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Dynamics and Regularization of the Kepler Problem on Surfaces of Constant Curvature

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Abstract. We classify and analyze the orbits of the Kepler problem on surfaces of constant curvature (both positive and negative, $\mathbb{S}^2$ and $\mathbb{H}^2$, respectively) as functions of the angular momentum and the energy. Hill's regions are characterized, and the problem of time-collision is studied. We also regularize the problem in Cartesian and intrinsic coordinates, depending on the constant angular momentum, and we describe the orbits of the regularized vector field. The phase portraits both for $\mathbb{S}^2$ and $\mathbb{H}^2$ are pointed out.

1 Introduction

The n-body problem in spaces of constant curvature has a long history, starting with the works of Lobachevsky and Bolyai, the co-discovers of the non Euclidean geometries, passing through Schering and Liebmann, and more recently Cariñena et al. [2] and by Kozlov and Harin [10]. In [7] the reader can find a nice chronology of this problem for $n = 2$. The study of this problem for $n \geq 3$ starts with the works of Diacu, Pérez-Chavela, and Santoprete [4–6]. After this breakthrough many authors have made important contributions on the subject; among others, we can cite [14, 21, 22].

In this paper we tackle the Kepler problem defined in a two-dimensional space of constant curvature that could be positive or negative. By the Kepler problem we understand that one particle of mass 1 (after normalization) is fixed and the other one, of arbitrary positive mass μ is moving with respect to it under the influence of the respective law of attraction. When the two particles are moving freely on the corresponding surface of constant curvature, it is called the 2-body problem. Contrary to the classical Newtonian 2-body problem where both problems are equivalent, due to the fact that the linear momentum is a first integral, this is not the case when the particles are moving on surfaces of constant curvature. In fact, here the Kepler problem is an integrable problem, but the two body problem is not [16].

As we will see later in this paper, it is enough to do the analysis on the surface $\mathbb{S}^2$ embedded in $\mathbb{R}^3$ (for positive curvature) and on the hyperbolic surface $\mathbb{H}^2$ embedded in the 3-dimensional Minkowski space (for curvature negative).

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Our contribution is to describe the Hill's regions for the Kepler problem on $\mathbb{S}^2$ and $\mathbb{H}^2$, showing how they depend of the sign of the total energy h, a first integral of this problem. Concerning the singularities of this problem we show that all them occur in finite time. For the case of positive curvature, we also show that in order to have collision, a necessary condition is that the angular momentum should be zero ($c = 0$).

By using spherical and hyperbolic coordinates in $\mathbb{S}^2$ and $\mathbb{H}^2$, respectively, we get the phase portrait of the Kepler problem on these surfaces, showing their dependence with respect to the total energy h and the angular momentum c. Doing a central projection for the solution curves of the Kepler problem on $\mathbb{S}^2$ and $\mathbb{H}^2$ we obtain a nice analogy between them and the solutions (conic curves) of the classical Newtonian Kepler problem.

Finally, we regularize all singularities in both cases (positive and negative curvature) by using a Levi–Civita type regularization. We do this in Cartesian coordinates and also in spherical and hyperbolic coordinates in order to have more elements for possible applications of this technique in more complicated problems. It is well known for people in the field that a Levi–Civita regularization of the Newtonian Kepler problem generates a harmonic oscillator, whereas in the case of the Kepler problem on positive curvature spaces and $c = 0$, it produces a Mathieu equation.

The paper is organized as follows. In Section 2 we review results and introduce the potential that we will use throughout the paper. We give the equations of motion for this problem. In Section 3 we study the first integrals of the problem and do a deep analysis of the Hill's regions in both cases for positive and negative curvature. In Section 4 we study the singularities of the problem. For positive and negative curvature, all collisions correspond to singularities. For positive curvature, we also have the "antipodal singularity" corresponding to antipodal positions of the two masses (this term was introduced in [4,7]). In Section 5 we analyse the phase portrait of the Kepler problem on $\mathbb{S}^2$ and $\mathbb{H}^2$. In Section 6 we study the central projection for the solutions of the Kepler problem on $\mathbb{S}^2$ and $\mathbb{H}^2$. Finally, in Section 7 we regularize all singularities of the problem, and in the small Section 8 we give the conclusions of this work.

2 Statement of the Problem

We know that all surfaces of constant curvature κ are characterized by the sign of the curvature as follows. If $\kappa > 0$, the surface is the 2 dimensional sphere of radius $R = 1/\sqrt{\kappa}$ denoted by

$$\mathbb{S}^2_\kappa = \{(x, y, z) \mid x^2 + y^2 + z^2 = \kappa^{-1}\},$$

embedded in the Euclidean space $\mathbb{R}^3$. If $\kappa = 0$, we recover the Euclidean space $\mathbb{R}^2$. If $\kappa < 0$, the surface is given by

$$\mathbb{H}^2_\kappa = \{(x, y, z) \mid x^2 + y^2 - z^2 = \kappa^{-1},\ z \geq \sqrt{-\kappa^{-1}}\},$$

which corresponds to the upper part of the hyperboloid $x^2 + y^2 - z^2 = \kappa^{-1}$ embedded in the 3-dimensional Minkowski space $\mathbb{R}^{2,1}$, *i.e.*, $\mathbb{R}^3$ endowed with the Lorentz inner product (see for instance [8,9]).

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Following [2], we can define a curved trigonometry in our surfaces, which we will call trigonometric κ-functions. The main utility of the κ-functions is that we can unify elliptical and hyperbolic trigonometry. We define the *κ-sine* as

$$\mathrm{sn}_\kappa(x) := \begin{cases} \kappa^{-1/2}\sin\kappa^{1/2}x & \text{if } \kappa>0,\\ (-\kappa)^{-1/2}\sinh(-\kappa)^{1/2}x & \text{if } \kappa<0, \end{cases}$$

the *κ-cosine* as

$$\mathrm{csn}_\kappa(x) := \begin{cases} \cos\kappa^{1/2}x & \text{if } \kappa>0,\\ \cosh(-\kappa)^{1/2}x & \text{if } \kappa<0, \end{cases}$$

as well as the *κ-tangent* and *κ-cotangent*, as

$$\mathrm{tn}_\kappa(x) := \frac{\mathrm{sn}_\kappa(x)}{\mathrm{csn}_\kappa(x)} \qquad \mathrm{ctn}_\kappa(x) := \frac{\mathrm{csn}_\kappa(x)}{\mathrm{sn}_\kappa(x)},$$

respectively. It is easy to verify the fundamental formula $\kappa\,\mathrm{sn}_\kappa^2(x)+\mathrm{csn}_\kappa^2(x)=1$.

If $\mathbf{a}=(a_x,a_y,a_z)$ and $\mathbf{b}=(b_x,b_y,b_z)$ in $\mathbb{R}^3$, we define $\mathbf{a}\odot\mathbf{b}$ as either of the inner products

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &:= a_xb_x+a_yb_y+a_zb_z, && \text{if } \kappa>0,\\ \mathbf{a}\boxdot\mathbf{b} &:= a_xb_x+a_yb_y-a_zb_z, && \text{if } \kappa<0. \end{aligned}$$

Also, we define an extended distance between two points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$ induced by the above scalar product:

$$d_\kappa(\mathbf{a},\mathbf{b}) := \begin{cases} \kappa^{-1/2}\cos^{-1}\Big(\dfrac{\kappa\mathbf{a}\cdot\mathbf{b}}{\sqrt{\kappa\mathbf{a}\cdot\mathbf{a}}\sqrt{\kappa\mathbf{b}\cdot\mathbf{b}}}\Big), & \kappa>0,\\ (-\kappa)^{-1/2}\cosh^{-1}\Big(\dfrac{\kappa\mathbf{a}\boxdot\mathbf{b}}{\sqrt{\kappa\mathbf{a}\boxdot\mathbf{a}}\sqrt{\kappa\mathbf{b}\boxdot\mathbf{b}}}\Big), & \kappa<0. \end{cases}$$

Notice that when $\mathbf{a}$ and $\mathbf{b}$ are points on the surface, the extended distance $d_\kappa(\mathbf{a},\mathbf{b})$ coincides with the geodesic distance on the corresponding surface.

Now we are going to define the Kepler problem on both surfaces $\mathbb{S}^2_\kappa$ and $\mathbb{H}^2_\kappa$. Let us consider a particle with fixed position $\widetilde{\mathbf{q}}=(0,0,(\sigma\kappa)^{-1/2})$, where $\sigma=\kappa/|\kappa|$, and mass $m_1>0$ and one free particle with position $\mathbf{q}=(x,y,z)$ remaining on one of the surfaces $\mathbb{S}^2_\kappa$ or $\mathbb{H}^2_\kappa$, with mass $m_2>0$. By normalizing the masses, we can suppose without loss of generality that $m_1=1$ and $m_2=\mu>0$ and choosing a suitable scale of units such that the universal gravitational constant is $\mathcal{G}=1$. Using the approach given in [2], we define the κ-dependent cotangent potential as $-U_\kappa(\mathbf{q})$, where

$$U_\kappa(\mathbf{q}) := \mu\,\mathrm{ctn}_\kappa\big[d_\kappa(\mathbf{q},\widetilde{\mathbf{q}})\big]$$

and the kinetic energy

$$T_\kappa(\mathbf{q},\dot{\mathbf{q}}) := \frac{1}{2}(\dot{\mathbf{q}}\cdot\dot{\mathbf{q}})(\kappa\mathbf{q}\cdot\mathbf{q}).$$

Using the above potential and kinetic energy, we obtain the Lagrangian

$$L_\kappa(\mathbf{q},\dot{\mathbf{q}}) := T_\kappa(\mathbf{q},\dot{\mathbf{q}})+U_\kappa(\mathbf{q}),$$

and the Hamiltonian

$$H_\kappa(\mathbf{q},\mathbf{p}) := T_\kappa(\mathbf{q},\mathbf{p}) - U_\kappa(\mathbf{q}), \tag{2.1}$$

where $\mathbf{p} = \dot{\mathbf{q}}$ is the momentum of the particle of unit mass, associated with the Kepler problem on a surface of constant curvature with a fixed point at $\widetilde{\mathbf{q}}$.

Now, we note that for $\kappa \neq 0$, we can consider the $(\sigma\kappa)^{1/4}$-symplectic linear transformation $E(\mathbf{q},\mathbf{p}) = ((\sigma\kappa)^{1/2}\mathbf{q}, (\sigma\kappa)^{-1/4}\mathbf{p}) = (Q,P)$, then the Hamiltonian (2.1) assumes the simplified form

$$\widehat{H}(Q,P) = (\sigma\kappa)^{1/4} H_\kappa(E^{-1}(Q,P)) = (\sigma\kappa)^{3/4} H_\sigma(Q,P).$$

By introducing the time-rescaling $\frac{dt}{d\tau} = (\sigma\kappa)^{-3/4}$, we observe that when κ is a nonzero constant, it is sufficient to study the cases $\kappa = 1$ and $\kappa = -1$. From now on, we just study such cases and keep the notation $\mathbf{q}$ and $\mathbf{p}$ for the position and linear momentum, respectively.

A straightforward computation shows that the potential can be written as

$$U_\sigma(\mathbf{q}) := \mu \frac{\dfrac{\sigma\widetilde{\mathbf{q}} \odot \mathbf{q}}{\sqrt{\sigma\mathbf{q} \odot \mathbf{q}}}}{\left[\sigma - \sigma\left(\dfrac{\sigma\widetilde{\mathbf{q}} \odot \mathbf{q}}{\sqrt{\sigma\mathbf{q} \odot \mathbf{q}}}\right)^2\right]^{1/2}}.$$

From the equations of Euler–Lagrange with constraints, we obtain that the equations of motion of the particle of unit mass on $\mathbb{S}^2$ ($\sigma = 1$) or $\mathbb{H}^2$ ($\sigma = -1$) are given by

$$\ddot{\mathbf{q}} = -\sigma(\dot{\mathbf{q}} \odot \dot{\mathbf{q}})\mathbf{q} + \mu \frac{\widetilde{\mathbf{q}} - (\sigma\widetilde{\mathbf{q}} \odot \mathbf{q})\mathbf{q}}{\left[\sigma - \sigma(\sigma\widetilde{\mathbf{q}} \odot \mathbf{q})^2\right]^{3/2}}, \quad \mathbf{q} \odot \mathbf{q} = \sigma, \quad \mathbf{q} \odot \dot{\mathbf{q}} = 0, \tag{2.2}$$

and the Hamiltonian equations are

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{p}, \\ \dot{\mathbf{p}} &= -\sigma(\mathbf{p} \odot \mathbf{p})\mathbf{q} + \mu \frac{\widetilde{\mathbf{q}} - (\sigma\widetilde{\mathbf{q}} \odot \mathbf{q})\mathbf{q}}{[\sigma - \sigma(\sigma\widetilde{\mathbf{q}} \odot \mathbf{q})^2]^{3/2}}, \quad \mathbf{q} \odot \mathbf{q} = \sigma, \quad \mathbf{q} \odot \mathbf{p} = 0. \end{aligned} \tag{2.3}$$

3 First Integrals and Hill's Region

The equations of motion (2.3) have energy integral

$$h = \frac{1}{2}(\mathbf{p} \odot \mathbf{p}) - \mu \frac{\sigma\widetilde{\mathbf{q}} \odot \mathbf{q}}{\sqrt{\sigma - \sigma(\sigma\widetilde{\mathbf{q}} \odot \mathbf{q})^2}}, \tag{3.1}$$

which can be written using Cartesian coordinates as

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \sigma\dot{z}^2) - \frac{\mu z}{\sqrt{\sigma - \sigma z^2}} = h.$$

Additionally, we define the angular momentum as the vector

$$\mathbf{c} := (c_1, c_2, c_3), \quad \text{with} \quad c_i = (\mathbf{q} \times \mathbf{p}) \odot e_i, \, i = 1,2,3.$$

Proposition 3.1 *The third component $c = c_3$ of the angular momentum is a first integral of the system (2.3).*

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Proof Let us write

$$\mathbf{c} = \sum_{i=1}^{3} \big[(\mathbf{q} \times \mathbf{p}) \odot e_i \big] e_i.$$

Then we have that

$$\dot{\mathbf{c}} = \sum_{i=1}^{3} \big[(\mathbf{q} \times \dot{\mathbf{p}}) \odot e_i \big] e_i = \frac{\mu}{(\sigma - \sigma z^2)^{3/2}} (y, -x, 0), \tag{3.2}$$

which completes the proof. ∎

Remark 3.2 The values of c_1 and c_2 can be obtained integrating equation (3.2). We observe that $\mathbf{c} \odot \mathbf{q} = 0$ and $\dot{\mathbf{c}} \odot \mathbf{q} = 0$, which means that the system has nonzero total rotation with respect to the origin only in the x-y plane.

3.1 Hill's Region of the Kepler Problem on $\mathbb{S}^2$

From equation (3.1) fixing the value of h for $\sigma = 1$, we have that the Hamiltonian function defines an invariant set given by

$$\Sigma_h = \{ (\mathbf{q}, \mathbf{p}) \in (\mathbb{S}^2 \smallsetminus N) \times \mathbb{S}^2 : H_1 = h \}$$

for each real constant h. Since the kinetic energy $\frac{1}{2}(\mathbf{p} \cdot \mathbf{p})$ is positive definite, in Σ_h we have that

$$U_1(\mathbf{q}) + h = \frac{1}{2}(\mathbf{p} \cdot \mathbf{p}) \geq 0.$$

In order to obtain the Hill's region that corresponds to the projection of Σ_h over the space of configuration (*i.e.,* positions), we can define it as the set

$$\mathcal{R}_h = \Big\{ \mathbf{q} \in \mathbb{S}^2 : U_1(\mathbf{q}) + h \geq 0 \Big\} = \Big\{ (x, y, z) \in \mathbb{S}^2 : \frac{\mu z}{\sqrt{1 - z^2}} + h \geq 0 \Big\}.$$

Thus, if $(\mathbf{q}(t), \mathbf{p}(t))$ is a solution of (2.3) with $H_1(\mathbf{q}(0), \mathbf{p}(0)) = h$, then $\mathbf{q}(t) \in \mathcal{R}_h$. We analyze the following cases for $\mathbf{q} = (x, y, z) \in \mathcal{R}_h$, the value $z = 1$ is not possible, it corresponds to a binary collision, a singularity in the energy relation that determines the Hill's region.

- If $h < 0$, then $z \in \big[\sqrt{h^2/(\mu^2 + h^2)}, 1 \big)$, thus the motion is restricted to a sphere cap contained in the upper hemisphere (see Figure 1(a)).
- If $h = 0$, then $z \geq 0$, thus the motion may occur on the whole upper hemisphere (see Figure 1(b)).
- If $h > 0$, then $z \in \big[-\sqrt{h^2/(\mu^2 + h^2)}, 1 \big)$, thus the motion may occur in a sphere cap which exceeds the Equator (see Figure 1(c)).

3.2 Hill's Region of the Kepler Problem on $\mathbb{H}^2$

The energy constant allows us to define a set that is not exactly the Hill's region defined as in the spherical case, but which gets information related to the region of motion. First, we note that the kinetic energy is not positive definite, from which we can only

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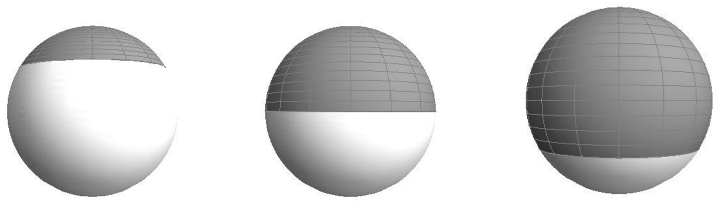


(a) (b) (c)

Figure 1: Hill's region $\mathcal{R}_h$ represented by the darkened region. (a) $h < 0$; (b) $h = 0$; (c) $h > 0$.

state the inequality

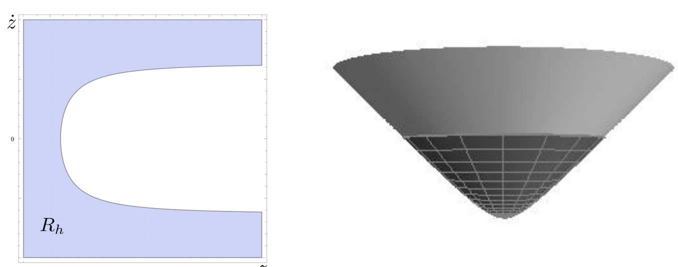
$$G_h(z, \dot{z}) := h + \frac{\dot{z}^2}{2} + \frac{\mu z}{\sqrt{z^2 - 1}} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) \geq 0,$$

which induce us to define the set

$$R_h = \left\{ (\mathbf{q}, \mathbf{p}) \in \mathbb{H}^2 \times \mathbb{R}^3 / \, G_h(z, \dot{z}) \geq 0 \right\}.$$

Thus, if $(\mathbf{q}(t), \mathbf{p}(t))$ is a solution of (2.3) with $H_{-1}(\mathbf{q}(0), \mathbf{p}(0)) = h$, then $\mathbf{q}(t) \in \mathcal{R}_h$. We analyze the following cases for $\mathbf{q} = (x, y, z) \in \mathcal{R}_h$.

- If $h \geq -\mu$, then $G_h(z, \dot{z}) \geq 0$ is always satisfied.
- If $h < -\mu$, we have that $G_h(z, \dot{z}) \geq 0$ implies that $1 \leq z < h/\sqrt{h^2 - \mu^2}$ and the Hill's region projected on the $(z, \dot{z})$ plane corresponds to the shaded region as in the Figure 2(a). Thus, the region of motion in $\mathbb{H}^2$ is as in Figure 2(b).



(a) (b)

Figure 2: The darkened region represents the projection of the Hill's region: (a) on the plane $(z, \dot{z})$ for $h < -\mu$, (b) on $\mathbb{H}^2$ for $h < -\mu$.

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4 Collisions

In the case of $\sigma = 1$, it is verified that equations (2.3) have singularities $\pm\widetilde{\mathbf{q}}$. The positive one is due to collision and the negative one is due to antipodal singularity, while for $\sigma = -1$, equation (2.3) has a unique singularity that is due to collision with $\widetilde{\mathbf{q}}$. Let us start by defining the moment of inertia of the system (2.3) as

$$I = \frac{1}{2}(\mathbf{q} - \widetilde{\mathbf{q}}) \odot (\mathbf{q} - \widetilde{\mathbf{q}}). \tag{4.1}$$

The first technical result is an adaptation of the Lagrange–Jacobi identity of the Newtonian Kepler problem (see [15]).

Lemma 4.1 *Let $\mathbf{q}(t)$ be a solution of (2.2). Then the following identity holds*

$$\ddot{I} = \sigma(\mathbf{p}\odot\mathbf{p})(\mathbf{q}\odot\widetilde{\mathbf{q}}) - \frac{1}{\sigma\mathbf{q}\odot\widetilde{\mathbf{q}}}U_\sigma(\mathbf{q}) = \left[2(\mathbf{q}\odot\widetilde{\mathbf{q}})^2 - 1\right]\frac{\mu}{\sqrt{\sigma - \sigma(\mathbf{q}\odot\widetilde{\mathbf{q}})^2}} + 2\sigma h(\mathbf{q}\odot\widetilde{\mathbf{q}}),$$

where h is the energy associated with the solution.

Proof By deriving twice in (4.1) along a solution and using the Euler formula and the energy integral, we have that

$$\begin{aligned}
\ddot{I} &= \mathbf{p}\odot\mathbf{p} + (\mathbf{q} - \widetilde{\mathbf{q}})\odot\dot{\mathbf{p}} \\
&= \mathbf{p}\odot\mathbf{p} + (\mathbf{q} - \widetilde{\mathbf{q}})\odot(-\sigma(\mathbf{p}\odot\mathbf{p})\mathbf{q} + \overline{\nabla}U_\sigma(\mathbf{q})) \\
&= \sigma(\mathbf{p}\odot\mathbf{p})(\mathbf{q}\odot\widetilde{\mathbf{q}}) - \frac{1}{\sigma\mathbf{q}\odot\widetilde{\mathbf{q}}}U_\sigma(\mathbf{q}) \\
&= 2\sigma(h + U_\sigma(\mathbf{q}))(\mathbf{q}\odot\widetilde{\mathbf{q}}) - \frac{1}{\sigma\mathbf{q}\odot\widetilde{\mathbf{q}}}U_\sigma(\mathbf{q}) \\
&= \left[2(\sigma\mathbf{q}\odot\widetilde{\mathbf{q}}) - \frac{1}{\sigma\mathbf{q}\odot\widetilde{\mathbf{q}}}\right]U_\sigma(\mathbf{q}) + 2\sigma h(\mathbf{q}\odot\widetilde{\mathbf{q}}) \\
&= \left[2(\mathbf{q}\odot\widetilde{\mathbf{q}})^2 - 1\right]\frac{\mu}{\sqrt{\sigma - \sigma(\mathbf{q}\odot\widetilde{\mathbf{q}})^2}} + 2\sigma h(\mathbf{q}\odot\widetilde{\mathbf{q}}).
\end{aligned}$$ ■

Theorem 4.2 *Let $\mathbf{q}(t)$ be a solution of (2.2) and suppose that $\mathbf{q}(t) \to \widetilde{\mathbf{q}}$ when $t \to t^*$; then $t^* \in \mathbb{R}$.*

Proof Note first that if $\mathbf{q}(t) \to \widetilde{\mathbf{q}}$, then $I \to 0$ when $t \to t^*$. In addition, from Lemma 4.1, it follows clearly that $\ddot{I} \to +\infty$ when $t \to t^*$. Suppose that $t^* = +\infty$; then, given $K > 0$, there exists $t_0 > 0$ such that $\ddot{I}(t) > K$, for all $t > t_0$. By integrating twice on both sides of this inequality, we get $I \geq \frac{K}{2}t^2 + At + B$ for all $t \geq t_0$. Thus, $I \to +\infty$ when $t \to \infty$, which contradicts the fact that $I \to 0$. ■

Remark 4.3 Note that in the case of $\mathbb{S}^2$, Theorem 4.2 applies to both, collision and antipodal singularities

The following result is an adaptation of the Sundman's inequality (see [15]). Here, the inequality is formulated for the case of $\mathbb{S}^2$.

Lemma 4.4 *Let $\mathbf{q} = \mathbf{q}(t)$ be a solution of (2.2) for $\sigma = 1$, with third component of the angular momentum $c = (\mathbf{q} - \widetilde{\mathbf{q}}) \times \mathbf{p}) \cdot \widetilde{\mathbf{q}}$ and energy h given by (3.1). Then, along this solution, the following inequality holds:*

$$c^2 \le \frac{4}{2(\mathbf{q}\cdot\widetilde{\mathbf{q}})^2 - 1} I\big((\mathbf{q}\cdot\widetilde{\mathbf{q}})\ddot{I} - h\big).$$

Proof It follows from Lemma 4.1 that

$$\ddot{I} = \|\mathbf{p}\|^2 \mathbf{q}\cdot\widetilde{\mathbf{q}} - \frac{1}{\mathbf{q}\cdot\widetilde{\mathbf{q}}}\Big(h - \frac{1}{2}\|\mathbf{p}\|^2\Big) = \frac{\|\mathbf{p}\|^2}{2\mathbf{q}\cdot\widetilde{\mathbf{q}}}\big(2(\mathbf{q}\cdot\widetilde{\mathbf{q}})^2 - 1\big) + \frac{h}{\mathbf{q}\cdot\widetilde{\mathbf{q}}},$$

which implies that

$$\|\mathbf{p}\|^2 = \frac{2\mathbf{q}\cdot\widetilde{\mathbf{q}}}{2(\mathbf{q}\cdot\widetilde{\mathbf{q}})^2 - 1}\Big(\ddot{I} - \frac{h}{\mathbf{q}\cdot\widetilde{\mathbf{q}}}\Big).$$

Using the Cauchy–Schwarz inequality, we obtain that

$$c^2 \le \|\mathbf{q} - \widetilde{\mathbf{q}}\|^2 \|\mathbf{p}\|^2 = \frac{4}{2(\mathbf{q}\cdot\widetilde{\mathbf{q}})^2 - 1} I\big((\widetilde{\mathbf{q}}\cdot\mathbf{q})\ddot{I} - h\big),$$

which completes the proof. ∎

Now, we are going to prove a modified version of the Sundman Theorem related to total collision (see [15,19]), which is adapted to the Kepler problem on $\mathbb{S}^2$.

Theorem 4.5 *Let $\mathbf{q}(t)$ be a solution of (2.2) for $\sigma = 1$. If $\mathbf{q}(t) \to \widetilde{\mathbf{q}}$ when $t \to t^* \in \mathbb{R}$, then $c = 0$.*

Proof Suppose that $\mathbf{q} \to \widetilde{\mathbf{q}}$ when $t \to t^* \in \mathbb{R}$, where we assume that $t^* > 0$ (analogously it is proved for $t^* < 0$). Then it follows from Lemma 4.1 that $\ddot{I}(t) > 0$ for $t \sim t^*$; therefore, $\dot{I}$ is increasing, which implies that $\lim_{t\to t^*} \dot{I}(t)$ exists. If $\dot{I} \to +\infty$ when $t \to t^*$, then I will be increasing for $t \sim t^*$. Moreover, $I \to 0$, which clearly forces $I < 0$ for $t \sim t^*$, which is impossible. Now, assume that $\dot{I} \to k > 0$, then $\dot{I} > 0$ for $t \sim t^*$, and by the same above argument, we obtain a contradiction. Therefore, $\lim_{t\to t^*} \dot{I}(t) \le 0$ and thus, $\dot{I}(t) \le 0$ for all $t \in J = (t^* - \epsilon, t^*]$, with ϵ small enough. Since $\mathbf{q}\cdot\widetilde{\mathbf{q}} \to 1$ when $t \to t^*$, then for ϵ small enough, we can assume that $\sqrt{3}/2 \le \mathbf{q}\cdot\widetilde{\mathbf{q}} \le 1$, for all $t \in J$, which implies that

$$1 \le \frac{1}{2(\mathbf{q}\cdot\widetilde{\mathbf{q}})^2 - 1} \le 2. \tag{4.2}$$

By using (4.2) to get an upper bound in the inequality given by Lemma 4.4, we obtain that $c^2 \le 8I(\ddot{I} - h)$, and multiplying this inequality by $-\dot{I}/I \ge 0$, we get that

$$-c^2 \frac{\dot{I}}{I} \le -8\ddot{I}\dot{I} + 8h\dot{I}, \quad \forall t \in J.$$

By taking $t^* - \epsilon < t_1 < s \le t^*$, with t_1 fixed and integrating in (t_1, s), it follows that

$$c^2 \ln\Big(\frac{I_1}{I(s)}\Big) \le -4(I(s) - I_1) + 8h(I(s) - I_1),$$

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where $I_1 = I(t_1)$. Notice that we can take ϵ small enough and s close enough to t^* to obtain that $\ln(I_1/I(s)) > 0$, thus

$$c^2 \leq -\frac{4(I(s)^2 - I_1^2)}{\ln(I_1/I(s))} + \frac{8h(I(s) - I_1)}{\ln(I_1/I(s))}. \tag{4.3}$$

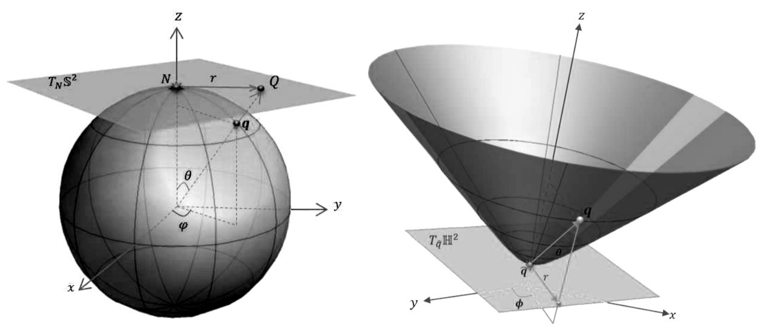
Finally, we make $s \to t^*$; then the right side of (4.3) tends to zero and $c = 0$ as claimed. ∎

5 Phase Portrait of the Kepler Problem on $\mathbb{S}^2$ and $\mathbb{H}^2$

The surfaces $\mathbb{S}^2$ and $\mathbb{H}^2$ can be parametrized by

$$x = \operatorname{sn}_\sigma \theta \cos\varphi, \quad y = \operatorname{sn}_\sigma \theta \sin\varphi, \quad z = \operatorname{csn}_\sigma \theta, \quad \varphi \in [0, 2\pi], \tag{5.1}$$

where $\theta \in [0, \pi]$ if $\sigma = 1$ and $\theta \in [0, +\infty)$ if $\sigma = -1$. In these coordinates, the equations



(a) (b)

Figure 3: (a) spherical coordinates and central projection of $\mathbb{S}^2$ onto $T_N\mathbb{S}^2$. (b) Hyperbolic coordinates and central projection of $\mathbb{H}^2$ onto $T_N\mathbb{H}^2$.

of motion become

$$\begin{aligned} \ddot\theta &= \dot\varphi^2 \operatorname{sn}_\sigma \theta \operatorname{csn}_\sigma \theta - \mu \operatorname{csc}_\sigma{}^2 \theta, \\ \ddot\varphi &= -2\dot\theta\dot\varphi \operatorname{ctn}_\sigma \theta, \end{aligned} \tag{5.2}$$

with $\varphi \in [0, \pi]$, $\theta \in (0, \pi)$ if $\sigma = 1$ and $\theta \in (0, +\infty)$ if $\sigma = -1$. Additionally, the constant of motion given by the angular momentum is expressed as $c = \dot\varphi \operatorname{sn}_\sigma{}^2 \theta$, thus it is sufficient to solve the first equation of (5.2), which becomes

$$\ddot\theta = c^2 \operatorname{ctn}_\sigma \theta \operatorname{csc}_\sigma{}^2 \theta - \mu \operatorname{csc}_\sigma{}^2 \theta. \tag{5.3}$$

By defining the variable $v = \dot\theta$, the equation (5.3) can be written as the Hamiltonian system of one degree of freedom

$$\begin{aligned} \dot\theta &= v, \\ \dot v &= c^2 \operatorname{csc}_\sigma{}^2 \theta \Big(\operatorname{ctn}_\sigma \theta - \frac{\mu}{c^2} \Big), \end{aligned} \tag{5.4}$$

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with Hamiltonian function

$$K_\sigma(\theta, \nu) = \frac{\nu^2}{2} + \frac{c^2}{2\,\mathrm{sn}_\sigma{}^2\,\theta} - \mu\,\mathrm{ctn}_\sigma\,\theta. \tag{5.5}$$

5.1 Phase Portrait for the Case $c \neq 0$

Let $h = K_\sigma$ a fixed energy level, and note that (5.5) induces the definition of the auxiliary function

$$F_h(\theta) = 2h + 2\mu\,\mathrm{ctn}_\sigma\,\theta - c^2 \csc_\sigma{}^2\,\theta, \tag{5.6}$$

which satisfies $F_h(\theta) \geq 0$, for all $\theta = \theta(t)$ solution of (5.3). Also, the orbits of the Hamiltonian system (5.4) are given by the curves $(\theta, \pm\sqrt{F_h(\theta)})$; thus, in order to determine the phase portrait of (5.4), it is sufficient to know the graph of the function $F_h(\theta)$. We start analyzing the phase portrait on $\mathbb{S}^2$. We do $\sigma = 1$ in each equation and definition depending on σ. The following theorem characterizes the orbits of the system (5.4) for $\sigma = 1$.

Theorem 5.1 *Let $c \neq 0$ and $h_* = -\frac{1}{2c^2}(\mu^2 - c^4)$. If $h \geq h_*$, then every orbit of the system* (5.3) *is periodic and its phase portrait in the plane $(\theta, \dot{\theta})$ is as in Figure* 4(b)*, in particular, the flow is complete. Moreover, there are values $\theta_1, \theta_2 \in (0, \pi)$ such that the solution $\theta(t)$ of* (5.3) *satisfies $\theta_1 \leq \theta(t) \leq \theta_2$. In particular, $[\theta_1, \theta_2] \subset (0, \pi/2)$ if and only if $h \in \left[h_*, \frac{c^2}{2}\right)$. Finally, there are no solutions of the system* (5.3) *if $h < h_*$.*

Proof It is verified that for $\sigma = 1$, the function $F_h(\theta)$ has an absolute maximum $\theta_* = \mathrm{arccot}(\mu/c^2)$, which is given by

$$F_h(\theta_*) = 2h + \frac{1}{c^2}(\mu^2 - c^4),$$

where, also, we get that $(\theta_*, 0)$ is an equilibrium solution, being a center of the system (5.4). Now, from the condition $F_h(\theta) \geq 0$, in order to assure the existence of the solution $\theta(t)$, we must impose that the constant of energy associated with such solution satisfies

$$h \geq -\frac{1}{2c^2}(\mu^2 - c^4) = h_*.$$

On the other hand, by defining the new variable $u = \cot\theta$ and by replacing in the equation $F_h(\theta) = 0$, we obtain the quadratic equation

$$u^2 - \frac{2\mu}{c^2}u + \frac{(c^2 - 2h)}{c^2} = 0,$$

whose solutions are

$$u_\pm = \frac{\mu}{c^2} \pm \frac{1}{|c|}\sqrt{2(h - h_*)},$$

both reals, since $h \geq h_*$ and $u_+ > u_- > 0$, if and only if, $h < \frac{c^2}{2}$, while $u_- > 0 > u_+$ if and only if $h > \frac{c^2}{2}$. Thus, the solutions of the equation $F_h(\theta) = 0$ are given by

$$\theta_1 = \mathrm{arccot}\,u_+, \qquad \theta_2 = \begin{cases} \mathrm{arccot}\,u_-, & \text{if } h \leq \frac{c^2}{2}, \\ \pi + \mathrm{arccot}\,u_-, & \text{if } h > \frac{c^2}{2}. \end{cases}$$

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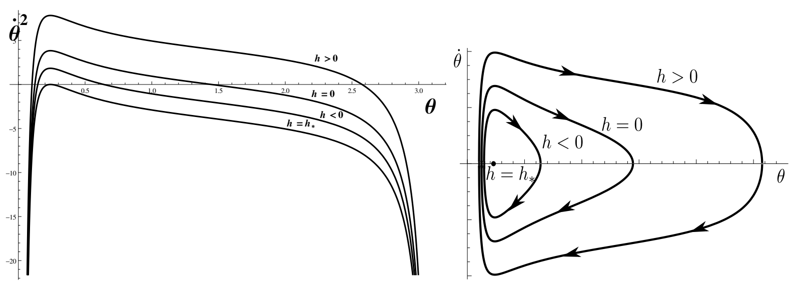
Therefore, we have proved the existence of values $\theta_1, \theta_2 \in (0, \pi)$ such that $\theta_1 \leq \theta(t) \leq \theta_2$, for all time t in which the solution is defined. From the above arguments, we conclude that if $h < \frac{c^2}{2}$, then $\theta(t)$ is contained on the upper hemisphere, since $\theta_1, \theta_2 \in (0, \pi/2)$, also $\lim_{h\to\infty} \theta_1 = 0$ and $\lim_{h\to\infty} \theta_2 = \pi$, so the range of θ approaches to the interval $(0, \pi)$ for large values of h. It follows from the above analysis that the graph of F_h is as in Figure 4(a), and therefore, the phase portrait given by the curves $(\theta, \pm F_h(\theta))$ is as in Figure 4. ■



(a) (b)

Figure 4: Graph of $F_h(\theta)$. (a) $\mu > c^2$; (b) $\mu \leq c^2$.

Remark 5.2 From the phase portrait in Figure 4, we observe that there are no solutions tending to collision, *i.e.,* $\theta(t) \nrightarrow 0$. This property agrees with Theorem 4.5. We observe also that there are not antipodal singularity either, *i.e.,* $\theta(t) \nrightarrow \pi$.

Now we analyze the phase portrait on $\mathbb{H}^2$. As in the previous case, here we set $\sigma = -1$ in each equation and definition depending on σ. The following result characterizes the orbits of the system (5.4) for $\sigma = -1$.

Theorem 5.3 *Let $h = K_{-1}$ be the constant of energy defined in (5.5), suppose that $c \neq 0$ and define the constant $h_* = -(c^4 + \mu^2)/2c^2$.*

If $c^2 < \mu$, then the following hold:

(i) *$(\theta_*, 0)$, with $\theta_* = \operatorname{arccoth}\left(\frac{\mu}{c^2}\right)$ is an equilibrium solution of the system (5.4) and it is a center stable in the sense of Lyapunov;*

(ii) *The motion is periodic if $h_* \leq h < -\mu$;*

(iii) *The motion is parabolic if $h = -\mu$ (*i.e., *$v \to 0$ when $\theta \to \infty$);*

(iv) *The motion is hyperbolic if $h > -\mu$ (*i.e., *$v \to v_0 \neq 0$ when $\theta \to \infty$);*

(v) *If $h \leq h_*$, there is no motion.*

If $c^2 \geq \mu$, then the following hold:

(vi) *The motion is hyperbolic if $h > -\mu$;*

(vii) *If $h \leq -\mu$, there is no motion.*

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Proof First we assume that $c^2 < \mu$. To prove (i), from equation (5.4), it follows that $(\theta_*, 0)$ is an equilibrium solution and it is well defined, because $\frac{\mu}{c^2} > 1$. By shifting the variables, we obtain the Hamiltonian $H(\theta + \theta_*, \nu)$, and its Taylor development up to order 2 around the origin has quadratic part

$$H_2 = \begin{pmatrix} \theta \\ \nu \end{pmatrix}^T \begin{pmatrix} \frac{(c^2-\mu)^2(c^2+\mu)^2}{c^6} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \theta \\ \nu \end{pmatrix},$$

which is positive definite, and therefore $(\theta_*, 0)$ is a center stable in the sense of Lyapunov. If we make $\sigma = -1$ in the auxiliary function (5.6), it is clear that if (θ, ν) is a solution of (5.4), then $|\nu| = \sqrt{F_h(\theta)}$. Thus, to prove (ii), (iii), (iv), and (v), we need only study the graph of F_h. By straightforward computations we get

$$\lim_{\theta \to 0^+} F_h(\theta) = -\infty \quad \text{and} \quad \lim_{\theta \to +\infty} F_h(\theta) = 2(\mu + h),$$

and the equation $F_h(\theta) = 0$ has solutions

$$\theta_\pm = \operatorname{arccoth}\left[\frac{\mu \pm \sqrt{4\mu^2 + 2c^2 h + c^4}}{c^2} \right]. \tag{5.7}$$

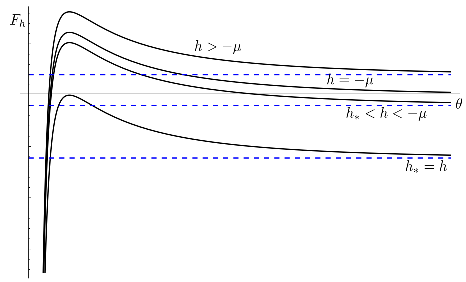
If $c^2 < \mu$, then $F_h(\theta)$ has an absolute maximum in θ_* with $F_h(\theta_*) = \frac{c^4+2c^2h+\mu^2}{c^2} \geq 0$ if and only if $h \geq h_*$. Now, if $h + \mu \geq 0$, then only θ_+ is real and $\theta_+ \leq \theta(t)$ for all time t in which the solution is defined. If $h + \mu < 0$ and $h \geq h_*$, then both values $\theta_\pm$ are reals and $\theta_+ \leq \theta(t) \leq \theta_-$ for all time t, where the solution is defined. Finally, if $h < h_*$, then $F_h(\theta) < 0$, which implies that there is no motion. From the above arguments, we conclude that for $c^2 < \mu$, the graph of F_h is as in Figure 5(a), while the orbits of the system (5.4) are the curves $(\theta, \pm\sqrt{F_h(\theta)})$ represented in Figure 5(b). If we make $\theta \to \infty$ in the equation $K(\theta, u) = h$, we get $\nu^2 \to 2(h + \mu)$, which implies that the motion is elliptic when $\mu + h < 0$, is parabolic if $\mu + h = 0$, and is hyperbolic if $\mu + h > 0$.

For the case $c^2 > \mu$, the system (5.4) does not have equilibrium solutions. On the other hand, the function $F_h(\theta)$ does not have critical points; it is always increasing, $F_h(\theta) \to 2(\mu + h)^-$ when $\theta \to \infty$, and it cuts the θ axis at a single point θ_+ given in (5.7), which exists if and only if $\mu + h > 0$. For $\mu + h \leq 0$ we have $F_h(\theta) < 0$, in which case there is no motion. From the above arguments, it follows that the graph of $F_h(\theta)$ is as in Figure 6(a), and the phase portrait is as in Figure 6(b). Similarly to the previous case, we get that if $\theta \to \infty$. Then ν tends to a positive value, and therefore, the motion is hyperbolic, which completes the proof. ■

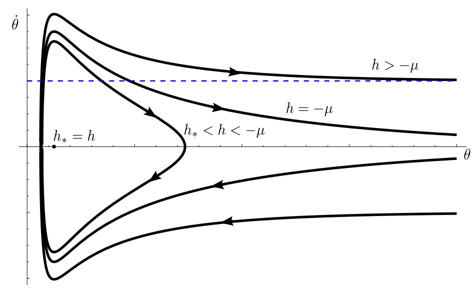
From Theorem 5.3 we obtain the following corollary.

Corollary 5.4 *If $c \neq 0$, then there are no solutions of* (5.4) *tending to collision.*

Remark 5.5 We observe that from Theorems 5.1 and 5.3 we get that all non-collision bounded orbits are periodic in both cases $\sigma = \pm 1$. This fact was already proved by H. Liebmann in 1902 [13].

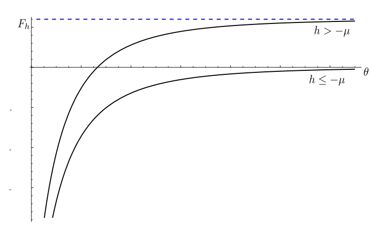


(a)

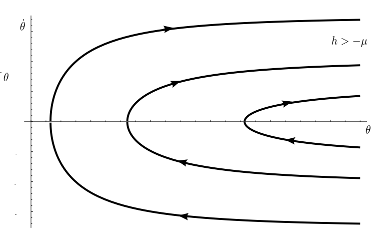


(b)

Figure 5: (a) Graph of $F_h(\theta)$ for $\sigma = -1$, $c^2 < \mu$ and different values of energy. (b) Phase portrait of the system (5.4) for $\sigma = -1$ and $c^2 < \mu$.



(a)



(b)

Figure 6: (a) Graph of $F_h(\theta)$ for $c^2 > \mu$ and different values of the energy parameter. (b) Phase portrait of the system (5.4) for $c^2 > \mu$.

5.2 Phase Portrait for the Case $c = 0$

It is clear that, for $c = 0$, the particle moves along of a geodesic containing the fixed and the free particle. In this case, (5.3)–(5.6) become

(5.8) $$\ddot\theta = -\mu \csc_\sigma{}^2\theta, \qquad h = \frac{v^2}{2} - \mu \operatorname{ctn}_\sigma \theta,$$

(5.9) $$\dot\theta = v, \qquad \dot v = -\mu \csc_\sigma{}^2\theta,$$

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and

$$F_h(\theta) = 2h + 2\mu \operatorname{ctn}_\sigma \theta, \tag{5.10}$$

with θ in its corresponding phase space.

5.2.1 Phase Portrait for $c = 0$ on $\mathbb{S}^2$

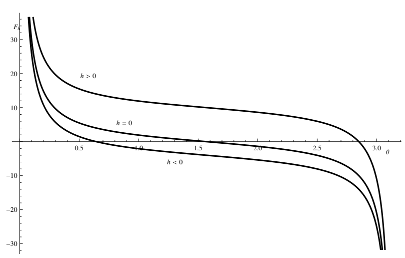
In this case, we take $\sigma = 1$ in (5.8)–(5.10), and we define

$$\theta_1 = \begin{cases} \pi + \operatorname{arccot}(-\frac{h}{\mu}) & \text{if } h > 0, \\ \operatorname{arccot}(-\frac{h}{\mu}) & \text{if } h \le 0. \end{cases} \tag{5.11}$$

Notice that $\lim_{h\to+\infty} \theta_1 = \pi$ and $\lim_{h\to-\infty} \theta_1 = 0$. It is easy to see that $F_h(\theta) \ge 0$ if and only if $\theta \in (0, \theta_1]$, so the graphic of F_h is as in Figure 7(a). The following result characterizes the orbits of the system (5.9).

Theorem 5.6 *If $c = 0$ and $\theta(t)$ is a solution of the first equation of (5.8), for $\sigma = 1$, then $\theta(t)$ remains in the interval $(0, \theta_1]$ for all time in which the solution is defined, with θ_1 defined in (5.11). If $h < 0$, then the orbit does not cross the equator,* i.e., $\theta_1 \in (0, \pi/2)$, *if $h = 0$, the orbit crosses the equator only once,* i.e., $\theta_1 = \pi/2$, *and if $h > 0$, the orbit crosses the equator twice,* i.e., $\theta_1 \in (\pi/2, \pi)$. *The phase portrait of the system (5.9) is as Figure 7(b).*

Proof Along a solution of (5.8), we have that $F_h(\theta(t)) \ge 0$, which occurs if and only if $\theta(t) \in (0, \theta_1]$ for all time in which the solution is defined. If we consider the graph of F_h given in Figure 7(a) and the fact that the solution satisfies $|\dot{\theta}| = \sqrt{F_h(\theta)}$, then we obtain that the phase portrait is as in Figure 7(b), which concludes the proof. ∎



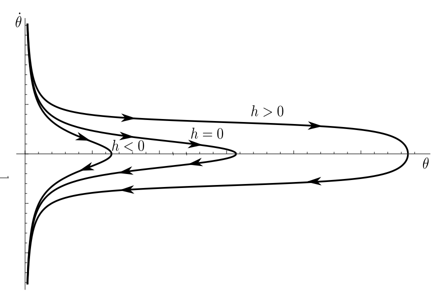


Figure 7: (a) Graph of $F_h(\theta)$. (b) Phase portrait for the system (5.9)

Remark 5.7 An immediate consequence obtained from Theorem 5.6 is that for $c = 0$, there are no solutions tending to antipodal singularity. This can easily be seen from the fact that all solution $\theta(t) \in (0, \theta_1]$, with $\theta_1 \in (0, \pi)$.

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Remark 5.8 An important difference between the classical Newtonian collinear Kepler problem and the Kepler problem on $\mathbb{S}^2$ for $c = 0$ is that in the last case every solution comes from a collision and goes to a collision, in both cases with equal velocity. But in the first one, this kind of behaviour holds only for negative energy, since for the nonnegative energy, the particle comes from (or tends to) a collision and tends to (or comes from) infinity with constant velocity.

5.2.2 Phase Portrait for $c = 0$ on $\mathbb{H}^2$.

The following result characterizes the phase portrait of the system (5.9) for $\sigma = -1$.

Theorem 5.9 *If $c = 0$ and $\theta(t)$ is a solution of the first equation of (5.8), for $\sigma = -1$, then the following statements hold:*

(i) *All solutions go to (or come from) collision.*
(ii) *If $h + \mu < 0$, then the motion is bounded, with $\theta \in (0, \operatorname{arccoth}(-h/\mu))$ and the orbits are elliptic.*
(iii) *If $h + \mu = 0$, then the orbit is parabolic.*
(iv) *If $h + \mu > 0$, then the orbits are hyperbolic.*

The phase portrait is as in Figure 8(b).

Proof It verifies that, for $\sigma = 1$ and each value of h, F_h is decreasing for all $\theta > 0$ and $F_h(\theta) \to +\infty$ when $\theta \to 0$, which implies that all orbits are collision ones. On the other hand, $F_h(\theta) \to 2(\mu + h)^+$ when $\theta \to \infty$. If $h + \mu \geq 0$, then $F_h(\theta) > 0$ for all $\theta > 0$. If $h + \mu < 0$, then $F_h(\theta) \geq 0$ if and only if $\theta \in (0, \operatorname{arccoth}(-h/\mu))$, which implies that the motion is bounded. Thus, the graph of F_h is as in Figure 8(a). Since the orbits are contained in the curves $(\theta, \pm\sqrt{F_h(\theta)})$, it follows that the phase portrait is as in Figure 8(b). Finally, if $\theta \to \infty$ in the second equation of (5.8), we obtain that $\dot{\theta} \to 2(h + \mu)$, from which it follows that the orbit is elliptic if $h + \mu < 0$, parabolic if $h + \mu = 0$ and hyperbolic if $h + \mu > 0$. ∎

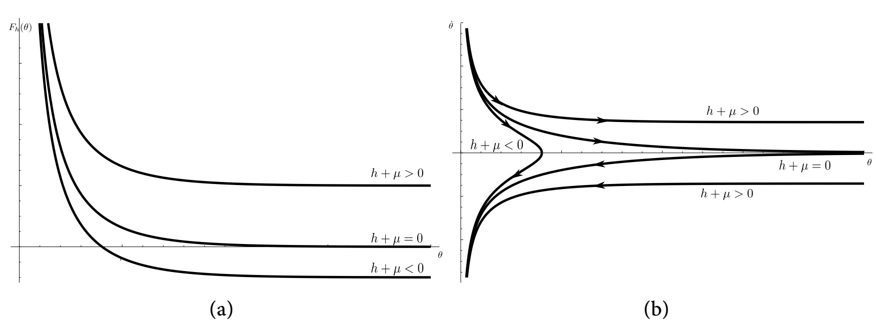

Figure 8: (a) Graph of $F_h(\theta)$. (b) Phase portrait of the system (5.9) for $\sigma = -1$

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6 Central Projection of the Kepler Problem on $\mathbb{S}^2$ and $\mathbb{H}^2$

In order to establish an analogy between the conic orbits of the Newtonian Kepler problem and the ones defined on the surfaces $\mathbb{S}^2$ and $\mathbb{H}^2$ as previously, we propose to identify every point of the surface with a point in the plane via central projection from the surface to the plane.

6.1 Central Projection for the Solutions of the Kepler Problem on $\mathbb{S}^2$

Points on the upper hemisphere are identified with a point in the tangent plane at the north pole by means of central projection, while points on the lower hemisphere are identified with their corresponding antipodal point which is on the upper hemisphere and then we proceed as previously described. In a similar way, we identify each point of the equator with its antipodal point and it is mapped with the infinity of $\mathbb{R}^2$. Notice that this correspondence is 2 to 1. Next, we write the equations of motion of the "projected problem" in suitable coordinates.

We start by establishing the correspondence between the points on $\mathbb{S}^2$ and $\mathbb{R}^2$ via inverse central projection (we call it simply central projection) defined from the upper hemisphere onto the tangent plane $T_N\mathbb{S}^2$, identified with $\mathbb{R}^2$. Such a transformation is given by

$$\begin{array}{rccc} \Phi^+\colon & H^+ & \longrightarrow & \mathbb{R}^2 \\ & (x,y,z) & \longmapsto & \left(\frac{x}{z},\frac{y}{z}\right). \end{array}$$

In order to extend Φ^+ to both hemispheres $H^+ \cup H^-$, we can identify every point $\mathbf{x} \in H^-$ with its antipodal $-\mathbf{x} \in H^+$ and thus, Φ^+ extends to the whole sphere as $\Phi\colon \mathbb{S}^2 \to \mathbb{R}^2 \cup \{\infty\}$; it is defined in spherical coordinates as

$$\Phi(\theta,\varphi) = \begin{cases} \Phi^+(\theta,\varphi) & \text{if } \theta \in (0,\pi/2), \\ \infty & \text{if } \theta = \pi/2, \\ \Phi^+(\pi-\theta,\pi+\varphi) & \text{if } \theta \in (\pi/2,\pi), \end{cases}$$

where $\Phi^+(\theta,\varphi) = \tan\theta(\cos\varphi,\sin\varphi)$. Thus, by means of central projection, we can project points from both the upper and lower hemisphere. We emphasize that points on the equator are in correspondence with the infinity of $\mathbb{R}^2$.

Now, we will show that if $(\theta(t),\varphi(t))$ is a solution of the system (5.2) (with $c \neq 0$ and $\sigma = 1$), then $\Phi(\theta(t),\varphi(t))$ is a conic orbit in $T_N\mathbf{S}^2$. For this purpose, we consider $\mathbf{q}$ the position of a particle of mass m and Q the point where the straight line passing through the origin and $\mathbf{q}$, intersects the plane $T_N S^2$ (see Figure 3(a)). Let θ be the angle determined by the vector $\overrightarrow{O\mathbf{q}}$ and the z axis. By defining $r = \tan\theta$, with $0 \leq \theta < \pi/2$, we have that $r = |N - Q|$. Thus, the point $\mathbf{q} = (\sin\theta\cos\theta, \sin\theta\sin\varphi, \cos\theta)$ becomes $Q = \Phi(\mathbf{q}) = (r\cos\varphi, r\sin\varphi)$. In these coordinates it is verified that the equation (5.3) is written as

$$\ddot{r} = -V'(r), \tag{6.1}$$

with $V(r) = V_{\mathrm{Kep}}(r) + \widetilde{V}(r)$, where

$$V_{\mathrm{Kep}}(r) = \frac{c^2}{2r^2} - \frac{\mu}{r} \tag{6.2}$$

is the effective potential of the planar Kepler problem, and

$$\widetilde{V}(r) = -2\mu r + \left(\frac{3c^2}{2} - 2h\right) r^2 - \mu r^3 + \left(\frac{c^2}{2} - h\right) r^4,$$

where h is the energy constant defined in (5.5). Note that the system (6.1) corresponds to the motion of a particle in a central field, and it has energy

$$\Gamma(r, \dot{r}) = \frac{1}{2}\dot{r}^2 + V(r). \tag{6.3}$$

The phase portrait of the Hamiltonian system associated to (6.3) can be seen in Figure 9.

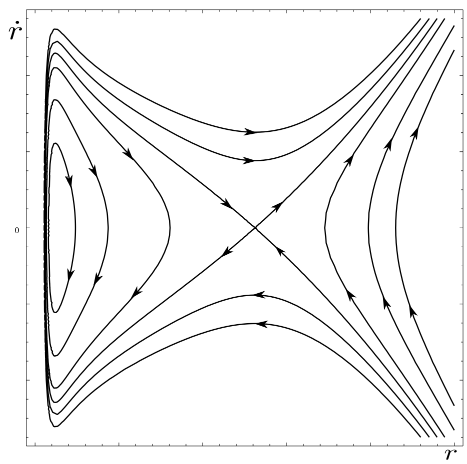

Figure 9: Phase portrait associated with the Hamiltonian Γ (equation (6.3)).

In order to describe how the trajectories are in the plane $T_N\mathbb{S}^2$, we must determine the trajectory $r(t)$ or some reparameterization of it. From the conservation of the third component of the angular momentum, it follows that φ varies monotonically with respect to the time; therefore, it can be considered as a new time that is denoted by $' \equiv \frac{d}{d\varphi}$. Let us define the new variable $\rho = 1/r$, and it is easy to check that

$$\rho' = -\frac{\dot{\theta}}{c}, \quad \rho'' = -\frac{\sin^2\theta\,\ddot{\theta}}{c^2}. \tag{6.4}$$

By replacing (6.4) in (5.3) (with $\sigma = 1$), we obtain that ρ satisfies the differential equation of a nonhomogeneous harmonic oscillator

$$\rho'' + \rho = \frac{\mu}{c^2}, \tag{6.5}$$

whose general solution is given by $\rho(\varphi) = \frac{\mu}{c^2} + \left(\rho_0 - \frac{\mu}{c^2}\right)\cos(\varphi - \varphi_0)$. Considering that $\rho = \cot\theta$, then by fixing an energy level h in (5.5) and from the first relation given

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in (6.4), we get that

$$r(\varphi) = \frac{p}{1 + e\,\cos(\varphi - \varphi_0)}, \tag{6.6}$$

with

$$p = \frac{c^2}{\mu} \quad \text{and} \quad e = \sqrt{1 + \frac{2c^2}{\mu^2}\Big(h - \frac{c^2}{2}\Big)}.$$

It is verified that in $T_N S^2$, (6.6) represents the polar equation of a conic with one focus at the origin and eccentricity e. From Theorem 5.1, we conclude the following.

(a) Elliptic orbits in $T_N S^2$ come from the orbits in $\mathbb{S}^2$ which are contained in H^+, since $0 \le e < 1$ if and only if $h_* \le h < \frac{c^2}{2}$ (see Figure 10(a)).
(b) Parabolic orbits in $T_N S^2$ come from the orbits in $\mathbb{S}^2$ which are tangent to the equator, since $e = 1$ if and only if $h = c^2/2$ (see Figure 10(b)).
(c) Hyperbolic orbits in $T_N S^2$ come from the orbits in $\mathbb{S}^2$ that cross the equator, since $e > 1$ if and only if $h > c^2/2$ (see Figure 10(c)).

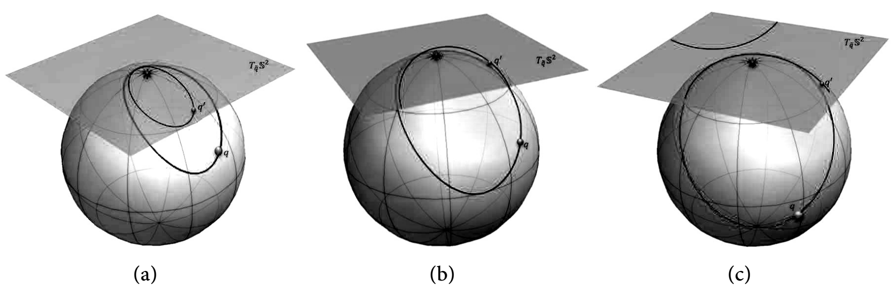

Figure 10: Central projection for the orbits of the Kepler problem on $\mathbb{S}^2$ onto $T_N\mathbb{S}^2$ for different values of the energy. (a) $h_* < h < \frac{c^2}{2}$. (b) $h = \frac{c^2}{2}$. (c) $h > h_*$.

6.2 Central Projection for the Solutions of the Kepler Problem on $\mathbb{H}^2$

We consider the straight line from the origin $O = (0,0,0)$ to a point $Q \in \mathbb{H}^2$. We define the central projection of the point $Q \in \mathbb{H}^2$ to be the point $P \in T_{O'}\mathbb{H}^2$ given by the intersection between the segment $\overline{OQ}$, and the plane $z = 1$ (here $O' = (0,0,1)$ is the origin of the plane $z = 1$). Consider the parametrization of $\mathbb{H}^2$ given in (5.1) and let $Q \in \mathbb{H}^2$ with the hyperbolic coordinates (θ, ϕ). Then we have that $\Phi\colon \mathbb{H}^2 \to \mathbb{D}^2$, where $\mathbb{D}^2$ is the Poincaré disk, is given by $Q \mapsto P = \Phi(\theta,\phi) = \tanh\theta(\cos\phi, \sin\phi)$. We define the variable $r = \tanh\theta$, then it is verified that $0 < d(O', P) = r < 1$, where $r = 1$ corresponds to the infinity. Thus, to describe the central projection of an orbit in $\mathbb{H}^2$, it will be sufficient to determine r. In these coordinates, the system (5.3), as well as in the case of $\mathbb{S}^2$, assumes the form (6.1), with $V(r) = V_{\text{Kep}}(r) + \widetilde{V}(r)$, where

$V_{\mathrm{Kep}}(r)$ is as in (6.2) and

$$\widetilde{V}(r) = 2\mu r + \Big(\frac{3c^2}{2} + 2h\Big)r^2 - \mu r^3 - \Big(\frac{c^2}{2} + h\Big)r^4.$$

The Hamiltonian function associated with the system is as in (6.3). Now, taking a fixed energy level $\gamma = \Gamma(r, \dot{r})$, we have that given c, μ and h, the level curves cut the line $r = 1$ (*i.e.,* the infinity) in two possible points $\dot{r} = \pm\sqrt{-3c^2 - 2h + 2\gamma}$. For the energy level $\gamma_0 = (3c^2 - 2h)/2$, the level curve cuts the straight line $r = 1$ at exactly one point and the r axis at

$$r = (-\mu + \sqrt{c^4 + 2c^2h + \mu})/(c^2 + 2h).$$

Also, since $\dot{r}$ cannot tend to infinity, because $0 < r < 1$ and γ is a real constant, then by symmetry of the Hamiltonian $\Gamma(r, \dot{r})$ with respect to the variable $\dot{r}$, we have that the level curve $\Gamma = \gamma_0$ must close at $(1, 0)$; thus, it must be similar to a homoclinic curve. For values of the energy $\gamma > \gamma_0$, the level curve cuts the straight line $r = 1$ in two points, while for $\gamma < \gamma_0$, the level curve does not cut the straight line $r = 1$. Then it corresponds to a closed orbit contained inside the curve $\Gamma = \gamma_0$. Thus, the phase portrait in the variables $(r, \dot{r})$ is as in Figure 11. We have shown that if values

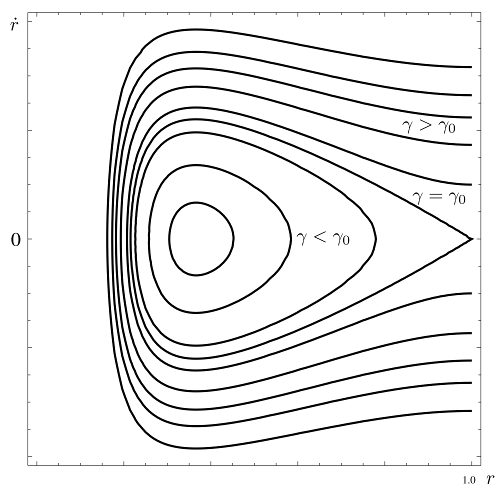

Figure 11: Phase portrait associated with the Hamiltonian $\Gamma(r, \dot{r})$ for the case $\sigma = -1$ (equation (6.3)).

of the parameters c, μ, and h are fixed, the projected system has elliptic orbits, which are identified with energy levels $\gamma < \gamma_0$, one parabolic orbit, identified with $\gamma = \gamma_0$, and hyperbolic orbits, which are identified with $\gamma > \gamma_0$. Nevertheless, this analysis can be more specific if we consider a new variable and also, a new time, as follows. We consider the variable $\rho = \coth\theta = 1/r$, and from the conservation of the third component of the angular momentum, we obtain that $\phi(t)$ is monotone; then it can

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be considered as a new time ϕ. By using prime to denote $\frac{d}{d\phi}$, we have that

$$\rho' = -\frac{\dot{\theta}}{c}, \quad \rho'' = -\frac{\sinh^2\theta\,\ddot{\theta}}{c^2}. \tag{6.7}$$

By replacing (6.7) in (5.3), we get that ρ satisfies the differential equation of a nonhomogeneous harmonic oscillator

$$\rho'' + \rho = \frac{\mu}{c^2}, \tag{6.8}$$

whose general solution is given by $\rho(\varphi) = \frac{\mu}{c^2} + \left(\rho_0 + \frac{\mu}{c^2}\right)\cos(\varphi - \varphi_0)$. Taking into account that $\rho = \cot\theta$, then fixing an energy level $K = h$ in (5.5) and using the relations given in (6.4), it follows that

$$r = \tanh\theta = \frac{p}{1 + e\,\cos(\varphi - \varphi_0)}, \tag{6.9}$$

with $p = c^2/\mu$ and $e = \sqrt{1 + 2c^2/\mu^2(h + c^2/2)}$. Equation (6.9) describes conics in $\mathbb{H}^2$. If $e = 0$, then the conic is a circumference, which in terms of the energy means $h = h_*$. If $e \geq 1$, then the conic is not bounded. To determine when the conic is an ellipse, we must require that it is contained inside the disk, *i.e.,* it does not cut the boundary, whereby we will determine the conditions so that such cuts do not exist. Where $r \to 1$ in (6.9), we get the equation

$$1 = \frac{p}{1 + e\cos(\varphi - \varphi_0)},$$

whose solutions are given by

$$\varphi_\pm = \varphi_0 \pm \arccos\left(\frac{c^2 - \mu}{\sqrt{c^4 + 2c^2h + \mu^2}}\right).$$

If $c^2 - \mu < 0$, then the above solutions exist if and only if $h > -\mu$; therefore, the conic must be an ellipse. In particular, it is a circumference when $h = h_*$. The conic will be tangent to the boundary of the disk if and only if $h = -\mu$, since in this case we have $\varphi_+ = \varphi_-$ and then the conic is an elliptic parabola. The conic will be a parabola if $h \in (-\mu, -\frac{c^2}{2})$, a hyperbolic parabola if $h = -c^2/2$ and a hyperbola if $h \in (-c^2/2, \infty)$ (see Figure 12(a)). For the case $c^2 - \mu \geq 0$, we know that solutions exist if and only if $h > -\mu$ and we obtain two different solutions $\theta_\pm$. If $\mu < c^2 \leq 2\mu$, then the conic will be a parabola if $h \in (-\mu, -c^2/2]$ and a hyperbola if $h \in (-c^2/2, \infty)$. If $c^2 > 2\mu$, then the conic will be a hyperbola, for all $h \in (-\mu, \infty)$ (see Figure 12 (b) and see [18]).

7 Regularization

In this section, we propose a regularization of Levi–Civita type, for the Kepler problem defined on both surfaces $\mathbb{S}^2$ and $\mathbb{H}^2$ (in fact, it is an adaptation of this type of Hamiltonian regularization [12]). Remember that a regularization of the Levi–Civita type consists in making a change of coordinates, a suitable time-rescaling and make use of the constant of energy, in order to obtain a regular field [3]. Once such a field is obtained, we can carry out a qualitative study of the regularized problem, as for example, in [11] for the Newtonian case and also by several other authors.

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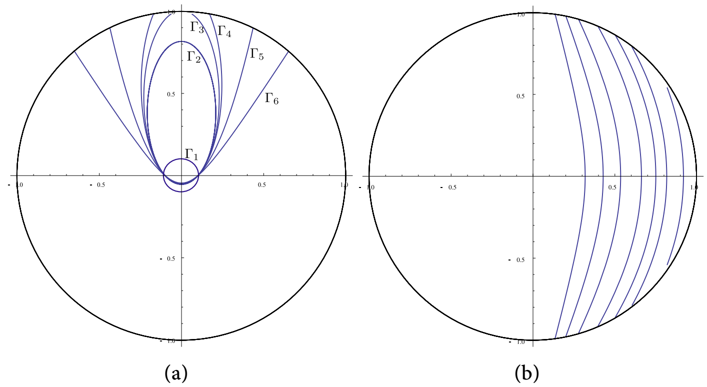

Figure 12: (a) Case $c^2 - \mu < 0$. Orbits for $\mu = 2.5$, $c = 0.5$, $\phi = 0$. $\Gamma_1 : h = h_*$; $\Gamma_2 : h \in (h_*, -\mu)$; $\Gamma_3 : h = -\mu$; $\Gamma_4 : h \in (-\mu, c^2/2)$; $\Gamma_5 : h = \frac{c^2}{2}$; $\Gamma_6 : h \in (c^2/2, \infty)$. (b) Case $c^2 - \mu \geq 0$. Orbits for $\mu = 2$, $c = \sqrt{3}$, $\phi = 0$.

By using Cartesian coordinates and introducing the energy constant defined in (3.1), we obtain the following system of second order ODE's

$$\begin{aligned} \ddot{x} &= \frac{\mu\sigma x z(2z^2 - 3)}{[\sigma(1 - z^2)]^{3/2}} - 2\sigma h x, \\ \ddot{y} &= \frac{\mu\sigma y z(2z^2 - 3)}{[\sigma(1 - z^2)]^{3/2}} - 2\sigma h y, \\ \ddot{z} &= \frac{\mu\sigma(1 - 2z^2)}{[\sigma(1 - z^2)]^{1/2}} - 2\sigma h z. \end{aligned} \tag{7.1}$$

7.1 Regularization of the Kepler Problem on $\mathbb{S}^2$

We know that for $\sigma = 1$, the system (2.3) has both for collision and antipodal singularities, which are given by the points $\mathbf{q} = \widetilde{\mathbf{q}}$ and $\mathbf{q} = -\widetilde{\mathbf{q}}$, respectively (either $\theta = 0$ or $\theta = \pi$, in spherical coordinates).

7.1.1 Regularization in Cartesian Coordinates

The following result globally regularizes the equations of motion of the Kepler problem defined on $\mathbb{S}^2$ in Cartesian coordinates given by (7.1); *i.e.,* both singularities are removed.

Theorem 7.1 *The Kepler problem on $\mathbb{S}^2$ in Cartesian coordinates given by (7.1) admits a Levi-Civita type regularization,* i.e., *the singularities are regularized by introducing the change of coordinates*

$$\zeta = (\zeta_1, \zeta_2, \zeta_3) = \Big(\frac{x}{\sqrt{1 - z^2}}, \frac{y}{\sqrt{1 - z^2}}, \frac{z}{\sqrt{1 - z^2}} \Big), \tag{7.2}$$

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and the new time τ given by

$$dt = (1 - z^2)^{3/2} d\tau. \tag{7.3}$$

Proof From (7.2) we have

$$\|\zeta\|^2 = \zeta_1^2 + \zeta_2^2 + \zeta_3^2 = \frac{1}{1 - z^2}; \tag{7.4}$$

thus, the following relations hold:

$$x = \frac{\zeta_1}{\|\zeta\|}, \quad y = \frac{\zeta_2}{\|\zeta\|}, \quad z = \frac{\zeta_3}{\|\zeta\|}. \tag{7.5}$$

Then the equations of motion (7.1) can be rewritten as

$$\begin{aligned} \ddot{x} &= \frac{\mu\zeta_1\zeta_3}{\|\zeta\|}(2\zeta_3^2 - 3\|\zeta\|^2) - 2h\frac{\zeta_1}{\|\zeta\|}, \\ \ddot{y} &= \frac{\mu\zeta_2\zeta_3}{\|\zeta\|}(2\zeta_3^2 - 3\|\zeta\|^2) - 2h\frac{\zeta_2}{\|\zeta\|}, \\ \ddot{z} &= -\frac{\mu}{\|\zeta\|}(2\zeta_3^2 - \|\zeta\|^2) - 2h\frac{\zeta_3}{\|\zeta\|}. \end{aligned} \tag{7.6}$$

Let prime denote $\frac{d}{d\tau}$. From (7.3) and (7.4), we obtain the relation

$$\dot{z} = z'\|\zeta\|^3. \tag{7.7}$$

On the other hand, from (7.2) we have that

$$\zeta_3' = \frac{d\zeta_3}{ds} = \frac{d\zeta_3}{dt}\frac{dt}{ds} = \dot{z}, \tag{7.8}$$

which combined with (7.7) yields

$$z' = \frac{\zeta_3'}{\|\zeta\|^3}.$$

Differentiating in (7.4) gives

$$\|\zeta\|' = \frac{\zeta_3\zeta_3'}{\|\zeta\|}. \tag{7.9}$$

Now, differentiating the first equation of (7.5) and using (7.9), we obtain the relation

$$\ddot{x} = \|\zeta\|^5\zeta'' + \zeta_3\zeta_1'\zeta_3'\|\zeta\|^3 - \zeta_1\zeta_3'^2\|\zeta\|^3 - \zeta_1\zeta_3\zeta_3''\|\zeta\|^3.$$

Substituting it into the first equation of (7.6) we obtain

$$\|\zeta\|^5\zeta'' + \zeta_3\zeta_1'\zeta_3'\|\zeta\|^3 - \zeta_1\zeta_3'^2\|\zeta\|^3 - \zeta_1\zeta_3\zeta_3''\|\zeta\|^3 = \frac{\mu\zeta_1\zeta_3}{\|\zeta\|}(2\zeta_3^2 - 3\|\zeta\|^2) - 2h\frac{\zeta_1}{\|\zeta\|},$$

or equivalently,

$$\zeta_1'' + \frac{\zeta_3\zeta_3'}{\|\zeta\|^2}\zeta_1' + \Big[\frac{2h}{\|\zeta\|^6} + \frac{3\mu\zeta_3}{\|\zeta\|^4} - \frac{2\mu\zeta_3^3}{\|\zeta\|^6} - \frac{\zeta_3'^2}{\|\zeta\|^2} - \frac{\zeta_3\zeta_3''}{\|\zeta\|^2}\Big]\zeta_1 = 0. \tag{7.10}$$

We proceed similarly for the second equation of (7.5) to obtain the relation

$$\ddot{y} = \|\zeta\|^5\zeta'' + \zeta_3\zeta_2'\zeta_3'\|\zeta\|^3 - \zeta_2\zeta_3'^2\|\zeta\|^3 - \zeta_2\zeta_3\zeta_3''\|\zeta\|^3,$$

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which being replaced into the second equation of (7.6) gives

$$\zeta_2'' + \frac{\zeta_3\zeta_3'}{\|\zeta\|^2}\zeta_2' + \Big[\frac{2h}{\|\zeta\|^6} + \frac{3\mu\zeta_3}{\|\zeta\|^4} - \frac{2\mu\zeta_3^3}{\|\zeta\|^6} - \frac{\zeta_3'^2}{\|\zeta\|^2} - \frac{\zeta_3\zeta_3''}{\|\zeta\|^2}\Big]\zeta_2 = 0. \tag{7.11}$$

It follows from (7.8) that $\ddot{z} = \zeta_3''\|\zeta\|^3$; then replacing into the third equation of (7.6) we obtain the third regularized equation

$$\zeta_3'' = \frac{\mu}{\|\zeta\|^2} - \frac{2\mu\zeta_3^2}{\|\zeta\|^4} - \frac{2h\zeta_3}{\|\zeta\|^4}. \tag{7.12}$$

Finally, replacing (7.12) into (7.10) and (7.11), the following regularized system is obtained:

$$\begin{aligned} \zeta_1'' + \frac{\zeta_3\zeta_3'}{\|\zeta\|^2}\zeta_1' + \Big[\frac{2h}{\|\zeta\|^4} + \frac{2\mu\zeta_3}{\|\zeta\|^4} - \frac{\zeta_3'^2}{\|\zeta\|^2}\Big]\zeta_1 &= 0,\\ \zeta_2'' + \frac{\zeta_3\zeta_3'}{\|\zeta\|^2}\zeta_2' + \Big[\frac{2h}{\|\zeta\|^4} + \frac{2\mu\zeta_3}{\|\zeta\|^4} - \frac{\zeta_3'^2}{\|\zeta\|^2}\Big]\zeta_2 &= 0,\\ \zeta_3'' + \frac{2\mu\zeta_3^2}{\|\zeta\|^4} + \frac{2h\zeta_3}{\|\zeta\|^4} - \frac{\mu}{\|\zeta\|^2} &= 0. \end{aligned} \tag{7.13}$$

It is clear that after the change of coordinates and the time-rescaling, the equations of motion (7.13) have no singularities, because for $(x,y,z) \in \mathbb{S}^2$, the term $\|\zeta\|$ defined in (7.4) is nonzero. In addition, since $1 - z^2 \not\to \infty$, $\|\zeta\| \not\to 0$, and the proof is concluded. ■

7.1.2 Pseudo-regularization in Spherical Coordinates and Angular Momentum $c \neq 0$

In this case we have used the term *pseudo-regularization*, because for $c \neq 0$ no singularities are encountered. Nevertheless, even though the solutions do not tend to collision, we get some kind of nice "regularized" formulation of the equations by doing a change of coordinates, a suitable time-rescaling, and making use of the constant of energy (Levi–Civita type regularization), in order to obtain a regularized formulation of the problem, which is equivalent to a harmonic oscillator.

Theorem 7.2 *The Kepler problem on $\mathbb{S}^2$ in spherical coordinates with the constant of motion given by the third component of the angular momentum $c \neq 0$, defined by the system (5.3) (for $\sigma = 1$), admits a pseudo-regularization, which is carried out by considering φ as the new time and introducing the change of coordinates*

$$\delta = \cot\theta, \qquad \theta \in [0,\pi].$$

Proof The proof follows directly from the process described in (6.4) and the obtained harmonic oscillator is given by (6.5), which has associated a regular vector field. ■

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We also note that the energy integral (5.5) becomes

$$\delta'^2 + \Big(\delta - \frac{\mu}{c^2}\Big)^2 = \frac{2(h-h_*)}{c^2}, \tag{7.14}$$

where h_* is the constant defined in Theorem 5.1. From here we get that the singularities in the new coordinates are determined by $\delta = \infty$, which occurs if and only if $\theta = 0$ or $\theta = \pi$. This proves that there are no solutions tending to collision in these variables. Therefore, from (7.14) it follows that the orbits of the pseudo-regularized system are as in Figure 13(a).

7.1.3 Regularization in Spherical Coordinates and Angular Momentum $c = 0$

In this case, the motion is carried out along a geodesic containing the poles. In contrast to the previous case, all orbits go to collision; therefore, singularities represent a big problem in obtaining the solutions.

Theorem 7.3 *The Kepler problem on $\mathbb{S}^2$ in spherical coordinates with $c = 0$, defined by (5.8) (for $\sigma = 1$) admits a Levi–Civita type regularization, which is carried out by considering the change of coordinates*

$$\theta = 2\arctan\psi^2 \tag{7.15}$$

and the time-rescaling

$$dt = \frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}\,d\tau. \tag{7.16}$$

Proof From (7.15) and (7.16), we obtain the following relations

$$\dot\theta = \frac{d\theta}{d\tau}\frac{d\tau}{dt} = \theta'\frac{\sec^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}}, \qquad \theta' = \frac{4\psi\psi'}{\sec^2\frac{\theta}{2}},$$

which imply that

$$\dot\theta = \frac{2\psi\psi'}{\tan\frac{\theta}{2}} = \frac{2\psi'}{\psi}. \tag{7.17}$$

Also, replacing (7.17) in the second equation of (5.8), we obtain that the constant of energy becomes

$$4\psi'^2 - \mu(1-\psi^4) = 2h\psi^2. \tag{7.18}$$

We compute the second derivative of θ,

$$\ddot\theta = \frac{d\dot\theta}{dt} = \frac{d\dot\theta}{d\tau}\frac{d\tau}{dt} = 2\Big(\frac{\psi''\psi - \psi'^2}{\psi^2}\Big)\frac{\sec^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}}.$$

Replacing the above relations into the first equation of (5.8), we have that

$$\Big(\frac{\psi''\psi - \psi'^2}{\psi^2}\Big)\frac{\sec^2\frac{\theta}{2}}{\tan\frac{\theta}{2}} = -\mu\csc^2\theta. \tag{7.19}$$

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Finally, replacing (7.18) into (7.19) we obtain the equation

$$\psi'' = \frac{h}{2}\psi - \frac{\mu}{2}\psi^3, \tag{7.20}$$

which is called the Mathieu equation. Thus, it is proved that through a change of coordinate and a time-rescaling, equation (5.8) is regularized, and the regularized equation is given by (7.20). ■

The phase portrait of (7.20) is shown in Figure 13(b). Note that it confirms the fact that all solutions do not have collisions, and the orbits are closed for any value of h.

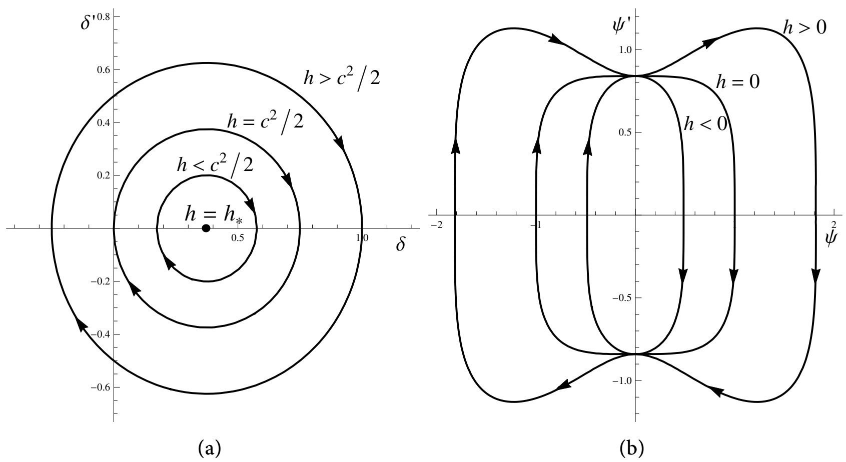

Figure 13: (a) Phase portrait generated by level curves associated to (7.14) for the regularized Kepler problem in $\mathbb{S}^2$ with angular momentum $c \neq 0$. (b) Phase portrait generated by level curves associated with (7.20) for the regularized Kepler problem in $\mathbb{S}^2$ with angular momentum $c = 0$.

Remark 7.4 The above theorem shows a significant difference with respect to the Newtonian case. In fact, it is known that in the Newtonian case, a Levi–Civita regularization gives a differential equation associated with a harmonic oscillator for any value of c ([1,11,12,17,20]), while in this case, for $c = 0$, we obtain a Mathieu equation.

7.2 Regularization of the Kepler Problem on $\mathbb{H}^2$

We already know that for $\sigma = -1$, the system (2.3) has a unique singularity that is due to collision at $\mathbf{q} = \widetilde{\mathbf{q}}$ (or $\theta = 0$).

7.2.1 Regularization in Cartesian Coordinates

An analogous result to Theorem 7.1 is verified for the hyperbolic case.

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Theorem 7.5 *The Kepler problem on $\mathbb{H}^2$ in Cartesian coordinates given by (2.3) allows a regularization of the Levi–Civita type,* i.e., *the singularities are regularized by introducing the change of coordinates*

$$\rho = (\rho_1, \rho_2, \rho_3) = \Big(\frac{x}{\sqrt{z^2-1}}, \frac{y}{\sqrt{z^2-1}}, \frac{z}{\sqrt{z^2-1}} \Big), \tag{7.21}$$

and the new time τ given by

$$dt = (z^2-1)^{3/2} d\tau. \tag{7.22}$$

Proof Let prime denote $\frac{d}{d\tau}$. From (7.21) we have

$$|||\rho|||^2 = -\rho \boxdot \rho = -\rho_1^2 - \rho_2^2 + \rho_3^2 = \frac{1}{z^2-1}; \tag{7.23}$$

thus, the following relations hold

$$x = \frac{\rho_1}{|||\rho|||}, \quad y = \frac{\rho_2}{|||\rho|||}, \quad z = \frac{\rho_3}{|||\rho|||}. \tag{7.24}$$

Then the equations of motion (2.3) can be rewritten as

$$\begin{aligned} \ddot{x} &= \frac{\mu\rho_1\rho_3}{|||\rho|||}(2\rho_3^2 - 3\,|||\,\rho|||^2) + 2h\frac{\rho_1}{|||\rho|||}, \\ \ddot{y} &= \frac{\mu\rho_2\rho_3}{|||\rho|||}(2\rho_3^2 - 3\,|||\,\rho\|^2) + 2h\frac{\rho_2}{|||\rho|||}, \\ \ddot{z} &= \frac{\mu}{|||\rho|||}(2\rho_3^2 - |||\rho|||^2) + 2h\frac{\rho_3}{|||\rho|||}. \end{aligned} \tag{7.25}$$

From (7.22) and (7.23), we obtain the relation

$$\dot{z} = \frac{dz}{dt} = \frac{dz}{d\tau}\frac{d\tau}{dt} = z'\,|||\,\rho\,|||^3\,. \tag{7.26}$$

On the other hand, from (7.21) we have that

$$\rho_3' = \frac{d\rho_3}{ds} = \frac{d\rho_3}{dt}\frac{dt}{ds} = -\dot{z}, \tag{7.27}$$

which, combined with (7.26) yields

$$z' = -\frac{\rho_3'}{|||\rho|||^3}.$$

Differentiating with respect to τ in (7.23) gives

$$|||\rho|||' = \frac{\rho_3\rho_3'}{|||\rho|||}, \tag{7.28}$$

and differentiating twice in the first equation of (7.24) and using (7.28) we obtain the relation

$$\ddot{x} = |||\rho\,|||^5\,\rho'' + \rho_3\rho_1'\rho_3'\,|||\,\rho\,|||^3 - \rho_1\rho_3'^2\,|||\,\rho\,|||^3 - \rho_1\rho_3\rho_3''\,|||\,\rho|||^3,$$

and substituting it into the first equation of (7.25) we obtain

$$|||\rho|||^5\rho'' + \rho_3\rho_1'\rho_3'|||\rho|||^3 - \rho_1\rho_3'^2|||\rho|||^3 - \rho_1\rho_3\rho_3''|||\rho|||^3 = \frac{\mu\rho_1\rho_3}{|||\rho|||}(2\rho_3^2 - 3|||\rho|||^2) + 2h\frac{\rho_1}{|||\rho|||},$$

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or equivalently

$$\rho_1'' + \frac{\rho_3\rho_3'}{|||\rho|||^2}\rho_1' + \Big[-\frac{2h}{|||\rho|||^6} + \frac{3\mu\rho_3}{|||\rho|||^4} - \frac{2\mu\rho_3^3}{|||\rho|||^6} - \frac{\rho_3'^2}{|||\rho|||^2} - \frac{\rho_3\rho_3''}{|||\rho|||^2}\Big]\rho_1 = 0. \tag{7.29}$$

We proceed similarly for the second equation of (7.24) to obtain the relation

$$\dot{y} = \rho_2' \,|||\,\rho\,|||^2 - \rho_2\rho_3\rho_3',$$
$$\ddot{y} = |||\rho\,|||^5\,\rho'' + \rho_3\rho_2'\rho_3' \,|||\,\rho\,|||^3 - \rho_2\rho_3'^2 \,|||\,\rho\,|||^3 - \rho_2\rho_3\rho_3'' \,|||\,\rho|||^3,$$

which being replaced into the second equation of (7.25) gives

$$\rho_2'' + \frac{\rho_3\rho_3'}{|||\rho|||^2}\rho_2' + \Big[-\frac{2h}{|||\rho|||^6} + \frac{3\mu\rho_3}{|||\rho|||^4} - \frac{2\mu\rho_3^3}{|||\rho|||^6} - \frac{\rho_3'^2}{|||\rho|||^2} - \frac{\rho_3\rho_3''}{|||\rho|||^2}\Big]\rho_2 = 0. \tag{7.30}$$

It follows form (7.27) that $\ddot{z} = -\rho_3'' \,|||\,\rho|||^3$; then replacing into the third equation of (7.25) we obtain the third regularized equation

$$\rho_3'' = \frac{\mu}{|||\rho|||^2} - \frac{2\mu\rho_3^2}{|||\rho|||^4} - \frac{2h\rho_3}{|||\rho|||^4}. \tag{7.31}$$

Finally, replacing (7.31) into (7.29) and (7.30), the following regularized system is obtained

$$\begin{aligned}
\rho_1'' + \frac{\rho_3\rho_3'}{|||\rho|||^2}\rho_1' + \Big[\frac{2h}{|||\rho|||^4} + \frac{2\mu\rho_3}{|||\rho|||^4} - \frac{\rho_3'^2}{|||\rho|||^2}\Big]\rho_1 &= 0,\\
\rho_2'' + \frac{\rho_3\rho_3'}{|||\rho|||^2}\rho_2' + \Big[\frac{2h}{|||\rho|||^4} + \frac{2\mu\rho_3}{|||\rho|||^4} - \frac{\rho_3'^2}{|||\rho|||^2}\Big]\rho_2 &= 0,\\
\rho_3'' + \frac{2\mu\rho_3^2}{|||\rho|||^4} + \frac{2h\rho_3}{|||\rho|||^4} - \frac{\mu}{|||\rho|||^2} &= 0,
\end{aligned}$$

which has no singularities, since the term $|||\rho|||$, defined in (7.23), is nonzero for any $\rho \in \mathbb{H}^2$, and the proof is concluded. ∎

7.2.2 Pseudo-regularization in Hyperbolic Coordinates and Angular Momentum $c \neq 0$.

We know that for $\sigma = -1$, the equation (5.3) has a singularity at $\theta = 0$, which corresponds to collision; however, there is no solution tending to collision if $c \neq 0$. The following results shows that, as in Subsection 7.1.2 for positive curvature, it is possible to give a pseudo-regularization that transforms equation (5.3) into a harmonic oscillator.

Theorem 7.6 *The Kepler problem on $\mathbb{H}^2$ in hyperbolic coordinates with the constant of motion given by the third component of the angular momentum $c \neq 0$, defined by the system (5.3), admits a pseudo-regularization, which is carried out by considering ϕ as the new time and introducing the change of coordinates $\xi = \coth\theta$.*

Proof The proof follows directly from the process described in (6.7), and the obtained harmonic oscillator is given by (6.8), which has associated a regular vector field. ∎

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We also note that the energy integral (5.5) becomes

$$\xi'^2 + \left(\xi - \frac{\mu}{c^2} \right)^2 = \frac{2(h - h_*)}{c^2},$$

where h_* is the constant defined in Theorem 5.3. Hence, it follows that the singularity in the new coordinates is determined by $\theta = 0$ if and only if $\xi = \infty$. Thus, the phase portrait is similar to the spherical case, and therefore, there are no solution tending to collision.

7.2.3 Regularization in Hyperbolic Coordinates and Angular Momentum $c = 0$

An analogous result to Theorem 7.3 for the spherical case is verified for $\mathbb{H}^2$.

Theorem 7.7 *The Kepler problem on $\mathbb{H}^2$ in hyperbolic coordinates with $c = 0$ defined by (5.8) admits a regularization of Levi–Civita type, which is carried out by considering the change of coordinates*

$$dt = \frac{\tanh\theta}{\operatorname{sech}^2\theta} d\tau, \tag{7.32}$$

and the time-rescaling

$$\theta = \operatorname{arg}\tanh \xi^2. \tag{7.33}$$

Proof We proceed as in proof of the Theorem 7.3 to obtain from the energy constant given in (5.8), the following relation

$$2\xi'^2 = h\xi^2 + \mu, \tag{7.34}$$

while the regularized equation is given by

$$\xi'' - \frac{h}{2}\xi = 0. \tag{7.35}$$

Thus, we have proved that by means of the time-rescaling (7.32) and the change of coordinates (7.33), the equation (5.8) (for $\sigma = -1$) becomes a regular differential equation (7.35). ∎

Similarly to the spherical case, we observe that the change of coordinates given by the transformation

$$(\theta, \dot\theta) = T(\xi, \xi') = \left(\operatorname{arg}\tanh \xi^2, \frac{2\xi'}{\xi} \right),$$

makes the Hamiltonian (5.5) (for $\sigma = -1$ and $c = 0$) assuming the following form

$$\widehat{H}(\xi, \xi') = \frac{2\xi'^2}{\xi^2} - \frac{\mu}{\xi^2}.$$

We observe that the transform T verifies

$$\left(\frac{\partial T(\xi, \xi')}{\partial(\xi, \xi')} \right)^T J \left(\frac{\partial T(\xi, \xi')}{\partial(\xi, \xi')} \right) = \frac{4}{1 - \xi^4} J,$$

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where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. So, in order to preserve the Hamiltonian structure, this transformation must be done along with a time-rescaling. From equation (7.34), we obtain that the phase portrait of the regularized problem is as in Figure 14, which is formed by three kind of orbits, essentially different from each other.

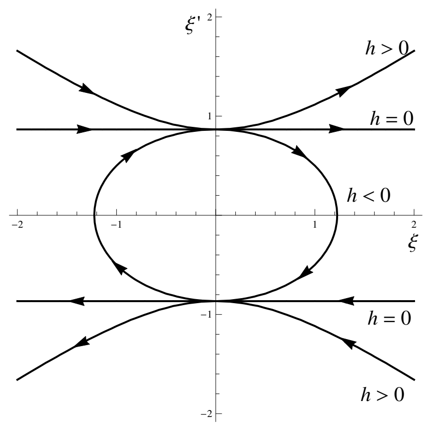

Figure 14: Phase portrait of the regularized Kepler problem defined on $\mathbb{H}^2$ with angular momentum $c = 0$.

Since $\theta \in]0, \infty[$, $\xi^2 \in]0, \infty[$ and therefore, our change of coordinates (7.33), is well defined, and also, the singularity (due to collision) in these variables is represented by $\xi = 0$.

(a) For a negative energy level, the orbit is elliptic. This orbit is periodic and bounded, and represents an elastic bounce between the free particle and the fixed body.
(b) For zero energy level, the orbit is parabolic and unbounded. The particle goes once to the attraction center with constant velocity in a finite time and then it escapes without return.
(c) For a positive energy level, the orbit is hyperbolic and the motion is similar to the zero energy.

8 Conclusions

We have studied the Kepler problem on surfaces of constant curvature, both positive and negative. We first show that it is enough to do the analysis on $\mathbb{S}^2$ an $\mathbb{H}^2$; then we get the equations of motion and obtain the first integrals of the problem, which allow us to get the Hill's regions. We prove that all singularities of this problem occur in finite time, and we have been able to regularize all of them. We show these results in Cartesian coordinates as well as in spherical and hyperbolic coordinates. For the case of positive curvature and zero angular momentum, we show that the Levi-Civita type regularization of the Kepler problem produces a Mathieu equation. By using the central projection of $\mathbb{S}^2$ and $\mathbb{H}^2$ we show a nice analogy between the solutions (conic orbits) of the classical Newtonian Kepler problem and the solution curves of the Kepler problem on $\mathbb{S}^2$ and $\mathbb{H}^2$.

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References

[1] D. Boccaletti and G. Puccaco, *Theory of orbits. Vol. 1. Integrable systems and non-perturbative methods.* Astronomy and Astrophysics Library, Springer-Verlag, Berlin, 1996.
[2] J. F. Cariñena and M. F. Rañada, and M. Santander, *Central potentials on spaces of constant curvature: the Kepler problem on the two-dimensional sphere S^2 and the hyperbolic plane H^2*. J. Math. Phys. **46**(2005), no. 5, 052702. http://dx.doi.org/10.1063/1.1893214
[3] A. Celletti *Basics of regularization theory*. In: Chaotic Worlds: From Order to Disorder in Gravitational N-Body Dynamical Systems, Springer, Netherlands, 2006, pp. 203-230.
[4] F. Diacu, E. Pérez-Chavela, and M. Santoprete, *The n-body problem in spaces of constant curvature.* arxiv:0807.1747
[5] ———, *The n-body problem in spaces of constant curvature. Part I: Relative equilibria.* J. Nonlinear Sci. **22**(2012), no. 2, 247–266. http://dx.doi.org/10.1007/s00332-011-9116-z
[6] ———, *The n-body problem in spaces of constant curvature. Part II: Singularities.* J. Nonlinear Sci. **22**(2012), no. 2, 267–275. http://dx.doi.org/10.1007/s00332-011-9117-y
[7] F. Diacu, *Relative equilibria of the curved N-body problem.* Atlantis Studies in Dynamical Systems, 1, Atlantis Press, Amsterdam, Paris, Beijing, 2012. http://dx.doi.org/10.2991/978-94-91216-68-8
[8] M. P. Do Carmo, *Differential geometry of curves and surfaces.* Prentice-Hall, Inc., Englewood Cliffs, NJ, 1976.
[9] B. A. Dubrovin, A. T. Fomenko, and P. Novikov, *Modern geometry—methods and applications. Part I. The geometry of surfaces, transformation groups, and fields.* Second ed. Graduate Texts in Mathematics, 93, Springer-Verlag, New York, 1992. http://dx.doi.org/10.1007/978-1-4612-4398-4
[10] V. V. Kozlov and A. O. Harin, *Kepler's problem in constant curvature spaces.* Celestial Mech. Dynam. Astronom. **54**(1992), no. 4, 393–399. http://dx.doi.org/10.1007/BF00049149
[11] E. A. Lacomba and G. Sienra, *Blow up techniques in the Kepler problem.* In: Holomorphic dynamics (Mexico, 1986), Lecture Notes in Math., 1345, Springer, Berlin, 1988, pp. 177–191. http://dx.doi.org/10.1007/BFb0081402
[12] T. Levi-Civita, *Sur la régularisation du problème des trois corps.* Acta Math. **42**(1920), 99–144. http://dx.doi.org/10.1007/BF02404404
[13] H. Liebmann, *Die Kegelschnitte und die Planetenbewegung im nichteuklidischen Raum.* Berichte Königl, Sächsischen Gesell. Wiss., Math. Phys. Klasse **54**(1902), 393–423.
[14] R. Martínez and C. Simó, *On the stability of the Lagrangian homographic solutions in a curved three body problem on $\mathbb{S}^2$.* Discrete Cont. Dyn. Syst. Ser. A. **33**(2013), no. 3, 1157–1175.
[15] H. Pollard, *Introduction to celestial mechanics.* The Carus Mathematical Monographs, 18, The Mathematical Association of America, Washington, D.C., 1976.
[16] A. V. Shchepetilov, *Nonintegrability of the two-body problem in constant curvature spaces.* J. Phys. A **39**(2006), no. 20, 5787–5806. http://dx.doi.org/10.1088/0305-4470/39/20/011
[17] E. L. Stiefel and G. Scheifele, *Linear and regular celestial mechanics. Perturbed two-body motion, numerical methods, canonical theory.* Die Grundlehren der mathematischen Wissenschaften, 174, Springer-Verlag, New York-Heidelberg, 1971.
[18] W. E. Story, *On non-euclidean properties of conics.* Amer. J. Math. **5**(1882), no. 1–4, 358–381. http://dx.doi.org/10.2307/2369551
[19] K. F. Sundman, *Mémoire sur le problème des trois corps.* Acta Math. **36**(1913), no. 1, 105–179. http://dx.doi.org/10.1007/BF02422379
[20] V. Szebehely, *Theory of orbits–the restricted problem of three bodies.* Academic Press, New York, 1967.
[21] P. Tibboel, *Polygonal homographic orbits in spaces of constant curvature.* Proc. Amer. Math. Soc. **141**(2013), no. 4, 1465–1471. http://dx.doi.org/10.1090/S0002-9939-2012-11410-8
[22] S. Zhu, *Eulerian relative equilibria of the curved* 3*-body problems in* $\mathbf{S}^2$. Proc. Amer. Math. Soc. **142**(2014), no. 8, 2837–2848. http://dx.doi.org/10.1090/S0002-9939-2014-11995-2

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