2. Waves and the Wave Equation
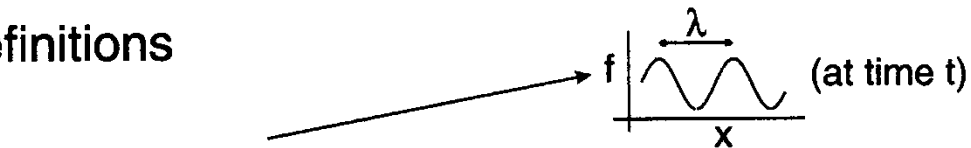
What is a wave?

Forward vs. backward propagating waves
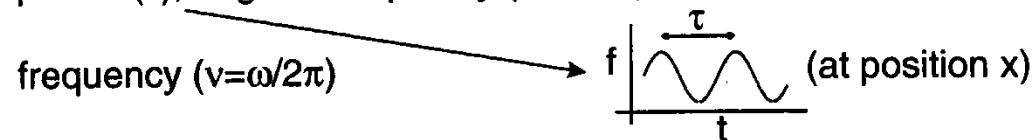
The one-dimensional wave equation

Harmonic waves
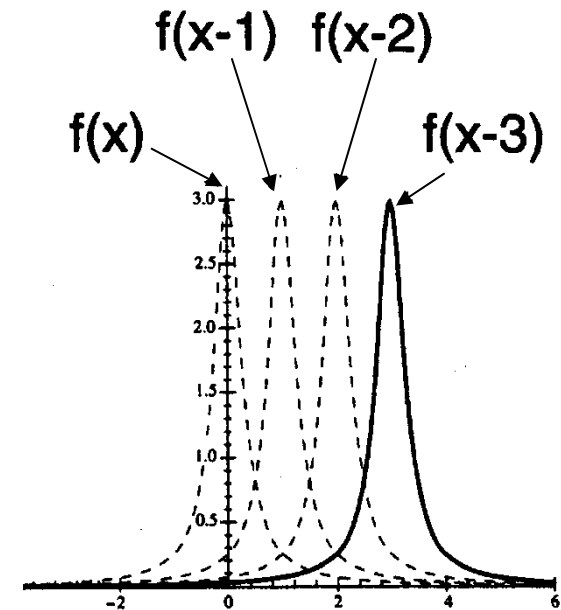
Definitions

f (at time t)

λ

x

Wavelength (λ), k-vector ($k=2\pi/\lambda$), period (τ), angular frequency ($\omega=2\pi/\tau$), frequency ($\nu=\omega/2\pi$)

f (at position x)

τ

t

Phase velocity

Reminders about complex numbers

The complex amplitude of a wave

f(x)   f(x-1)   f(x-2)   f(x-3)

3.0
2.5
2.0
1.5
1.0
0.5

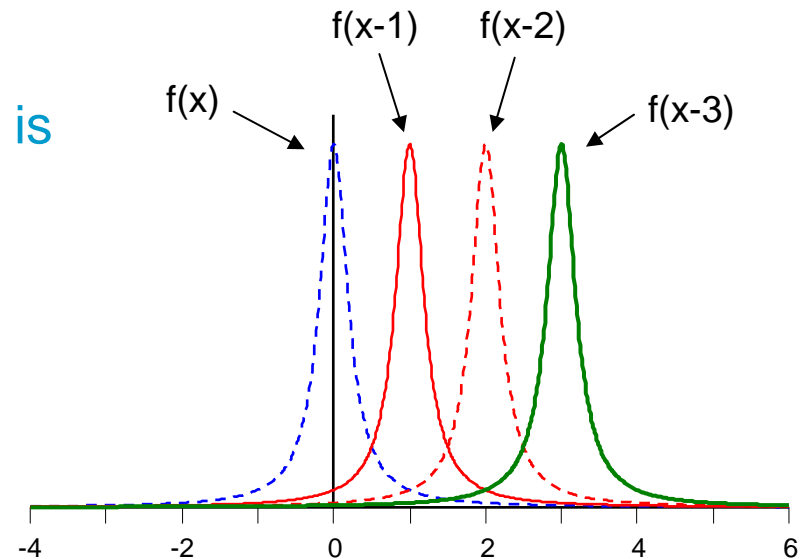-2   0   2   4   6

What is a wave?

In the mathematical sense, a wave is any function that moves.

To displace any function f(x) to the right, just change its argument from x to $x-x_0$, where x_0 is a positive number.

If we let $x_0 = v\,t$, where v is positive and t is time, then the displacement increases with increasing time.

So f(x-vt) represents a rightward, or forward, propagating wave.

Similarly, f(x+vt) represents a leftward, or backward, propagating wave.

v is the velocity of the wave.

The wave equation in one dimension

Later, we will derive the wave equation from Maxwell's equations.

Here it is, in its one-dimensional form for scalar (i.e., non-vector) functions, f.

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$
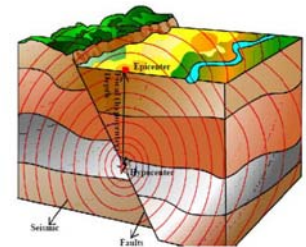
This equation determines the properties of most wave phenomena, not only light waves.

water wave   air wave   earth wave

In many real-world situations, the velocity of a wave depends on its amplitude, so $v = v(f)$. In this case, the solutions can be hard to determine.

Fortunately, this is not the case for electromagnetic waves.

The wave equation is linear:
The principle of "Superposition" holds.

If $f_1(x,t)$ and $f_2(x,t)$ are solutions to the wave equation,
then their sum $f_1(x,t) + f_2(x,t)$ is also a solution.

Proof:
$$\frac{\partial^2 (f_1 + f_2)}{\partial x^2} = \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 (f_1 + f_2)}{\partial t^2} = \frac{\partial^2 f_1}{\partial t^2} + \frac{\partial^2 f_2}{\partial t^2}$$

$$\frac{\partial^2 (f_1 + f_2)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 (f_1 + f_2)}{\partial t^2} = \left\{ \frac{\partial^2 f_1}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f_1}{\partial t^2} \right\} + \left\{ \frac{\partial^2 f_2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f_2}{\partial t^2} \right\} = 0$$

This has important consequences for light waves. It means that light beams can pass through each other without altering each other.

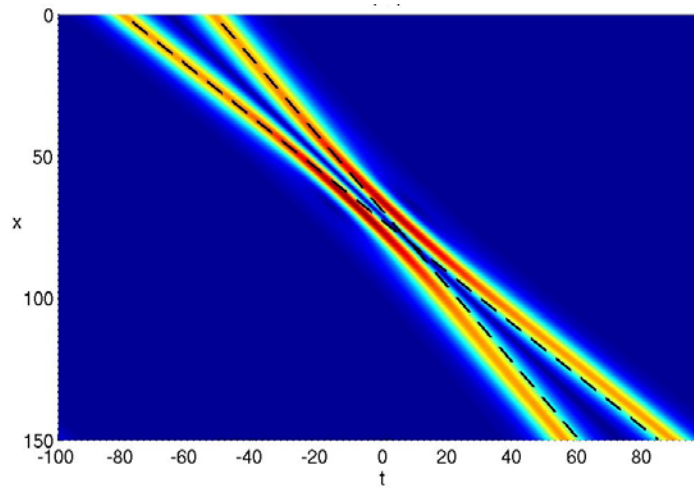It also means that waves can constructively or destructively interfere.

What if superposition wasn't true?

That would mean that two waves would interact with each other when passing through each other. This leads to some truly odd behaviors.
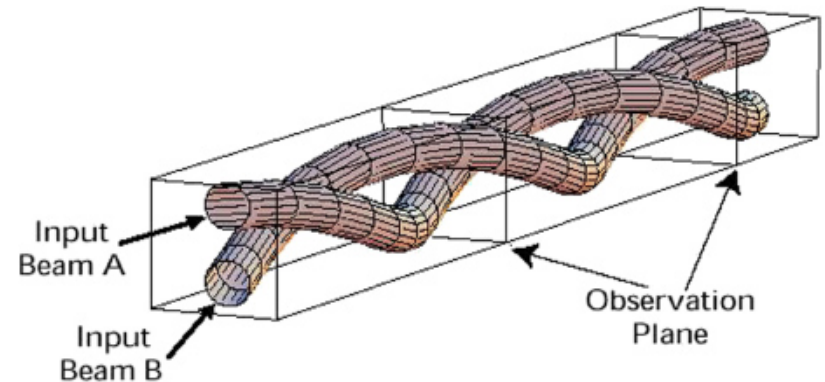
waves anti-crossing

strange wave collisions
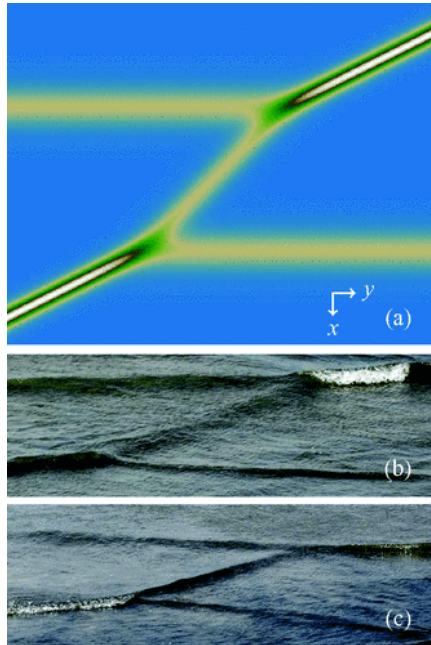
waves spiraling around each other

The solution to the one-dimensional wave equation

The wave equation has the simple solution:

$$f(x,t) = f(x \pm vt)$$

where $f(u)$ can be <u>any</u> twice-differentiable function.

If this is a "solution" to the equation, it seems pretty vague…
Is it at all useful?

First, let's prove that it <u>is</u> a solution.

Proof that $f(x \pm vt)$ solves the wave equation

Write $f(x \pm vt)$ as $f(u)$, where $u = x \pm vt$. So $\dfrac{\partial u}{\partial x} = 1$ and $\dfrac{\partial u}{\partial t} = \pm v$

Now, use the chain rule: $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x}$ $\qquad \dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial t}$

So $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial u} \Rightarrow \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial u^2}$ and $\dfrac{\partial f}{\partial t} = \pm v \dfrac{\partial f}{\partial u} \Rightarrow \dfrac{\partial^2 f}{\partial t^2} = v^2 \dfrac{\partial^2 f}{\partial u^2}$

Substituting into the wave equation:

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \; = \; \frac{\partial^2 f}{\partial u^2} - \frac{1}{v^2}\left\{ v^2 \frac{\partial^2 f}{\partial u^2} \right\} \; = \; 0$$

QED

The 1D wave equation for light waves

$$\frac{\partial^2 E}{\partial x^2} - \mu\varepsilon\frac{\partial^2 E}{\partial t^2} = 0$$

where:

$E(x,t)$ is the electric field
μ is the magnetic permeability
ε is the dielectric permittivity

This is a linear, second-order, homogeneous differential equation.

<u>A useful thing to know about such equations:</u>
   The most general solution has two unknown constants, which cannot be determined without some additional information about the problem (e.g., initial conditions or boundary conditions).
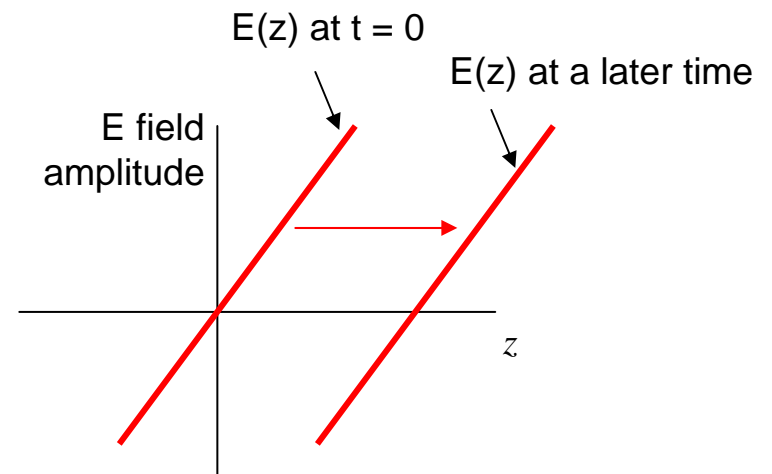
<u>And:</u>
   We might expect that oscillatory solutions (sines and cosines) will be very relevant for light waves.
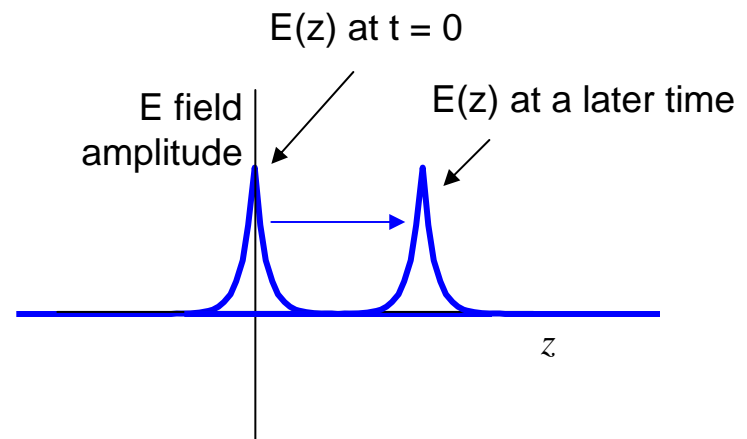
1D wave equation: some solutions

We showed that any twice-differentiable function can be a solution, as long as z and t appear in the right combination.

So this is a solution: $E(z,t) = 5z - 3t$

And this is a solution: $E(z,t) = e^{-|z-6t|}$

But these are not really very useful solutions.

E(z) at t = 0
E(z) at a later time
E field amplitude
z

E(z) at t = 0
E(z) at a later time
E field amplitude
z

A more useful form for the solution

It is more useful to use cosine- and sine-wave solutions:

$$E(z,t) = B \cos[k(z \pm ct)] + C \sin[k(z \pm ct)]$$

$$\downarrow$$

$$kz \pm (kc)t$$

$$\downarrow$$

$$E(z,t) = B \cos(kz \pm \omega t) + C \sin(kz \pm \omega t)$$

where:

$$\frac{\omega}{k} = c = \frac{1}{\sqrt{\mu \varepsilon}}$$

Note: this is the Greek letter omega, 'ω'. **Do not** get it confused with 'w'.

Even more useful form for the solution

$$E(z,t) = \underline{B}\,\cos(kz - \omega t) + \underline{C}\,\sin(kz - \omega t)$$

For simplicity, we'll just use the forward-propagating wave for now, so no $\pm$.

Instead of the unknown constants B and C, let's use two different unknown constants, A and δ. We define them so that:

$$A\cos(\delta) = B \quad \text{and} \quad A\sin(\delta) = C$$

Thus our solution to the wave equation becomes:

$$E(z,t) = \underline{A\cos\delta}\,\cos(kz - \omega t) + \underline{A\sin\delta}\,\sin(kz - \omega t)$$

Now we can rewrite this in another form, using a trigonometric identity:

$$\cos(x{-}y) = \cos(x)\,\cos(y) + \sin(x)\,\sin(y)$$

With this identity, our solution becomes:

$$E(z,t) = A\,\cos[(kz - \omega t) - \delta]$$

Definitions: Amplitude and Absolute phase

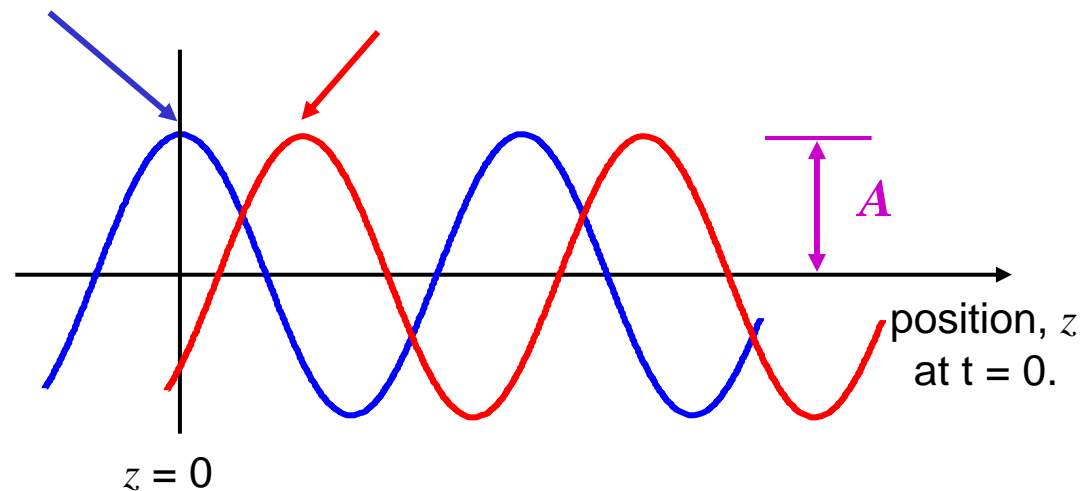This is a common way of writing the solution to the wave equation:

$$E(z,t) = A\cos(kz - \omega t - \delta)$$

A = **Amplitude** (we will see that this is related to the wave's energy)

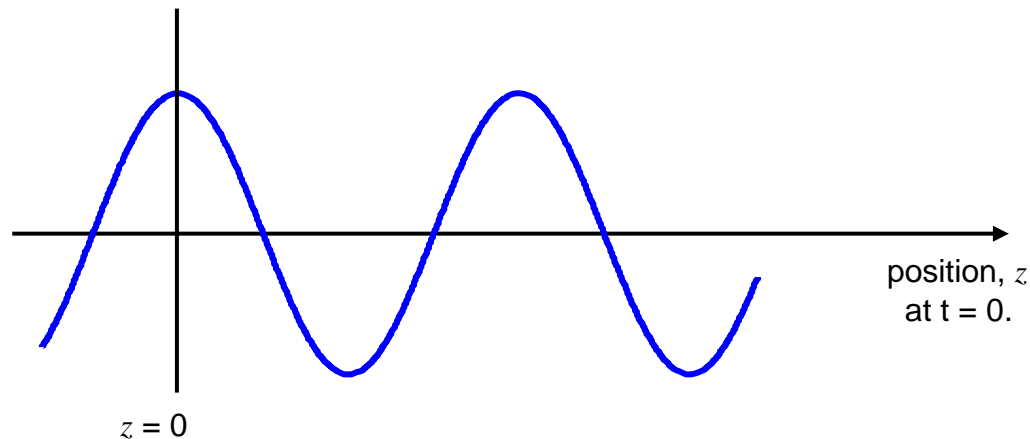δ = **Absolute phase** (or "initial" phase: the phase when z = t = 0)


Absolute phase = 0

Absolute phase = $2\pi/3$

A

position, z
at t = 0.

$z = 0$

Clarification: What does this graph mean?

Just as an illustration, here is the plot again for Absolute phase = 0

position, z
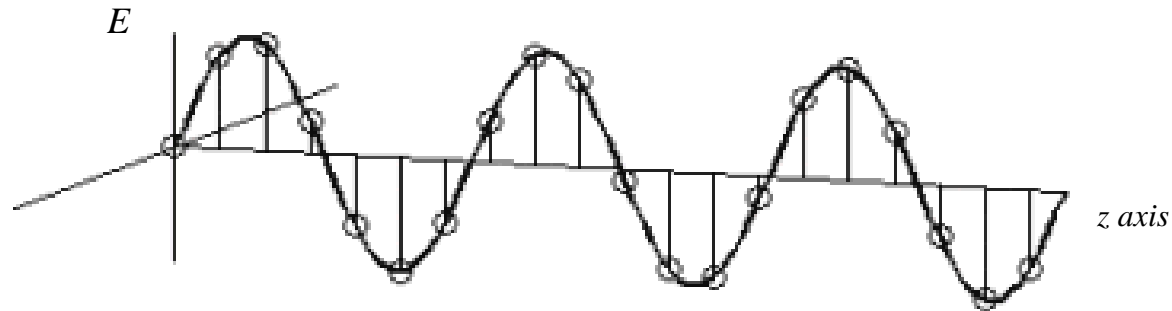at t = 0.

$z = 0$

What are we plotting here? Be sure you understand this.

Amplitude of E field vector

This picture contains no information about which way the E field is pointing! Only about the length of the vector at any point on the z axis, not about its direction.

position, z
at t = 0.

$z = 0$
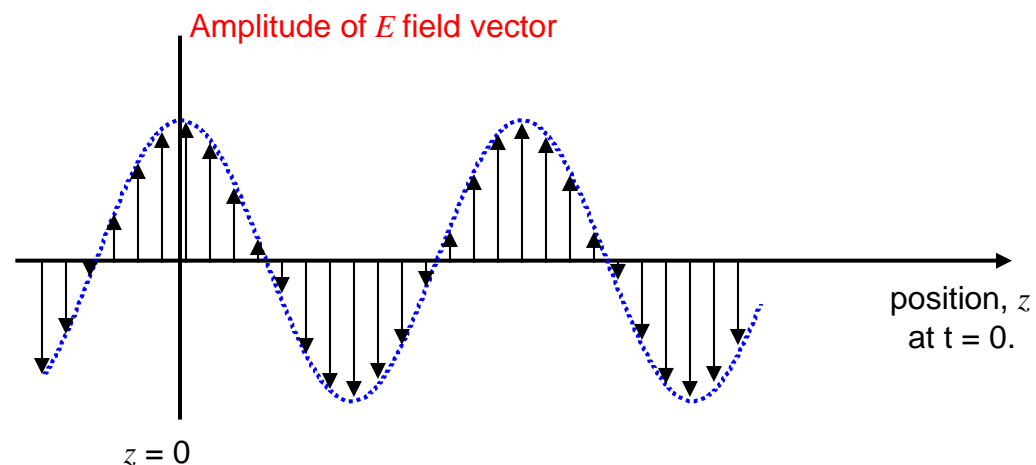
A function of both z and t
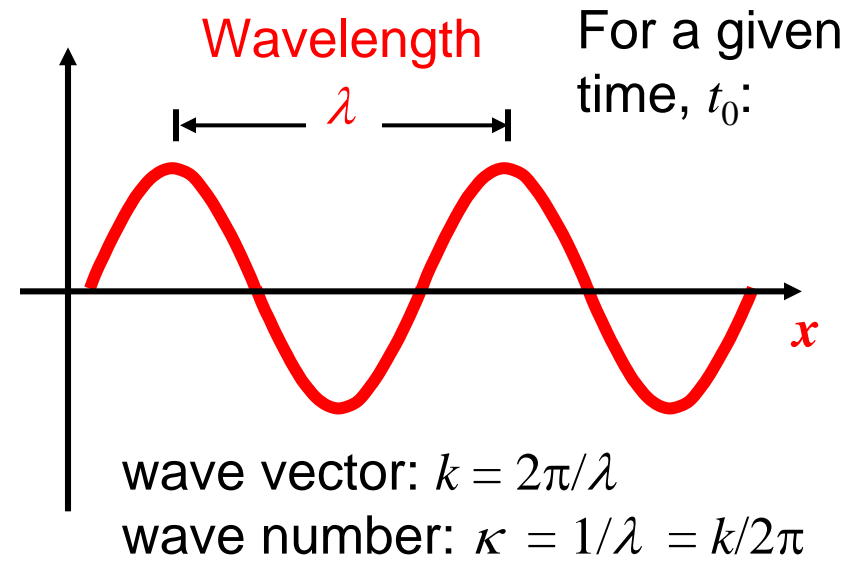
$$E(z,t) = A\cos(kz - \omega t + \delta)$$

Note: if you take a snapshot at any instant of time, the magnitude of E oscillates as a function of z.

And: if you sit at any location z, the magnitude of E oscillates as a function of time.
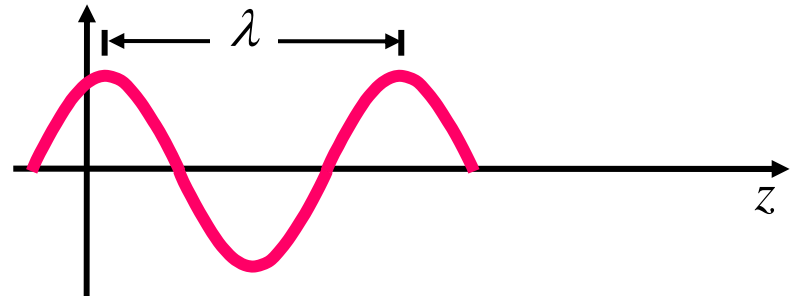
Definitions

Spatial quantities:

Wavelength

For a given time, t_0:

wave vector: $k = 2\pi/\lambda$

wave number: $\kappa = 1/\lambda = k/2\pi$

The Velocity

How fast is the wave traveling?

The wave moves one wavelength, λ, in one period, τ.

Velocity is a reference distance divided by a reference time.

The velocity is the wavelength / period:

$$c \;=\; \lambda \,/\, \tau \;=\; \lambda f$$

In terms of the k-vector, $k = 2\pi \,/\, \lambda$, and the angular frequency, $\omega = 2\pi f$, this is:

$$c \;=\; \omega \,/\, k$$

Do you need to memorize the value of the speed of light in empty space?

YES.

Approximately: $c_0 = 3 \times 10^8$ meters / second

Note: this is the only constant I expect you to memorize.

The Phase of a Wave

The phase, φ, is everything inside the cosine.

$$E(x,t) = A \cos(\varphi), \text{ where } \varphi = kx - \omega t - \theta$$

Don't confuse "the phase" with "the absolute phase" (or "initial phase").

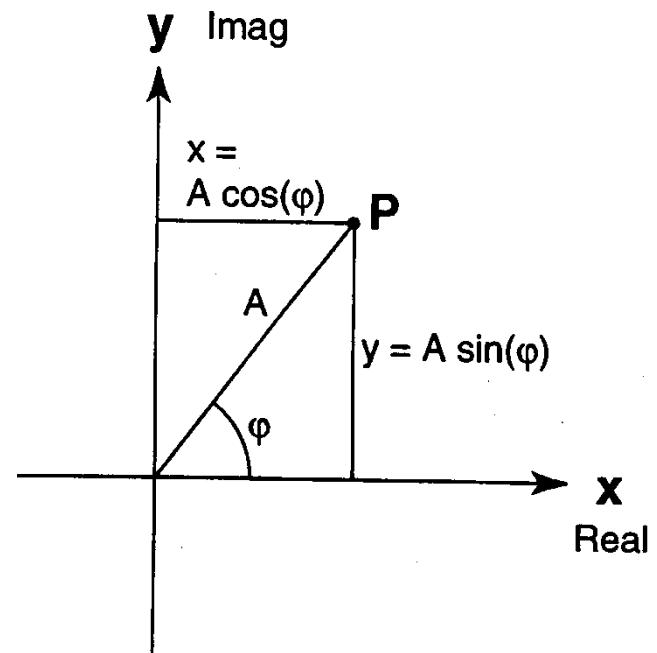The angular frequency and wave vector can be expressed as derivatives of the phase:

$$\omega = -\partial\varphi/\partial t$$

$$k = \partial\varphi/\partial x$$

Complex numbers

Consider a point, P = (x,y), on a 2D Cartesian grid.

Let the x-coordinate be the real part and the y-coordinate the imaginary part of a complex number.

So instead of using an ordered pair, we write:

$$P = x + j\,y$$

where: $j = \sqrt{-1}$

The controversy: i vs j

In physics: $i = \sqrt{-1}$

        and j = current density

In engineering: $j = \sqrt{-1}$

        and i = current

In this class:
      We strive to be engineers and use j…

(don't worry… we won't see current too often.)

Euler's Formula

$$e^{j\varphi} = \cos\varphi + j\sin\varphi$$

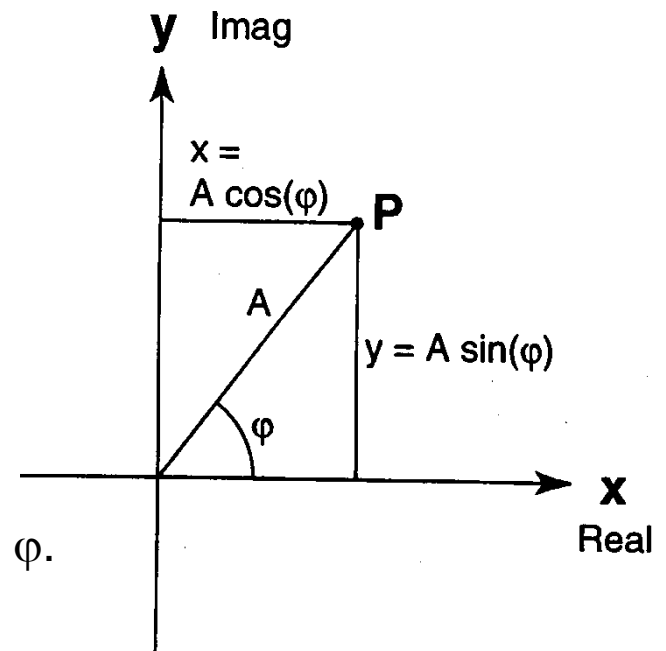Truly one of the most important equations in all the world…

so the point, $P = A\cos(\varphi) + jA\sin(\varphi)$, can be written:

$$P = A\exp(j\varphi)$$

where: A = Amplitude

φ = Phase

Reminder: $|\exp(j\varphi)| = 1$, for any (real) value of φ.

Waves using complex numbers

We have seen that the electric field of a light wave can be written:

$$E(x,t) = A\cos(kx - \omega t - \theta)$$

Since $\exp(j\varphi) = \cos(\varphi) + j\sin(\varphi)$, $E(x,t)$ can also be written:

$$E(x,t) = \mathrm{Re}\left\{A\exp\left[j(kx - \omega t - \theta)\right]\right\}$$

or

$$E(x,t) = \frac{1}{2}A\exp\left[j(kx - \omega t - \theta)\right] + c.c.$$

We often write these expressions without the ½, Re, or +c.c.

where "$+ c.c.$" means "plus the complex conjugate of everything before the plus sign."

Waves using complex amplitudes

We can let the amplitude be complex:

$$E(x,t) = A\exp\left[\,j(kx - \omega t - \theta)\right]$$

$$E(x,t) = \left\{A\exp(-j\theta)\right\} \times \left\{\exp\left[\,j(kx - \omega t)\right]\right\}$$

where we've separated the constant stuff from the rapidly changing stuff.

The resulting "complex amplitude" is:

$$\underset{\sim}{E}_0 = A\exp(-j\theta) \qquad \leftarrow \text{(note the " ~ " symbol)}$$

$$\boxed{\underset{\sim}{E}(x,t) = \underset{\sim}{E}_0 \exp\left[\,j(kx - \omega t)\right]} \quad \underset{\sim}{E}_0 \text{ does not depend on x or t.}$$

How do you know if E_0 is real or complex?

Sometimes people use the "~", but not always.
So always assume it's complex unless told otherwise.

The complex amplitude of a wave

Remember, nothing measurable ever contains j.

Complex numbers are merely a useful bookkeeping tool for tracking the phase of a quantity.   They don't appear in measurements.

$$\underset{\sim}{E}(x,t) = \underset{\sim}{E}_0 \exp\left[\, j\left(kx - \omega t\right)\right]$$

where

$$\underset{\sim}{E}_0 = A\exp\left(-j\theta\right)$$

The amplitude of an electric field like this one is a quantity that (in principle) we can measure. It has units: volts/meter.

This measurable quantity is never *ever* complex.

In principle, we can also measure the phase of this quantity (in radians). This value is also *always* a real number.

$$\left|\underset{\sim}{E}_0\right| = A \qquad \angle\,\underset{\sim}{E}_0 = -\theta$$