THE NOTION OF THE GREEN'S FUNCTION IN THE THEORY OF INTEGRO-DIFFERENTIAL EQUATIONS*

BY

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Introduction

In 1835 Duhamel in his thermo-mechanical studies discussed the integro-differential system†

$$\frac{d^2u}{dx^2} + \rho^2 u = c\rho^2 x \int_a^b u(x)x\,dx,$$

$$u(a) = 0, \quad u(b) = 0,$$

where ρ denotes a parameter and c a given constant. Since then the problems of this kind do not seem to have attracted much attention, in spite of, or perhaps because of, the great development of the general theory of integral and integro-differential equations by Volterra, Fredholm, Hilbert and their followers. In 1914, however, there appeared an interesting paper by L. Lichtenstein‡, which contains an independent treatment of the boundary problem

$$\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + [q(x) + \lambda k(x)]y(x) + \lambda \int_0^\pi M(x,\xi)y(\xi)d\xi = h(x),$$

$$y(0) = 0, \quad y(\pi) = 0,$$

by means of the theory of quadratic forms in infinitely many variables. It was remarked by the author in 1917§ (without knowledge of Lichtenstein's

* Presented to the Society, January 1, 1926; received by the editors in October, 1926. Part IV added in May, 1927.

† *Sur les phénomènes thermo-mécaniques*, Journal de l'Ecole Polytechnique, vol. 15 (1835), cahier 25, pp. 1–57; pp. 44–48.

‡ *Ueber eine Integro-Differential Gleichung und die Entwickelung willkürlicher Funktionen nach deren Eigenfunktionen*, Schwarz's Festschrift, Berlin, 1914, pp. 274–285. Cf. also L. Koschmieder, *Anwendung der Integralgleichungen auf eine thermo-elastische Aufgabe*, Crelle's Journal, vol. 143 (1913), pp. 285–293.

§ *On some general problems of the theory of ordinary linear differential equations and expansion of an arbitrary function*, Petrograd, 1917 (in Russian). This paper contains as special cases most of the results of M. Stone's important paper *A comparison of the series of Fourier and Birkhoff*, these Transactions, vol. 28 (1926), pp. 695–761, concerning the theory of equiconvergence and convergence of Birkhoff series and obtained by Stone without knowledge of author's paper. The theory of the derived Birkhoff series successfully treated by Stone was not touched in our paper in question.

paper) that Duhamel's problem may yield to treatment by the method used by G. D. Birkhoff* and by the author (loc. cit.) in the theory of ordinary differential boundary problems. Some special cases of integro-differential boundary problems (mostly with constant coefficients) have been recently treated.†

The present paper sketches a more or less general theory of the integro-differential boundary problem originated by Duhamel. The characteristic feature of the paper lies in the constant use of the notion (which appears to be new) of the Green's function of the integro-differential problem. This function is closely related to the Green's function of a certain corresponding differential problem. It is found that the treatment of the integro-differential problem requires no methods other than those which have been used in the treatment of the differential boundary problem; from certain points of view the two problems are equivalent. The general theory as developed in Parts I–III admits of an interesting application to Fredholm integral equations with discontinuous kernels (Part IV).

A detailed discussion of the differential boundary problem and of the properties of the corresponding Green's function is embodied in the previous papers by the author.‡ The second of these (reference to which will be indicated merely by the symbol "D") contains the proofs of some propositions which are merely stated in the present paper.

I. Definition of the Green's function

1. We consider the integro-differential equation

$$(\bigstar)\qquad u^{(k)}(x)+r_1(x)u^{(k-1)}(x)+\cdots+r_k(x)u(x)=r(x)\sum_{\sigma=0}^{m}\int_a^b u^{(\sigma)}(\xi)d_\xi R_\sigma(x,\xi)$$

in which the integrals are of the Stieltjes type, and in which the functions $R_\sigma(x,\xi)$, together with their partial derivatives (to a certain order) with respect to ξ are continuous in ξ, except along a finite number of lines

$$\xi=\text{constant}=\alpha$$

* *Boundary value and expansion problems of ordinary linear differential equations*, these Transactions, vol. 9 (1908), pp. 373–395.

† W. Jaroshek, *Entwickelung willkürlicher Funktionen* (Diss.), Breslau, 1918; H. Laudien, *Entwickelung willkürlicher Funktionen bei einem thermo-elastischen Problem*, Crelle's Journal, vol. 148 (1918), pp. 79–87; A. Kneser, *Die Integralgleichungen*, 2d edition, Braunschweig, 1922, pp. 199–214; R. Krzenziessa, *Thermo-elastischen Randwertaufgaben*, Mathematische Zeitschrift, vol. 25 (1926), pp. 209–260; the last paper discusses some cases of variable coefficients, which are reducible to Bessel functions.

‡ Loc. cit. and *Some general problems of the theory of ordinary linear differential equations and expansion of an arbitrary function in series of fundamental functions*, Mathematische Zeitschrift.

and possibly, along the line

$$\xi = x.$$

The discontinuities are all to be of the first kind. Under these hypotheses the equation (★) may be reduced by a suitable number of integrations by parts to the form

$$\text{(I)} \qquad L(u) = f(x) + \sum_{j=1}^{\mu} l_j(u)\phi_j(x) + \int_a^b h(x,\xi)u(\xi)d\xi$$

where $L(u)$ is a linear differential operator

$$\text{(1)} \qquad L(u) \equiv u^{(n)} + p_1(x)u^{(n-1)} + \cdots + p_n(x)u,$$

the $l_j(u)$ are linear forms in u and its derivatives taken at the points a, b, α, and $\phi_j(x)$, $h(x, \xi)$ are known functions.

In the following we confine the discussion to the equation (I) either in the form above or in the form

$$\text{(I}') \quad L^{(0)}(u) = f(x) + \sum_{j=1}^{\mu} l_j(u)\phi_j(x) + \int_a^b h(x,\xi)u(\xi)d\xi + L^{(1)}(u),$$

where $L^{(1)}(u)$ is any linear differential operator of order $(n-1)$,

$$\text{(2)} \qquad L^{(1)}(u) \equiv p_1^{(1)}(x)u^{(n-1)} + \cdots + p_n^{(1)}(x)u,$$

and

$$L^{(0)}(u) \equiv L(u) - L^{(1)}(u).$$

The boundary conditions to be imposed on the unknown function $u(x)$ are

$$\text{(II)} \qquad L_i(u) = 0 \qquad (i = 1, 2, \cdots, n)$$

where

$$\text{(3)} \qquad \begin{gathered} L_i(u) \equiv A_i(u) + B_i(u) + \int_a^b \alpha_i(x)u(x)dx, \\ A_i(u) \equiv \sum_{k=1}^{n} a_{ik}u^{(k-1)}(a)\ ; \ B_i(u) \equiv \sum_{k=1}^{n} b_{ik}u^{(k-1)}(b).* \end{gathered}$$

The integro-differential problem consisting of (I) and (II) we shall designate as "problem (★)". The problem obtained by setting in (I′)

$$\phi_j(x) = h(x,\xi) = L^{(1)}(u) \equiv 0$$

we shall designate as the "differential problem (★)."

* The boundary operators used in D, 5, are somewhat more general than those used here.

The hypotheses on the functions involved in (I′) and (II) are as follows:

i. The functions

$$p_i(x), \quad p_i^{(1)}(x)$$

possess continuous derivatives to order $(n-i)$ on (a, b).

ii. The functions

$$f(x), \quad \phi_j(x) \qquad\qquad (j = 1,2, \cdots ,\mu)$$

are integrable (in the sense of Lebesgue) on (a, b).

iii. The function $h(x, \xi)$ is bounded and is integrable as a function of both variables as well as in x, for every value of ξ, and in ξ, for every value of x.

iv. The functions

$$\alpha_i(x)$$

are integrable on (a, b).

v. The operators

$$A_i(u) + B_i(u), \quad l_j(u)$$

are linearly independent, and the operators $l_j(u)$ contain no derivatives of order higher than $(n-1)$.*

2. THEOREM 1. *Under the hypotheses* (i-v), *two alternative cases are possible as follows:*

(1) *The non-homogeneous problem* (★) *admits of a uniquely determined solution† for an arbitrary $f(x)$, and this solution can be represented in form of a definite integral*

$$\text{(III)} \qquad u(x) = \int_a^b \Gamma(x,t)f(t)dt.$$

The function $\Gamma(x, t)$ is called the Green's function of the problem (★). *It is uniquely determined at its points of continuity.*

(2) *The non-homogeneous problem* (★) *is not possible for an arbitrary $f(x)$. In this case the Green's function $\Gamma(x, t)$ does not exist, but the homogeneous problem* (★) *admits of at least one solution not identically zero on (a, b).*‡

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* This excludes the cases in which some of the operators $L_i(u)$ reduce to the integral terms only. It is obvious also that condition (v) implies the linear independence of the operators $L_i(u)$ and $l_j(u)$.

† The term "solution" is to designate a function $u(x)$ which possesses an absolutely continuous derivative of order $(n-1)$ and which satisfies (I′) (almost everywhere) and (II).

‡ In this case we shall say simply that the homogeneous problem (★) is possible. If the homogeneous problem (★) admits of no solution other than $u(x)\equiv 0$, we shall say the problem is impossible.

3. We may assume without loss of generality that the Green's function $G(x, t)$ of the differential problem ($\bigstar$) exists, namely that

$$\Delta(y) = \begin{vmatrix} L_1(y_1) \cdots L_1(y_n) \\ \cdots\cdots\cdots \\ L_n(y_1) \cdots L_n(y_n) \end{vmatrix} \neq 0$$

where $y_1(x), \cdots, y_n(x)$ is a fundamental system of solutions of the equation

$$L^{(0)}(y) = 0.$$

This assumption is permissible since we may alter $L^{(0)}(y)$, if necessary, by drawing into it any portion of the operator $L^{(1)}(y)$ without thereby changing the content of equation (I′).

4. It is well known (D, 5–8) that the solution of the non-homogeneous differential problem

$$L^{(0)}(u) = \phi(x), \quad L_i(u) = 0 \qquad (i = 1,2, \cdots ,n) \tag{4}$$

is given by

$$u(x) = \int_a^b G(x,t)\phi(t)dt. \tag{5}$$

Replacing $\phi(x)$ in this by the right-hand member of (I′) we obtain for that equation the form

$$u(x) = F(x) + \sum_{j=1}^{\mu} l_j(u)\Phi_j(x) + \int_a^b H(x,\xi)u(\xi)d\xi + \int_a^b G(x,t)L^{(1)}(u)_t dt^* \tag{6}$$

where

$$F(x) = \int_a^b G(x,t)f(t)dt\ ; \ \Phi_j(x) = \int_a^b G(x,t)\phi_j(t)dt \tag{7}$$

and

$$H(x,\xi) = \int_a^b G(x,t)h(t,\xi)dt. \tag{8}$$

Under the hypotheses concerning the functions $p_i(x)$, the Green's function $G(x, t)$, as function of t possesses a continuous derivative of order $(n-2)$ and a derivative of order $(n-1)$ which is continuous except along the line $x=t$, where it has a discontinuity of the first kind (D, 11). By a suitable number of integrations by parts the final integral in (6) may be thrown into the form

$$\int_a^b H'(x,t)u(t)dt + l(u,x)$$

* The subscript t on $L^{(1)}$ is used here to indicate the variable in which the operator is written.

where $l(u, x)$ is a linear form in $u(x)$ and its derivatives to order $(n-2)$ taken at the points $x=a$ and $x=b$, the coefficients of the form being functions of x. Portions of this form may possibly be eliminated by the use of relations (II), and others may be expressible in terms of the quantities

$$l_j(u) \qquad (j = 1,2, \cdots ,\mu).$$

Thus we obtain finally

$$u(x) = F(x) + \sum_{j=1}^{\nu} l_j(u)\Phi_j'(x) + \int_a^b K'(x,\xi)u(\xi)d\xi \qquad (\nu \geqq \mu), \tag{9}$$

where the operators

$$l_j(u) \qquad (j = \mu + 1, \cdots ,\nu)$$

are of the same type as those in (I′) and the set

$$l_1(u), \quad l_2(u), \cdots , l_\nu(u) \tag{10}$$

may be considered as linearly independent and independent of $L_i(u)$. In other words, if $u(x)$ is considered arbitrary subject to (II), the quantities (10) may assume any prescribed set of values.

The functions $\Phi'_j(x)$, $K'(x, \xi)$ obviously possess the same properties of integrability as the functions $\Phi_j(x)$, $H(x, \xi)$ respectively.

On setting

$$l_j(u) = c_j \qquad (j = 1,2, \cdots ,\nu), \tag{11}$$

$$F(x) + \sum_{j=1}^{\nu} c_j\Phi_j'(x) = \psi(x), \tag{12}$$

we reduce our problem to the Fredholm integral equation

$$u(x) = \psi(x) + \int_a^b K'(x,\xi)u(\xi)d\xi. \tag{13}$$

5. We suppose, in order to include the general case, that the corresponding homogeneous integral equation

$$u(x) = \int_a^b K'(x,\xi)u(\xi)d\xi$$

admits of σ linearly independent solutions

$$\omega_1(x), \quad \omega_2(x), \cdots , \omega_\sigma(x). \tag{14}$$

In this case the associated integral equation

$$v(x) = \int_a^b K'(\xi, x)v(\xi)d\xi$$

has the same number of linearly independent solutions:*

(15) $$\omega_1'(x), \ \omega_2'(x), \ \cdots, \ \omega_\sigma'(x).$$

Either of the sets (14), (15) may be supposed to be orthogonal and normalized. The necessary and sufficient condition that (13) have a solution is given by the system of equations

(16) $$\int_a^b \psi(x)\omega_k'(x)dx = 0 \qquad (k = 1,2, \cdots, \sigma)$$

and if these conditions are satisfied, the most general solution of (13) is

(17) $$u(x) = \psi(x) - \int_a^b \mathfrak{K}'(x,\xi)\psi(\xi)d\xi + \sum_{k=1}^{\sigma} c_k'\omega_k(x)$$

where the c_k' are arbitrary constants, and $\mathfrak{K}'(x, \xi)$ is a so called "pseudo-resolvent kernel" of the kernel $K'(x, \xi)$.

The case where the kernel $K'(x, \xi)$ possesses a reciprocal, may be considered as the special case in which

$$\omega_k(x) = \omega_k'(x) = c_k' = 0 \qquad (k = 1,2, \cdots, \sigma)$$

and $\mathfrak{K}'(x, \xi)$ coincides with this reciprocal.

6. The constants c_j, c_k' remain to be determined. This can be done by substituting (17) into (11) and eliminating $\psi(x)$ from the resulting equations and the relations (12) and (16). Thus we obtain a system of $(\nu+\sigma)$ linear equations for the $(\nu+\sigma)$ unknowns c_j, c_k' as follows:

(18) $$\lambda_j(c) = l_j\left\{ F(x) - \int_a^b \mathfrak{K}'(x,\xi)F(\xi)d\xi \right\} \qquad (j = 1,2, \cdots, \nu),$$
$$\lambda_k'(c) = -\int_a^b F(x)\omega_k'(x)dx \qquad (k = 1,2, \cdots, \sigma).$$

The $\lambda_j(c)$ and $\lambda_k'(c)$ are linear forms in c's and c''s whose coefficients are constants which do not depend upon the function $f(x)$. Denoting by D_0 the determinant of this system, we must consider the two cases

$$D_0 \neq 0; \qquad D_0 = 0.$$

* E. W. Hobson, *On the linear integral equation*, Proceedings of the London Mathematical Society, (2), vol. 13 (1914), pp. 307–340. The classical results of the Fredholm theory are extended in this paper to more general kernels, of which ours is a particular case. Cf. also our paper *On Fredholm's integral equations whose kernels are analytic in a parameter*, Annals of Mathematics, (2), vol. 28 (1927), pp. 127–152.

(a) Suppose that $D_0 \neq 0$. In this case system (18) determines uniquely the constants c_j, c_k', and each of these constants is a linear combination of integrals of the form

$$\int_a^b f(t)\beta(t)dt.$$

Substituting these values into (17), we obtain the required expression for $u(x)$:

$$\text{(III)} \qquad u(x) = \int_a^b \Gamma(x,t)f(t)dt.$$

Conversely, the preceding operations being reversible, it is readily shown that (17) represents a solution of the problem (★). This solution is uniquely determined and by familiar reasoning it is proved that the function $\Gamma(x, t)$ is uniquely determined at all its points of continuity.

Under the following supplementary restrictions:

iii′. The discontinuities of the function $h(x, \xi)$ are regularly distributed, i.e., $h(x, \xi)$ is continuous with respect to ξ for almost all values of x, and continuous in x for almost all values of ξ,*

v′. The operators $l_j(u)$ involve the values of $u(x)$ and its derivatives only at the end points $x=a$, $x=b$,

it can be shown by means of some results of Hobson that $\Gamma(x, t)$ is continuous everywhere on (a, b), except, possibly, along the line $x=t$, where it may have a discontinuity of the first kind.

7. (b) Suppose now that $D_0=0$. We shall prove first that the homogeneous problem (★) has at least one solution which is not identically zero on (a, b). The homogeneous system

$$\text{(19)} \qquad \lambda_j(c) = 0 \quad (j = 1,2, \cdots ,\nu)\,; \qquad \lambda_k'(c) = 0 \qquad (k = 1,2, \cdots ,\sigma)$$

admits of at least one solution in which not all the constants c_j, c_k' are zero. We substitute these values of c_j, c_k' into (17) and set $f(x)\equiv 0$. The corresponding function $u_0(x)$ is certainly a solution of the homogeneous problem (★), and it only remains to prove that $u_0(x)\not\equiv 0$.

Suppose that $u_0(x)\equiv 0$. Then from (11) it follows that

$$c_j = l_j(u_0) = 0 \qquad (j = 1,2, \cdots ,\nu),$$

and, substituting in (12)

$$c_j = 0, \quad F(x) \equiv 0,$$

* In this case the solution of the problem (★), with continuous $f(x)$ and $\phi_i(x)$, possesses a continuous derivative of order n and satisfies (I′) everywhere.

we have $\psi(x)=0$. Hence

$$\sum_{k=1}^{\sigma} c_k' \omega_k(x) = u_0(x) = 0, \quad c_k' = 0 \quad (k = 1,2, \cdots, \sigma),$$

contrary to our supposition.

8. We shall prove now that the right-hand members of equations (18) may assume any given set of values if $f(x)$ is suitably chosen. Since this linear system (18) will admit of a solution for the c's only for special values of the right-hand members, it will follow that no solution for a general $f(x)$ is possible and that in this case the Green's function $\Gamma(x, t)$ does not exist.

It follows from the classical theory of the Green's function $G(x, t)$ that, if $F(x)$ is an arbitrary function which possesses a continuous derivative of order n and satisfies the boundary conditions

$$L_i(F) = 0 \qquad (i = 1,2, \cdots, n), \tag{20}$$

then the function

$$f(x) \equiv L^{(0)}(F)$$

satisfies the relation

$$F(x) = \int_a^b G(x,t)f(t)dt.$$

The question is reduced, therefore, to the proof that there always exists a function $F(x)$ which satisfies conditions (20) and for which the expressions

$$l_j\left\{F(x) - \int_a^b \mathfrak{K}'(x,\xi)F(\xi)d\xi\right\} \qquad (j = 1,2, \cdots, \nu);$$

$$\int_a^b F(x)\omega_k'(x)dx \qquad (k = 1,2, \cdots, \sigma)$$

take on arbitrary prescribed values. Let $\chi(x)$ be a function subsequently to be determined and determine $F(x)$ from the integral equation

$$\chi(x) = F(x) - \int_a^b \mathfrak{K}'(x,\xi)F(\xi)d\xi.$$

This is always possible, if we use the special form of the pseudo-resolvent kernel $\mathfrak{K}'(x, \xi)$ as given by W. A. Hurwitz.* This kernel $\mathfrak{K}'(x, \xi)$ is defined as the reciprocal of the function

$$K'(x,\xi) - \sum_{k=1}^{\sigma} \omega_k'(x)\omega_k(\xi).$$

* *On the pseudo-resolvent to the kernel of an integral equation*, these Transactions, vol. 13 (1912), pp. 405–418; pp. 405–409.

Hence this latter function is in turn the reciprocal of the kernel $\mathfrak{K}'(x, \xi)$ and we have

$$F(x) = \chi(x) - \int_a^b K'(x,\xi)\chi(\xi)d\xi + \sum_{k=1}^{\sigma} \omega_k'(x) \int_a^b \chi(\xi)\omega_k(\xi)d\xi.$$

On multiplying this by $\omega_s'(x)$ and integrating, we obtain in virtue of the relation

$$\omega_s'(\xi) = \int_a^b K'(x,\xi)\omega_s'(x)dx$$

the result

$$\int_a^b F(x)\omega_s'(x)dx = \int_a^b \chi(\xi)\omega_s(\xi)d\xi \qquad (s = 1,2, \cdots, \sigma).$$

Now, as functions of x, $G(x, t)$ and therefore $H(x, t)$, $H'(x, t)$ and $K'(x, t)$ satisfy conditions (II), so that

$$L_i(F) = L_i(\chi) + \sum_{k=1}^{\sigma} L_i(\omega_k') \int_a^b \chi(\xi)\omega_k(\xi)d\xi.$$

It is readily seen that, under the conditions (v), § 1, imposed upon the operators $L_i(u)$, $l_j(u)$, the function $\chi(x)$ can always be chosen so that

$$L_i(F) = 0 \qquad (i = 1,2, \cdots, n)$$

while the expressions

$$l_j(\chi) \quad (j = 1,2, \cdots, \nu); \qquad \int_a^b \chi(\xi)\omega_k(\xi)d\xi \qquad (k = 1,2, \cdots, \sigma)$$

assume the prescribed values. This completes the proof of Theorem 1.

9. Theorem 2. *Suppose that, under the conditions of Theorem* 1, *the coefficients of the operators of the problem* (★) *depend on a parameter* ρ *and are analytic in an open region* $(\mathfrak{D})$ *of the* ρ*-plane. Suppose that the Green's function* $\Gamma(x, t, \rho)$ *of the problem* (★) *exists for an infinite set of values of* ρ, *which possesses at least one limiting point interior to* $(\mathfrak{D})$. *Then* $\Gamma(x, t, \rho)$ *is a meromorphic function in* $(\mathfrak{D})$. *The homogeneous problem* (★) *is possible when and only when* ρ *is equal to one of the poles of* $\Gamma(x, t, \rho)$ (*characteristic values of the problem* (★)). *For such values of* ρ *the non-homogeneous problem* (★) *is not possible with an arbitrary* $f(x)$, *and the Green's function does not exist.*

To prove this theorem, we shall use another method for obtaining the Green's function $\Gamma(x, t, \rho)$.* We observe that a particular solution of the equation

$$L(u) = \phi(x) \tag{21}$$

is given by

$$Y(x) = \int_a^b g(x,t)\phi(t)dt$$

where

$$g(x,t) = \pm \frac{1}{2\delta(t)} \begin{vmatrix} y_1(x) & ,\cdots, & y_n(x) \\ y_1^{(n-2)}(t), & \cdots & ,y_n^{(n-2)}(t) \\ \cdot & \cdots & \cdot \\ y_1(t) & ,\cdots, & y_n(t) \end{vmatrix} \qquad \begin{cases} + \text{ if } x > t, \\ - \text{ if } x < t, \end{cases}$$

$$\delta(t) = \begin{vmatrix} y_1^{(n-1)}(t), & \cdots & ,y_n^{(n-1)}(t) \\ y_1^{(n-2)}(t), & \cdots & ,y_n^{(n-2)}(t) \\ \cdot & \cdots & \cdot \\ y_1(t) & ,\cdots, & y_n(t) \end{vmatrix}$$

and $y_1(x), \cdots, y_n(x)$† denotes any fundamental system of solutions of the equation $L(u)=0$.

The general solution of (21) is then

$$u(x) = \sum_{i=1}^{n} c_i'' y_i(x) + Y(x),$$

where $c_1'', \cdots, c_n''$ are arbitrary constants. If we set

$$l_j(u) = c_j \qquad (j = 1,2,\cdots,\mu),$$

$$\psi(x) = \sum_{i=1}^{n} c_i'' y_i(x) + \sum_{j=1}^{\mu} c_j \int_a^b g(x,t)\phi_j(t)dt + \int_a^b g(x,t)f(t)dt,$$

$$H''(x,\xi) = \int_a^b g(x,t)h(t,\xi)dt,$$

we easily obtain from (I) the following integral equation for $u(x)$:

$$u(x) = \psi(x) - \int_a^b H''(x,\xi)\psi(\xi)d\xi.$$

* For the sake of brevity the letter ρ sometimes will be omitted.

† The fundamental system $y_1(x), \cdots, y_n(x)$ may always be chosen so that $\delta(t)$, which depends on ρ, is $\neq 0$ for all values of ρ.

The kernel $H''(x, \xi)$ is a function of ρ, which is analytic in $(\mathfrak{D})$, and by the classical Fredholm theory it possesses a reciprocal $\mathfrak{H}''(x, \xi)$ which is meromorphic in $(\mathfrak{D})$. Suppose that ρ is not a pole of $\mathfrak{H}''(x, \xi)$. Then we have

$$u(x) = \psi(x) + \int_a^b \mathfrak{H}''(x,\xi)u(\xi)d\xi,$$

and it only remains to determine the constants c_j, c_i'' from the equations

$$L_i(u) = 0 \quad (i = 1,2, \cdots ,n); \qquad l_j(u) = c_j \quad (j = 1,2, \cdots ,\mu).$$

The determinant $D(\rho)$ of this system is a meromorphic function in $(\mathfrak{D})$, and if it is not identically zero the resulting expression for $\Gamma(x, t, \rho)$ shows that $\Gamma(x, t, \rho)$ is also a meromorphic function in $(\mathfrak{D})$. Thus we obtain the solution of the non-homogeneous problem $(\bigstar)$ for all values of ρ which are different from the poles of either of the functions $\mathfrak{H}''(x, t)$, $\Gamma(x, t, \rho)$ in the form

$$u(x) = \int_a^b \Gamma(x,t,\rho)f(t)dt. \tag{22}$$

Expression (22) represents a solution of the problem $(\bigstar)$:

$$L(u) = f(x) + \sum_{j=1}^{\mu} l_j(u)\phi_j(x) + \int_a^b h(x,\xi)u(\xi)d\xi, \tag{I}$$

$$L_i(u) = 0 \qquad (i = 1,2, \cdots ,n), \tag{II}$$

even if ρ is a pole of the function $\mathfrak{H}''(x, \xi)$ but not a pole of the function $\Gamma(x, t, \rho)$, for if we exclude from $(\mathfrak{D})$ the immediate vicinities of the poles of $\Gamma(x, t, \rho)$ the left and right-hand members of equations (I), (II) are analytic throughout the remaining region. Since they have been proved to be equal in a part of this region it follows that they must be equal throughout the whole of it. Now, under the conditions of Theorem 2, we can prove that the determinant $D(\rho)$ cannot be identically zero in $(\mathfrak{D})$, for if it were, the homogeneous problem $(\bigstar)$ would have a solution for all values of ρ which are not poles of $\mathfrak{H}''(x, \xi)$. By virtue of Theorem 1, then the Green's function of the problem $(\bigstar)$ would not exist, except possibly for ρ one of the excepted values. Since this set of values has no limiting point in the interior of $(\mathfrak{D})$ the conditions of Theorem 2 would be contradicted. This completes the proof of Theorem 2.*

* There are examples where the Green's function $\Gamma(x, t, \rho)$ exists for no values of ρ, for instance the system $u'+\rho u=f(x)+\rho\int_0^1 u dx$; $u(1)=u(0)$. In this connection it would be of interest to determine whether the requirement of Theorem 2 concerning the set of points of existence of $\Gamma(x, t, \rho)$ might be replaced by a less restrictive one, as for instance that the Green's function $\Gamma(x, t, \rho)$ exist at a finite number of points, or, even, at a single point of the region $(\mathfrak{D})$. This can be done in the case of a differential problem.

II. The Green's function of the differential problem

10. We shall specialize our problem (★) now by introducing certain restrictions upon the coefficients of the operators involved. Under these restrictions, which will be enunciated as they become necessary, we have at hand established facts from the theory of differential equations. With the use of these we shall derive formulas which reveal the structural features of the Green's function $G(x, t, \rho)$. Thus we make the following assumptions:

(A) i. The operators of the differential problem (★) are polynomials in ρ, i.e. our problem is of the form

$$L(u) \equiv u^{(n)} + P_1(x,\rho)u^{(n-1)} + \cdots + P_n(x,\rho)u,$$

$$L_i(u) \equiv \sum_{s=0}^{n} \rho^s L_i^{(s)}(u) \equiv A_i(u,\rho) + B_i(u,\rho) + \int_a^b \alpha_i(x,\rho)u(x)dx \qquad (i = 1,2, \cdots ,n),$$

where

$$P_i(x,\rho) \equiv \rho^i \sum_{j=0}^{i} \rho^{-j} p_{ij}(x),$$

$$L_i^{(s)}(u) \equiv A_i^{(s)}(u) + B_i^{(s)}(u) + \int_a^b \alpha_{is}(x)u(x)dx,$$

$$A_i^{(s)}(u) \equiv \sum_{k=1}^{n} a_{iks}u^{(k-1)}(a) \; ; \quad B_i^{(s)}(u) \equiv \sum_{k=1}^{n} b_{iks}u^{(k-1)}(b),$$

$$A_i(u,\rho) \equiv \sum_{s=0}^{n} \rho^s A_i^{(s)}(u) \; ; \quad B_i(u,\rho) \equiv \sum_{s=0}^{n} \rho^s B_i^{(s)}(u),$$

$$\alpha_i(x,\rho) \equiv \sum_{s=0}^{n} \rho^s \alpha_{is}(x).$$

ii. The functions

$$p_{ij}(x) \qquad (i = 1,2, \cdots ,n; j = 0,1, \cdots ,i)$$

are of bounded variation on (a, b) and possess continuous derivatives of order $(n-i)$. The functions

$$\frac{d^2 p_{i0}(x)}{dx^2}, \quad \frac{dp_{i1}(x)}{dx}$$

are of bounded variation on (a, b).

iii. The "characteristic" equation

$$\mathfrak{F}(\theta) \equiv \theta^n + p_{10}(x)\theta^{n-1} + \cdots + p_{n0}(x) = 0$$

has simple roots for all values of x in (a, b). These roots

$$\theta_1(x), \cdots, \theta_n(x)$$

are of the form

$$\theta_i(x) = \pi_i q_i(x) \quad (\pi_i \text{ constant}; \ i = 1, 2, \cdots, n) \tag{1}$$

where either

$$\pi_i = \pm \pi_0 \quad (\pi_0 \text{ real or complex and } \neq 0) \text{ and } q_i(x) \geqq q_0 > 0 \text{ on } (a, b),$$

or else

$$\pi_i \neq 0 \text{ and } q_1(x) \equiv q_2(x) \equiv \cdots \equiv q_n(x) \equiv q(x) \geqq q_0 > 0 \text{ on } (a, b).$$

iv. The operators

$$A_i(u, \rho) + B_i(u, \rho) \qquad (i = 1, 2, \cdots, n)$$

are linearly independent for all values of ρ.

v. The functions

$$\alpha_{is}(x)$$

possess continuous derivatives of the first order which are of bounded variation on (a, b).*

11. It is well known under the conditions stated that the complex ρ-plane may be divided into a finite number of sectors $(\mathfrak{R})$, such that, in each sector the numbers

$$w_i = \int_a^b \theta_i(x)dx = \pi_i \int_a^b q_i(x)dx \qquad (i = 1, 2, \cdots, n) \tag{2}$$

may be ordered as follows:†

$$\operatorname{Re}\rho w_1 \leqq \cdots \leqq \operatorname{Re}\rho w_r \leqq 0 \leqq \operatorname{Re}\rho w_{r+1} \leqq \cdots \leqq \operatorname{Re}\rho w_n \text{ in } \mathfrak{R}, \tag{3}$$

and further that there exists a fundamental system of solutions of the equation $L(y) = 0$, which, in any $\mathfrak{R}$, is of the form‡

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* The integral terms of the boundary operators used in D(17–19, 26) are $\int_a^b \alpha_i(x, \rho)u^{(n-1)}(x)dx$ while those here are $\int_a^b \alpha_i(x, \rho)\, u(x)\, dx$. This explains a slight difference between the following results and those of D.

† These sectors may always be so constructed that none of them contain more than one of the rays $\operatorname{Re}\rho w_i = \operatorname{Re}\rho w_k$, and none of these rays serve as a boundary.

‡ The symbol E, $E(\rho)$, $E(\rho, \cdots)$ is used as a generic notation to designate functions of ρ (and other variables), which remain bounded for large $|\rho|$. The symbol ϵ, $\epsilon(\rho)$, $\epsilon(\rho, \cdots)$ will be used to designate functions of ρ (and other variables) which tend uniformly to zero as ρ becomes infinite.

$$
(4)\qquad \begin{aligned}
y_i(x,\rho) &= e^{\rho\int_a^x \theta_i(x)dx}\left\{\eta_i(x) + \frac{1}{\rho}\eta_{i0}(x) + \frac{E}{\rho^2}\right\} = e^{\rho\int_a^x \theta_i(x)dx}\left\{\eta_i(x) + \frac{E}{\rho}\right\},\\
\frac{d^s y_i(x,\rho)}{dx^s} &= e^{\rho\int_a^x \theta_i(x)dx}\rho^s\{\theta_i(x)\}^s\left\{\eta_i(x) + \frac{1}{\rho}\eta_{is}(x) + \frac{E}{\rho^2}\right\}
\end{aligned}
$$

$$(i = 1,2, \cdots ,n\,;\ s = 0,1, \cdots ,n-1).$$

The functions*

$$\eta_i(x)$$

possess derivatives of the second order, and the functions

$$\eta_{is}(x)$$

possess derivatives of the first order which are continuous and of bounded variation on (a, b).

Lastly it is known that if $Y_k(t, \rho)$ denotes the cofactor of $y_k^{(n-1)}(t, \rho)$ in the determinant

$$\delta(t,\rho) \equiv \begin{vmatrix} y_1^{(n-1)}(t,\rho) & \cdots & y_n^{(n-1)}(t,\rho) \\ \cdot & \cdots & \cdot \\ y_1(t,\rho) & \cdots & y_n(t,\rho) \end{vmatrix},$$

then the adjoint equation $L'(y)=0$ admits of a fundamental system of solutions

$$z_k(t,\rho) = \frac{Y_k(t,\rho)}{\delta(t,\rho)} \qquad (k = 1,2, \cdots ,n),$$

and in any $\mathfrak{R}$

$$
(5)\qquad \begin{aligned}
z_k(t,\rho) &= e^{-\rho\int_a^t \theta_k(x)dx}\rho^{-n+1}\left\{\zeta_k(t) + \frac{1}{\rho}\zeta_{k0}(t) + \frac{E}{\rho^2}\right\},\\
\frac{d^s z_k(t,\rho)}{dt^s} &= e^{-\rho\int_a^t \theta_k(x)dx}\rho^{-n+1+s}\left\{\zeta_k(t) + \frac{1}{\rho}\zeta_{ks}(t) + \frac{E}{\rho^2}\right\}(-1)^s\{\theta_k(t)\}^s
\end{aligned}
$$

$$(k = 1,2, \cdots ,n\,;\ s = 0,1, \cdots ,n-1),$$

where

$$\zeta_k(t) = (\eta_k(t)\mathfrak{F}'\{\theta_k(t)\})^{-1}$$

---

* It can be proved (D, 2) that

$$\eta_i(x) = \frac{1}{(\mathfrak{F}'(\theta_i))^{1/2}}\exp\left\{-\int_a^x (\mathfrak{F}_1(\theta_i)/\mathfrak{F}'(\theta_i))dx\right\}$$

where

$$\mathfrak{F}'(\theta) \equiv \frac{d\mathfrak{F}(\theta)}{d(\theta)}\,;\ \mathfrak{F}_1(\theta) \equiv p_{11}(x)\theta^{n-1} + \cdots + p_{n-1,1}(x)\theta + p_{n1}(x).$$

and where $\zeta_k(t)$, $\zeta_{ks}(t)$ possess the same properties of continuity as $\eta_i(x)$, $\eta_{is}(x)$ respectively.

12. The Green's function of the differential problem (★) is given by

$$G(x,t,\rho) = \frac{(-1)^n\Delta(x,t,\rho)}{\Delta(\rho)}, \tag{6}$$

where

$$\Delta(\rho) = \begin{vmatrix} u_{11}(\rho) & \cdots & u_{1n}(\rho) \\ \cdot\ \cdot\ \cdot & \cdots & \cdot\ \cdot \\ u_{n1}(\rho) & \cdots & u_{nn}(\rho) \end{vmatrix}, \tag{7}$$

$$u_{ik}(\rho) = L_i(y_k), \tag{8}$$

$$\Delta(x,t,\rho) = \begin{vmatrix} y_1(x,\rho) & \cdots & y_n(x,\rho) & g_0(x,t,\rho) \\ u_{11}(\rho) & \cdots & u_{1n}(\rho) & g_1(t,\rho) \\ \cdot\ \cdot\ \cdot & \cdots & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ u_{n1}(\rho) & \cdots & u_{nn}(\rho) & g_n(t,\rho) \end{vmatrix}, \tag{9}$$

$$g_0(x,t,\rho) = \begin{cases} \displaystyle\sum_{k=1}^{\tau} y_k(x,\rho)z_k(t,\rho) \text{ if } x > t, \\ \displaystyle -\sum_{k=\tau+1}^{n} y_k(x,\rho)z_k(t,\rho) \text{ if } x < t, \end{cases} \tag{10}$$

$$\begin{aligned} g_i(t,\rho) = &-\sum_{k=\tau+1}^{n} A_i(y_k)z_k(t,\rho) + \sum_{k=1}^{\tau} B_i(y_k)z_k(t,\rho) \\ &-\sum_{k=\tau+1}^{n} z_k(t,\rho)\int_a^t \alpha_i(x,\rho)y_k(x,\rho)dx \\ &+\sum_{k=1}^{\tau} z_k(t,\rho)\int_t^b \alpha_i(x,\rho)y_k(x,\rho)dx. \end{aligned} \tag{11}$$

Let l_{is} denote the greatest value of the index l for which, with i and s fixed, at least one of the coefficients a_{ils}, b_{ils} is different from zero, so that

$$a_{ils} = 0, \quad b_{ils} = 0 \text{ if } l > l_{is},$$
$$|a_{ils}| + |b_{ils}| > 0 \text{ if } l = l_{is}.$$

If

$$a_{ils} = b_{ils} = 0 \text{ for } l = 1,2,\cdots,n,$$

we set $l_{is}=0$.

Also let l_i be the greatest number of the set

$$l_{is} + s - 1 \qquad (s = 0,1,\cdots,n)$$

with $l_{is}>0$. Then obviously

$$(12)\qquad A_i(y_k) = \rho^{l_i}[A_{ik}]\ ;\ B_i(y_k) = \rho^{l_i}e^{\rho w_k}[B_{ik}]^*$$

where

$$(13)\qquad \begin{aligned} A_{ik} &= \eta_k(a)\sum_{(s)} a_{ils}\{\theta_k(a)\}^{l_{is}-1}, \\ B_{ik} &= \eta_k(b)\sum_{(s)} b_{ils}\{\theta_k(b)\}^{l_{is}-1}, \qquad l = l_{is}, \end{aligned}$$

and the summation is taken over all values of s for which $l_{is}+s-1=l_i$. We have further

$$(14)\quad \int_a^b \alpha_{is}(x)y_k(x,\rho)dx = \frac{1}{\rho}\{[-\alpha_{is}(a)\eta_k(a)\theta_k(a)^{-1}] + e^{\rho w_k}[\alpha_{is}(b)\eta_k(b)\theta_k(b)^{-1}]\}$$

which may be easily proved by integration by parts, if we take into account condition (**A**, v) and the fact that if c is any constant different from zero, and $\psi(z)$ is any function of bounded variation on $(0, Z)$, then

$$(15)\qquad \int_0^Z e^{c\rho z}\psi(z)dz = \frac{E}{\rho} + \frac{E}{\rho}e^{c\rho Z}.$$

In the same way we obtain

(16)

$$\int_a^t \alpha_{is}(x)y_k(x,\rho)dx = \frac{1}{\rho}\{[-\alpha_{is}(a)\eta_k(a)\theta_k(a)^{-1}] + e^{\rho\int_a^t\theta_k(x)dx}[\alpha_{is}(t)\eta_k(t)\theta_k(t)^{-1}]\},$$

$$\int_t^b \alpha_{is}(x)y_k(x,\rho)dx = \frac{1}{\rho}e^{\rho w_k}\{[\alpha_{is}(b)\eta_k(b)\theta_k(b)^{-1}] - e^{\rho\int_a^t\theta_k(x)dx}[\alpha_{is}(t)\eta_k(t)\theta_k(t)^{-1}]\}.$$

13. We shall now make the following assumptions:

(**B**) i. If s_i denotes the greatest value of the index s for which $\alpha_{is}(x)\not\equiv 0$, so that

$$\alpha_{is}(x)\equiv 0 \text{ if } s > s_i\ ;\ \alpha_{is}(x)\equiv\alpha^{(i)}(x)\not\equiv 0 \text{ on } (a,b) \text{ if } s = s_i,$$

then

$$s_i < l_i + 1 \qquad\qquad (i = 1,2,\cdots,n).$$

Under this condition we find immediately that

$$(17)\qquad u_{ik}(\rho) = \rho^{l_i}\{[A_{ik}] + e^{\rho w_k}[B_{ik}]\}.$$

---

* The notation $[\phi]$ is due to G. D. Birkhoff and designates the expression of the form $\phi+E/\rho$.

Hence from (7)

$$(18)\qquad \Delta(\rho) = \rho^l \begin{vmatrix} [A_{11}] + e^{\rho w_1}[B_{11}] \dots [A_{1n}] + e^{\rho w_n}[B_{1n}] \\ \dots\dots\dots\dots\dots\dots \\ [A_{n1}] + e^{\rho w_1}[B_{n1}] \dots [A_{nn}] + e^{\rho w_n}[B_{nn}] \end{vmatrix}$$

where

$$l = l_1 + l_2 + \cdots + l_n$$

and, if we denote by $\Delta_{ik}(\rho)$ the cofactor of $u_{ik}(\rho)$ in the determinant $\Delta(\rho)$ we have

$$(19)\qquad \Delta_{ji}(\rho) = \begin{cases} \rho^{l-l_i} e^{\rho w} E_{ji}(\rho) & \text{if } i = 1, 2, \cdots, \tau, \\ \rho^{l-l_i} e^{\rho(w-w_i)} E_{ji}(\rho) & \text{if } i = \tau+1, \cdots, n, \end{cases}$$

where

$$w = w_{\tau+1} + w_{\tau+2} + \cdots + w_n .$$

For the sake of brevity we set

$$(20)\quad x_i = \int_a^x q_i(x)dx\ ;\ \xi_i = \int_a^t q_i(x)dx\ ;\ X_i = \int_a^b q_i(x)dx\ ;\ w_i = \pi_i X_i .$$

A simple computation gives then

$$(21)\qquad g_0(x,t,\rho) = \begin{cases} \rho^{-n+1} \sum_{k=1}^{\tau} e^{\rho\pi_k(x_k-\xi_k)} \left\{ \eta_k(x)\zeta_k(t) + \dfrac{\mathrm{H}_k(x,t)}{\rho} + \dfrac{E}{\rho^2} \right\} \text{if } x > t, \\ -\rho^{-n+1} \sum_{k=\tau+1}^{n} e^{\rho\pi_k(x_k-\xi_k)} \left\{ \eta_k(x)\zeta_k(t) + \dfrac{\mathrm{H}_k(x,t)}{\rho} + \dfrac{E}{\rho^2} \right\} \text{if } x < t, \end{cases}$$

where the functions $\mathrm{H}_k(x, t)$ are polynomials in

$$\eta_i(x),\ \eta_{i0}(x),\ \zeta_i(t),\ \zeta_{i0}(t),$$

and, a fortiori, possess derivatives of the first order which are continuous and of bounded variation in each of the variables x and t, the total variation being uniformly bounded on (a, b).

Also

$$(22)\qquad \begin{aligned} g_i(t,\rho) = \rho^{l_i-n+1} \Big\{ &- \sum_{k=\tau+1}^{n} e^{-\rho\pi_k\xi_k} [A_{ik}\zeta_k(t)] \\ &+ \sum_{k=1}^{\tau} e^{\rho\pi_k(X_k-\xi_k)} [B_{ik}\zeta_k(t)] + \frac{1}{\rho}\left[\frac{a_i(t)}{p_{n0}(t)}\right] \Big\} \end{aligned}$$

* In deducing (22) from (11), (12), (16) we use the relations which are satisfied by the roots $z_1, z_2, \cdots, z_n$ of any algebraic equation

$$P(z) \equiv z^n + A_1 z^{n-1} + \cdots + A_{n-1} z + A_n = 0,$$

namely

$$\sum_{i=1}^{n} (z_i)^s / (P'(z_i)) = \begin{cases} 0 \text{ if } s = 0, 1, \cdots, n-2, \\ 1 \text{ if } s = n-1, \\ -1/A_n \text{ if } s = -1. \end{cases}$$

where

$$a_i(t) = \begin{cases} \alpha^{(i}(t) \text{ if } s_i = l_i \\ 0 \quad \text{if } s_i < l_i. \end{cases} \tag{23}$$

A further restriction now is the following:

(**B**) ii. If σ is the largest integer for which Re $\rho w_\sigma < 0$ throughout $(\mathfrak{R})$ then all the constants

$$M_1 = \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1\sigma} & B_{1\sigma+1} & \dots & B_{1n} \\ A_{21} & A_{22} & \dots & A_{2\sigma} & B_{2\sigma+1} & \dots & B_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A_{n2} & \dots & A_{n\sigma} & B_{n\sigma+1} & \dots & B_{nn} \end{vmatrix},$$

$$M_2 = \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1\tau} & B_{1\tau+1} & \dots & B_{1n} \\ A_{21} & A_{22} & \dots & A_{2\tau} & B_{2\tau+1} & \dots & B_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A_{n2} & \dots & A_{n\tau} & B_{n\tau+1} & \dots & B_{nn} \end{vmatrix}$$

are to be different from zero.*

Under the conditions (**A**), (**B**) there exist infinitely many poles of the Green's function $G(x, t, \rho)$ which are the characteristic values of the differential problem (★) (D, 20–26). Moreover, if the interiors of small circles with a common radius δ around these roots are all excluded from the ρ-plane, then in the remaining part $(\mathfrak{R}_\delta)$ of the sector $(\mathfrak{R})$

$$\left|\Delta(\rho)e^{-\rho w}\rho^{-l}\right| \geqq N_\delta > 0 \tag{24}$$

where N_δ is a positive constant which does not depend on ρ, and may be taken the same for all the sectors $(\mathfrak{R})$.

14. These results we shall apply now to the discussion of the behavior of the Green's function $G(x, t, \rho)$ for large values of ρ in $(\mathfrak{R}_\delta)$. We have from (6), (7), and (9)

$$G(x,t,\rho) = g_0(x,\tau,\rho) + \sum_{i,j=1}^{n} \frac{\Delta_{ji}(\rho)y_i(x,\rho)g_j(t,\rho)}{\Delta(\rho)}. \tag{25}$$

Hence if we set

$$\begin{aligned} &e^{\rho\pi_i x_i} = \varpi_i'\,(i = 1,2, \cdots,\tau)\,;\; e^{-\rho\pi_i(X_i - x_i)} = \varpi_i' \quad (i = \tau+1, \cdots, n), \\ &e^{\rho\pi_i(X_i - \xi_i)} = \varpi_i''\,(i = 1,2, \cdots,\tau)\,; e^{-\rho\pi_i\xi_i} = \varpi_i'' \quad (i = \tau+1, \cdots, n), \end{aligned} \tag{26}$$

---

* It should be noted that the constants τ, σ, A_{ik}, B_{ik} and hence M_1 and M_2 depend on the sector $(\mathfrak{R})$.

we obtain

$$G(x,t,\rho) = g_0(x,t,\rho) + \rho^{-n+1}Q_0(x,t,\rho) + \rho^{-n}R_0(x,t,\rho). \tag{27}$$

Here $Q_0(x, t, \rho)$ is a bilinear form in two sets of quantities ϖ_i' and ϖ_i''; i.e.,

$$Q_0(x,t,\rho) = \sum_{i,k=1}^{n} Q_{ik0}(x,t,\rho)\varpi_i'\varpi_k'', \tag{28}$$

the coefficients Q_{ik0} being of the form

$$Q_{ik0}(x,t,\rho) = [\omega_{ik0}(x,t)]E_{ik}(\rho). \tag{29}$$

The quantities $E_{ik}(\rho)$ depend only on ρ and remain bounded in $(\mathfrak{R}_\delta)$, while the functions $\omega_{ik0}(x, t)$ depend only on x and t, each being a sum of products of a function of x by a function of t whose second derivatives are continuous on (a, b).

$R_0(x, t, \rho)$ is a bilinear form in ϖ_i' and $a_i(t)$:

$$R_0(x,t,\rho) = \sum_{i,k=1}^{n} R_{ik0}(x,t,\rho)\varpi_i a_k(t), \tag{30}$$

$$R_{ik0}(x,t,\rho) = [r_{i0}(x)/p_{n0}(t)]E_{ik}(\rho),^* \tag{31}$$

where the functions $r_{i0}(x)$ depend only on x and possess second derivatives which are continuous on (a, b).

If $s_i < l_i (i=1, 2, \cdots, n)$, all the functions $a_i(t)$ vanish and the form $R_0(x, t, \rho)$ reduces to zero.

In exactly the same fashion it can be proved that

$$\frac{\partial^s G(x,t,\rho)}{\partial x^s} = g_{0s}(x,t,\rho) + \rho^{-n+s+1}Q_s(x,t,\rho) + \rho^{-n+s}R_s(x,t,\rho) \tag{32}$$

where

$$g_{0s}(x,t,\rho) \equiv \frac{\partial^s g_0(x,t,\rho)}{\partial x^s}$$

$$= \begin{cases} \rho^{-n+s+1} \sum_{k=1}^{\tau} e^{\rho\pi_k(x_k-\xi_k)}\left\{(\theta_k(x))^s\eta_k(x)\zeta_k(t) + \dfrac{\mathrm{H}_{ks}(x,t)}{\rho} + \dfrac{E}{\rho^2}\right\}, \\ -\rho^{-n+s+1} \sum_{k=\tau+1}^{n} e^{\rho\pi_k(x_k-\xi_k)}\left\{(\theta_k(x))^s\eta_k(x)\zeta_k(t) + \dfrac{\mathrm{H}_{ks}(x,t)}{\rho} + \dfrac{E}{\rho^2}\right\}, \end{cases} \tag{33}$$

according as $x > t$ or $x < t$;

---

* The coefficients $E_{ik}(\rho)$ are of course not the same as in (29).

$$(34)\quad Q_s(x,t,\rho) = \sum_{i,k=1}^{n} Q_{iks}(x,t,\rho)\varpi_i'\varpi_k''; \; Q_{iks}(x,t,\rho) = [\omega_{iks}(x,t)]E_{ik}(\rho),$$

$$(35)\quad R_s(x,t,\rho) = \sum_{i,k=1}^{n} R_{iks}(x,t,\rho)\varpi_i' a_k(t) \; ; \; R_{iks}(x,t,\rho) = [r_{is}(x)/p_{n0}(t)]E_{ik}(\rho),$$

and $\omega_{iks}(x, t)$, $r_{is}(x)$ are functions analogous to $\omega_{ik0}(x, t)$, $r_{i0}(x)$.

15. Let us suppose now that conditions (**A**), (**B**) are satisfied and set

$$(36)\qquad T_1(f,x) \equiv \rho^{n-1}\int_a^b G(x,t,\rho)p_{n0}(t)f(t)dt,$$

$$(37)\qquad T_2(f,t) \equiv \rho^{n-1}\int_a^b G(x,t,\rho)p_{n0}(x)f(x)dx.$$

Further let us denote by $(\mathfrak{D}_\delta)$ that part of the ρ-plane which is outside (or on the boundaries) of the circles with a common small radius δ around the poles of the Green's function $G(x, t, \rho)$. Then the representation (27) of the Green's function $G(x, t, \rho)$ enables us to prove the following

LEMMA 1. (1) *If $f(x)$ is any function integrable on (a, b), then the integrals* (36) *and* (37) *tend to zero uniformly on (a, b), as ρ becomes infinite, remaining in $(\mathfrak{D}_\delta)$. The same is true of the product*

$$\rho^{-s}\frac{d^sT_1(f,x)}{dx^s}\qquad (s = 1,2,\cdots,n-1).$$

(2) *If $f(x)$ is any function of bounded variation on (a, b) we have in $(\mathfrak{D}_\delta)$*

$$(38)\qquad |T_1(f,x)| < \frac{NV_f}{|\rho|};\qquad |T_2(f,t)| < \frac{NV_f}{|\rho|}$$

where N denotes a positive constant which depends only on δ (and on the coefficients of the problem (★)) and V_f is the greater of the two numbers, namely the upper limit of $f(x)$ and the total variation of $f(x)$ on (a, b). The same is true of the product

$$\rho^{-s}\frac{d^sT_1(f,x)}{dx^s}\qquad (s = 1,2,\cdots,n-1).$$

(3) *If $f(x)$ is any function which possesses a first derivative of bounded variation on (a, b), we have in $(\mathfrak{D}_\delta)$*

$$(39)\qquad T_1(f,x) = \frac{1}{\rho}f(x) + \frac{1}{\rho}\Lambda_1(\varpi_i') + \frac{E}{\rho^2},$$

$$(40)\qquad T_2(f,t) = \frac{1}{\rho}f(t) + \frac{1}{\rho}\Lambda_2(\varpi_i'') + \frac{E}{\rho^2},$$

where $\Lambda_1(\varpi_i')$, $\Lambda_2(\varpi_i'')$ *are linear forms in* ϖ_i', ϖ_i'' *respectively*:*

$$\Lambda_1(\varpi_i') = \sum_{i=1}^{n} [\omega_{i1}(x)]\varpi_i' E_i(\rho),$$

$$\Lambda_2(\varpi_i'') = \sum_{i=1}^{n} [\omega_{i2}(x)]\varpi_i'' E_i(\rho),$$

the coefficients $\omega_{i1}(x)$, $\omega_{i2}(x)$ *being continuous and of bounded variation on* (a, b).

16. The proof is based upon the following two simple lemmas:

LEMMA 2. *If* $\psi(t)$ *is any function integrable on* (a_0, b_0), *and* c *is any constant* $\neq 0$, *then the integral*

$$\int_\alpha^\beta e^{c\rho t}\psi(t)dt \qquad (0 \leqq a_0 \leqq \alpha \leqq \beta \leqq b_0)$$

tends uniformly to zero, as $\rho \to \infty$ *remaining in the half-plane* $\operatorname{Re} c\rho \leqq 0$ (D, 38).

LEMMA 3. *If* $\psi(t)$ *is any function of bounded variation on* (a_0, b_0) *and* c *is any constant* $\neq 0$, *then in the half-plane* $\operatorname{Re} c\rho \leqq 0$,

$$\int_\alpha^\beta e^{c\rho t}\psi(t)dt = \frac{V_\psi}{\rho}E \qquad (0 \leqq a_0 \leqq \alpha \leqq \beta \leqq b_0),$$

where E *is a bounded function whose upper bound does not depend on* ψ, α *or* β.

The proof of Lemma 3 follows immediately from the second Law of the Mean, if we represent the function $\psi(t)$ (or its real and imaginary parts in case it is complex) as a difference of two monotonic functions.

17. To prove Lemma 1, we use (27) and (32). These give

$$(41) \qquad T_1(f,x) = \rho^{n-1}\int_a^b g_0(x,t,\rho)p_{n0}(t)f(t)dt + \int_a^b Q_0(x,t,\rho)p_{n0}(t)f(t)dt + \frac{1}{\rho}\int_a^b R_0(x,t,\rho)p_{n0}(t)f(t)dt,$$

$$(42) \qquad \rho^{-s}\frac{d^sT_1(f,x)}{dx^s} = \rho^{n-s-1}\int_a^b g_{0s}(x,t,\rho)p_{n0}(t)f(t)dt + \int_a^b Q_s(x,t,\rho)p_{n0}(t)f(t)dt + \frac{1}{\rho}\int_a^b R_s(x,t,\rho)p_{n0}(t)f(t)dt.$$

* In the following, the symbol Λ will be used to designate any linear form in $\bar\omega_i'$ or $\bar\omega_i''$, of the type Λ_1 or Λ_2.

The statement (1) of Lemma 1 follows immediately from Lemma 2 and the statement (2) follows from Lemma 3. In order to prove the statement (3), we observe first that, by virtue of Lemma 3, the sum of the second and the third terms in (41) is of the form

$$\frac{1}{\rho}\Lambda(\varpi_i') + \frac{E}{\rho^2}.$$

On substituting (21) into the first term of (41) we can rewrite it in the form

$$\sum_{k=1}^{\tau}\int_a^x e^{\rho\pi_k(x_k-\xi_k)}\eta_k(x)\zeta_k(t)p_{n0}(t)f(t)dt$$
$$-\sum_{k=\tau+1}^{n}\int_x^b e^{\rho\pi_k(x_k-\xi_k)}\eta_k(x)\zeta_k(t)p_{n0}(t)f(t)dt$$
$$+\frac{1}{\rho}\sum_{k=1}^{\tau}\int_a^x e^{\rho\pi_k(x_k-\xi_k)}\mathrm{H}_k(x,t)p_{n0}(t)f(t)dt$$
$$-\frac{1}{\rho}\sum_{k=\tau+1}^{n}\int_a^b e^{\rho\pi_k(x_k-\xi_k)}\mathrm{H}_k(x,t)p_{n0}(t)f(t)dt + \frac{E}{\rho^2}.$$

The application of Lemma 3 shows that the coefficient of $1/\rho$ here is also E/ρ.

An integration by parts in the two first terms yields

$$-\frac{1}{\rho}\sum_{k=1}^{n}\frac{\eta_k(x)\zeta_k(x)p_{n0}(x)f(x)}{\theta_k(x)} + \frac{1}{\rho}\Lambda(\varpi_i')$$
$$+\frac{1}{\rho}\sum_{k=1}^{\tau}\int_a^x e^{\rho\pi_k(x_k-\xi_k)}\eta_k(x)\frac{d}{dt}\,\frac{\zeta_k(t)p_{n0}(t)f(t)}{\theta_k(t)}dt$$
$$-\frac{1}{\rho}\sum_{k=\tau+1}^{n}\int_x^b e^{\rho\pi_k(x_k-\xi_k)}\eta_k(x)\frac{d}{dt}\,\frac{\zeta_k(t)p_{n0}(t)f(t)}{\theta_k(t)}dt,$$

which by virtue of the footnote on page 772 and Lemma 3 is equal to

$$\frac{1}{\rho}f(x) + \frac{1}{\rho}\Lambda(\varpi_i') + \frac{E}{\rho^2}.$$

On collecting this material we have the complete proof of (36). An analogous proof may be carried through for (37).

LEMMA 4. *If $f_1(x)$ and $f_2(x)$ are any two functions whose first derivatives are of bounded variation on (a, b), then under the conditions of Lemma* 1,

$$\rho^n \int_a^b \int_a^b G(x,t,\rho) f_1(x) f_2(t) p_{n0}(t) dx dt = \left[\int_a^b f_1(x) f_2(x) dx \right], \tag{43}$$

$$\rho^n \int_a^b \int_a^b G(x,t,\rho) f_1(t) f_2(x) p_{n0}(x) dx dt = \left[\int_a^b f_1(x) f_2(x) dx \right]. \tag{44}$$

The proof is given by (39) if we observe that, by virtue of Lemma 3,

$$\int_a^b \Lambda_1(\varpi_i') f_1(x) dx = \frac{E}{\rho} \text{ on } (\mathfrak{D}_\delta).$$

18. **Remark** 1. Lemmas 1 and 4 hold true if the functions $f(x), f_1(x), f_2(x)$ depend on any number of other variables, provided that they are uniformly bounded or else that their total variation or that of their derivatives is uniformly bounded.

Remark 2. It follows immediately from (27) and (32) that, in $(\mathfrak{D}_\delta)$,

$$|G(x,t,\rho)| < N|\rho|^{-n+1}; \quad \left| \frac{\partial^s G(x,t,\rho)}{\partial x^s} \right| < N|\rho|^{-n+1+s}$$

where N may be taken the same as in (38).

III. The equiconvergence theorem

19. We return now to the general problem

$$L(u) = f(x) + \sum_{j=1}^{\mu} l_j(u)\phi_j(x,\rho) + \int_a^b h(x,\xi,\rho)u(\xi)d\xi, \tag{I}$$

$$L_i(u) = 0 \qquad (i = 1,2,\cdots,n). \tag{II}$$

In addition to the conditions (**A**), (**B**) we assume also the following:

(**C**) i. The functions

$$h(x,\xi,\rho),\ \phi_j(x,\rho) \qquad (j = 1,2,\cdots,\mu)$$

are polynomials in ρ, of at most the degree $(n-1)$.

ii. The operators

$$l_j(u) \qquad (j = 1,2,\cdots,n)$$

do not depend on ρ.

iii. If m_j denotes the degree of $\phi_j(x, \rho)$ and λ_j denotes the order of the highest derivative which occurs in the operator $l_j(u)$, then

$$\lambda_k + m_j \leqq n - 1 \qquad (k,j = 1,2,\cdots,\mu). \tag{1}$$

iv. If $\lambda_k+m_j=n-1$, then the coefficient of the highest power of ρ in $\phi_j(x, \rho)$ is of bounded variation on (a, b). The coefficient of ρ^{n-1} in $h(x, \xi, \rho)$

is of bounded variation in ξ for each value of x, the total variation being uniformly bounded on (a, b). The remaining coefficients in $h(x, \xi, \rho)$, $\phi_j(x, \rho)$ are integrable on (a, b), with respect either to x, or to ξ, or with respect to x and ξ.

v. The operators

$$l_j(u),\ A_i(u) + B_i(u) \qquad (j = 1,2, \cdots, \mu\ ;\ i = 1,2, \cdots, n)$$

are linearly independent for all values of ρ.

20. We see at once that the conditions of Theorems 1 and 2 are fulfilled in this case, and now we proceed to the explicit expression of the Green's function $\Gamma(x, t, \rho)$ of the problem $(\bigstar)$.

Using the method indicated in §§4–6 we set

$$F(x,\rho) \equiv \int_a^b G(x,t,\rho)f(t)dt, \tag{2}$$

$$\Phi_j(x,\rho) \equiv \int_a^b G(x,t,\rho)\phi_j(t,\rho)dt, \tag{3}$$

$$H(x,\xi,\rho) \equiv \int_a^b G(x,t,\rho)\,h(t,\xi,\rho)dt, \tag{4}$$

$$l_j(u) = c_j \qquad (j = 1,2, \cdots, \mu), \tag{5}$$

$$\Psi(x,\rho) \equiv \sum_{j=1}^{\mu} c_j\,\Phi_j(x,\rho) + F(x,\rho). \tag{6}$$

Suppose for a moment that ρ is not a pole of $G(x, t, \rho)$, nor of $\mathfrak{H}(x, \xi, \rho)$, the reciprocal of the kernel $H(x, \xi, \rho)$. The problem $(\bigstar)$ is equivalent to the integral equation

$$u(x,\rho) = \Psi(x,\rho) + \int_a^b H(x,\xi,\rho)u(\xi,\rho)d\xi, \tag{7}$$

together with the conditions (5). Hence

$$u(x,\rho) = \Psi(x,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)\Psi(\xi,\rho)d\xi = \Omega(x,\rho) + \sum_{j=1}^{\mu} c_j\Omega_j(x,\rho), \tag{8}$$

where

$$\Omega(x,\rho) \equiv F(x,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)F(\xi,\rho)d\xi, \tag{9}$$

$$\Omega_j(x,\rho) \equiv \Phi_j(x,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)\,\Phi_j(\xi,\rho)d\xi. \tag{10}$$

If we set

(11) $$\chi_{ik}(\rho) \equiv \begin{cases} -\ \ l_i(\Omega_k) \text{ if } k \neq i\,; \\ 1 - l_i(\Omega_i) \text{ if } k = i\,; \end{cases} \quad \chi_i(\rho) \equiv l_i(\Omega) \qquad (i, k = 1, 2, \cdots, \mu),$$

equations (5) can be written as follows:

(12) $$\sum_{k=1}^{\mu} c_k \chi_{ik}(\rho) = \chi_i(\rho) \qquad (i = 1, 2, \cdots, \mu).$$

Denote by $D(\rho)$ the determinant of the system (12) and by

(13) $$D_{ij}(\rho)$$

the quotient of the cofactor of the element $\chi_{ij}(\rho)$ in $D(\rho)$ by $D(\rho)$ (supposing of course that $D(\rho) \neq 0$). Then

(14) $$c_j = \sum_{k=1}^{\mu} D_{kj}(\rho)\chi_k(\rho) \qquad (j = 1, 2, \cdots, \mu)$$

and substitution in (8) gives

$$\begin{aligned} u(x,\rho) &= \Omega(x,\rho) + \sum_{k,j=1}^{\mu} D_{kj}(\rho)\chi_k(\rho)\Omega_j(x,\rho) \\ &= \int_a^b dtf(t) \Big\{ G(x,t,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)G(\xi,t,\rho)d\xi \\ &\quad + \sum_{k,j=1}^{\mu} D_{kj}(\rho)\Omega_j(x,\rho) l_k \Big(G(x,t,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)G(\xi,t,\rho)d\xi \Big)_x \Big\}^*. \end{aligned}$$

Hence

(15) $$\begin{aligned} \Gamma(x,t,\rho) &= G(x,t,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)G(\xi,t,\rho)d\xi \\ &\quad + \sum_{k,j=1}^{\mu} D_{kj}(\rho)\Omega_j(x,\rho) l_k \Big(G(x,t,\rho) - \int_a^b \mathfrak{H}(x,\xi,\rho)G(\xi,t,\rho)d\xi \Big)_x . \end{aligned}$$

21. We shall justify these formal operations by proving that, for sufficiently large values of $|\rho|$ in $(\mathfrak{T}_\delta)$, the reciprocal $\mathfrak{H}(x, \xi, \rho)$ exists and the determinant $D(\rho) \neq 0$. By virtue of Theorem 2, then, it will follow that the Green's function $\Gamma(x, t, \rho)$ is meromorphic in ρ, and that the poles of $\Gamma(x, t, \rho)$, for ρ sufficiently large, if there are such, must lie within a distance less than δ from the corresponding poles of $G(x, t, \rho)$. The justification in question is based on the fact that in $(\mathfrak{T}_\delta)$ the function $H(x, \xi, \rho)$ is of order

* The subscript x indicates the independent variable of the operation l_k.

$O(1/\rho)$. This follows immediately from Lemma 1. Let R_δ be chosen so large that for ρ in $(\mathfrak{D}_\delta)$ and $|\rho| \geqq R_\delta$

$$|H(x,\xi,\rho)| \leqq h_0 < 1 .$$

The Neumann series for the reciprocal of the kernel $H(x, \xi, \rho)$, i.e.

$$(16)\qquad \begin{gathered} \mathfrak{H}(x,\xi,\rho) = - \sum_{\nu=0}^{\infty} H_\nu(x,\xi,\rho) ; \\ H_0(x,\xi,\rho) = H(x,\xi,\rho) ; \ H_{\nu+1}(x,\xi,\rho) = \int_a^b H(x,\eta,\rho)H_\nu(\eta,\xi,\rho)d\eta \end{gathered}$$

converges uniformly in x, ξ on (a, b) and ρ in $(\mathfrak{D}_\delta)$ and $|\rho| \geqq R_\delta$. In the following $(\mathfrak{D})$ will designate that part of $(\mathfrak{D}_\delta)$ which is outside the circle $|\rho| = R_\delta$.

Expression (16) shows that in $(\mathfrak{D})$

$$(17)\qquad \mathfrak{H}(x,t,\rho) = \frac{E}{\rho} \cdot$$

Further we observe that $\Phi_j(x, \rho)$ is a combination of integrals of the type T_1 of Lemma 1, whence an easy application of this Lemma shows, by virtue of conditions (**C**, i–iv) and formula (17),

$$(18)\qquad \begin{gathered} \Phi_j(x,\rho) = E\rho^{-n+m_j} ; \ \Omega_j(x,\rho) = E\rho^{-n+m_j}, \\ l_i(\Omega_k) = \frac{E}{\rho} ; \ \chi_{ik}(\rho) = [\delta_{ik}] ; \ \delta_{ik} = \begin{cases} 0 \text{ if } i \neq k, \\ 1 \text{ if } i = k, \end{cases} (i,k,j = 1,2, \cdots,\mu). \end{gathered}$$

Hence $D(\rho)$ tends uniformly to 1, as ρ becomes infinite in $(\mathfrak{D})$. Taking R_δ sufficiently large we have then

$$(19)\qquad |D(\rho)| \geqq N_\delta > 0 \text{ in } (\mathfrak{D}).$$

Lastly we see that all the $D_{ik}(\rho)$ are bounded in $(\mathfrak{D})$. Thus the following theorem is proved:

THEOREM 3. *Under the conditions* (**A**), (**B**), (**C**), *the Green's function* $\Gamma(x, t, \rho)$ *of the problem* (★), *which is given by* (15), *is meromorphic in* ρ, *and in* $(\mathfrak{D})$ *we have*

$$\Gamma(x,t,\rho) - G(x,t,\rho) = O(1/\rho^n).$$

22. The preceding evaluations enable us to prove the following fundamental proposition:

THEOREM 4. *Under the conditions* (A), (B), (C), *if* (C_R) *denotes a circle* $|\rho|=R$ *in* $(\mathfrak{D})$, *the difference of the two Birkhoff integrals*

$$
(20)\qquad \begin{aligned} I_R(f) = &\frac{1}{2\pi i}\int_{(C_R)}\rho^{n-1}d\rho\int_a^b \Gamma(x,t,\rho)p_{n0}(t)f(t)dt \\ &-\frac{1}{2\pi i}\int_{(C_R)}\rho^{n-1}d\rho\int_a^b G(x,t,\rho)p_{n0}(t)f(t)dt \end{aligned}
$$

corresponding to the integro-differential and the differential problems $(\bigstar)$ *tends to zero as* $R\to\infty$, *for any integrable function* $f(x)$, *and uniformly in* x *on* (a, b).

Using the notation of §15 we obviously have

$$
(21)\qquad \begin{aligned} I_R(f) = &-\frac{1}{2\pi i}\int d\rho\int_a^b \mathfrak{H}(x,\xi,\rho)T_1(f,\xi)d\xi \\ &+\frac{1}{2\pi i}\sum_{k,j=1}^{\mu}\int_{(C_R)} d\rho D_{kj}(\rho)\Omega_j(x,\rho)l_k\Big\{T_1(f,x) \\ &-\int_a^b \mathfrak{H}(x,\xi,\rho)T_1(f,\xi)d\xi\Big\}_x. \end{aligned}
$$

The application of Lemma 1 and the preceding inequalities show that each term under the sign of integral $\int_{(C_R)}$ is of the form* $\epsilon(x,\rho)/\rho$. Hence

$$
I_R(f) = \int_{(C_R)}\frac{\epsilon(x,\rho)}{\rho}\rho d \to 0 \text{ as } R\to\infty .
$$

Thus, the "equiconvergence theorem" for the integro-differential and differential problems $(\bigstar)$ is established; in other words it is proved that both problems are equivalent in so far as questions of the conditions of convergence, uniform convergence, divergence, or summability by any regular method are concerned.

23. THEOREM 5. *Under the conditions* (A), (B), (C) *the problem* $(\bigstar)$ *possesses infinitely many characteristic values and fundamental functions. If* δ *is any positive number, arbitrarily small, each characteristic value of the problem* $(\bigstar)$, *which is sufficiently large in absolute value, is distant by less than* δ *from a characteristic value of the differential problem* $(\bigstar)$.

---

* See third foot note, p. 768.

The second assertion of the theorem has already been proved. To prove the first, let us suppose that the function $\Gamma(x, t, \rho)$ has but a finite number of poles

$$\rho_1, \rho_2, \cdots, \rho_m.$$

Let

$$\Gamma_1(x,t), \Gamma_2(x,t), \cdots, \Gamma_m(x,t)$$

be the corresponding residues of the function $\rho^{n-1}\Gamma(x, t, \rho)$. The integral

$$\frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b \Gamma(x,t,\rho)p_{n0}(t)f(t)dt$$

has, then, the same value for all values of R sufficiently large. Hence, in virtue of Theorem 3,

$$\begin{aligned}\frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b \Gamma(x,t,\rho)p_{n0}(t)f(t)dt \\ = \sum_{\nu=1}^{m} \int_a^b \Gamma_\nu(x,t)p_{n0}(t)f(t)dt \\ = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b G(x,t,\rho)p_{n0}(t)f(t)dt = f(x),\end{aligned}$$

$f(x)$ being an arbitrary function satisfying certain conditions of continuity. On setting

$$k(x,t) \equiv \sum_{\nu=1}^{m} \Gamma_\nu(x,t)p_{n0}(t)$$

this relation takes the form of an integral equation

$$f(x) = \int_a^b k(x,t)f(t)dt$$

whose kernel is bounded and integrable and which is possessed of infinitely many linearly independent solutions, which is in contradiction to the classical Fredholm theory.

24. Theorem 6. *Assuming that the problem* ($\bigstar$) *satisfies the conditions* (A), (B), (C), *let* ($\bigstar\bigstar$) *be any differential problem of the type* ($\bigstar$) *which satisfies the conditions* (A) *and* (B) *and whose coefficients* $p_{i0}(x)$, $p_{i1}(x)$, A_{ik}, B_{ik} *and indices* l_{in} *are the same as those of the problem* ($\bigstar$). *Denote by* $G'(x, t, \rho)$ *the Green's function of the problem* ($\bigstar\bigstar$). *Let* $f(x)$ *be any function integrable on* (a, b) *and*

$$I_R'(f) \equiv \frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b \{\Gamma(x,t,\rho) - G'(x,t,\rho)\} p_{n0}(t)f(t)dt. \tag{22}$$

Then, as $R \to \infty$,

$$I_R'(f) \to \begin{cases} 0 \text{ if } a < x < b, \\ \sum_{i=1}^{n} \int_a^b \{a_i(t) - a_i'(t)\}\Theta_{ai}(t)f(t)dt \text{ if } x = a, \\ \sum_{i=1}^{n} \int_a^b \{a_i(t) - a_i'(t)\}\Theta_{bi}(t)f(t)dt \text{ if } x = b, \end{cases} \tag{23}$$

where $a_i'(x)$ *are the coefficients of the problem* ($\bigstar\bigstar$), *which correspond to the coefficients* $a_i(x)$ *of the problem* ($\bigstar$) *and* Θ_{ai}, Θ_{bi} *are known functions which depend merely on the coefficients of the problem* ($\bigstar$) *and do not depend on* $f(x)$. *The convergence of* $I_R'(f)$ *to zero is uniform on every interval interior to* (a, b). *If the functions* $a_i(x)$, $a_i'(x)$ *coincide, then*

$$I_R'(f) \to 0 \quad as \quad R \to \infty,$$

uniformly on (0, 1).

It is readily proved that the characteristic values of the differential problem ($\bigstar\bigstar$) are asymptotic to those of the differential problem ($\bigstar$) in the sense that if δ is any fixed positive number, arbitrarily small, an R_δ is available which is so large that outside of the circle $|\rho| = R_\delta$ all the characteristic values of ($\bigstar\bigstar$) lie in the interiors of the circles of radius δ around the characteristic values of ($\bigstar$) and vice versa (D, 26). Hence the integral

$$\frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b G'(x,t,\rho)p_{n0}(t)f(t)dt$$

exists, and Theorem 6 is proved if the assertions of the theorem are proved for the integral

$$j_R(f) \equiv \frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_a^b \{G(x,t,\rho) - G'(x,t,\rho)\} p_{n0}(t)f(t)dt.$$

Now, it is readily seen that in the expressions of the Green's functions $G(x, t, \rho)$ and $G'(x, t, \rho)$, as given by Part II, (27), the exponential factors and the leading terms are the same, provided the conditions of Theorem 6 are satisfied. Hence the integral $j_R(f)$ reduces to a sum of terms of the form

$$\int \frac{d\rho}{\rho} \int_a^x e^{\rho\tau_k(x_k-\xi_k)}[f_1(t)]E(\rho)dt \qquad (k = 1,2, \cdots ,\tau),$$

(24)

$$\int \frac{d\rho}{\rho} \int_x^b e^{\rho\tau_k(x_k-\xi_k)}[f_1(t)]E(\rho)dt \qquad (k = \tau + 1, \cdots ,n) ;$$

(25)
$$\int \frac{\varpi_i'}{\rho} d\rho \int_a^b \varpi_k''[f_1(t)]E(\rho)dt \qquad (i,k = 1,2, \cdots ,n) ;$$

(26)
$$\int \frac{\varpi_i'}{\rho} d\rho \int_a^b [a_k(t) - a_k'(t)]\Theta_{ik}(t)f(t)dt \qquad (i,k = 1,2, \cdots ,n).$$

The integration with respect to ρ is taken over those parts of (C_R) which are cut off by the various sectors $(\Re)$ and $f_1(t)$ designates an integrable function. In virtue of Lemma 2, integrals (24) and (25) $\to 0$ as $R \to \infty$, uniformly on (a, b). Integrals (26) are absent if the last condition of Theorem 6 is satisfied. Otherwise integrals (26) $\to \infty$ as $R \to \infty$, uniformly on every interval interior to (a, b), as is shown by

LEMMA 5. *Let (γ) be any part of the circle $|\rho| = r$ which lies in the half-plane* Re $c\rho \leqq 0$ *(c is a constant $\neq 0$). If $\epsilon(\rho, z)$ tends uniformly to zero as r assumes a given set of values tending to ∞, the integral*

$$\int_{(\gamma)} \epsilon(\rho, z) e^{c\rho z} d\rho$$

*tends to zero uniformly on every interval $0 < \alpha \leqq z \leqq \beta$.**

Finally, if $x = a$ or $x = b$, integrals (26) obviously tend to limits of the type indicated in (23).

25. COROLLARY. *Let*

$$\rho_1, \rho_2, \cdots, \rho_\nu, \cdots \qquad (|\rho_1| \leqq |\rho_2| \leqq \cdots)$$

be the set of the characteristic values of the problem (★). Under the conditions **(A)**, **(B)**, **(C)** *the series*

$$\sum_{(\nu)} \int_a^b \operatorname*{Res}_{\rho=\rho_\nu} \{\rho^{n-1}\Gamma(x,t,\rho)\} p_{n0}(t)f(t)dt$$

converges, for any function $f(x)$ of bounded variation on (a, b), to the values

$$\tfrac{1}{2}\{f(x+0) + f(x-0)\} \quad \text{if} \quad a < x < b,$$

$$A_a f(a+0) + B_a f(b-0) + \sum_{i=1}^n \int_a^b \Theta_{ai}(t) a_i(t) f(t) dt \quad \text{if} \quad x = a,$$

$$A_b f(a+0) + B_b f(b-0) + \sum_{i=1}^n \int_a^b \Theta_{bi}(t) a_i(t) f(t) dt \quad \text{if} \quad x = b,$$

* See D, 38.

where A_a, B_a, A_b, B_b are constants which depend merely on the coefficients of the problem (★) *but not on $f(x)$. The convergence is uniform on every interval of continuity of $f(x)$, which is interior to (a, b).*

This follows immediately from Theorem 6 and from the known properties of the integral*

$$\frac{1}{2\pi i}\int_{(C_R)}\rho^{n-1}d\rho\int_a^b G'(x,t,\rho)p_{n0}(t)f(t)dt.$$

Better results concerning the uniformity of convergence on the whole of (a, b) and the values of the series at the end points a, b may be obtained by using another integral in place of the Birkhoff integral. We shall return to this question elsewhere.

26. THEOREM 7. *In addition to* (A), (B), (C) *let us assume that the operator $L(u)$ is of the form*

$$L(u) \equiv u^{(n)} + p_1(x)u^{(n-1)} + \cdots + \{p_n(x) + \rho^n\}u$$

and the operators $A_i(u)+B_i(u)$ $(i=1, 2, \cdots, n)$ are independent of ρ. Suppose furthermore that all the poles of the Green's function $G'(x, t, \rho)$ which are sufficiently large in absolute value are simple.

Then the characteristic values of the integro-differential problem (★) *are asymptotic to those of the differential problem* (★) (*and also to those of any of the problems* (★★)) *in the sense above.*

Let ρ_0 be any one of the poles of the function $G(x, t, \rho)$ (in the region $|\rho|>R_\delta$). Denote by (c) the circle of radius δ around ρ_0. We suppose R_δ so large that all the poles (in the region $|\rho|>R_\delta$) of either of the functions $\Gamma(x, t, \rho)$, $G'(x, t, \rho)$ are interior to the circles (c). To prove that $\Gamma(x, t, \rho)$ has at least one pole in each of the circles (c) it is sufficient to prove that the integral

$$J_c(\Gamma) = \frac{1}{2\pi i}\int_a^b dx\int_{(c)}\Gamma(x,x-0,\rho)d\rho \tag{27}$$

is always different from zero, for if $\Gamma(x, t, \rho)$ has no poles in any one of the circles (c), the integral (27) must vanish. Now, Theorem 3 shows, on the one hand,

$$|J_c(\Gamma - G)| < Ar^{-n},$$

* We may take as the problem (★★) the differential problem whose operators are $L(u)$ and $A_i(u)+B_i(u)$.

where A is a positive constant which depends only on δ, and r denotes the shortest distance from the origin to the contour of (c). On the other hand, we see from Part II, (27), that

$$G(x,t,\rho) - G'(x,t,\rho) = O(r^{-n}) \text{ on } (c),$$
$$|J_c(G - G')| < Ar^{-n}.$$

Hence if we suppose that

$$J_c(\Gamma) = 0,$$

then

$$|J_c(G')| \leqq |J_c(G)| + |J_c(G - G')| = |J_c(\Gamma - G)| + |J_c(G - G')| < 2Ar^{-n}. \tag{28}$$

Now, under the conditions of Theorem 7, it has been proved* that the residue of the function $G'(x, t, \rho)$ corresponding to any simple pole ρ' is

$$\frac{1}{n(\rho')^{n-1}} \sum_{k=1}^{\sigma} u_k(x)v_k(t)$$

where the sets

$$u_1(x), \cdots, u_\sigma(x) \text{ and } v_1(x), v_2(x), \cdots, v_\sigma(x) \tag{29}$$

are composed respectively of the fundamental functions of the problem (★★) and of the adjoint problem (★★) and are biorthogonal and normalized. Supposing that the pole ρ' is within the circle (c) we see that

$$J_c(G') = \frac{\sigma}{n(\rho')^{n-1}},$$

which is in contradiction to (28). We know, however, that no poles of $\Gamma(x, t, \rho)$ (greater than R_δ in absolute value) are outside the circles (c). Hence Theorem 7 is proved.†

27. The method used above can be considerably generalized. It can be immediately applied to the case where the function $h(x, \xi, \rho)$ is replaced by

$$\rho^n h_n(x,\xi) + h(x,\xi,\rho)$$

where $h_n(x, \xi)$ is the sum of a finite number of products of a function of x by a function of ξ. The method can be extended further to the general case where $h_n(x, \xi)$ is any function of the two variables x and ξ, which satisfies certain

* Birkhoff, these Transactions, vol. 9 (1908), pp. 377–380; D, 15.

† It is very probable that Theorem 7 holds true under the more general conditions of Theorem 4. This question, however, requires more detailed knowledge of the structure of the Green's function $G(x, t, \rho)$ in the vicinity of a pole, than that which is available in D.

conditions of continuity and possesses a reciprocal.* This extension (which includes L. Lichtenstein's case) will be treated in another paper by the author. It is sufficient to remark here that Theorem 3 is true in this more general case, but that in Theorem 4, the Birkhoff integral must be replaced by the following one:

$$\frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho\left\{\int_a^b f(t)dt\left(\Gamma(x,t,\rho)p_{n0}(t)\right.\right.$$
$$\left.\left. - \int_a^b \Gamma(x,\xi,\rho)h_n(\xi,t)p_{n0}(\xi)d\xi\right)\right\}\cdot$$

IV. An application to the theory of Fredholm's integral equations†

28. The general theory as indicated above admits of an important application to the theory of Fredholm's integral equations with discontinuous kernels.‡

Let us first develop the formal side of the method; it will be then an easy matter to state all the conditions under which our formal operations are justified.

Consider the integral equation

$$y(t) = \lambda\int_\alpha^\beta k(t,\tau)y(\tau)d\tau \tag{1}$$

whose kernel $k(t, \tau)$ has a finite jump along the line $\tau = t$:

$$k(t,t^+) - k(t,t^-) = k(t,\tau)\Big]_{t^-}^{t^+} = \phi(t) \neq 0.$$

* This condition is essential. For instance, in the example of the foot note on p. 766 all values of ρ are characteristic values of the integro-differential problem (★) while the corresponding differential problem admits only $\rho = 2k\pi$. Another curious example is the following: $u''+\rho^2u=\rho^2\int_0^1\xi u(\xi)d\xi$; $u(0)=0, \int_0^1 u(\xi)d\xi=0$. The characteristic values of the integro-differential problem are asymptotic to the roots of the equation $\cos\rho = 1/3$, while those of the differential problem are $\rho=2k\pi(k=\pm 1, \cdots)$. The characteristic values of the integro-differential problem $u''+\rho^2u=2\rho^2\int_0^1\xi u(\xi)d\xi$; $u(0)=0, \int_0^1 u(\xi)d\xi=0$, are, however, asymptotic to those of the problem

$$u'' + \rho^2 u = 0;\ u(0) = 0,\ \int_0^1 u(\xi)d\xi = 0.$$

† This chapter was added during the revision of the paper, in May 1927.

‡ An independent treatment of this question has been recently given in the interesting paper by R. E. Langer, *On the theory of integral equations with discontinuous kernels*, these Transactions, vol. 28 (1926), pp. 585–639; in the sequel this paper is denoted by "L." The method which we propose here appears to present some advantage of greater simplicity and generality. We consider here the case only where the kernel itself is discontinuous. The general case of a kernel which is continuous but possesses discontinuous partial derivatives is discussed in a joint paper by Langer and the author.

We assume that, except on the line $\tau = t$, $k(t, \tau)$ is continuous on the square

$$(\mathfrak{S}_0) \qquad \alpha \leqq t \leqq \beta \,;\, \alpha \leqq \tau \leqq \beta,$$

together with its partial derivatives up to a certain order which will be specified later on, and that all these functions tend to finite limiting values as the point (t, τ) tends to any point on the line $\tau = t$, without crossing this line. It is readily seen that equation (1) can be "normalized" by means of the substitutions (L, Chapter 2)

$$lx = \int_\alpha^t \phi(t)dt \,;\, l\xi = \int_\alpha^\tau \phi(\tau)d\tau \,;\, l = \int_\alpha^\beta \phi(t)dt \neq 0 \,;\, \lambda l = \rho \,;$$

$$\Omega(x,\xi) = k(t,\tau)/\phi(\tau) \,;\quad \psi(x) = \Omega_x(x,\xi)\Big]_{x^+}^{x^-},$$

$$K(x,\xi) \equiv \Omega(x,\xi)e^{\int_\xi^x \psi(x)dx},$$

$$u(x) \equiv y(t)e^{\int_0^x \psi(x)dx}.$$

This brings (1) to the form

$$(2) \qquad u(x) = \rho \int_0^1 K(x,\xi)u(\xi)d\xi.$$

The function $K(x, \xi)$ possesses the same properties of continuity on the square

$$(\mathfrak{S}) \qquad 0 \leqq x \leqq 1 \,;\, 0 \leqq \xi \leqq 1,$$

as the function $k(t, \tau)$ on the square $(\mathfrak{S}_0)$. But, in addition we have now

$$(3) \qquad K(x,\xi)\Big]_{x^-}^{x^+} = 1 \,;\quad K_x(x,\xi)\Big]_{x^-}^{x^+} = 0.$$

So far the function $K_x(x, \xi)$ is defined for $x \neq \xi$. On setting

$$K_x(x,\xi)\,\big|_{\xi=x} = K_x(x,x^-) = K_x(x,x^+)$$

we obtain a function which is defined and continuous on the whole of the square $(\mathfrak{S})$.

From now on we shall deal exclusively with the normalized equation (2)*.

29. If we differentiate (2) we get, on account of (3),

$$(4) \qquad u(x) = -\,u'(x)/\rho + \int_0^1 K_x(x,\xi)u(\xi)d\xi.$$

* The normalization of (1) is not at all necessary for our purposes; we might discuss (1) without any transformation as well. It is only for the sake of simplicity of computations that we prefer to deal with the normalized equation (2).

Here we consider $-u'(x)/\rho$ as a known function and assume that the kernel $K_x(x, \xi)$ possesses a reciprocal, $\mathfrak{G}(x, \xi)$, which is defined by

$$(5) \qquad K_x(x,\xi) + \mathfrak{G}(x,\xi) = \int_0^1 K_x(x,s)\,\mathfrak{G}(s,\xi)ds = \int_0^1 \mathfrak{G}(x,s)K_s(s,\xi)ds.$$

The function $\mathfrak{G}(x, \xi)$ has on $(\mathfrak{S})$ the same properties of continuity as the function $K_x(x, \xi)$; in particular, $\mathfrak{G}(x, \xi)$ is continuous on $(\mathfrak{S})$. We see at once that

$$\rho u(x) = - u'(x) + \int_0^1 \mathfrak{G}(x,\xi)u'(\xi)d\xi,$$

and an integration by parts gives the integro-differential equation which is satisfied by $u(x)$:

$$(6) \qquad u'(x) + \rho u(x) = u(1)\,\mathfrak{G}(x,1) - u(0)\,\mathfrak{G}(x,0) - \int_0^1 \mathfrak{G}_\xi(x,\xi)u(\xi)d\xi.$$

It remains now to find the boundary condition which is satisfied by $u(x)$. This can be done by substituting $x=0$ into (2):

$$u(0) = \rho \int_0^1 K(0,\xi)u(\xi)d\xi.$$

The parameter ρ may be eliminated from here by means of (6). If we substitute $\rho u(x)$ from (6) and integrate by parts, we easily find

$$(7) \qquad au(0) + bu(1) = \int_0^1 \alpha(\xi)u(\xi)d\xi$$

where

$$a = 1 - K(0,0^+) + \int_0^1 K(0,\xi)\,\mathfrak{G}(\xi,0)d\xi,$$

$$(8) \qquad b = K(0,1) - \int_0^1 K(0,\xi)\,\mathfrak{G}(\xi,1)d\xi,$$

$$\alpha(\xi) = K_\xi(0,\xi) - \int_0^1 K(0,s)\,\mathfrak{G}_\xi(s,\xi)ds.$$

Thus we have proved that every solution of (2) is a solution of the homogeneous integro-differential problem $(\bigstar)$ consisting of (6) and (7). The complete equivalence of those two problems will be proved if we can show that, conversely, every solution of (6), (7) is a solution of (2).

30. It is desirable to answer a more general question, namely, what is the integral equation which is satisfied by a solution of the system consisting of the *non-homogeneous* integro-differential equation

$$(9)\quad u'(x) + \rho u(x) = f(x) + u(1)\mathfrak{G}(x,1) - u(0)\mathfrak{G}(x,0) - \int_0^1 \mathfrak{G}_\xi(x,\xi)u(\xi)d\xi,$$

and the boundary condition (7), $f(x)$ being an arbitrary continuous function.

First, we have from (9)

$$u'(x) = f(x) - \rho u(x) + \int_0^1 \mathfrak{G}(x,\xi)u'(\xi)d\xi,$$

whence, $K_x(x, \xi)$ being the reciprocal of $\mathfrak{G}(x, \xi)$,

$$u'(x) = f(x) - \rho u(x) - \int_0^1 K_x(x,\xi)\{f(\xi) - \rho u(\xi)\}d\xi,$$

or else

$$u'(x) = f(x) - \int_0^1 K_x(x,\xi)f(\xi)d\xi + \rho\left\{ - u(x) + \int_0^1 K_x(x,\xi)u(\xi)d\xi \right\}$$

$$= \frac{d}{dx}\left\{\rho \int_0^1 K(x,\xi)u(\xi)d\xi\right\} - \frac{d}{dx}\left\{ \int_0^1 K(x,\xi)f(\xi)d\xi \right\},$$

and finally

$$u(x) = - \int_0^1 K(x,\xi)f(\xi)d\xi + \rho \int_0^1 K(x,\xi)u(\xi)d\xi + C,$$

where C is a constant.

Now, to find C, we rewrite (7) as follows:

$$u(0) = - \int_0^1 K(0,\xi)u'(\xi)d\xi - \int_0^1 u(\xi)d\xi \int_0^1 K(0,s)\mathfrak{G}_\xi(s,\xi)ds$$

$$+ u(1) \int_0^1 K(0,\xi)\mathfrak{G}(\xi,1)d\xi - u(0) \int_0^1 K(0,\xi)\mathfrak{G}(\xi,0)d\xi,$$

and, substituting $-u'(\xi)$ from (9),

$$u(0) = \rho \int_0^1 K(0,\xi)u(\xi)d\xi - \int_0^1 K(0,\xi)f(\xi)d\xi.$$

Hence, $C=0$ and $u(x)$ is a solution of the non-homogeneous integral equation

$$(10)\qquad u(x) = F(x) + \rho \int_0^1 K(x,\xi)u(\xi)d\xi$$

where

$$F(x) = -\int_0^1 K(x,\xi)f(\xi)d\xi. \tag{11}$$

On setting $f(x)\equiv 0$, we obtain the desired proof of the equivalence of the integral equation (2) and of the homogeneous integro-differential problem (★).

31. Using the results of §30 we can establish the important fact that the Green's function $\Gamma(x, t, \rho)$ of the integro-differential problem (★) and the resolvent kernel $\mathfrak{R}(x, t, \rho)$ of (2) are identical. The function $\rho\mathfrak{R}(x, t, \rho)$ is defined as the reciprocal of the kernel $\rho K(x, \xi)$ so that*

$$K(x,\xi) + \mathfrak{R}(x,\xi,\rho) = \rho\int_0^1 K(x,s)\mathfrak{R}(s,\xi,\rho)ds = \rho\int_0^1 \mathfrak{R}(x,s,\rho)K(s,\xi)ds.$$

The homogeneous problems (2) and (★) being equivalent, they have the same set of characteristic values, say

$$\rho_1,\rho_2,\rho_3,\cdots,\rho_\nu,\cdots.$$

Let ρ be different from any ρ_ν. The solution of the non-homogeneous problem (★), then, is uniquely determined and given by

$$u(x) = \int_0^1 \Gamma(x,t,\rho)f(t)dt.$$

On the other hand, from (10), (11) we have

$$\begin{aligned} u(x) &= F(x) - \rho\int_0^1 \mathfrak{R}(x,\xi,\rho)F(\xi)d\xi \\ &= \int_0^1 f(t)dt\left\{-K(x,t) + \rho\int_0^1 \mathfrak{R}(x,\xi,\rho)K(\xi,t)d\xi\right\} \\ &= \int_0^1 \mathfrak{R}(x,t,\rho)f(t)dt, \end{aligned}$$

which, in view of the arbitrariness of $f(x)$, shows that

$$\Gamma(x,t,\rho) \equiv \mathfrak{R}(x,t,\rho).$$

This being stated, the integral

$$\frac{1}{2\pi i}\int_{(C_R)} \rho^{n-1}d\rho \int_0^1 \Gamma(x,t,\rho)p_{n0}(t)f(t)dt$$

* Cf. Lalesco, *Introduction à la Théorie des Equations Intégrales*, Paris, 1912, pp. 23–24. It should be noted that our kernel $K(x, \xi)$ corresponds to the kernel $-N(x, \xi)$ of Lalesco.

used in Part III reduces in the present case to

$$(12) \qquad \frac{1}{2\pi i}\int_{(C_R)} d\rho \int_0^1 \mathfrak{K}(x,t,\rho)f(t)dt = \sum_{\nu=1}^{N} \int_0^1 f(t) \operatorname*{Res}_{\rho=\rho_\nu} \mathfrak{K}(x,t,\rho)dt$$

where $\rho_1, \rho_2, \cdots, \rho_N$ are the characteristic values of (2) within (C_R). It is well known, however, that the principal part of $\mathfrak{K}(x, t, \rho)$ corresponding to a pole ρ_0, of multiplicity m, is

$$\frac{\phi_m(x,t)}{(\rho-\rho_0)^m} + \cdots + \frac{\phi_1(x,t)}{\rho-\rho_0}$$

where

$$\phi_1(x,t) = \sum_{(i)} \Phi_i(x)\Psi_i(t)$$

and

$$\Phi_1(x), \cdots; \ \Psi_1(x), \cdots$$

are respectively the *fundamental functions* of the equation (2) and of the associated equation

$$(13) \qquad v(x) = \rho \int_0^1 K(\xi,x)v(\xi)d\xi$$

corresponding to the pole ρ_0.* The functions $\Phi_1(x), \Psi_1(x), \cdots$ constitute a biorthogonal system of function which reduces to that of the *fundamental solutions* of (2) and (13), when and only when the pole ρ_0 is simple. Now, let

$$u_1(x), v_1(x) \ ; u_2(x), v_2(x) \ ; \cdots, u_\nu(x), v_\nu(x), \cdots$$

be the complete set of fundamental functions of the equations (2) and (13) which, as we know, is biorthogonal. Let ρ_ν denote the pole of $\mathfrak{K}(x, t, \rho)$ which corresponds to the pair of functions $u_\nu(x), v_\nu(x)$, each pole being counted as many times as there are linearly independent fundamental functions corresponding to this pole, and

$$\sum(f) \equiv \sum_{\nu=1}^{\infty} u_\nu(x) \int_0^1 f(t)v_\nu(t)dt.$$

The integral

$$\mathfrak{J}(f) \equiv \frac{1}{2\pi i}\int_{(C_R)} d\rho \int_0^1 \Gamma(x,t,\rho)f(t)d\rho$$

becomes then

$$\sum_{(C_R)} u_\nu(x) \int_0^1 f(t)v_\nu(t)dt \equiv \sum_{(C_R)} (f)$$

* Lalesco, loc. cit., pp. 46–55.

which is exactly the sum of the N_R first terms of the expansion of the arbitrary function $f(x)$ in the series related to the integral equation (2), N_R denoting the number of poles ρ_ν within (C_R). Our expansion above includes as a special case that discussed by Langer (L, Chapter 12); it reduces to the latter one when and only when all the poles of the resolvent kernel $\mathfrak{K}(x, \xi, \rho)$ are simple.

32. So far our work has been of more or less formal character. Now it is obvious that the reasonings of §§28-30 can be carried through if we assume that the partial derivatives of $K(x, \xi)$ of the second order are continuous on $(\mathfrak{S})$, except on the line $\xi=x$, and that they tend to finite limiting values when the point (x, ξ) tends to any point on the line $\xi=x$ in $(\mathfrak{S})$, without crossing this line. After this, we have to take care of the conditions **(A)**, **(B)** and **(C)**, §§ 10, 13, 19. Conditions **(A)** in the present case reduce to the requirement that neither of the coefficients a, b in (7) is zero, and that the function $\alpha(\xi)$ is possessed of a continuous derivative which is of bounded variation on $(0,1)$. To meet the latter of these conditions it suffices to assume that the partial derivatives of $K(x, \xi)$ of the third order are continuous on $(\mathfrak{S})$ except on the line $\xi=x$, and that they are of bounded variation in ξ for fixed x, as well as in x for fixed ξ, the total variation being uniformly bounded on $(0, 1)$. Let us turn now to the conditions **(C)**. Equation (9) is not of the form required by **(C**, v) since the operators $u(0), u(1)$, $au(0)+bu(1)$ are not linearly independent. However, we can bring (9) to the desired form by eliminating one of the arguments $u(0)$, $u(1)$ by means of (7) which is possible since $a\neq 0$ and $b\neq 0$. Then, under the assumptions previously made, all the conditions **(C)** are satisfied.

33. On collecting our results we may now formulate the following proposition:

THEOREM 8. *Given the integral equation*

$$u(x) = \rho \int_0^1 K(x,\xi)u(\xi)d\xi \tag{2}$$

whose kernel $K(x, \xi)$ satisfies the following conditions:

(D) i. *$K(x,\xi)$ is continuous on the region*

$$0 \leqq x \leqq 1\,;\, 0 \leqq \xi \leqq 1\,;\, x \neq \xi$$

together with its partial derivatives up to the third order inclusive.

ii. *The partial derivatives of the third order are of bounded variation in ξ for fixed x, and in x for fixed ξ, the total variation being uniformly bounded on* $(0, 1)$.*

* Restrictions i–ii ensure the finiteness and even the continuity of the limiting values of $K(x, \xi)$ and of those of its partial derivatives up to the second order inclusive on the line $x=\xi$.

iii. *The kernel* $K(x, \xi)$ *is normalized, that is*

$$K(x,x^+) - K(x,x^-) = 1\ ;\ K_x(x,x^+) - K_x(x,x^-) = 0.$$

iv. *The kernel* $K_x(x, \xi)$, *where*

$$K_x(x,x) \equiv K_x(x,x^+) = K_x(x,x^-),$$

is possessed of a reciprocal, $\mathfrak{C}(x, \xi)$.*

v. *The constants* a *and* b *defined by*

$$a = 1 - K(0,0^+) + \int_0^1 K(0,\xi)\,\mathfrak{C}(\xi,0)d\xi, \tag{14}$$

$$b = K(0,1) - \int_0^1 K(0,\xi)\,\mathfrak{C}(\xi,1)d\xi,$$

are different from zero.

Under these hypotheses we have the following conclusions:

(I) *The integral equation* (2) *has infinitely many characteristic values. Let*

$$\rho_k' = \log(-b/a) + 2k\pi i \qquad (k = 0, \pm 1, \pm 2, \cdots) \tag{15}$$

be the characteristic values of the differential boundary problem

$$u'(x) + \rho u(x) = 0\ ;\ au(0) + bu(1) = 0. \tag{★★}$$

Then, if δ *is any positive number arbitrarily small but fixed, an* R_δ *is available which is so large that, outside the circle* $|\rho| = R_\delta$, *all the characteristic values of* (2) *lie in the interiors of the circles of radius* δ *around the points* (15), *each circle containing one and only one characteristic value of* (2).

(II) *Outside the circle* $|\rho| = R_\delta$ *all the characteristic values of* (2) *are simple poles of the resolvent kernel* $\mathfrak{K}(x, \xi, \rho)$ *and to each of them there corresponds a single pair* $u(x)$, $v(x)$ *of the fundamental solutions of* (2) *and of the associated integral equation* (13).

(III) *Let*

$$u_1(x),\ v_1(x),\ u_2(x),\ v_2(x),\ \cdots,\ u_\nu(x),\ v_\nu(x),\ \cdots \tag{16}$$

be the complete biorthogonal set of the fundamental functions of (2), (13) *and*

$$u_k^{(1)}(x),\ v_k^{(1)}(x) \qquad (k = 0, \pm 1, \pm 2, \cdots)$$

the complete biorthogonal set of the fundamental solutions of (★★) *and of the adjoint problem*

$$v'(x) - \rho v(x) = 0\ ;\ bv(0) + av(1) = 0. \tag{★★'}$$

*This condition is somewhat less general than the corresponding Langer's condition (v) (L, p. 592). We expect to discuss in another paper the case where $K_x(x, \xi)$ has no reciprocal.

Let $f(x)$ be any function integrable on $(0, 1)$. *If we set*

$$(17)\quad \sum(f) \equiv \sum_{\nu=1}^{\infty} u_\nu(x) \int_0^1 f(t)v_\nu(t)dt\,;\ \sum\nolimits_N(f) \equiv \sum_{\nu=1}^{N} u_\nu(x) \int_0^1 f(t)v_\nu(t)dt,$$

$$(18)\quad S(f) \equiv \sum_{\nu=-\infty}^{\infty} u_\nu^{(1)}(x) \int_0^1 f(t)v_\nu^{(1)}(t)dt;\ S_N(f) \equiv \sum_{\nu=-N}^{N} u_\nu^{(1)}(x) \int_0^1 f(t)u_\nu^{(1)}(t)dt,$$

the series $\sum(f)$ and $S(f)$ are equiconvergent on the interior of $(0, 1)$, *that is*

$$(19)\qquad \sum\nolimits_N(f) - S_N(f) \to 0 \quad as \quad N \to \infty,$$

uniformly on every interval interior to $(0, 1)$. *At the end points* 0, 1 *the difference* (19) *tends respectively to*

$$(20)\qquad C_0 \int_0^1 f(t)\alpha(t)dt, \qquad C_1 \int_0^1 f(t)\alpha(t)dt$$

where

$$(21)\qquad \alpha(\xi) = K_\xi(0,\xi) - \int_0^1 K(0,s)\,\mathfrak{G}_\xi(s,\xi)ds,$$

and C_0, C_1 are constant factors which depend only on $K(x, \xi)$ and do not depend on $f(x)$.

(IV) *No modifications are necessary in the statements* (I) *and* (II) *if the differential problem* (★★) *is replaced by*

$$(\bigstar\bigstar\bigstar)\qquad u'(x) + \rho u(x) = 0\ ;\ au(0) + bu(1) = \int_0^1 \alpha(\xi)u(\xi)d\xi.$$

Let $G(x, t, \rho)$ be the Green's function of the problem (★★★) *and*

$$(22)\qquad \rho_1'', \rho_2'', \cdots, \rho_\nu'', \cdots$$

the set of the characteristic values of (★★★). *Let*

$$(C_R)\qquad |\rho| = R$$

be a circle around the origin, which does not pass through any of the points ρ_ν, ρ_ν', ρ_ν''. If $\rho_1, \rho_2, \cdots, \rho_N$ are the characteristic values of (2) *within* (C_R), *then*

$$(23)\qquad \lim_{R\to\infty} \left\{ \sum\nolimits_N(f) - \frac{1}{2\pi i} \int_{(C_R)} d\rho \int_0^1 G(x,t,\rho)f(t)dt \right\} = 0,$$

uniformly on $(0, 1)$.

34. The second part of the statement (IV) follows immediately from Theorem 4 since the integro-differential problem (★) satisfies the conditions (A), (B), (C) and the Green's function $\Gamma(x, t, \rho)$ of this problem coincides with the resolvent kernel $\mathfrak{K}(x, t, \rho)$.

In order to prove the statements (I), (II) and the first part of (IV) let ρ_0' be any pole of $G'(x, t, \rho)$ outside $|\rho| = R_\delta$. If R_δ is sufficiently large, then to each ρ_0' there corresponds one and only one pole ρ_0'' of (★★★) which is within the circle (c) of radius δ around ρ_0' and vice versa. Let $G'(x, t, \rho)$ be the Green's function of (★★). Take the integrals

$$J_c(\Gamma), \; J_c(G'), \; J_c(G)$$

which have been used already in § 26. The principal part of $G'(x, t, \rho)$ corresponding to the pole $\rho = \rho_0'$ is

$$U_0(x)V_0(t)/(\rho - \rho_0') \; ; \; \int_0^1 U_0(x)V_0(x)dx = 1,$$

where $U_0(x)$, $V_0(x)$ are respectively the fundamental solutions of the problem (★★) and of the adjoint problem (★★') for $\rho = \rho_0'$ (D, 15). Hence

$$J_c(G') = 1.$$

On the other hand, the same argument being applied to $\Gamma(x, t, \rho) \equiv \mathfrak{K}(x, t, \rho)$ shows

$$J_c(\Gamma) = \sigma$$

where σ denotes the total number of pairs u_ν, v_ν in (16) which correspond to the characteristic values of (2) within (c). We have, however,

$$\sigma - 1 = J_c(\Gamma - G') = J_c(\Gamma - G) + J_c(G - G') = O(1/r),$$

where r is the shortest distance from the origin to the contour of (c) (§ 26). Hence $\sigma = 1$, provided R_δ is sufficiently large. Thus the statements (I), (II), (IV) are proved.

The statement (III) follows immediately from Theorem 6; we observe that in the present case

$$i = 1, \; a_i(x) \equiv \alpha(x) \; ; \; a_i'(x) \equiv 0 \; ;$$

the functions Θ_{ai}, Θ_{bi} in (23) reduce to constants. Finally, the expressions

$$\frac{1}{2\pi i}\int_{(C_R)} d\rho \int_0^1 G'(x,t,\rho)f(t)dt \;\text{ and }\; S_N(f)$$

may differ but by a finite sum of terms of the form

$$\pm\, u_\nu^{(1)}(x) \int_0^1 f(t)v_\nu^{(1)}(t)dt$$

for which either $|\rho_\nu'|>R$ or $|\nu|>N$, and whose number does not exceed the number of the characteristic values (15) within $|\rho|=R_s$. It is readily seen that each of these terms $\to 0$ as $R\to\infty$, uniformly on (0, 1).

35. THEOREM 9. *Under the conditions of Theorem* 8 *the resolvent kernel* $\mathfrak{R}(x, \xi, \rho)$ *of* (2) *admits of an expansion* (ρ *is not a characteristic value*)

$$\mathfrak{R}(x,\xi,\rho) = \sum_{\nu=1}^{\infty} \mathfrak{R}_\nu(x,\xi,\rho) \tag{24}$$

where $\mathfrak{R}_\nu(x, \xi, \rho)$ *denotes the principal part of* $\mathfrak{R}(x, \xi, \rho)$ *corresponding to the pole* ρ_ν. *Accordingly, for the kernel* $K(x, \xi)$ *itself we have*

$$K(x,\xi) = - \sum_{\nu=1}^{\infty} \mathfrak{R}_\nu(x,\xi,0), \tag{25}$$

the series of the left-hand member of (24) *and* (25) *being uniformly convergent in* (x, ξ) *on every closed part of the square* ($\mathfrak{S}$), *which does not contain any of the points* (0, 1), (1, 0) *and has no points in common with the line* $x=\xi$.

The solution of the non-homogeneous integral equation

$$u(x) = f(x) + \rho \int_0^1 K(x,\xi)u(\xi)d\xi \tag{26}$$

admits of the expansion

$$u(x) = f(x) - \rho \sum_{\nu=1}^{\infty} \int_0^1 \mathfrak{R}_\nu(x,\xi,\rho)f(\xi)d\xi \tag{27}$$

which is uniformly convergent on (0, 1) *for any integrable* $f(x)$.

An easy application of the Cauchy fundamental theorem shows

$$\begin{aligned} \mathfrak{R}(x,\xi,\rho) - G(x,\xi,\rho) - \sum_{(C_R)} \mathfrak{R}_\nu(x,\xi,\rho) + \sum_{(C_R)} G_\nu(x,\xi,\rho) \\ = \int_{(C_R)} \frac{\Gamma(x,\xi,\zeta) - G(x,\xi,\zeta)}{\zeta - \rho} d\zeta \end{aligned} \tag{28}$$

where $G_\nu(x, \xi, \rho)$ denotes the principal part of $G(x, \xi, \rho)$ corresponding to the pole ρ_ν''. Since on (C_R)

$$\Gamma(x,\xi,\zeta) - G(x,\xi,\zeta) = O(1/\zeta)$$

(Theorem 3) the right-hand member of (28)$\to 0$ as $R\to\infty$, uniformly on (0, 1). It is known from the general theory of the Green's function (D, 27) and may be proved directly in the present case that

$$G(x,\xi,\rho) = \sum_{(\nu)} G_\nu(x,\xi,\rho),$$

the series of the right-hand member being uniformly convergent on any region of the type mentioned in the statement of Theorem 9. Hence the same is true of the expansion (24). Expansion (25) follows from (24), for $\rho=0$, since $K(x, \xi) = -\mathfrak{K}(x, \xi, 0)$.

Now, let $f(x)$ be any integrable function. For any fixed ρ which is not a characteristic value of (26), the solution of (26) is given by the formula

$$(29)\quad u(x) = f(x) - \rho\int_0^1 \mathfrak{K}(x,\xi,\rho)f(\xi)d\xi = f(x) - \rho\sum_{\nu=1}^{\infty}\int_0^1 \mathfrak{K}_\nu(x,\xi,\rho)f(\xi)d\xi,$$

the term by term integration being permissible by virtue of a known theorem of Lebesgue. To prove the uniform convergence of (29), take the difference

$$\begin{aligned} u(x) - f(x) + \rho\sum_{(C_R)}\int_0^1 \mathfrak{K}_\nu(x,\xi,\rho)f(\xi)d\xi &= -\frac{1}{2\pi i}\int_{(C_R)}\frac{d\zeta}{\zeta-\rho}\int_0^1 \mathfrak{K}(x,\xi,\zeta)f(\xi)d\xi \\ &= -\frac{1}{2\pi i}\int_{(C_R)}\frac{d\zeta}{\zeta-\rho}\int_0^1 \{\Gamma(x,\xi,\zeta) - G(x,\xi,\zeta)\}f(\xi)d\xi \\ &\quad -\frac{1}{2\pi i}\int_{(C_R)}\frac{d\zeta}{\zeta-\rho}\int_0^1 G(x,\xi,\zeta)f(\xi)d\xi. \end{aligned}$$

On account of Theorem 3 and Lemma 1 each term of the right side here is of the form

$$\int_{(C_R)}\frac{\epsilon d\zeta}{\zeta-\rho}$$

and hence $\to 0$, as $R\to\infty$, uniformly on (0,1).

It should be noted that, in the case where all the poles of $\mathfrak{K}(x, \xi, \rho)$ are simple, formulas (24), (25), (27) reduce to well known expansions

$$\mathfrak{K}(x,\xi,\rho) = \sum_{\nu=1}^{\infty}\frac{u_\nu(x)v_\nu(\xi)}{\rho-\rho_\nu},$$

$$K(x,\xi) = \sum_{\nu=1}^{\infty}\frac{1}{\rho_\nu}u_\nu(x)v_\nu(\xi).$$

$$u(x) = f(x) - \rho\sum_{\nu=1}^{\infty}\frac{f_\nu u_\nu(x)}{\rho-\rho_\nu},\quad f_\nu = \int_0^1 f(\xi)v_\nu(\xi)d\xi,$$

which, however, never before have been proved under our general conditions concerning the kernel $K(x, \xi)$.

We may leave to the reader the computation of the approximate expressions for the fundamental functions $u_\nu(x)$, $v_\nu(x)$ for ν large, as well as the applications of the results above to the integral equation (1).

36. To illustrate our general theory take the integral equation

$$(30) \qquad u(x) = \rho \int_0^1 K(x,\xi)u(\xi)d\xi \; ; \; K(x,\xi) = \begin{cases} mx^2\xi^2 + 2 \text{ if } \xi > x, \\ mx^2\xi^2 + 1 \text{ if } \xi < x, \end{cases}$$

which is analogous to that considered by Langer (L, pp. 638–639). An easy computation gives

$$\mathfrak{G}(x,\xi) = -\frac{4mx\xi^2}{2-m} \; ; \; a = -1 \; ; \; b = 2\frac{2+m}{2-m} \; ; \; \alpha(x) = \frac{8mx}{2-m} \cdot$$

Hence our theory can be applied to (30) unless $m = \pm 2$.

A direct computation shows that the characteristic values of (30) are the roots ($\neq 0$) of the transcendental equation

$$e^\rho\{(6-3m)\rho^4 + 4m\rho^3 + 24m(\rho-1)^2\} = \rho^4(6m+12) + 4m(2\rho^3 - 3\rho^2 - 6\rho + 6)$$

and hence they are asymptotic to the roots of the equation

$$e^\rho = 2\frac{2+m}{2-m} = -\frac{b}{a} \qquad (m \neq \pm 2)$$

which agrees with Theorem 8.

The situation changes substantially if $m = \pm 2$. In this case the characteristic values of (30) are determined by the equations

$$e^\rho = \rho\left[\frac{3}{4}\right] \text{ if } m = 2,$$

$$e^\rho = \frac{1}{\rho}\left[-\frac{8}{3}\right] \text{ if } m = -2.$$

The asymptotic formula for ρ_ν involves logarithmic terms which are absent in the case where $m = \pm 2$. It should be noted that Langer's method also fails when $m = \pm 2$, since the asymptotic formula for ρ_ν as given by Langer does not contain logarithmic terms either.*

* It was not until recently that we noticed that a problem in some respects more general than that of Lichtenstein was treated by Mrs. Anna Pell-Wheeler in 1910. Cf. *Applications of biorthogonal systems of functions to the theory of integral equations*, these Transactions, vol. 12 (1911), pp. 165–180 (p. 176, Ex. 4). However, Mrs. Pell-Wheeler's problem does not include our problem (★) as a special case.

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