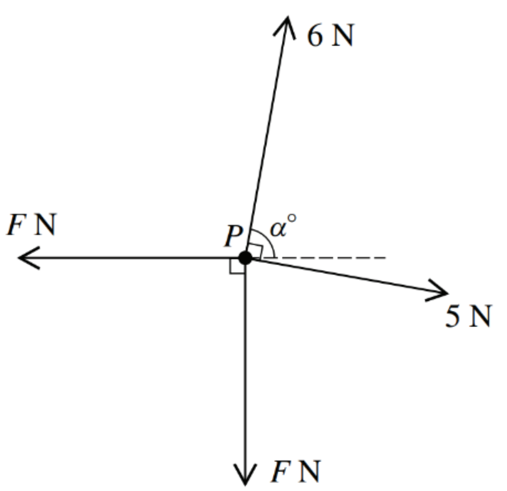
for $0 \le \theta \le \frac{\pi}{2}$ and $x > 0$. It is given that $x = 1$ when $\theta = \frac{\pi}{4}$.

(i) Show that $\frac{d}{d\theta}(\tan^2\theta) = \frac{2\tan\theta}{\cos^2\theta}$. 1

(ii) Solve the differential equation :

$$x \cos^2\theta \, \frac{dx}{d\theta} = 1 + 2\tan\theta \, .$$ 3

(d) A particle P is in equilibrium under the action of four forces of magnitude 6 N , 5 N , F N and F N acting in the directions shown in the diagram below.

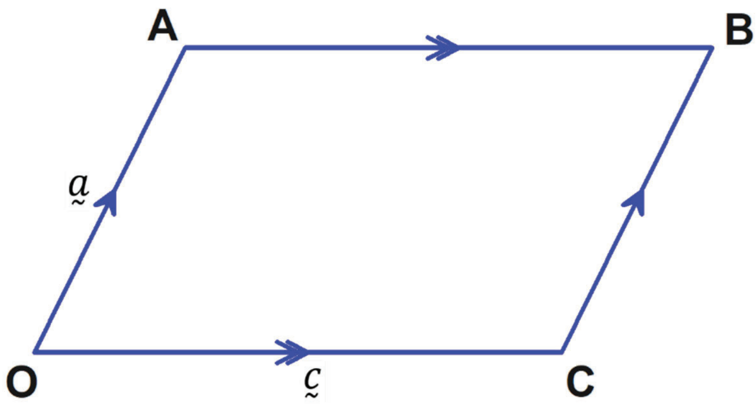


(i) By considering the vertical components of the forces, find an expression for F in terms of α. 1

(ii) Hence, determine the value of α , correct to the nearest degree. 2

Question 12 (continued)

(e) In the diagram below, $OABC$ is a parallelogram where $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.



(i) X is the midpoint of the line AC. Explain why X is also the midpoint of OB. **1**

(ii) OCD is a straight line so that $OC : CD = k : 1$.

Given that $\overrightarrow{XD} = 3\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}$, find the value of k. **3**

Question 13 (15 marks)

Use a new sheet of paper

(a)

A ball is thrown horizontally with speed $5\ \mathrm{ms}^{-1}$ from a point O on the roof of a building. At time t seconds after projection, the horizontal and vertical displacements of the ball from O are x metres and y metres respectively.

Note that after the ball is projected , $x > 0$ and $y < 0$.

Use $10\ \mathrm{m/s^2}$ for the acceleration due to gravity.

(i) Derive the expressions for x and y in terms of t. 2

(ii) Hence, show that the Cartesian equation of the trajectory of the ball is

$$y = -0.2x^2.$$ 1

The ball strikes the horizontal ground which surrounds the building at a point A.

(iii) Given that $OA = 18$ m, calculate the value of x at A. 2

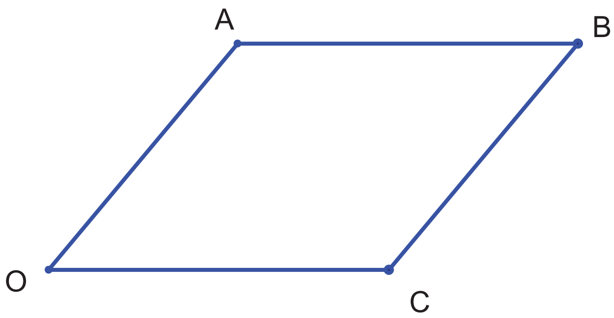
Give your answer correct to 2 decimal places.

(iv) Hence, calculate the speed of the ball as it strikes the ground at A. 2

Give your answer correct to 2 decimal places.

Question 13 (continued)

(b) Consider a rhombus $OABC$ where $\overrightarrow{OA} = \underset{\sim}{a} + \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{b}$.



(i) Find an expression for the vector $\overrightarrow{CA}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. 1

(ii) Find an expression for the scalar product $\overrightarrow{OB} \cdot \overrightarrow{CA}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. 1

(iii) Use the scalar product $\overrightarrow{OB} \cdot \overrightarrow{CA}$ to prove the diagonals are perpendicular. 2

(c) A rectangular reservoir has a horizontal base of area 1000 m^2 . At time $t = 0$, the reservoir is empty and water begins to flow into it at a constant rate of 30 m^3 s^{-1} . At the same time, water begins to flow out at a rate proportional to $\sqrt{h}$, where h m is the depth of the water at time t seconds. When $h = 1, \frac{dh}{dt} = 0.02$.

(i) Show that h satisfies the differential equation : 2

$$\frac{dh}{dt} = 0.01\left(3 - \sqrt{h}\right).$$

(ii) The differential equation is transformed using the substitution $x = 3 - \sqrt{h}$. 1
Show that the resulting differential equation is :

$$(x-3)\frac{dx}{dt} = 0.005x \; .$$

(iii) Given that $x = 3$ when $t = 0$, find t as a function of x. 1

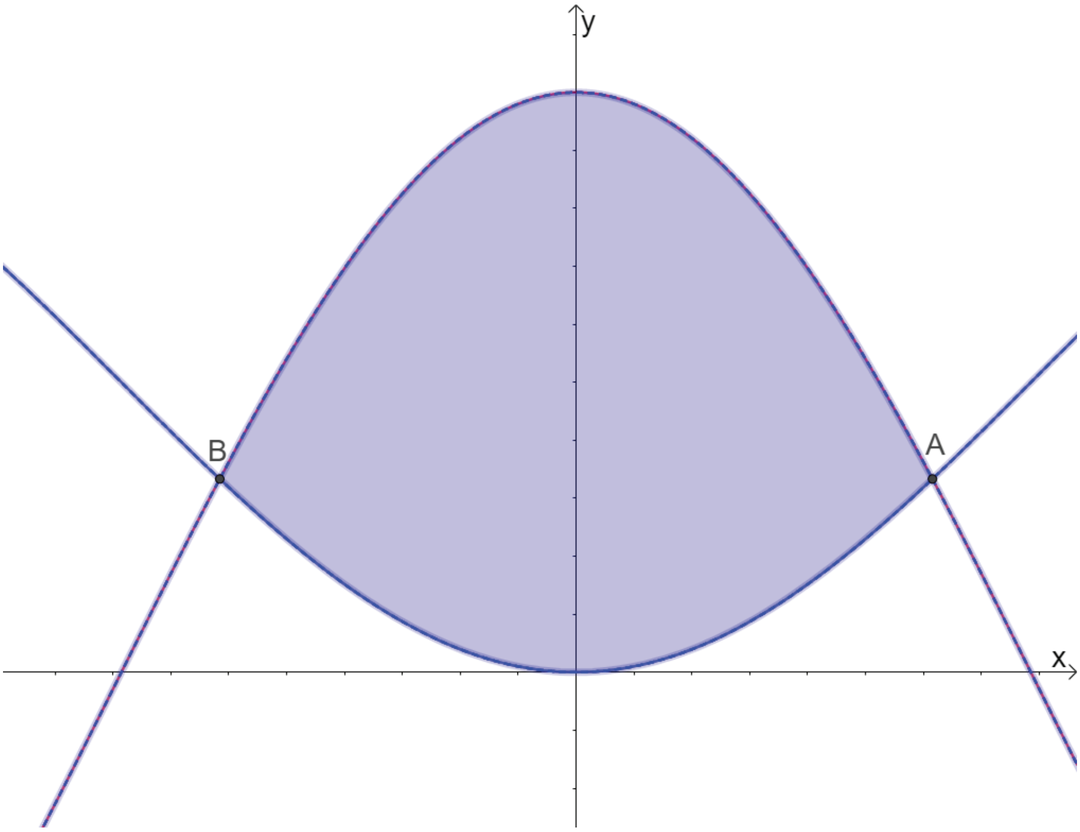
Question 14 (15marks)

Use a NEW sheet of paper

(a) Use mathematical induction to prove that, for all positive integers n , 3

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = \frac{(-1)^{n-1}n(n+1)}{2} .$$

(b) In the diagram below, the two curves shown are $y = \cos 2x$ and $y = \sin^2 x$.



(i) The two curves intersect at the points A and B.

Determine the coordinates of A. 2

(ii) The shaded region in the diagram is the area between the two curves $y = \cos 2x$ and $y = \sin^2 x$. This area is rotated about the x −axis to form a solid. Find the volume of this solid.

Give your answer correct to two decimal places. 3

Question 14 (continued)

(c) Solve the following inequation :

$$3^x < 4|1 - 3^x|$$

Express your answers correct to 3 decimal places. 3

(d) Find the Cartesian equation of the curve with the parametric equations : 2

$$\left.\begin{array}{l} x = 4\cos\theta - \sin\theta \\ y = 5\cos\theta + \sin\theta \end{array}\right\}$$

(e) Consider the function $f(x) = x + e^{5x} + 2$ and its inverse function $g(x)$.

Use the derivative of $f\big(g(x)\big)$ to determine the gradient of the tangent to the curve $y = g(x)$, at the point where $x = 3$. 2

End of paper

Sydney Girls High School
Mathematics Faculty

Multiple Choice Answer Sheet
Trial HSC Mathematics

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

correct

A ~~●~~ B ~~●~~ C ○ D ○

Student Number: EXTENSION 1

Completely fill the response oval representing the most correct answer.

1. A ○ B ○ C ● D ○
2. A ● B ○ C ○ D ○
3. A ○ B ○ C ● D ○
4. A ● B ○ C ○ D ○
5. A ○ B ○ C ● D ○
6. A ○ B ○ C ○ D ●
7. A ○ B ● C ○ D ○
8. A ○ B ● C ○ D ○
9. A ○ B ○ C ○ D ●
10. A ○ B ○ C ○ D ●

Question 11 (15 marks) - Yr 12 Trial - Ext 1 - 2020

a) $y = x^3 \tan^{-1}(2x)$

let $u = x^3$, $v = \tan^{-1}(2x)$

$\frac{du}{dx} = 3x^2$, $\frac{dv}{dx} = \frac{2}{1+(2x)^2}$

$= \frac{2}{1+4x^2}$

(Most students completed this very well)

$\therefore \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$= 3x^2 . \tan^{-1}(2x) + \frac{2x^3}{1+4x^2}$ 2

b) $\int \frac{dx}{\sqrt{12-4x^2}} = \int \frac{dx}{\sqrt{4\left(\frac{12}{4}-x^2\right)}}$

$= \frac{1}{2}\int \frac{dx}{\sqrt{3-x^2}}$

$= \frac{1}{2}\int \frac{dx}{\sqrt{(\sqrt{3})^2-x^2}}$

$= \frac{1}{2}\sin^{-1}\frac{x}{\sqrt{3}} + C$ 2

(Most student completed this very well.)

- 1 mark was given to slight errors.

c) $x = \cos\theta$

$$\frac{dx}{d\theta} = -\sin\theta$$

$$dx = -\sin\theta . d\theta$$

$$\int \sqrt{\frac{x^2}{1-x^2}}\, dx = \int \sqrt{\frac{\cos^2\theta}{1-\cos^2\theta}} \times -\sin\theta\, d\theta$$

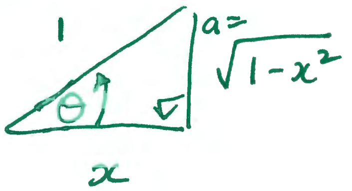
$$= \int \sqrt{\frac{\cos^2\theta}{\sin^2\theta}} \times -\sin\theta\, d\theta$$

$$= \int \frac{\cos\theta}{\cancel{\sin\theta}} \times -\cancel{\sin\theta}\, d\theta$$

$$= \int -\cos\theta\, d\theta$$

$\cos\theta = x$

$$= -\sin\theta + C \quad \cdots\ 2$$



$$= -\sqrt{1-x^2} + C \quad \cdots\ 1$$

$a^2 = 1^2 - x^2$

$a = \sqrt{1-x^2}$

$0 < \theta < \frac{\pi}{2}$

3

(This section was completed poorly)

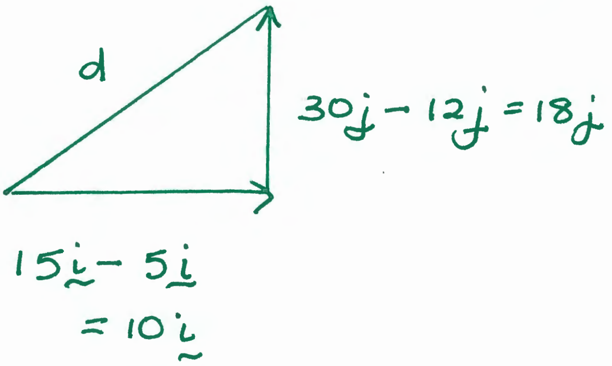
- 1 mark was given $\rightarrow \int -\cos\theta\, d\theta$

- 1 mark for correct integration $= -\sin\theta +$

- 1 mark for expressing in terms of x

* Most students didn't express final answer in terms of x.

d) $\text{Speed} = \dfrac{\text{distance travelled}}{\text{time taken}}$



(This section was completed well.) However, many students didn't express final answer in km.

$\therefore\ d^2 = 10^2 + 18^2$

$d = \sqrt{424}$ units $\times 100\text{m}$

$= 2059\text{ m}$

$\therefore$ distance $= 2.06$ km (2 d.p.).

2

e) i) $R = \sqrt{a^2+b^2}$

$= \sqrt{1^2+(\sqrt{3})^2}$

$= \sqrt{4}$

$\therefore R = 2$

$R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$

Now, $\cos x - \sqrt{3}\sin x = 2\cos x\cos\alpha - 2\sin x\sin\alpha$

$\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x = \cos x\cos\alpha - \sin x\sin\alpha$

$\therefore \left.\begin{aligned}\cos\alpha &= \tfrac{1}{2}\\ \sin\alpha &= \tfrac{\sqrt{3}}{2}\end{aligned}\right]$ 1st quadrant.

(This part was completed very well with most students getting correct answer)

$\therefore\ \tan\alpha = \frac{\sqrt{3}}{2} \div \frac{1}{2}$

$\tan\alpha = \sqrt{3}$

$\alpha = \tan^{-1}\sqrt{3}$

$\alpha = \frac{\pi}{3}$ $(60°)$

$\therefore\ \cos x - \sqrt{3}\sin x \equiv R\cos(x+\alpha)$

$\equiv 2\cos(x + \pi/3)$

2

e) ii) Greatest value of

$$f(x) = \cos x - \sqrt{3}\sin x + 10$$

Since from i) $\cos x - \sqrt{3}\sin x \equiv 2\cos(x + \frac{\pi}{3})$

Amplitude $= 2$

$\therefore$ Heightest Value $= 2$

Now: $f(x) = \cos x - \sqrt{3}\sin x + 10$

$$= 2\cos(x + \frac{\pi}{3}) + 10$$

Hence, greatest value of function

$= 2 + 10$

$= 12$ units.

1

(This section was completed very poorly)

- Most students could not get correct answer and complicated concept.

f) $u = e^{x}$ $\quad u = e^{\ln 5}$ $\quad u = e^{\ln 3}$

$\frac{du}{dx} = e^{x}$ $\quad u = 5$ $\quad u = 3$

$du = e^{x} \cdot dx \rightarrow dx = \frac{du}{e^{x}}$

also, $u = e^{x}$

$u^2 = (e^{x})^2$

$u^2 = e^{2x}$

$$\therefore \int_{\ln 3}^{\ln 5} \frac{2e^{x}}{2 + e^{2x}}\, dx = 2\int_{3}^{5} \frac{du}{2 + u^2}$$

$$= 2 \times \left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}}\right]_3^5$$

$$= \frac{2}{\sqrt{2}} \left[\tan^{-1} \frac{5}{\sqrt{2}} - \tan^{-1} \frac{3}{\sqrt{2}}\right]$$

$$= \frac{2}{\sqrt{2}} \times 0.16486$$

$$= 0.233 \quad (3\ d.p).$$

- This part was completed well. $\quad$ 3
- 1 mark was given to $2\int_{3}^{5} \frac{du}{2+u^2}$)
- 1 mark was given to integration $= 2\left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}}\right]_3^5$
- 1 mark was given to correct answer in radians.

* Most students lost a mark because they didn't find answer in radians.

Ext 1 2020

12) a)

$$\text{Proj}_{b}\, a = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|}\cdot\frac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$= \frac{8-5}{\sqrt{9}}\cdot\frac{2i+\sqrt{5}j}{\sqrt{9}}$$

$$= \frac{3}{9}(2i+\sqrt{5}j) \qquad = \frac{1}{3}(2i+\sqrt{5}j)$$

✓✓

To get 2 marks students had to have perfect solutions.

b)

$$\sin 5x + \sin x = 0$$

$$\sin(3x+2x) + \sin(3x-2x) = 0$$

$$2\sin 3x \cos 2x = 0$$

$$\sin 3x = 0 \quad \text{or} \quad \cos 2x = 0$$

✓

$$3x = 0, \pi, 2\pi, 3\pi, 4\pi \ldots \qquad 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$x = 0, \frac{\pi}{3}. \qquad x = \frac{\pi}{4}.$$

✓

c) i)

$$\frac{d}{d\theta}(\tan\theta)^2 = 2\tan\theta\cdot\sec^2\theta$$

$$= 2\tan\theta\cdot\frac{1}{\cos^2\theta}$$

$$= \frac{2\tan\theta}{\cos^2\theta}$$

✓

ii)

$$x\cos^2\theta\,\frac{dx}{d\theta} = 1 + 2\tan\theta$$

$$\int x\,dx = \int \frac{1+2\tan\theta}{\cos^2\theta}\,d\theta$$ ✓

$$\frac{x^2}{2} = \int \frac{1}{\cos^2\theta} + \frac{2\tan\theta}{\cos^2\theta}\,d\theta$$ ✓

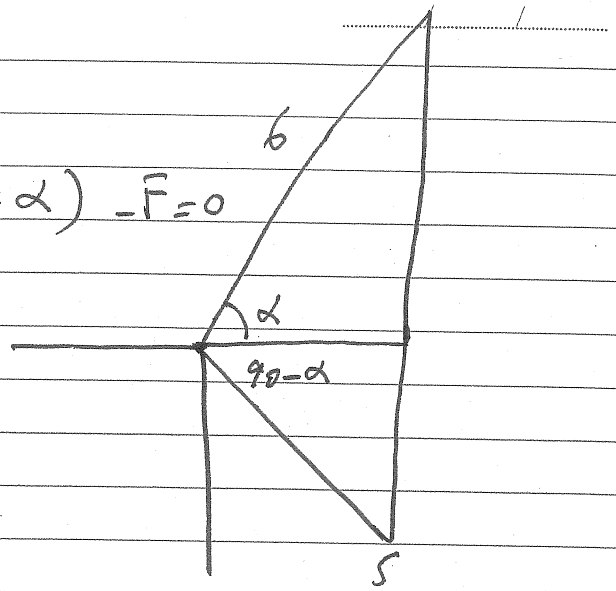
many students didn't find C correctly

$$\frac{x^2}{2} = \tan\theta + \tan^2\theta + C$$ ✓

$$x = 1 \rightarrow \theta = \frac{\pi}{4} \quad \therefore C = -\frac{3}{2}$$

$$\boxed{x^2 = 2\tan\theta + 2\tan^2\theta - 3}$$

d) i)



V: $6\sin\alpha - 5\sin(90-\alpha) - F = 0$

$6\sin\alpha - 5\cos\alpha - F = 0$

✓

H: $6\cos\alpha + 5\sin\alpha - F = 0$

$6\sin\alpha - 5\cos\alpha - F = 0$ −

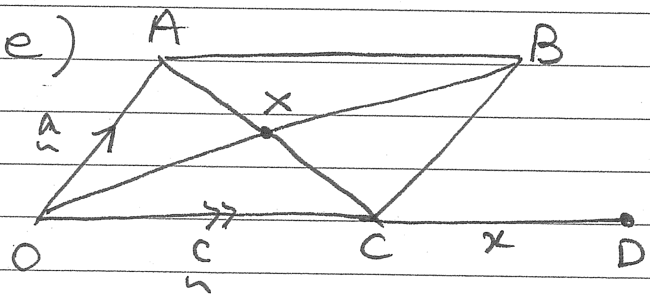
$11\cos\alpha - \sin\alpha = 0$

$\sin\alpha = 11\cos\alpha$

$\tan\alpha = 11$

$\alpha = 85°$ ✓✓

e)



Many students waste time to do this quest.

i) Diagonals of a parallelogram bisect each other ✓

ii) $OC : CD = k : 1$

$\overrightarrow{XD} = 3\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}$

$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$

$\underset{\sim}{a} + \overrightarrow{AC} = \underset{\sim}{c}$ ✓

$\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$

$\overrightarrow{AX} = \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{a})$

$\overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$

$\underset{\sim}{c} + \underset{\sim}{a} = \overrightarrow{OB}$ ✓

$\overrightarrow{OX} = \frac{1}{2}\overrightarrow{OB}$

$\overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$

$\overrightarrow{OB} = \underset{\sim}{c} + \underset{\sim}{a}$

$\overrightarrow{OX} = \frac{1}{2}(\underset{\sim}{c} + \underset{\sim}{a})$

$\frac{c}{x} = \frac{k}{1}$

$x = \frac{c}{k}$

$\overrightarrow{OX} + \overrightarrow{XD} = \overrightarrow{OD}$

$\frac{1}{2}(\underset{\sim}{c} + \underset{\sim}{a}) + 3\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a} = \underset{\sim}{c} + \overrightarrow{CD}$

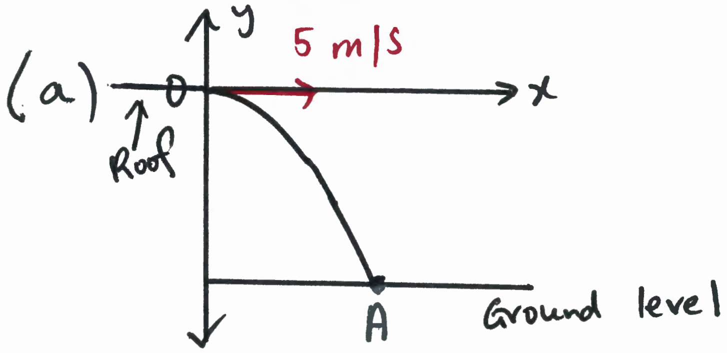
$\frac{1}{2}\underset{\sim}{c} + 3\underset{\sim}{c} = \underset{\sim}{c} + \frac{c}{k}$ ✓

$\frac{1}{2}\underset{\sim}{c} + 2\underset{\sim}{c} = \frac{c}{k}$

$\frac{5\underset{\sim}{c}}{2} = \frac{c}{k}$ $\therefore$ $\boxed{k = \frac{2}{5}}$

Question 13 (a)

(a)



(i)

Horizontally

$\ddot{x} = 0$

$\dot{x} = C_1$

Vertically

$\ddot{y} = -10$

$\dot{y} = -10t + C_2$

when $t = 0$, $\dot{x} = 5$, $\dot{y} = 0$

$5 = C_1$

$\dot{x} = 5$

$x = 5t + C_3$

$0 = C_2$

$\dot{y} = -10t$

$y = -5t^2 + C_4$

when $t = 0$, $x = 0$, $y = 0$

$0 = C_3$

$0 = C_4$

$\therefore \quad x = 5t \quad$ and $\quad y = -5t^2$

To derive the expressions for x and y, students were expected to start with the acceleration and integrate, showing evaluation of constants.
An alternative approach was to work with vector notation.
Some students had "θ" in their working but $\theta = 0$ since the projection is horizontal.

Question 13 (a) continued

(ii) Since $x = 5t$, $t = \frac{x}{5}$.

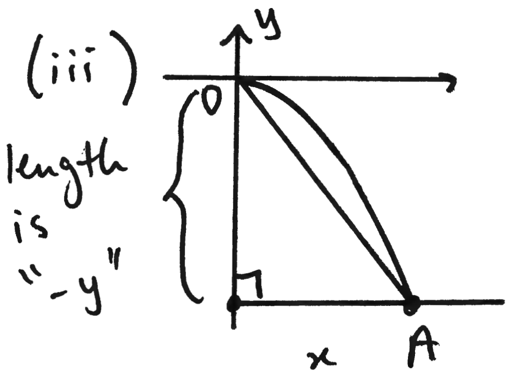
$$y = -5 \times \left(\frac{x}{5}\right)^2$$

$$= -5 \times \frac{x^2}{25}$$

$$\therefore y = -0.2x^2$$

Well done across the cohort.

(iii)



$$OA^2 = x^2 + (-y)^2 \qquad x^2 = \frac{y}{-0.2}$$

$$18^2 = \frac{y}{-0.2} + y^2$$

$$y^2 - 5y - 324 = 0$$

$$y = \frac{5 \pm \sqrt{25 - 4(-324)}}{2}$$

$$= \frac{5 \pm \sqrt{1321}}{2}$$

Observe that at A, $y < 0$.

$$\therefore y = \frac{5 - \sqrt{1321}}{2}$$

$$x^2 = -\frac{1}{0.2} \times \frac{5 - \sqrt{1321}}{2}$$

$$\doteqdot 78.36$$

$$x = \sqrt{78.36} \quad (x > 0)$$

$$\therefore x \doteqdot 8.85$$

Generally, done well but a number of students did not seem to process that OA is the hypotenuse.

Alternative approaches included use of reducible quadratic equations in terms of x and t.

Question 13(a) continued

(iv) $t = \frac{x}{5}$

$= \frac{8.85}{5}$

$t \doteqdot 1.77$

Most students who did (iii) correctly, also did (iv) correctly. However, some students attempted to use (incorrectly) : $\sqrt{x^2+y^2}$ for velocity.

Velocity at this time : $\dot{x} = 5$

$\dot{y} = -10 \times 1.77$

$= -17.77$

Resultant speed $= \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \sqrt{5^2 + (-17.77)^2}$

$\doteqdot 18.39$ m/s

(b) (i) $\vec{OC} + \vec{CA} = \vec{OA}$

$\vec{CA} = \vec{OA} - \vec{OC}$

$= \underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{b}$

$\therefore \vec{CA} = \underset{\sim}{a}$

Most students did this correctly.

(ii) $\vec{OB} \cdot \vec{CA} = (\vec{OA} + \vec{AB}) \cdot \vec{CA}$

$= (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{b}) \cdot \underset{\sim}{a}$

$= (\underset{\sim}{a} + 2\underset{\sim}{b}) \cdot \underset{\sim}{a}$

Mostly well done but some students need to be more careful with the vector notation. Showing the • is important and necessary.

Question 13 (b) continued

(iii) $|\vec{OA}| = |\vec{OC}|$ since OA and OC are sides of a rhombus.

$$|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{b}|$$

$$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{b} \cdot \underset{\sim}{b}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{b}$$

$$\therefore \underset{\sim}{a} \cdot \underset{\sim}{a} + 2\underset{\sim}{a} \cdot \underset{\sim}{b} = 0 \quad (*)$$

From (ii) $\vec{OB} \cdot \vec{CA} = (\underset{\sim}{a} + 2\underset{\sim}{b}) \cdot \underset{\sim}{a}$

$= \underset{\sim}{a} \cdot \underset{\sim}{a} + 2\underset{\sim}{a} \cdot \underset{\sim}{b}$ ← 1st mark allocated for this expansion

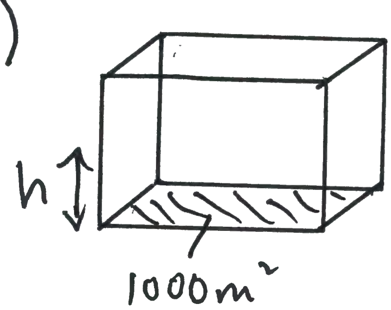
$= 0$ using $(*)$

$\therefore$ OB $\perp$ CA i.e. the diagonals are perpendicular.

Success varied across the grade. Again, some students have not mastered the use of vector notation.

Question 13 (c)

(i)



$V = 1000h \qquad \dfrac{dV}{dh} = 1000$

$$\frac{dV}{dt} = 30 - k\sqrt{h}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{1000} \times (30 - k\sqrt{h})$$

when $h = 1$, $\dfrac{dh}{dt} = 0.02$

$$0.02 = \frac{30-k}{1000}$$

$$30 - k = 20$$

$$k = 10$$

$$\therefore \frac{dh}{dt} = \frac{1}{1000} \times (30 - 10\sqrt{h})$$

$$= \frac{3-\sqrt{h}}{100} \quad \text{or} \quad 0.01(3-\sqrt{h})$$

Poorly done but most students - many "solutions" simply showed that the equation worked for the case $h=1$, $\frac{dh}{dt} = 0.02$. You needed to show it worked for <u>all</u> h. Showing the one case worked received no mark.

Question 13 (c) continued

(ii) $x = 3 - \sqrt{h}$ $\quad \therefore \dfrac{dh}{dt} = 0.01x$

$$\sqrt{h} = 3 - x$$

$$h = (3-x)^2$$

$$\frac{dh}{dx} = 2(3-x) \times -1$$

$$\frac{dh}{dt} = \frac{dh}{dx} \times \frac{dx}{dt}$$

$$0.01x = -2(3-x) \times \frac{dx}{dt}$$

$$= 2(x-3)\frac{dx}{dt}$$

$$(x-3)\frac{dx}{dt} = \frac{0.01x}{2}$$

i.e. $(x-3)\dfrac{dx}{dt} = 0.005x$

The quality of solutions varied and some students did not show clearly.

(iii) $$\int \frac{(x-3)\,dx}{x} = \int 0.005\,dt$$

$$\int \left(1 - \frac{3}{x}\right) dx = \frac{1}{200}\int dt$$

$$x - 3\ln|x| = \frac{1}{200}t + c$$

Many students did this question correctly.

when $x = 3$, $t = 0$

$$3 - 3\ln 3 = c$$

$$\therefore \quad x - 3\ln|x| = \frac{1}{200}t + 3 - 3\ln 3$$

i.e. $t = 200\left(x - 3\ln|x| - 3 + 3\ln 3\right)$

Q14

a) Prove it is true for $n = 1$.

$LHS = (-1)^0 \times 1 = 1$

$RHS = \frac{(-1)^0 \times 1(2)}{2} = 1$

$LHS = RHS = 1$ ✓

Assume it is true for $n = k$

$$1^2 - 2^2 + 3^2 - \dots + (-1)^{k-1} \times k^2 = \frac{(-1)^{k-1} \times k(k+1)}{2}$$

prove it is true for $n = k+1$

$$1^2 - 2^2 + 3^2 - \dots + (-1)^{k-1} \cdot k^2 + (-1)^k (k+1)^2 = \frac{(-1)^k (k+1)(k+2)}{2}$$

$$\frac{(-1)^{k-1} \cdot k \cdot (k+1)}{2} + (-1)^k \cdot (k+1)^2 = \frac{(-1)^k (k+1)(k+2)}{2}$$ ✓

$$LHS = \frac{(-1)^{k-1} \cdot k(k+1) + 2(-1)^k (k+1)^2}{2}$$

$$= (k+1)\left[\frac{k(-1)^k \cdot (-1)^{-1} + 2(k+1)(-1)^k}{2}\right]$$

$$= (k+1)\left[\frac{-k \cdot (-1)^k + (-1)^k (2k+2)}{2}\right]$$

$$= (k+1)\left[\frac{(-1)^k (2k+2-k)}{2}\right]$$

$$= \frac{(k+1)(k+2)(-1)^k}{2} = RHS$$ ✓

This is true for $n = k+1$. It is proven by Mathematical Induction.

Q14

b) $y = \cos 2x \quad , \quad y = \sin^2 x$

i) $\sin^2 x = \cos 2x$

$\sin^2 x - (1 - 2\sin^2 x) = 0$

$\sin^2 x = \frac{1}{3} \therefore \sin x = \pm \frac{1}{\sqrt{3}}$

At A: $\sin x = \frac{1}{\sqrt{3}}$

$x = 0.6155$ Rad

$y = 0.3333$ Rad

$A(0.6155, 0.3333)$ OR $A\left(\sin^{-1}\left(\frac{1}{\sqrt{3}}\right), \cos 2\left(\sin^{-1}\frac{1}{\sqrt{3}}\right)\right)$

ii) $V = 2\pi \int_0^{0.6155} \left[(\cos 2x)^2 - (\sin^2 x)^2\right] dx$

$V = 2\pi \int_0^{0.6155} (\cos^2 2x - \sin^4 x)\, dx$

$V = 2\pi \int_0^{0.6155} \left[\frac{1}{2}\cos 4x + \frac{1}{2} - \frac{1}{4}(1 - \cos 2x)^2\right] dx$

$V = 2\pi \int_0^{0.6155} \left(\frac{1}{2}\cos 4x + \frac{1}{2} - \frac{1}{4} + \frac{1}{2}\cos 2x - \frac{1}{4}\cos^2 2x\right) dx$

$V = 2\pi \int_0^{0.6155} \left[\frac{1}{2}\cos 4x + \frac{1}{4} + \frac{1}{2}\cos 2x - \frac{1}{4}\left(\frac{1}{2}\cos 4x + \frac{1}{2}\right)\right] dx$

Q14

b/ii)

$$V = 2\pi \int_0^{0.6155} \left(\frac{1}{2}\cos 4x + \frac{1}{4} + \frac{1}{2}\cos 2x - \frac{1}{8}\cos 4x - \frac{1}{8} \right) dx$$

$$V = 2\pi \int_0^{0.6155} \left(\frac{3}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{1}{8} \right) dx$$

$$V = 2\pi \left[\frac{3}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{x}{8} \right]_0^{0.6155}$$

$$V = 2\pi \left[\frac{3}{32}\sin(4 \times 0.6155) + \frac{1}{4}\sin 2(0.6155) + \frac{0.6155}{8} \right]$$

$$\underline{V = 2.52 \text{ units}^3}$$

Note: some students have $x = 0.6155$ (Rad) but the Mode was on Deg leads to incorrect Ans.

c) $3^x < 4|1 - 3^x|$

$$|1 - 3^x| > \frac{3^x}{4}$$

$$\left[\begin{array}{l} 1 - 3^x > \frac{3^x}{4} \\ 1 - 3^x < -\frac{3^x}{4} \end{array}\right. \quad \therefore \left[\begin{array}{l} 4 - 4 \cdot 3^x - 3^x > 0 \\ 4 - 4 \cdot 3^x + 3^x < 0 \end{array}\right.$$

$$\left[\begin{array}{l} 3^x < \frac{4}{5} \\ 3^x > \frac{4}{3} \end{array}\right. \quad \therefore \left[\begin{array}{l} x < \frac{\ln(4/5)}{\ln 3} \\ x > \frac{\ln(4/3)}{\ln 3} \end{array}\right. \quad \therefore \left[\begin{array}{l} x < -0.203 \\ x > 0.262 \end{array}\right.$$

$\boxed{Q14}$

d) $f(x) = x + e^{5x} + 2$

$f^{-1}(x) = g(x)$ OR $f[g(x)] = x$

For $f(0,3) \Longrightarrow g(3,0)$

Also $f'(x) = 1 + 5e^{5x}$

$f'[g(x)] = f'[g(x)] \cdot g'(x) = 1$

$f'[g(3)] \cdot g'(3) = 1$

$f'(0) \cdot g'(3) = 1$

But $f'(0) = 1 + 5e^{5\times 0} = 6$

$g'(3) = \dfrac{1}{f'(0)} = \dfrac{1}{6}$

$\boxed{14e}$ $x = 4\cos\theta - \sin\theta$ (1)

$y = 5\cos\theta + \sin\theta$ (2)

1 Mark for having either eqs (a) or (b)

(1) + (2) : $\boxed{x + y = 9\cos\theta}$ (a)

(1) x 5 : $5x = 20\cos\theta - 5\sin\theta$ (3)

(2) x 4 : $4y = 20\cos\theta + 4\sin\theta$ (4)

(4) - (3) : $\boxed{4y - 5x = 9\sin\theta}$ (b)

Adding square (a) and square (b)

$x^2 + 2xy + y^2 + 16y^2 - 40xy + 25x^2 = 81$

OR $26x^2 + 17y^2 - 38xy - 81 = 0$