

A General Method of Approximation. Part I

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Abstract. In this paper we study two families of functions, viz. F and H , and show how to approximate the functions considered in the interval $[0, 1]$. The functions are assumed to be real when the argument is real.

We define

$$F = \{f; \text{(i) } f(\frac{1}{2} + x) = f(\frac{1}{2} - x), \text{ (ii) } f(0) = f(1) = 0,$$

$$\text{(iii) } f(x) \text{ is analytic in a sufficiently large neighborhood of } x = 0\},$$

$$H = \{h; \text{(j) } h(\frac{1}{2} + x) = -h(\frac{1}{2} - x), \text{ (ij) } h(0) = h(1) = 0,$$

$$\text{(ijj) } h(x) \text{ is analytic in a sufficiently large neighborhood of } x = 0\}.$$

The approximations are defined in the interval $[0, 1]$ by

$$\min \int_0^1 \left(f(x) - \sum_{n=1}^k c_{n,k} [x(1-x)]^n \right)^2 x^q (1-x)^q dx$$

and

$$\min \int_0^1 \left(h(x) - (1-2x) \sum_{n=1}^k C_{n,k} [x(1-x)]^n \right)^2 x^q (1-x)^q dx,$$

where $q \in \{0, 1, 2, \dots\}$.

The associated matrices are analyzed using the theory of orthogonal polynomials, especially the Jacobi polynomials $G_n(p, q, x)$. We apply the general theory to the basic trigonometric functions $\sin(x)$ and $\cos(x)$.

Introduction. This paper traces its origin from a wish to determine simple, accurate and rapid approximations of the basic trigonometric functions $\sin(x)$ and $\cos(x)$. We encountered this problem when repeatedly applying the Box-Müller transformation for generating bivariate normally distributed pseudo-random variates. But as is often the case when starting with an analysis of a special example one discovers an underlying more general pattern. The method found by us in [6], when approximating $\sin(x)$ and $\cos(x)$, could thus be applied to a much wider class of functions; see [7].

When measuring the "distance" between the functions and their approximations we use the L_2 -norm. The required coefficients can then be determined from the resulting linear equation system.

The calculation of accurate values of the required coefficients is difficult. The associated Hankel matrices are as usual almost singular. We solve this problem by explicit calculation of the inverse matrices. The necessary numerical values of the associated integrals are determined using high-precision techniques.

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Once recognized as "moment-matrices" the resulting Hankel matrices can be analyzed using the theory of orthogonal polynomials (especially the Jacobi-polynomials $G_n(p, q, x)$ with associated weight function $w(x) = (1-x)^p x^q$, $x \in [0, 1]$). Empirically observed peculiarities can then be proved, and results can be generalized.

An even more thorough analysis of the methods used in [7] revealed that Bernstein polynomials were involved. That made it possible to further generalize the methods in [7]. The results thus found (see [8]) will be presented in part II.

The approximation method presented here consists of two parts. One can be solved once and for all (i.e., inversion of the associated matrices), the other part involves calculation of some integrals associated with the approximated functions.

1. Certain General Expansions.

a. *The Symmetric Case.* We state the main result of this section in the following

THEOREM 1. *Let $f(x)$ be a function with the following characteristics:*

- (i) $f(\frac{1}{2} + x) = f(\frac{1}{2} - x)$, $f(x)$ is real when x is real,
- (ii) $f(0) = f(1) = 0$,
- (iii) $f(x)$ may be expanded in a Taylor series around $x = 0$, and the radius of convergence is greater than 1.

Then $f(x)$ has an expansion of the form $f(x) = \sum_{m=1}^{\infty} a_m [x(1-x)]^m$ valid at least in the interval $[0, 1]$. Expressions for the coefficient a_m are given by Eqs. (1.3), (1.6), and (1.9).

To prove Theorem 1 we expand $f(x)$ in a Taylor series around $x = \frac{1}{2}$, i.e.,

$$(1.1) \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(2n)}(\frac{1}{2})}{(2n)!} (x - \frac{1}{2})^{2n},$$

where we have

$$(x - \frac{1}{2})^{2n} = \frac{1}{4^n} [1 - 4x(1-x)]^n = \frac{1}{4^n} \sum_{k=0}^n \binom{n}{k} (-1)^k 4^k [x(1-x)]^k.$$

Putting

$$(1.2) \quad f(x) = \sum_{m=1}^{\infty} a_m [x(1-x)]^m,$$

we get

$$(1.3) \quad a_m = (-1)^m \sum_{v=m}^{\infty} \frac{f^{(2v)}(\frac{1}{2})}{(2v)!} \binom{v}{m} 4^{m-v}.$$

To proceed we also need the identity

$$(1.4) \quad (1-x)^n + x^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} (-1)^k [x(1-x)]^k.$$

(1.4) is easily proved using Lagrange's inversion formula; see [6, p. 13].

From (i) we conclude that

$$(1.5) \quad f(x) + f(1-x) = 2f(x).$$

Using (1.5) and the expansion in (1.4) we get

$$2f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} (-1)^k [x(1-x)]^k.$$

An alternative expression of the coefficient a_m is then

$$(1.6) \quad a_m = \frac{1}{2} \sum_{n=2m}^{\infty} \frac{n}{n-m} \binom{n-m}{m} (-1)^m \frac{f^{(n)}(0)}{n!}.$$

Writing

$$f(x) = \sum_{v=1}^{\infty} \frac{f^{(v)}(0)}{v!} x^v = \sum_{m=1}^{\infty} a_m [x(1-x)]^m$$

and identifying corresponding powers of x , we get the equation system

$$(1.7) \quad \frac{f^{(n)}(0)}{n!} = \sum_{v=0}^{[n/2]} \binom{n-v}{n-2v} (-1)^v a_{n-v}, \quad n = 1, 2, 3, \dots$$

The solution of this equation system may be obtained using Lagrange's inversion formula. Thus

$$(1.8) \quad a_m = \frac{1}{m!} \left\{ \frac{d^{m-1}}{dx^{m-1}} \left[\frac{f(x)}{(1-x)^m} \right] \right\}_{x=0},$$

which may be written

$$(1.9) \quad a_m = \sum_{v=0}^{m-1} \binom{m-1}{v} \frac{(m+v-1)!}{m!(m-1)!} f^{(m-v)}(0).$$

Now suppose that $g(x)$ is any analytic function satisfying only the condition (iii). Then we may put $f(x) = g(x - \frac{1}{2}) + g(\frac{1}{2} - x) - g(-\frac{1}{2}) - g(\frac{1}{2})$. We notice that $f(x)$ is symmetric around $x = \frac{1}{2}$, and $f(0) = f(1) = 0$.

Example. We may expand

$$(1.10) \quad e^{t(x-1/2)} + e^{-t(x-1/2)} = e^{t/2} + e^{-t/2} + \sum_{m=1}^{\infty} a_m [x(1-x)]^m,$$

where the coefficients a_m satisfy the recurrence relation

$$(1.11) \quad (n+2)(n+1)a_{n+2} = (2n+2)(2n+1)a_{n+1} + t^2 a_n,$$

with starting values $a_0 = 2 \cosh(\frac{1}{2}t)$, $a_1 = -2t \sinh(\frac{1}{2}t)$. \square

Another more general possibility is to consider

$$f(x) = G(g(x), g(1-x)) - G(g(0), g(1)),$$

where $G(x, y)$ is any regular and symmetric function (i.e. $G(x, y) = G(y, x)$). Evidently we have $f(\frac{1}{2} + x) = f(\frac{1}{2} - x)$ and an expansion of the form

$$(1.12) \quad f(x) = \sum_{m=1}^{\infty} a_m [x(1-x)]^m.$$

Suppose that $g(x)$ satisfies a functional relation of the form $g(1-x) = H(g(x), F(x))$ where $H(x, y)$ and $F(x)$ are "simple" functions. Approximating in

the L_2 -norm, as indicated in Section 2a, we get a relation of the form

$$G(g(x), H(g(x), F(x))) - G(g(0), g(1)) \approx \sum_{n=1}^k c_{n,k} [x(1-x)]^n.$$

If $G(x, y)$ and $H(x, y)$ are simple functions, we may sometimes solve the “identity” with respect to $g(x)$, which enables us to calculate $g(x)$ even if the function does not satisfy the conditions (i) and (ii).

Example. We put $G(x, y) = x + y$ and $g(x) = 1/\Gamma(1 + x)$. Then $g(x)$ satisfies the relation $g(x)g(1-x) = \sin(\pi x)/\pi x(1-x)$.

Approximating in the L_2 -norm we get a relation of the form

$$(1.13) \quad \frac{1}{\Gamma(1+x)} + \frac{\Gamma(1+x)\sin(\pi x)}{\pi x(1-x)} \approx 2 + \sum_{n=1}^k c_{n,k} [x(1-x)]^n. \quad \square$$

b. *The Antisymmetric Case.* The theorem corresponding to Theorem 1 is

THEOREM 2. *Let $h(x)$ be a function satisfying the conditions*

(j) $h(\frac{1}{2} + x) = -h(\frac{1}{2} - x)$, $h(x)$ is real when x is real,

(jj) $h(0) = h(1) = 0$,

(jjj) $h(x)$ may be expanded in a Taylor series around $x = 0$, and the radius of convergence is greater than 1.

Then $h(x)$ has an expansion of the form $h(x) = (1 - 2x) \sum_{m=1}^{\infty} b_m [x(1-x)]^m$ valid at least in the interval $[0, 1]$. Different expressions for the coefficients b_m are given by Eqs. (1.16) and (1.19).

To prove Theorem 2 we expand $h(x)$ in a Taylor series around $x = \frac{1}{2}$, i.e.,

$$(1.14) \quad h(x) = \sum_{n=0}^{\infty} \frac{h^{(2n+1)}(\frac{1}{2})}{(2n+1)!} (x - \frac{1}{2})^{2n+1},$$

where we have

$$(x - \frac{1}{2})^{2n+1} = -\frac{1}{2}(1 - 2x) \frac{1}{4^n} \sum_{k=0}^n \binom{n}{k} (-1)^k 4^k [x(1-x)]^k.$$

Putting

$$(1.15) \quad h(x) = (1 - 2x) \sum_{m=1}^{\infty} b_m [x(1-x)]^m,$$

we get

$$(1.16) \quad b_m = -\frac{1}{2}(-1)^m \sum_{v=m}^{\infty} \frac{h^{(2v+1)}(\frac{1}{2})}{(2v+1)!} \binom{v}{m} 4^{m-v}.$$

To proceed we will need an identity similar to (1.4), viz.

$$(1.17) \quad (1-x)^n - x^n = (1-2x) \sum_{k=0}^{[(n-1)/2]} \binom{n-k-1}{k} (-1)^k [x(1-x)]^k.$$

(1.17) follows easily from (1.4) when differentiating with respect to x .

From (j) we conclude that

$$(1.18) \quad h(x) - h(1-x) = 2h(x).$$

Using (jjj) and (1.18) we get

$$2h(x) = - \sum_{n=1}^{\infty} \frac{h^{(n)}(0)}{n!} (1-2x)^{\lfloor (n-1)/2 \rfloor} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k} (-1)^k [x(1-x)]^k.$$

The coefficient b_m in the expansion (1.15) is then given by

$$(1.19) \quad b_m = (-1)^{m+1} \frac{1}{2} \sum_{n=2m+1}^{\infty} \frac{h^{(n)}(0)}{n!} \binom{n-m-1}{m}.$$

Multiplying (1.15) with $(1-2x)$ we get, using Lagrange's inversion formula,

$$(1.20) \quad b_m - 4b_{m-1} = \frac{1}{m!} \left\{ \frac{d^{m-1}}{dx^{m-1}} \left[\frac{(1-2x)h(x) - 2h(x)}{(1-x)^m} \right] \right\}_{x=0}.$$

When the function $g(x)$ does not satisfy the conditions (j) and (jj) we may put $h(x) = g(x - \frac{1}{2}) - g(-x + \frac{1}{2}) + (1-2x)[g(\frac{1}{2}) - g(-\frac{1}{2})]$. Then $h(x)$ is antisymmetric around $x = \frac{1}{2}$ and $h(0) = h(\frac{1}{2}) = h(1) = 0$.

If (jj) is not satisfied, then we have $h(0) = -h(1) \neq 0$ (which is the case for $h(x) = \cos(\pi x)$). We may avoid this difficulty by substituting $h_1(x) = h(x) - (1-2x)h(0)$.

Finally let

$$h(x) = G(g(x), g(1-x)) - (1-2x)G(g(0), g(1)),$$

where $G(x, y)$ is a regular, antisymmetric function of x and y . Assuming that $G(x, y)$ is "regular enough", then $h(x)$ satisfies the conditions (j), (jj), and (jjj). All the remarks occurring in the end of Section 1a are relevant also in this case.

c. *The General Case.* From a study of the proofs of Theorems 1 and 2 it is obvious that we may formulate

THEOREM 3. *Let $g(x)$ be a function satisfying the conditions*

(k) $g(x)$ is neither symmetric nor antisymmetric but real when x is real,

(kk) $g(0) = g(1) = 0$,

(kkk) $g(x)$ may be expanded in a Taylor series around $x = 0$, and the radius of convergence is greater than 1.

Then $g(x)$ has an expansion of the form

$$g(x) = \sum_{m=1}^{\infty} a_m [x(1-x)]^m + (1-2x) \sum_{m=1}^{\infty} b_m [x(1-x)]^m$$

valid at least in the interval $[0, 1]$. Expressions for the coefficients a_m and b_m are given by obvious generalizations of Eqs. (1.3) and (1.16).

When $g(x)$ does not satisfy the condition (kk) we simply substitute

$$g_1(x) = g(x) - \left[\frac{1}{2}(g(0) + g(1)) + \frac{1}{2}(g(0) - g(1))(1-2x) \right].$$

2. Approximation in the L_2 -norm.

a. *The Symmetric Case.* Let $f(x)$ be a function which satisfies the conditions (i), (ii), and (iii) in Section 1a. In many cases already the truncated series in Eq. (1.2) may be used for computational purposes. We may however improve this result by

approximating in the L_2 -norm, i.e., we consider $\min D_1^2(w)$, where

$$(2.1) \quad D_1^2(w) = \int_0^1 \left\{ f(x) - \sum_{n=1}^k c_{n,k} [x(1-x)]^n \right\}^2 x^q (1-x)^q dx$$

and $q \in \{0, 1, 2, \dots\}$.

The resulting linear equation system yielding the coefficients $c_{n,k}$ will be (with $r = 1, 2, \dots, k$)

$$(2.2) \quad \sum_{n=1}^k c_{n,k} u_{n,r+q} = \int_0^1 f(x) [x(1-x)]^{r+q} dx,$$

where

$$(2.3) \quad u_{i,j} = \int_0^1 [x(1-x)]^{i+j} dx = \frac{1}{(2i+2j+1) \binom{2i+2j}{i+j}}.$$

The inversion of the matrices $\mathbf{D}_{n,s} = \{u_{i,j}\}$ (with $i = 0, 1, 2, \dots, n; j = s, s+1, \dots, s+n; s \in \{0, 1, 2, \dots\}$) will be treated in Sections 4 and 5.

A remaining problem, with many solutions, is how to determine numerical high precision values of the integrals

$$(2.4) \quad I_j = \int_0^1 f(x) [x(1-x)]^j dx; \quad j = 1, 2, 3, \dots$$

That problem will be treated in Section 3.

b. *The Antisymmetric Case.* Let $h(x)$ be a function satisfying the conditions (j), (ij), and (ijj) in Section 1b. We consider $\min D_2^2(w)$, where

$$(2.5) \quad D_2^2(w) = \int_0^1 \left\{ h(x) - (1-2x) \sum_{n=1}^k C_{n,k} [x(1-x)]^n \right\}^2 x^q (1-x)^q dx$$

and $q \in \{0, 1, 2, \dots\}$.

The resulting linear equation system yielding the coefficients $C_{n,k}$ will be (with $r = 1, 2, \dots, k$)

$$(2.6) \quad \sum_{n=1}^k C_{n,k} U_{n,r+q} = \int_0^1 h(x) (1-2x) [x(1-x)]^{r+q} dx,$$

where

$$(2.7) \quad U_{i,j} = \int_0^1 (1-2x)^2 [x(1-x)]^{i+j} dx \\ = \frac{1}{(2i+2j+3)(2i+2j+1) \binom{2i+2j}{i+j}}.$$

The inversion of the matrices $\mathbf{E}_{n,t} = \{U_{i,j}\}$ (with $i = 0, 1, 2, \dots, n; j = t, t+1, \dots, t+n; t \in \{0, 1, 2, \dots\}$) will be treated in Sections 4 and 5.

In Section 3 we will consider different techniques to determine numerical values of the integrals

$$(2.8) \quad I'_j = \int_0^1 h(x) (1-2x) [x(1-x)]^j dx, \quad j = 1, 2, \dots$$

c. *The General Case.* Let the function $g(x)$ satisfy the conditions (k), (kk), and (kkk) in Section 1c and consider $\min D_3^2(w)$, where $q \in \{0, 1, 2, \dots\}$ and

$$(2.9) \quad D_3^2(w) = \int_0^1 \left\{ g(x) - \sum_{n=1}^k (c'_{n,k} + (1-2x)C'_{n,k}) [x(1-x)]^n \right\}^2 x^q(1-x)^q dx.$$

The resulting equation systems yielding the coefficients $c'_{n,k}$ and $C'_{n,k}$ will be (with $r = 1, 2, \dots, k$)

$$(2.10) \quad \sum_{n=1}^k c'_{n,k} u_{n,r+q} = \int_0^1 g(x) [x(1-x)]^{r+q} dx$$

and

$$(2.11) \quad \sum_{n=1}^k C'_{n,k} U_{n,r+q} = \int_0^1 g(x) (1-2x) [x(1-x)]^{r+q} dx.$$

The equations (2.10) and (2.11) will be considered in detail in part II in connection with Bernstein polynomials.

d. *Approximations in L_2 With Certain Restrictions.* In many cases it is natural to consider $\min D_1^2(w)$, $\min D_2^2(w)$, and $\min D_3^2(w)$ given certain restrictions. We may, e.g., consider

$$(2.12) \quad \begin{cases} \min \int_0^1 \left\{ f(x) - \sum_{n=1}^k c_{n,k} [x(1-x)]^n \right\}^2 x^q(1-x)^q dx \\ \sum_{n=1}^k c_{n,k} \frac{1}{4^n} = f\left(\frac{1}{2}\right). \end{cases}$$

When $f(x)$ is a probability density function it is natural to consider

$$(2.13) \quad \begin{cases} \min \int_0^1 \left\{ f(x) - \sum_{n=1}^k c_{n,k} [x(1-x)]^n \right\}^2 x^q(1-x)^q dx \\ \sum_{n=1}^k c_{n,k} \frac{1}{(2n+1) \binom{2n}{n}} = 1. \end{cases}$$

We will consider (2.12) and (2.13) in part II, when approximating $f(x) = \sin(\pi x)$ with restrictions.

Another natural approximation technique would be to equalize all the moments. We then get the equation system

$$(2.14) \quad \int_0^1 x^j f(x) dx = \sum_{n=1}^k c_{n,k} \frac{1}{(2n+j+1) \binom{2n+j}{n}}; \quad j = 0, 1, 2, \dots, k-1.$$

3. Calculation of the Associated Integrals. We will consider different techniques to determine numerical values of the associated integrals occurring in (2.4) and (2.8). We start with

a. *The Symmetric Case.* In some cases we may use a recursion formula for I_j or a closed expression (See Application in Section 6, (6.22).) When it is easy to determine numerically the coefficients a_m defined by (1.2), (e.g., by using a recursion formula) then

$$(3.1) \quad I_j = \sum_{m=1}^{\infty} a_m \frac{1}{(2m+2j+1) \binom{2m+2j}{m+j}}.$$

Expanding $f(x)$ in a Taylor series around $x = 0$, we get

$$(3.2) \quad I_j = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot \frac{1}{(n+2j+1) \binom{n+2j}{j}}.$$

Putting $u = x(1-x)$ and making use of the symmetry, we get

$$(3.3) \quad I_j = \frac{1}{2^{2j+1}} \int_0^1 f\left(\frac{1}{2} - \frac{1}{2}(1-u)^{1/2}\right) \frac{u^j}{(1-u)^{1/2}} du.$$

Making an obvious expansion of $f(\frac{1}{2} - \frac{1}{2}(1-u)^{1/2})$ in (3.3), we obtain

$$(3.4) \quad I_j = \sum_{n=0}^{\infty} \frac{f^{(2n)}(\frac{1}{2})}{(2n)!} \cdot \frac{\binom{2n}{n}}{2^{2n-1} \binom{2n+2j+2}{n+j+1} \binom{n+j+1}{n+1} (n+1)}.$$

In some cases we may also use a Fourier expansion. For reasons of symmetry we must have

$$(3.5) \quad f(x) = \beta_0 + \sum_{n=1}^{\infty} [\alpha_n \sin((2n-1)\pi x) + \beta_n \cos(2n\pi x)],$$

where $\beta_0 + \sum_{n=1}^{\infty} \beta_n = 0$ and $x \in [0, 1]$.

The expansion in (3.5) is of special importance when α_n and β_n converge fast towards zero. That is, e.g., the case when $f(x) = \operatorname{sn}(2K(k)x, k)$; see [7, p. 25].

To calculate I_j , making use of the expansion (3.5), we must know how to calculate the integrals

$$(3.6) \quad S_{2n,j} = \int_0^1 \cos(2n\pi x) [x(1-x)]^j dx$$

and

$$(3.7) \quad T_{2n-1,j} = \int_0^1 \sin((2n-1)\pi x) [x(1-x)]^j dx.$$

However $S_{2n,j}$ and $T_{2n-1,j}$ satisfy the recurrence relations

$$(3.8) \quad S_{2n,j} = \frac{2j(2j-1)}{(2n\pi)^2} S_{2n,j-1} - \frac{j(j-1)}{(2n\pi)^2} S_{2n,j-2}$$

and

$$(3.9) \quad T_{2n-1,j} = \frac{2j(2j-1)}{(2n-1)^2\pi^2} T_{2n-1,j-1} - \frac{j(j-1)}{(2n-1)^2\pi^2} T_{2n-1,j-2}.$$

Starting values are given by

$$S_{2n,0} = 0, \quad S_{2n,1} = -\frac{2}{(2n\pi)^2},$$

$$T_{2n-1,0} = \frac{2}{(2n-1)\pi}, \quad T_{2n-1,1} = \frac{4}{(2n-1)^3\pi^3}.$$

c. *The Antisymmetric Case.* We consider next the integrals I'_j defined by (2.8). When it is easy to calculate the coefficients b_m in the expansion (1.15) we easily get

$$(3.10) \quad I'_j = \sum_{m=1}^{\infty} b_m \frac{1}{(2m+2j+3)(2m+2j+1) \binom{2m+2j}{m+j}}.$$

But we may also use the expansion in (1.14). After a routine calculation we get

$$(3.11) \quad I'_j = - \sum_{n=0}^{\infty} \frac{h^{(2n+1)}(\frac{1}{2})}{(2n+1)!} \cdot \frac{\binom{2n+2}{n+1}}{2^{2n}(n+2) \binom{2n+2j+4}{n+j+2} \binom{n+j+2}{j}}.$$

Expanding $h(x)$ in a Taylor series around $x = 0$, we obtain

$$(3.12) \quad I'_j = \sum_{n=1}^{\infty} \frac{h^{(n)}(0)}{n!(n+2j+2)} \left[\frac{1}{\binom{n+2j+1}{j+1}} - \frac{1}{\binom{n+2j+1}{j}} \right].$$

Even in this case we may sometimes use a Fourier series. For reasons of asymmetry we must have

$$(3.13) \quad h(x) = \sum_{n=1}^{\infty} [\gamma_n \sin(2n\pi x) + \delta_n \cos((2n-1)\pi x)],$$

where $\sum_{n=1}^{\infty} \delta_n = 0$ and $x \in [0, 1]$.

To calculate I'_j , using the expansion (3.13), we must be able to calculate

$$(3.14) \quad Q_{2n-1,j} = \int_0^1 (1-2x)\cos((2n-1)\pi x)[x(1-x)]^j dx$$

and

$$(3.15) \quad R_{2n,j} = \int_0^1 (1-2x)\sin(2n\pi x)[x(1-x)]^j dx.$$

However, $Q_{2n-1,j}$ and $R_{2n,j}$ satisfy the recurrence relations

$$(3.16) \quad Q_{2n-1,j} = \frac{2j(2j+1)}{(2n-1)^2\pi^2} Q_{2n-1,j-1} - \frac{j(j-1)}{(2n-1)^2\pi^2} Q_{2n-1,j-2}$$

and

$$(3.17) \quad R_{2n,j} = \frac{2j(2j+1)}{(2n\pi)^2} R_{2n,j-1} - \frac{j(j-1)}{(2n\pi)^2} R_{2n,j-2}.$$

Starting values are given by

$$Q_{2n-1,0} = \frac{4}{(2n-1)^2\pi^2}, \quad Q_{2n-1,1} = \frac{24}{(2n-1)^4\pi^4} - \frac{2}{(2n-1)^2\pi^2},$$

$$R_{2n,0} = \frac{1}{n\pi}, \quad R_{2n,1} = \frac{3}{2n^3\pi^3}.$$

4. Analysis of the Associated Matrices and Determinants. We now turn to the most difficult section of this paper, namely the analysis of the associated Hankel matrices occurring in (2.2) and (2.6). A well-known and well-analyzed example of a Hankel matrix is the Hilbert matrix \mathbf{H}_n with general element $h_{jk} = (j+k-1)^{-1}$ ($j, k = 1, 2, \dots, n$). Both the determinant $\det \mathbf{H}_n$ and the inverse matrix \mathbf{H}_n^{-1} are exactly known. (See Savage and Lukacs [5].) The Hilbert matrix has a settled bad reputation in regard to numerical difficulties. The matrices occurring in (2.2) and (2.6) are no better.

Consider the finite square Hankel matrix $\mathbf{D}_{n,s}$ of order $(n+1)$ defined by

$$(4.1) \quad \mathbf{D}_{n,s} = \begin{pmatrix} u_{0,s} & u_{0,s+1} & \cdots & u_{0,s+n} \\ u_{1,s} & u_{1,s+1} & \cdots & u_{1,s+n} \\ u_{2,s} & u_{2,s+1} & \cdots & u_{2,s+n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n,s} & u_{n,s+1} & \cdots & u_{n,s+n} \end{pmatrix}; \quad s \in \{0, 1, 2, \dots\},$$

where

$$u_{i,j} = \int_0^1 [x(1-x)]^{i+j} dx = \frac{1}{(2i+2j+1) \binom{2i+2j}{i+j}}.$$

We wish to calculate explicitly $\mathbf{D}_{n,s}^{-1} = \{d_{j,k}\}$ ($j, k = 0, 1, \dots, n$) and $D_{n,s} = \text{Det}(\mathbf{D}_{n,s})$. This may be done noting that the Hankel matrix $\mathbf{D}_{n,s}$ is also a moment matrix. To see this we rewrite the coefficients $u_{i,j}$ in the following way, putting $x(1-x) = z$,

$$(4.2) \quad u_{i,j} = \int_0^1 [x(1-x)]^{i+j} dx = 2 \int_0^{1/2} [x(1-x)]^{i+j} dx$$

$$= \int_0^{1/4} \frac{z^{i+j-s} z^s}{(\frac{1}{4}-z)^{1/2}} dz.$$

The coefficients $u_{i,j}$ may thus be interpreted as moments belonging to the weight function $w(z) = z^s(\frac{1}{4}-z)^{-1/2}$; $z \in [0, \frac{1}{4}]$, $i+j \geq s$.

Let $Q_{n,s}(x)$ and $q_{n,s}(x)$ be the associated orthogonal and orthonormal polynomials defined by the determinants (see, e.g., Cramér [3, p. 132])

$$(4.3) \quad Q_{n,s}(x) = \begin{vmatrix} u_{0,s} & u_{0,s+1} & \cdots & u_{0,s+n} \\ u_{1,s} & u_{1,s+1} & \cdots & u_{1,s+n} \\ \vdots & & & \vdots \\ u_{n-1,s} & & \cdots & u_{n-1,s+n} \\ 1 & x & & x^n \end{vmatrix},$$

$$(4.4) \quad q_{n,s}(x) = \frac{Q_{n,s}(x)}{(D_{n,s}D_{n-1,s})^{1/2}}; \quad D_{n,s} = \text{Det}(\mathbf{D}_{n,s}).$$

The polynomials $Q_{n,s}(x)$ and $q_{n,s}(x)$ satisfy the following equations, where $i \neq j$,

$$(4.5) \quad 0 = \int_0^{1/4} Q_{i,s}(x)Q_{j,s}(x) \frac{x^s}{(\frac{1}{4}-x)^{1/2}} dx = \int_0^1 Q_{i,s}\left(\frac{x}{4}\right)Q_{j,s}\left(\frac{x}{4}\right) \frac{x^s}{(1-x)^{1/2}} dx.$$

$$(4.6) \quad 1 = \int_0^{1/4} q_{i,s}^2(x) \frac{x^s}{(\frac{1}{4}-x)^{1/2}} dx = \frac{1}{2^{2s+1}} \int_0^1 q_{i,s}^2\left(\frac{x}{4}\right) \frac{x^s}{(1-x)^{1/2}} dx.$$

The orthogonal polynomials associated with the weight function $w(x) = (1-x)^{p-q}x^{q-1}$ are known as Jacobi polynomials and are denoted by $G_n(p, q, x)$. They may be standardized so that the coefficients of x^n equal 1 [1, p. 774]. The explicit expression is then

$$(4.7) \quad G_n(p, q, x) = \frac{\Gamma(q+n)}{\Gamma(p+2n)} \sum_{m=0}^n (-1)^m \binom{n}{m} \frac{\Gamma(p+2n-m)}{\Gamma(q+n-m)} x^{n-m}.$$

The normalizing constant h_n is in this case determined by

$$(4.8) \quad h_n = \int_0^1 G_n^2(p, q, x)w(x) dx = \frac{n!\Gamma(n+q)\Gamma(n+p)\Gamma(n+p-q+1)}{(2n+p)\Gamma^2(2n+p)}.$$

Using (4.5) and (4.6), we may identify the polynomials and get

$$(4.9) \quad G_n\left(s + \frac{1}{2}, s + 1, x\right) = \frac{4^n}{D_{n-1,s}} Q_{n,s}\left(\frac{x}{4}\right),$$

$$(4.10) \quad \frac{1}{2^{s+1/2}} q_{n,s}\left(\frac{x}{4}\right) = \frac{Q_{n,s}(x/4)}{2^{s+1/2}(D_{n,s}D_{n-1,s})^{1/2}} = \frac{1}{h_n^{1/2}} G_n\left(s + \frac{1}{2}, s + 1, x\right).$$

An identification of the coefficients of x^n in (4.10) yields

$$(4.11) \quad \frac{1}{2^{2s+1}4^{2n}} \frac{D_{n-1,s}}{D_{n,s}} = \frac{(2n+s+\frac{1}{2})\Gamma^2(2n+s+\frac{1}{2})}{n!\Gamma(n+s+1)\Gamma(n+s+\frac{1}{2})\Gamma(n+\frac{1}{2})}.$$

We note the special case $s = 2$, which corresponds to $q = 0$ in (2.1) ($w(x) = x^q(1-x)^q = 1$). We get, after a little algebra,

$$(4.12) \quad D_{n,2} = D_{n-1,2} \frac{1}{(2n+3) \binom{4n+5}{2n+2} \binom{4n+4}{2n}},$$

i.e.,

$$(4.13) \quad D_{n,2}^{-1} = \prod_{r=0}^n (2r + 3) \binom{4r + 5}{2r + 2} \binom{4r + 4}{2r}.$$

We will now show how to find explicit expressions of the elements of $\mathbf{D}_{n,s}^{-1}$. Of special importance is the case $s = 2$, which we will use to find approximations of $\sin(\pi x)$ when $x \in [0, 1]$; see Section 6.

Some of the elements of $\mathbf{D}_{n,s}^{-1}$ may be obtained quite easily. Let $D_{n,s}^{i,j}$ be the first minor obtained by deleting the i th row and the j th column of the determinant $D_{n,s}$. We expand the polynomial $Q_{n,s}(x)$ (defined by the determinant in (4.3)) with respect to the last row, i.e., with respect to powers of x . Then

$$(4.14) \quad Q_{n,s}(x) = \sum_{v=0}^n (-1)^{n+v} D_{n,s}^{n+1,v+1} x^v; \quad D_{n,s}^{n+1,n+1} = D_{n-1,s}.$$

Using (4.9) and (4.11), we get, upon identifying corresponding powers of x ,

$$(4.15) \quad \frac{D_{n,s}^{n+1,k+1}}{D_{n,s}} = \frac{(4n + 2s + 1)4^{s+k+n}\Gamma(2n + s + \frac{1}{2})\Gamma(s + \frac{1}{2} + n + k)}{n!\Gamma(n + s + \frac{1}{2})\Gamma(n + \frac{1}{2})\Gamma(s + 1 + k)} \binom{n}{k}.$$

Now define

$$(4.16) \quad \begin{pmatrix} q_{0,s}(x) \\ q_{1,s}(x) \\ q_{2,s}(x) \\ \vdots \\ q_{n,s}(x) \end{pmatrix} = \begin{pmatrix} z_{00} & 0 & 0 & \cdots & 0 \\ z_{10} & z_{11} & 0 & \cdots & 0 \\ z_{20} & z_{21} & z_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ z_{n0} & z_{n1} & z_{n2} & \cdots & z_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{pmatrix}.$$

Putting

$$(4.17) \quad \mathbf{L}_{n,s}^{-1} = \{z_{ij}\},$$

we may use the Choleski factorization [6, p. 8] and get the important relation

$$(4.18) \quad \mathbf{D}_{n,s}^{-1} = \mathbf{L}_{n,s}^{-T} \mathbf{L}_{n,s}^{-1}.$$

Let the general element of $\mathbf{D}_{n,s}^{-1}$ be $d_{j,k}$ ($j, k = 0, 1, 2, \dots, n$). Then

$$(4.19) \quad d_{j,k} = \sum_{r=\max(j,k)}^n z_{rj} z_{rk}.$$

We therefore have to determine the elements z_{rk} . This may be done using (4.4) and (4.15). After some calculation we get

$$(4.20) \quad z_{nk} = \left(\frac{n!\Gamma(n + s + 1)(4n + 2s + 1)}{\Gamma(n + s + \frac{1}{2})\Gamma(n + \frac{1}{2})} \right)^{1/2} \frac{2^{2k+s}\Gamma(s + \frac{1}{2} + n + k)}{\Gamma(s + 1 + k)n!} \binom{n}{k} (-1)^{n+k}.$$

Inserted in (4.19), this yields

$$(4.21) \quad d_{j,k} = \sum_{r=\max(j,k)}^n \frac{\Gamma(r + s + 1)(4r + 2s + 1)4^{k+j+s}\Gamma(s + \frac{1}{2} + r + k)\Gamma(s + \frac{1}{2} + r + j)}{\Gamma(r + s + \frac{1}{2})\Gamma(r + \frac{1}{2})\Gamma(s + 1 + k)\Gamma(s + 1 + j)r!} \times \binom{r}{k} \binom{r}{j} (-1)^{j+k}.$$

For $s = 2$ the elements $d_{j,k}$ (and $d_{k,j}$) are given by

$$(4.22) \quad d_{j,k} = (-1)^{j+k} \sum_{r=\max(j,k)}^n (4r+5) \times \frac{\binom{r+k+2}{k+2} \binom{r+j+2}{j+2} \binom{2r+2k+4}{r+k+2} \binom{2r+2j+4}{r+j+2} \binom{r}{k} \binom{r}{j}}{\binom{2r}{r} \binom{2r+4}{r+2}}.$$

(Accurate values of $D_{n,2}^{-1}$ ($n = 0, 1, \dots, 6$) and $\mathbf{D}_{n,2}^{-1}$ ($n = 0, 1, \dots, 9$) are given in [7, pp. 30–34].) In Table I we give the matrices $\mathbf{D}_{n,2}^{-1}$ for $n = 0, 1, \dots, 5$.

TABLE I
Table of the matrices $\mathbf{D}_{n,2}^{-1}$

$$\mathbf{D}_{5,2} = \begin{pmatrix} \frac{1}{30} & \frac{1}{140} & \frac{1}{630} & \frac{1}{2772} & \frac{1}{12012} & \frac{1}{51480} \\ \frac{1}{140} & \frac{1}{630} & \frac{1}{2772} & \frac{1}{12012} & \frac{1}{51480} & \frac{1}{218790} \\ \frac{1}{630} & \frac{1}{2772} & \frac{1}{12012} & \frac{1}{51480} & \frac{1}{218790} & \frac{1}{923780} \\ \frac{1}{2772} & \frac{1}{12012} & \frac{1}{51480} & \frac{1}{218790} & \frac{1}{923780} & \frac{1}{3879876} \\ \frac{1}{12012} & \frac{1}{51480} & \frac{1}{218790} & \frac{1}{923780} & \frac{1}{3879876} & \frac{1}{16224936} \\ \frac{1}{51480} & \frac{1}{218790} & \frac{1}{923780} & \frac{1}{3879876} & \frac{1}{16224936} & \frac{1}{67603900} \end{pmatrix}$$

$$\mathbf{D}_{0,2}^{-1} = (30)$$

$$\mathbf{D}_{1,2}^{-1} = \begin{pmatrix} 840 & -3780 \\ -3780 & 17640 \end{pmatrix}$$

$$\mathbf{D}_{2,2}^{-1} = \begin{pmatrix} 6300 & -69300 & 180180 \\ -69300 & 803880 & -2162160 \\ 180180 & -2162160 & 5945940 \end{pmatrix}$$

$$\mathbf{D}_{3,2}^{-1} = \begin{pmatrix} 27720 & -540540 & 3243240 & -6126120 \\ -540540 & 11171160 & -69549480 & 134774640 \\ 3243240 & -69549480 & 443963520 & -876035160 \\ -6126120 & 134774640 & -876035160 & 1752070320 \end{pmatrix}$$

$$\mathbf{D}_{4,2}^{-1} = \begin{pmatrix} 90090 & -2702700 & 27567540 & -116396280 & 174594420 \\ -2702700 & 86126040 & -912791880 & 3957473520 & -6052606560 \\ 27567540 & -912791880 & 9930440520 & -43881397560 & 68091823800 \\ -116396280 & 3957473520 & -43881397560 & 196709713200 & -308682934560 \\ 174594420 & -6052606560 & 68091823800 & -308682934560 & 488747979720 \end{pmatrix}$$

$$\mathbf{D}_{5,2}^{-1} = \begin{pmatrix} 240240 & -10210200 & 155195040 & -1086365280 & 3569485920 & -4461857400 \\ -10210200 & 461501040 & -7294166880 & 52455923520 & -175797181560 & 223092870000 \\ 155195040 & -7294166880 & 118413815520 & -868355047560 & 2953749598800 & -3792578790000 \\ -1086365280 & 52455923520 & -868355047560 & 6462709453200 & -22239682024560 & 28823598804000 \\ 3569485920 & -175797181560 & 2953749598800 & -22239682024560 & 77247244794720 & -100882595814000 \\ -4461857400 & 223092870000 & -3792578790000 & 28823598804000 & -100882595814000 & 132588554498400 \end{pmatrix}$$

In a similar way we consider the matrix $\mathbf{E}_{n,t}$ defined by

$$(4.23) \quad \mathbf{E}_{n,t} = \begin{pmatrix} U_{0,t} & U_{0,t+1} & \cdots & U_{0,t+n} \\ U_{1,t} & U_{1,t+1} & \cdots & U_{1,t+n} \\ U_{2,t} & U_{2,t+1} & \cdots & U_{2,t+n} \\ \vdots & \vdots & \cdots & \vdots \\ U_{n,t} & U_{n,t+1} & \cdots & U_{n,t+n} \end{pmatrix}, \quad t \in \{0, 1, \dots\},$$

where

$$U_{i,j} = \int_0^1 (1-2x)^2 [x(1-x)]^{i+j} dx = \frac{1}{(2i+2j+3)(2i+2j+1) \binom{2i+2j}{i+j}}.$$

Let the general element of $E_{n,2}^{-1}$ be $e_{j,k}$ ($j, k = 0, 1, 2, \dots, n$). We may then prove, in a similar way as for $D_{n,2}^{-1}$, that

$$(4.24) \quad e_{j,k} = (-1)^{j+k} \sum_{r=\max(j,k)}^n \frac{(4r+7)(r+1) \binom{2r+2k+6}{r+k+3} \binom{2r+2j+6}{r+j+3} \binom{r+k+3}{k+2} \binom{r+j+3}{j+2} \binom{r}{k} \binom{r}{j}}{(r+3) \binom{2r+6}{r+3} \binom{2r+2}{r+1}}$$

(Accurate values of $E_{n,2}^{-1}$ ($n = 0, 1, \dots, 4$) and $E_{n,2}^{-1}$ ($n = 0, 1, \dots, 9$) are given in [7, pp. 35–40].)

In Table II we give the matrices $E_{n,2}^{-1}$ for $n = 0, 1, \dots, 5$. Those interested in details may read Wrigge, Fransen and Borenius [6, pp. 20–22, 26–28].

TABLE II
Table of the matrices $E_{n,2}^{-1}$

$$E_{5,2} = \begin{pmatrix} \frac{1}{210} & \frac{1}{1260} & \frac{1}{6930} & \frac{1}{36036} & \frac{1}{180180} & \frac{1}{875160} \\ \frac{1}{1260} & \frac{1}{6930} & \frac{1}{36036} & \frac{1}{180180} & \frac{1}{875160} & \frac{1}{4157010} \\ \frac{1}{6930} & \frac{1}{36036} & \frac{1}{180180} & \frac{1}{875160} & \frac{1}{4157010} & \frac{1}{19399380} \\ \frac{1}{36036} & \frac{1}{180180} & \frac{1}{875160} & \frac{1}{4157010} & \frac{1}{19399380} & \frac{1}{89237148} \\ \frac{1}{180180} & \frac{1}{875160} & \frac{1}{4157010} & \frac{1}{19399380} & \frac{1}{89237148} & \frac{1}{405623400} \\ \frac{1}{875160} & \frac{1}{4157010} & \frac{1}{19399380} & \frac{1}{89237148} & \frac{1}{405623400} & \frac{1}{1825305300} \end{pmatrix}$$

$$E_{0,2}^{-1} = (210)$$

$$E_{1,2}^{-1} = \begin{pmatrix} 2520 & -13860 \\ -13860 & 83160 \end{pmatrix}$$

$$E_{2,2}^{-1} = \begin{pmatrix} 13860 & -180180 & 540540 \\ -180180 & 2522520 & -7927920 \\ 540540 & -7927920 & 25765740 \end{pmatrix}$$

$$E_{3,2}^{-1} = \begin{pmatrix} 51480 & -1158300 & 7876440 & -16628040 \\ -1158300 & 27953640 & -198661320 & 432329040 \\ 7876440 & -198661320 & 1456266240 & -3242467800 \\ -16628040 & 432329040 & -3242467800 & 7349593680 \end{pmatrix}$$

$$E_{4,2}^{-1} = \begin{pmatrix} 150150 & -5105100 & 58198140 & -271591320 & 446185740 \\ -5105100 & 185825640 & -2211529320 & 10630860240 & -17847429600 \\ 58198140 & -2211529320 & 27120333240 & -133273740600 & 227554727400 \\ -271591320 & 10630860240 & -133273740600 & 666174709200 & -1152943952160 \\ 446185740 & -17847429600 & 227554727400 & -1152943952160 & 2017651916280 \end{pmatrix}$$

$$E_{5,2}^{-1} = \begin{pmatrix} 371280 & -17635800 & 296281440 & -2271491040 & 8112468000 & -10951831800 \\ -17635800 & 895898640 & -15702916320 & 123958511040 & -452270091000 & 620603802000 \\ 296281440 & -15702916320 & 283456686240 & -2286499105800 & 8481585294000 & -11791472238000 \\ -2271491040 & 123958511040 & -2286499105800 & 18753267776880 & -70486800711600 & 99048366799200 \\ 8112468000 & -452270091000 & 8481585294000 & -70486800711600 & 267797436160800 & -379685406063600 \\ -10951831800 & 620603802000 & -11791472238000 & 99048366799200 & -379685406063600 & 542407722948000 \end{pmatrix}$$

The result of this section may be summed up in

THEOREM 4. Let $u_{i,j} = \int_0^1 [x(1-x)]^{i+j} dx$ and $U_{i,j} = \int_0^1 (1-2x)^2 [x(1-x)]^{i+j} dx$. Define the matrices $D_{n,s} = \{(u_{i,j})\}$, $i = 0, 1, 2, \dots, n$; $j = s, s+1, \dots, s+n$, and $E_{n,t} = \{(U_{i,j})\}$, $i = 0, 1, \dots, n$; $j = t, t+1, \dots, t+n$; $s, t \in \{0, 1, 2, \dots, n\}$. Let $d_{j,k}^{n,s}$ and

$e_{j,k}^{n,t}$ be the general elements of $\mathbf{D}_{n,s}^{-1}$ and $\mathbf{E}_{n,t}^{-1}$, with $j, k = 0, 1, 2, \dots, n$. Then

$$d_{j,k}^{n,s} = \sum_{r=\max(j,k)}^n \frac{\Gamma(r+s+1)(4r+2s+1)4^{k+j+s}\Gamma(s+\frac{1}{2}+r+k)\Gamma(s+\frac{1}{2}+r+j)}{\Gamma(r+s+\frac{1}{2})\Gamma(r+\frac{1}{2})\Gamma(s+1+k)\Gamma(s+1+j)r!}$$

$$\times \binom{r}{k} \binom{r}{j} (-1)^{k+j},$$

$$e_{j,k}^{n,t} = \sum_{r=\max(j,k)}^n \frac{\Gamma(r+t+1)(4r+2t+3)4^{k+j+t}\Gamma(t+3/2+r+k)\Gamma(t+3/2+r+j)}{\Gamma(r+t+3/2)\Gamma(r+3/2)\Gamma(t+1+k)\Gamma(t+1+j)r!}$$

$$\times \binom{r}{k} \binom{r}{j} (-1)^{k+j}.$$

Define the determinants $D_{n,s} = \text{Det}(\mathbf{D}_{n,s})$ and $E_{n,t} = \text{Det}(\mathbf{E}_{n,t})$. Then

$$D_{n,s} = \frac{n!\Gamma(n+s+1)\Gamma(n+s+\frac{1}{2})\Gamma(n+\frac{1}{2})}{2^{2s+1}4^{2n}(2n+s+\frac{1}{2})\Gamma^2(2n+s+\frac{1}{2})} D_{n-1,s},$$

$$E_{n,t} = \frac{n!\Gamma(n+t+1)\Gamma(n+t+3/2)\Gamma(n+3/2)}{2^{2t+1}4^{2n}(2n+t+3/2)\Gamma^2(2n+t+3/2)} E_{n-1,t}.$$

5. Calculation of Determinants and Inverse Matrices. The determinants $D_{n,2}$ and $E_{n,2}$ are calculated using the formulae

$$(5.1) \quad D_{n,2}^{-1} = \prod_{r=0}^n (2r+3) \binom{4r+5}{2r+2} \binom{4r+4}{2r},$$

$$(5.2) \quad E_{n,2}^{-1} = \prod_{r=0}^n (r+2) \binom{4r+7}{2r+2} \binom{4r+5}{2r+1}.$$

See [6, p. 22].

When calculating the inverse matrices we may choose one of several methods. We may, e.g., use a recursion technique (i.e., Householder's method).

Let $\mathbf{C}_{n,2}$ denote any one of the square matrices $\mathbf{D}_{n,2}$ and $\mathbf{E}_{n,2}$ of order $(n+1)$. We start making a partition of $\mathbf{C}_{n,2}$, i.e.,

$$(5.3) \quad \mathbf{C}_{n,2} = \begin{bmatrix} \mathbf{C}_{n-1,2} & \hat{c}_n \\ \hat{c}_n^T & \gamma_n \end{bmatrix}.$$

Here \hat{c}_n is a column vector (of order n) and γ_n a rational number. Denoting $C_{n,2} = \det \mathbf{C}_{n,2}$, we finally get the recursion formula

$$(5.4) \quad \mathbf{C}_{n,2}^{-1} = \begin{bmatrix} \mathbf{C}_{n-1,2}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \frac{\mathbf{C}_{n-1,2}}{C_{n,2}} \begin{bmatrix} -\mathbf{C}_{n-1,2}^{-1} \hat{c}_n \\ 1 \end{bmatrix} \begin{bmatrix} -\hat{c}_n^T \mathbf{C}_{n-1,2}^{-1} & 1 \end{bmatrix}.$$

(See e.g. F. Ayres [2, pp. 56–58].)

Thus we may start with $\mathbf{C}_{1,2}^{-1} = \mathbf{D}_{1,2}^{-1}$ or $\mathbf{E}_{1,2}^{-1}$. A repeated use of (5.4) then yields the matrices $\mathbf{D}_{k,2}^{-1}$ and $\mathbf{E}_{k,2}^{-1}$ ($k = 2, 3, \dots, n$). (See [7, pp. 29–40], where $n = 9$.)

Alternatively, we may also use the formulae (4.22) and (4.24) which give explicit expressions of the general elements of $\mathbf{D}_{n,2}^{-1}$ and $\mathbf{E}_{n,2}^{-1}$.

But we can do better. The drawback of the recursion method given by (5.4) is that to calculate, e.g., $\mathbf{D}_{6,2}^{-1}$ we have to calculate all $\mathbf{D}_{k,2}^{-1}$ with $k \leq 5$. Therefore a direct method is to be preferred. Such a method may be obtained by a clever use of the Christoffel-Darboux formula [1, p. 785], which applied to the orthonormal functions

$q_{n,s}(x)$ (Eq. (4.4)) may be stated as

$$(5.5) \quad \sum_{m=0}^n q_{m,s}(x)q_{m,s}(y) = \frac{z_{n,n}}{z_{n+1,n+1}} \frac{[q_{n+1,s}(x)q_{n,s}(y) - q_{n,s}(x)q_{n+1,s}(y)]}{x - y}.$$

Multiplying both sides of (5.5) with $(x - y)$ and identifying the coefficients of $x^j y^k$, we get

$$(5.6) \quad d_{j-1,k}^{n,s} - d_{j,k-1}^{n,s} = \frac{z_{n,n}}{z_{n+1,n+1}} (z_{n+1,j}z_{n,k} - z_{n,j}z_{n+1,k}),$$

where we have written $d_{j,k} = d_{j,k}^{n,s}$ to point out the dependence of the parameters n and s . Putting $s = 2$, we get

$$(5.7) \quad d_{j-1,k} - d_{j,k-1} = (-1)^{k+j} \times \frac{(k-j)(n+1)(n+3) \binom{n}{j} \binom{n}{k} \binom{2n+2j+4}{n+j+2} \binom{2n+2k+4}{n+k+2} \binom{n+j+2}{n} \binom{n+k+2}{n}}{2(n+1-j)(n+1-k) \binom{2n+4}{n+2} \binom{2n}{n}}.$$

The corresponding formula for $e_{j,k}$ (with $t = 2$) is

$$(5.8) \quad e_{j-1,k} - e_{j,k-1} = (-1)^{k+j} \times \frac{(k-j)(n+1)^2 \binom{n}{j} \binom{n}{k} \binom{2n+2j+6}{n+j+3} \binom{2n+2k+6}{n+k+3} \binom{n+j+3}{j+2} \binom{n+k+3}{k+2}}{2(n+1-j)(n+1-k) \binom{2n+6}{n+3} \binom{2n+2}{n+1}}.$$

The calculation of $d_{j,k}$ may be given by

Algorithm D. Let the general matrix element of $\mathbf{D}_{n,2}^{-1}$ be $d_{j,k}$ ($j, k = 0, 1, 2, \dots, n$). The steps in the calculation of $d_{j,k}$ are then as follows:

I.

$$d_{n,k+1} = -\frac{2(n-k)(2n+2k+5)}{(k+1)(k+3)} d_{n,k}; \quad k = 0, 1, 2, \dots, n-1.$$

Starting values are given by

$$d_{n,0} = (-1)^n \frac{(4n+5) \binom{4n+4}{2n+2} \binom{2n+2}{n+2} \binom{n+2}{2}}{\binom{2n}{n}}.$$

II. Calculate the elements of the lower right-half of the matrix using

$$d_{j-1,k} - d_{j,k-1} = (-1)^{k+j} \times \frac{(k-j)(n+1)(n+3) \binom{n}{j} \binom{n}{k} \binom{2n+2j+4}{n+j+2} \binom{2n+2k+4}{n+k+2} \binom{n+j+2}{n} \binom{n+k+2}{n}}{2(n+1-j)(n+1-k) \binom{2n+4}{n+2} \binom{2n}{n}}.$$

III. Calculate $d_{j,0} = d_{0,j}$ ($j = 0, 1, 2, \dots, n$) using

$$d_{j,0} = (-1)^j \frac{\prod_{i=-1}^{j+1} (2n+3+2i)(2n+4-2i)}{4(j+3)!j!}.$$

IV. Use step II to calculate the elements of the upper left-half of the matrix.

V. In all steps we make use of the symmetry $d_{j,k} = d_{k,j}$.

A similar scheme of calculation exists for the elements $e_{j,k}$ of the matrix $\mathbf{E}_{n,2}^{-1}$, viz,

Algorithm E. Let the general matrix element of $\mathbf{E}_{n,2}^{-1}$ be $e_{j,k}$ ($j, k = 0, 1, 2, \dots, n$). The steps in the calculation of $e_{j,k}$ are then as follows:

I.

$$e_{n,k+1} = -\frac{2(n-k)(2n+2k+7)}{(k+1)(k+3)} e_{n,k}, \quad k = 0, 1, 2, \dots, n-1.$$

Starting values are given by

$$e_{n,0} = (-1)^n \frac{(4n+7) \binom{4n+6}{2n+3} \binom{2n+3}{n+3} \binom{n+3}{2}}{\binom{2n+2}{n+1}}.$$

II. Calculate the elements of the lower right-half of the matrix using

$$e_{j-1,k} - e_{j,k-1} = (-1)^{k+j} \times \frac{(k-j)(n+1)^2 \binom{n}{j} \binom{n}{k} \binom{2n+2j+6}{n+j+3} \binom{2n+2k+6}{n+k+3} \binom{n+j+3}{j+2} \binom{n+k+3}{k+2}}{2(n+1-j)(n+1-k) \binom{2n+6}{n+3} \binom{2n+2}{n+1}}.$$

III. Calculate $e_{j,0} = e_{0,j}$ using

$$e_{j,0} = (-1)^j \frac{\prod_{i=-1}^{j+1} (2n+5+2i)(2n+4-2i)}{4j!(j+3)!}.$$

IV. Use step II to calculate the elements of the upper left-half of the matrix.

V. In all steps we make use of the symmetry $e_{j,k} = e_{k,j}$.

We managed to find still another method to calculate the elements of $\mathbf{D}_{n,2}^{-1}$ and $\mathbf{E}_{n,2}^{-1}$ making use of a special partition technique. The great advantage of this method is that we can describe both cases with one formula using an idempotent number β (i.e. $\beta^2 = \beta$). This result may be summarized in

THEOREM 5. Let $\mathbf{C}_{n,2}^{-1}$ denote any one of the matrices $\mathbf{D}_{n,2}^{-1}$ and $\mathbf{E}_{n,2}^{-1}$, and let the general element of $\mathbf{C}_{n,2}^{-1}$ be $y_{j,k}$ ($j, k = 0, 1, 2, \dots, n$). Then

$$y_{j,k} = (-1)^{j+k} \frac{\prod_{i=-1}^{j+1} (2n+3+2\beta+2i)(2n+4-2i)}{2(j+k+1)!(j+k+3)!k!(k+2)!} x_{j,k},$$

where

$$x_{j,k} = \sum_{v=0}^k R_v T_v (j-k+1+2v) \times \prod_{i=v+1}^k (j+1+i)(j+3+i) \prod_{i=0}^{v-1} (k-i)(k+2-i)$$

and

$$R_m = \prod_{i=j+2}^{j+1+m} (2n+3+2\beta+2i)(2n+4-2i);$$

$$T_m = \prod_{i=2}^{k+1-m} (2n+1+2\beta+2i)(2n+6-2i).$$

The case $\beta = 0$ corresponds to $\mathbf{D}_{n,2}^{-1}$, and the case $\beta = 1$ corresponds to $\mathbf{E}_{n,2}^{-1}$.

(The products in Theorem 5 are interpreted as 1 if the upper index is less than the lower.)

6. Application. In this section we will apply our method to the case $f(x) = \sin(\pi x)$, $f(x) = \cos(2\pi x)$ and $h(x) = \sin(2\pi x)$, $h(x) = \cos(\pi x)$; $x \in [0, 1]$. (In [7, pp. 23–26] we successfully approximated the Jacobian elliptic function $\operatorname{sn}(2K(k)x, k)$ for $0 \leq k \leq 0.25$.) Consider the expansions

$$(6.1) \quad \sin(\pi x) = \sum_{n=1}^{\infty} a_n [x(1-x)]^n;$$

(see Lyusternik et al. [4, p. 82]) and

$$(6.2) \quad \sin(2\pi x) = (1-2x) \sum_{n=1}^{\infty} b_n [x(1-x)]^n.$$

We will also need the coefficients A_n defined by

$$(6.3) \quad \sin^2(\pi x) = \sum_{n=2}^{\infty} A_n [x(1-x)]^n.$$

Since $d^2 \sin(\pi x)/dx^2 = -\pi^2 \sin(\pi x)$, we get, differentiating (6.1) twice and identifying the coefficients of $[x(1-x)]^n$, the recurrence relation

$$(6.4) \quad (n+2)(n+1)a_{n+2} = (2n+2)(2n+1)a_{n+1} - \pi^2 a_n; \quad a_1 = a_2 = \pi.$$

Exact and approximate values of a_i ($i = 1(1)10$ and $i = 1(1)20$) are given in [6, p. 35] and [7, p. 41]. Note that a_n converges fast towards zero. We have, e.g., $a_{20} \approx 9.1 \times 10^{-29}$. We solve the recurrence relation (6.4) using (1.9). Thus

$$(6.5) \quad a_{2n+1} = \frac{\pi}{2n+1} \sum_{v=0}^n \binom{2n+2v}{2v} \frac{(-1)^{n-v} \pi^{2n-2v}}{(2n-2v)!},$$

and

$$(6.6) \quad a_{2n+2} = \frac{\pi}{2n+2} \sum_{v=0}^n \binom{2n+2v+2}{2v+1} \frac{(-1)^{n-v} \pi^{2n-2v}}{(2n-2v)!}.$$

Using (1.3) we obtain

$$(6.7) \quad a_n = (-1)^n \sum_{k=n}^{\infty} \pi^{2k} \binom{k}{n} \frac{(-1)^k 4^{n-k}}{(2k)!}.$$

(1.6) yields the alternative form

$$(6.8) \quad a_n = \frac{(-1)^n}{2 \cdot n!} \sum_{k=n}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k)(2k-1) \cdots (2k-n+1)(2k-2n+1)!}.$$

To obtain a recurrence relation for the coefficients b_n we use the coefficients A_n defined by (6.3). We note that

$$\frac{d^2}{dx^2} \sin^2(\pi x) = 2\pi^2 - 4\pi^2 \sin^2(\pi x).$$

Then

$$(6.9) \quad (n+2)(n+1)A_{n+2} = (2n+2)(2n+1)A_{n+1} - 4\pi^2 A_n; \\ A_2 = \pi^2, \quad A_3 = 2\pi^2.$$

Differentiating (6.3) with respect to x and identifying coefficients, we get

$$(6.10) \quad b_{n-1} = \frac{n}{\pi} A_n.$$

b_n thus satisfies the recurrence relation

$$(6.11) \quad n(n+1)b_{n+1} = 2n(2n+1)b_n - 4\pi^2 b_{n-1}; \quad b_1 = 2\pi, \quad b_2 = 6\pi.$$

Using (1.8), we get

$$(6.12) \quad A_{2n+1} = \frac{\pi}{2n+1} \sum_{v=1}^n \binom{2n+2v-1}{2v-1} \frac{(-1)^{n-v} (2\pi)^{2n-2v+1}}{(2n-2v+1)!}$$

and

$$(6.13) \quad A_{2n+2} = \frac{\pi}{2n+2} \sum_{v=0}^n \binom{2n+2v+1}{2v} \frac{(-1)^{n-v} (2\pi)^{2n-2v+1}}{(2n-2v+1)!}.$$

From (6.10), (6.12), and (6.13) we get finite expressions for b_{2n} and b_{2n+1} .

The expansions for $\cos(\pi x)$ and $\cos(2\pi x)$ are deduced in a similar way. We put

$$(6.14) \quad \cos(\pi x) = (1-2x) \sum_{n=0}^{\infty} \beta_n [x(1-x)]^n$$

and

$$(6.15) \quad \cos(2\pi x) = \sum_{n=0}^{\infty} \alpha_n [x(1-x)]^n.$$

Differentiating (6.1) with respect to x , we get

$$(6.16) \quad \beta_{n-1} = n\alpha_n/\pi.$$

Using (6.4), it is easy to prove that β_n satisfies the recurrence relation

$$(6.17) \quad n(n+1)\beta_{n+1} = 2n(2n+1)\beta_n - \pi^2 \beta_{n-1}; \quad \beta_0 = 1, \quad \beta_1 = 2.$$

A differentiation of (6.15) with respect to x yields

$$(6.18) \quad b_{n-1} = -n\alpha_n/2\pi.$$

Using (6.11), we get the recurrence relation

$$(6.19) \quad (n+2)(n+1)\alpha_{n+2} = (2n+2)(2n+1)\alpha_{n+1} - 4\pi^2 \alpha_n; \quad \alpha_0 = 1, \quad \alpha_1 = 0.$$

We next turn our interest to approximations in L_2 , i.e., we consider $\min D_1^2(w)$ and $\min D_2^2(w)$, where

$$(6.20) \quad D_1^2(w) = \int_0^1 \left\{ \sin(\pi x) - \sum_{n=1}^k c_{n,k} [x(1-x)]^n \right\}^2 dx$$

and

$$(6.21) \quad D_2^2(w) = \int_0^1 \left\{ \sin(2\pi x) - (1-2x) \sum_{n=1}^k C_{n,k} [x(1-x)]^n \right\}^2 dx.$$

The associated integrals

$$I_j = \int_0^1 \sin(\pi x) [x(1-x)]^j dx$$

and

$$I'_j = \int_0^1 (1 - 2x) \sin(2\pi x) [x(1 - x)]^j dx$$

are calculated from the recurrence formulae

$$(6.22) \quad I_j = \frac{2j(2j - 1)}{\pi^2} I_{j-1} - \frac{j(j - 1)}{\pi^2} I_{j-2}; \quad I_0 = \frac{2}{\pi}, \quad I_1 = \frac{4}{\pi^3},$$

and

$$(6.23) \quad I'_j = \frac{2j(2j + 1)}{4\pi^2} I'_{j-1} - \frac{j(j - 1)}{4\pi^2} I'_{j-2}; \quad I'_0 = \frac{1}{\pi}, \quad I'_1 = \frac{3}{2\pi^3}.$$

Solving the equation systems (2.2) and (2.6) by inverting the matrices as described in Sections 4 and 5 we may formulate

THEOREM 6. *Let $x \in [0, 1]$, then we have the approximations*

$$\sin(\pi x) \approx \begin{cases} 3.141583993[x(1 - x)] + 3.141891945[x(1 - x)]^2 \\ + 1.112123058[x(1 - x)]^3 + 0.219850867[x(1 - x)]^4 \end{cases}$$

and

$$\sin(\pi x) \approx \begin{cases} 3.141592715257[x(1 - x)] + 3.141589575603[x(1 - x)]^2 \\ + 1.115524716287[x(1 - x)]^3 + 0.204430015076[x(1 - x)]^4 \\ + 0.024416348195[x(1 - x)]^5. \end{cases}$$

The absolute errors are less than 8×10^{-8} , respectively 4×10^{-10} .

The corresponding relations for $\sin(2\pi x)$ are formulated in

THEOREM 7. *Let $x \in [0, 1]$, then we have the approximations*

$$\sin(2\pi x) \approx \begin{cases} (1 - 2x)\{6.281856[x(1 - x)] + 18.902201[x(1 - x)]^2\} \\ + (1 - 2x)\{20.829857[x(1 - x)]^3 + 16.439719[x(1 - x)]^4\} \end{cases}$$

and

$$\sin(2\pi x) \approx \begin{cases} (1 - 2x)\{6.283217166[x(1 - x)] + 18.847760765[x(1 - x)]^2\} \\ + (1 - 2x)21.523970874[x(1 - x)]^3 \\ + (1 - 2x)\{12.922874461[x(1 - x)]^4 + 6.154478369[x(1 - x)]^5\}. \end{cases}$$

The absolute errors are less than 10^{-5} , respectively 2×10^{-7}

To carry through the corresponding approximations for $\cos(\pi x)$ and $\cos(2\pi x)$ we must calculate the integrals

$$(6.24) \quad I_j = \int_0^1 [\cos(2\pi x) - 1] [x(1 - x)]^j dx$$

and

$$(6.25) \quad I'_j = \int_0^1 (1 - 2x) [\cos(\pi x) - (1 - 2x)] [x(1 - x)]^j dx.$$

We introduce the complementary integrals J_j and J'_j defined by

$$(6.26) \quad J_j = \int_0^1 \cos(2\pi x)[x(1-x)]^j dx$$

and

$$(6.27) \quad J'_j = \int_0^1 (1-2x)\cos(\pi x)[x(1-x)]^j dx.$$

The integrals J_j and J'_j satisfy the recurrence relations

$$(6.28) \quad J_j = \frac{2j(2j-1)}{4\pi^2} J_{j-1} - \frac{j(j-1)}{4\pi^2} J_{j-2}$$

and

$$(6.29) \quad J'_j = \frac{2j(2j+1)}{\pi^2} J'_{j-1} - \frac{j(j-1)}{\pi^2} J'_{j-2}.$$

Starting values are given by

$$J_0 = 0, \quad J_1 = -\frac{1}{2\pi^2} \quad \text{and} \quad J'_0 = \frac{4}{\pi^2}, \quad J'_1 = \frac{24}{\pi^4} - \frac{2}{\pi^2}.$$

We finally get the following expressions for the integrals I_j and I'_j

$$(6.30) \quad I_j = J_j - \frac{1}{(2j+1)\binom{2j}{j}}; \quad I'_j = J'_j - \frac{1}{(2j+3)(2j+1)\binom{2j}{j}}.$$

The approximations for $\cos(\pi x)$ are given in

THEOREM 8. *Let $x \in [0, 1]$, then*

$$\cos(\pi x) \approx \left\{ \begin{aligned} &(1-2x)\{1 + 1.999999230[x(1-x)] + 1.065228532[x(1-x)]^2\} \\ &+ (1-2x)\{0.260400939[x(1-x)]^3 + 0.038640515[x(1-x)]^4\} \end{aligned} \right\}$$

and

$$\begin{aligned} &\cos(\pi x) \\ &\approx \left\{ \begin{aligned} &(1-2x)\{1 + 2.000000004489[x(1-x)] + 1.065197545425[x(1-x)]^2\} \\ &+ (1-2x)\{0.260796014285[x(1-x)]^3 + 0.036638801083[x(1-x)]^4\} \\ &+ (1-2x)0.003502999395[x(1-x)]^5. \end{aligned} \right. \end{aligned}$$

The absolute errors are less than 0.6×10^{-8} , respectively 0.3×10^{-10} .

We have found the approximations of $\cos(2\pi x)$ of less value and prefer to compute $\cos(2\pi x)$ from the formula $\cos(2\pi x) = 2 \cos^2(\pi x) - 1$, thereby using Theorem 8 to compute $\cos(\pi x)$.

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