

Bounds on Iterated Coerror Functions and Their Ratios

By D. E. Amos

Abstract. Upper and lower bounds on $y_n = i^n \operatorname{erfc}(x)$ and $r_n = y_n/y_{n-1}$, $n \geq 1$, $-\infty < x < \infty$, are established in terms of elementary functions. Numerical procedures for refining these bounds are presented so that r_n and y_k , $k = 0, 1, \dots, n$, can be computed to a specified accuracy. Some relations establishing bounds on r'_n and r''_n are also derived.

Simple Bounds. Let $y_n(x) = i^n \operatorname{erfc}(x)$, $n = -1, 0, 1, \dots$. The basic inequality (see (31))

$$(1) \quad P_n(x) = y_{n-2}(x)y_n(x) - y_{n-1}^2(x) < 0, \quad n \geq 1,$$

expressing monotone decreasing behavior of $r_n(x) = y_n(x)/y_{n-1}(x)$, $n \geq 1$, in both n and x ,

$$(2) \quad r'_n = \frac{P_n(x)}{y_{n-1}^2(x)} = \frac{r_n - r_{n-1}}{r_{n-1}} < 0,$$

is derived in the Appendix. The utility of this relation follows from the recurrence formulae for the iterated coerror function,

$$(3) \quad y_{-1}(x) = i^{-1} \operatorname{erfc}(x) = 2e^{-x^2}/\pi^{1/2}, \quad y_0(x) = i^0 \operatorname{erfc}(x) = \operatorname{erfc}(x), \\ y_{n-2} = 2ny_n + 2xy_{n-1}, \quad n = 1, 2, \dots,$$

to yield

$$(4) \quad r'_n = 2nr_n^2 + 2xr_n - 1 < 0, \quad n \geq 1.$$

This establishes bounds on the ratios $r_n = y_n/y_{n-1}$,

$$0 < r_n < \frac{-x + (x^2 + 2n)^{1/2}}{2n} = B_n(x)$$

since the parabola $v = 2nt^2 + 2xt - 1$ is negative between its roots. The upper bound is of most interest and we write $B_n(x)$ in the form

$$(5) \quad B_n(x) = \frac{-x + (x^2 + 2n)^{1/2}}{2n}, \quad x < 0, \\ n \geq 1, \\ = \frac{1}{x + (x^2 + 2n)^{1/2}}, \quad x \geq 0,$$

Received October 29, 1971, revised May 23, 1972.

AMS (MOS) subject classifications (1970). Primary 65D20, 33A20; Secondary 41-04.

Key words and phrases. Iterated coerror function, error function, coerror function, Mill's ratio, probability integral.

to avoid losses of significance in computation when x is large and positive. Furthermore, if we divide (3) by y_{n-1} and solve for r_{n-1} , we get

$$(6) \quad r_{n-1} = \frac{1}{2x + 2nr_n} > \frac{1}{2x + 2nB_n(x)} = B_n(x), \quad n \geq 1.$$

Thus, we have (with a shift of index)

$$(7) \quad C_n(x) < r_n < D_n(x),$$

and recursion on (7) in the form $C_n y_{n-1} < y_n < D_n y_{n-1}$ produces

$$(8) \quad \operatorname{erfc}(x) \prod_{k=1}^n C_k(x) < y_n < \operatorname{erfc}(x) \prod_{k=1}^n D_k(x)$$

where, in this context,

$$(9) \quad C_k(x) = B_{k+1}(x) \quad \text{and} \quad D_k(x) = B_k(x).$$

Table 1 gives some numerical results for these bounds. Both relative and absolute errors can be assessed. The number of significant digits in y_n is approximately the minimum number in agreement between C_k and D_k as k ranges from 1 to n .

TABLE 1
C_n and D_n of (9) compared with r_n

n \ x	-10	-5	0	5	10
1	1.005E+01 1.000E+01 5.050E+00	5.098E+00 5.000E+00 2.596E+00	7.071E-01 5.642E-01 5.000E-01	9.808E-02 9.635E-02 9.629E-02	4.975E-02 4.951E-02 4.951E-02
10	1.048E+00 1.043E+00 9.566E-01	5.854E-01 5.788E-01 5.389E-01	2.236E-01 2.181E-01 2.132E-01	8.541E-02 8.449E-02 8.435E-02	4.772E-02 4.753E-02 4.752E-02
20	5.458E-01 5.438E-01 5.218E-01	3.266E-01 3.240E-01 3.139E-01	1.581E-01 1.562E-01 1.543E-01	7.656E-02 7.598E-02 7.584E-02	4.580E-02 4.564E-02 4.563E-02
50	2.414E-01 2.408E-01 2.374E-01	1.618E-01 1.612E-01 1.595E-01	1.000E-01 9.950E-02 9.901E-02	6.180E-02 6.156E-02 6.146E-02	4.142E-02 4.132E-02 4.130E-02
100	1.366E-01 1.364E-01 1.355E-01	1.000E-01 9.978E-02 9.934E-02	7.071E-02 7.053E-02 7.035E-02	5.000E-02 4.989E-02 4.983E-02	3.660E-02 3.654E-02 3.653E-02
200	8.090E-02 8.082E-02 8.061E-02	6.404E-02 6.396E-02 6.384E-02	5.000E-02 4.994E-02 4.988E-02	3.904E-02 3.899E-02 3.897E-02	3.090E-02 3.087E-02 3.086E-02

For $x \geq 0$, the upper bound of (7) immediately implies

$$(10) \quad y_n(x) < \frac{y_{n-1}(x)}{x + (x^2 + 2n)^{1/2}} \leq \frac{y_{n-1}(x)}{(2n)^{1/2}} < y_{n-1}(x), \quad n \geq 1, x \geq 0.$$

$$y_n(x) \leq \frac{y_{n-1}(x)}{x + (x^2 + 2)^{1/2}},$$

$y_n < y_{n-1}$ can also be established for $n \geq 0$ by induction on

$$(11) \quad y_n(x) = \int_x^\infty y_{n-1}(t) dt, \quad n \geq 0,$$

and the upper bound in Mill's ratio [4], [14, p. 343]

$$(12) \quad m(x) = \frac{\pi/2}{(\pi - 1)x + (x^2 + \pi)^{1/2}} \leq \frac{i^0 \operatorname{erfc}(x)}{i^{-1} \operatorname{erfc}(x)}$$

$$\leq \frac{\pi/2}{2x + [(\pi - 2)x]^2 + \pi)^{1/2}} = M(x), \quad x \geq 0.$$

(See references for other results on Mill's ratio.) With these inequalities, purely elementary bounds on y_n can be given:

$$(13) \quad \frac{2m(x)e^{-x^2}}{\pi^{1/2}} \leq y_0 \leq \frac{2M(x)e^{-x^2}}{\pi^{1/2}}, \quad x \geq 0,$$

$$\frac{2m(x)e^{-x^2}}{\pi^{1/2}} \prod_{k=1}^n C_k(x) < y_n < \frac{2M(x)e^{-x^2}}{\pi^{1/2}} \prod_{k=1}^n D_k(x), \quad x \geq 0, n \geq 1.$$

Elementary bounds for $x < 0$ follow from the identity $\operatorname{erfc}(x) = 2 - \operatorname{erfc}(-x)$. An upper bound on $y_n(x)$ for $x \geq 0$ can be obtained from (11) and (12) by induction

$$(14) \quad y_n(x) \leq \frac{2}{\pi^{1/2}} [M(x)]^{n+1} e^{-x^2}, \quad n \geq 0, x \geq 0,$$

using $M(t) \leq M(x)$ for $t \geq x$. Simple backward recursion in (10), followed by (12), gives complementary forms

$$(15) \quad y_n(x) \leq \frac{2/\pi^{1/2} M(x)e^{-x^2}}{2^{n/2} (n!)^{1/2}}, \quad n \geq 0, x \geq 0.$$

$$y_n(x) \leq \frac{2/\pi^{1/2} M(x)e^{-x^2}}{(x + (x^2 + 2)^{1/2})^n},$$

While the bounds in (7) were obtained from $r'_n < 0$ and are best for large positive x , Eq. (36), $r'_n > -1/n$, represents the other extreme, and bounds from this inequality would be best for large negative x . They are (7) with

$$(16) \quad C_n(x) = \frac{-x + (x^2 + 2n - 2)^{1/2}}{2n}, \quad x < 0,$$

$$= \frac{1 - 1/n}{x + (x^2 + 2n - 2)^{1/2}}, \quad x \geq 0,$$

$$D_n(x) = B_n(x).$$

Table 2 shows the results for these bounds.

TABLE 2
 C_n and D_n of (16) compared with r_n

$n \backslash x$	-10	-5	0	5	10
1	1.005E+01	5.098E+00	7.071E-01	9.808E-02	4.975E-02
	1.000E+01	5.000E+00	5.642E-01	9.635E-02	4.951E-02
	1.000E+01	5.000E+00	0.	0.	0.
10	1.048E+00	5.854E-01	2.236E-01	8.541E-02	4.772E-02
	1.043E+00	5.788E-01	2.181E-01	8.449E-02	4.753E-02
	1.043E+00	5.779E-01	2.121E-01	7.787E-02	4.314E-02
20	5.458E-01	3.266E-01	1.581E-01	7.656E-02	4.580E-02
	5.438E-01	3.240E-01	1.562E-01	7.598E-02	4.564E-02
	5.437E-01	3.234E-01	1.541E-01	7.343E-02	4.368E-02
50	2.414E-01	1.618E-01	1.000E-01	6.180E-02	4.142E-02
	2.408E-01	1.612E-01	9.950E-02	6.156E-02	4.132E-02
	2.407E-01	1.609E-01	9.899E-02	6.091E-02	4.071E-02
100	1.366E-01	1.000E-01	7.071E-02	5.000E-02	3.660E-02
	1.364E-01	9.978E-02	7.053E-02	4.989E-02	3.654E-02
	1.363E-01	9.967E-02	7.036E-02	4.967E-02	3.631E-02
200	8.090E-02	6.404E-02	5.000E-02	3.904E-02	3.090E-02
	8.082E-02	6.396E-02	4.994E-02	3.899E-02	3.087E-02
	8.079E-02	6.392E-02	4.987E-02	3.892E-02	3.079E-02

Bounds on y_n/y_{n-k} can be obtained by bounding each term of

$$y_n/y_{n-k} = r_n r_{n-1} \cdots r_{n-k+1}.$$

Improved Bounds. The simplicity of the previous bounds is appealing; however, they are not very sharp near $x = 0$. The results of this section improve this situation. Relation (35) in the Appendix is

$$r'_n = 2nr_n^2 + 2xr_n - 1 < -r_n^2, \quad x \geq 0, \quad n \geq 1, \\ < -r_n^2 \exp\{-x^2\}, \quad x < 0,$$

Following through as before, we get (7) with the bounds

$$(17) \quad C_n(x) = \frac{2n + 2 + \exp\{-x^2\}}{2x \exp\{-x^2\} + (2n + 2)/D_{n+1}(x)} > D_{n+1}(x), \quad x < 0, \\ = \frac{2n + 3}{(2n + 4)x + (2n + 2)(x^2 + 2n + 3)^{1/2}}, \quad x \geq 0, \\ D_n(x) = \frac{-x + (x^2 + 2n + \exp\{-x^2\})^{1/2}}{2n + \exp\{-x^2\}}, \quad x < 0, \\ = \frac{1}{x + (x^2 + 2n + 1)^{1/2}}, \quad x \geq 0,$$

TABLE 3
 C_n and D_n of (17) compared with r_n

$n \backslash x$	-10	-5	0	5	10
1	1.005E+01	5.098E+00	5.774E-01	9.717E-02	4.963E-02
	1.000E+01	5.000E+00	5.642E-01	9.635E-02	4.951E-02
	5.050E+00	2.596E+00	5.590E-01	9.632E-02	4.951E-02
10	1.048E+00	5.854E-01	2.182E-01	8.487E-02	4.762E-02
	1.043E+00	5.788E-01	2.181E-01	8.449E-02	4.753E-02
	9.566E-01	5.389E-01	2.180E-01	8.443E-02	4.752E-02
20	5.458E-01	3.266E-01	1.562E-01	7.620E-02	4.572E-02
	5.438E-01	3.240E-01	1.562E-01	7.598E-02	4.564E-02
	5.218E-01	3.139E-01	1.561E-01	7.593E-02	4.564E-02
50	2.414E-01	1.618E-01	9.950E-02	6.163E-02	4.136E-02
	2.408E-01	1.612E-01	9.950E-02	6.156E-02	4.132E-02
	2.374E-01	1.595E-01	9.950E-02	6.153E-02	4.131E-02
100	1.366E-01	1.000E-01	7.053E-02	4.992E-02	3.656E-02
	1.364E-01	9.978E-02	7.053E-02	4.989E-02	3.654E-02
	1.355E-01	9.934E-02	7.053E-02	4.988E-02	3.654E-02
200	8.090E-02	6.404E-02	4.994E-02	3.900E-02	3.088E-02
	8.082E-02	6.396E-02	4.994E-02	3.899E-02	3.087E-02
	8.061E-02	6.384E-02	4.994E-02	3.899E-02	3.087E-02

whose derivatives of order 2 or greater are discontinuous at $x = 0$. (The other half of (35) leads to $C_n(x)$ in (9).) Table 3 shows these results for some numerical values. Notice also that for $x < 0$, D_{n+1} is a lower bound on r_n . While these bounds are relatively good at $x = 0$, they can be made sharp by observing that

$$(18) \quad r'_n = 2nr_n^2 + 2xr_n - 1 \geq r'_n(0) = 2nr_n^2(0) - 1, \quad x \geq 0$$

is exact at $x = 0$, where

$$(19) \quad r_n(0) = \frac{\Gamma((n+1)/2)}{2\Gamma(n/2+1)}, \quad n \geq 0.$$

The inequalities in (18) follow from (38) which shows that r'_n is monotone increasing in x . The roots give

$$(20a) \quad D_n(x) = \frac{-x + (x^2 + 2na_n)^{1/2}}{2n}, \quad x < 0,$$

$$C_n(x) = \frac{a_n}{x + (x^2 + 2na_n)^{1/2}}, \quad x \geq 0,$$

where $a_n = 2nr_n^2(0)$ and, with (6),

$$(20b) \quad C_n(x) = \frac{-x + (x^2 + 2(n+1)a_{n+1})^{1/2}}{2(n+1)a_{n+1}}, \quad x < 0,$$

$$D_n(x) = \frac{1}{x + (x^2 + 2(n+1)a_{n+1})^{1/2}}, \quad x \geq 0.$$

Table 4 shows the partial improvement over Tables 1, 2 and 3. Bounds analogous to those in (10) and (15) can be formed for $x \geq 0$ by setting $x = 0$ or $n = 1$ in $D_n(x)$ of (20b). The largest lower bound and the smallest upper bound among the formulae listed would be the best over the range of interest in n and x .

Iterative improvements, generating upper and lower bounds at each step, can be made by recurring backward on (6), starting the continued fraction approximants with some C_n and D_n (see also the next section on numerical computations).

The connection between these bounds and Mill's ratio can be established by taking (6) with $n = 1$ and applying the expressions for $C_1(x)$ or $D_1(x)$. (20a) for $x \geq 0$ gives the upper bound quoted in the NBS handbook [1, p. 298] while (5) for $x \geq 0$ gives the lower bound in the same reference. (20b) gives Boyd's [4] lower bound (12).

The bounds in (12) are fairly sharp with maximum relative errors of 1.17% and 0.91%, respectively. Best results are obtained with $M(x)$ for $x \leq 0.85$ and $m(x)$ for $x > 0.85$, with an overall maximum relative error for this combination of about 0.86%.

TABLE 4
 C_n and D_n of (20) compared with r_n

$n \backslash x$	-10	-5	0	5	10
1	1.003E+01	5.063E+00	5.642E-01	9.704E-02	4.961E-02
	1.000E+01	5.000E+00	5.642E-01	9.635E-02	4.951E-02
	6.416E+00	3.280E+00	5.642E-01	6.287E-02	3.173E-02
10	1.045E+00	5.818E-01	2.181E-01	8.486E-02	4.762E-02
	1.043E+00	5.788E-01	2.181E-01	8.449E-02	4.753E-02
	9.989E-01	5.605E-01	2.181E-01	8.176E-02	4.550E-02
20	5.448E-01	3.250E-01	1.562E-01	7.619E-02	4.571E-02
	5.438E-01	3.240E-01	1.562E-01	7.598E-02	4.564E-02
	5.334E-01	3.200E-01	1.562E-01	7.502E-02	4.476E-02
50	2.411E-01	1.614E-01	9.950E-02	6.163E-02	4.136E-02
	2.408E-01	1.612E-01	9.950E-02	6.156E-02	4.132E-02
	2.394E-01	1.606E-01	9.950E-02	6.136E-02	4.107E-02
100	1.365E-01	9.983E-02	7.053E-02	4.992E-02	3.656E-02
	1.364E-01	9.978E-02	7.053E-02	4.989E-02	3.654E-02
	1.361E-01	9.967E-02	7.053E-02	4.983E-02	3.646E-02
200	8.085E-02	6.398E-02	4.994E-02	3.900E-02	3.088E-02
	8.082E-02	6.396E-02	4.994E-02	3.899E-02	3.087E-02
	8.076E-02	6.394E-02	4.994E-02	3.898E-02	3.085E-02

Numerical Computation of r_n and y_k , $k = 0, 1, \dots, n$. In [5] and [6], Gautschi shows that forward recursion on (3) is appropriate for $x \leq 0$ while an iterative backward technique on (6) (which generates continued fraction approximants) is appropriate for $x > 0$ for stability. It would be hard to improve on the simplicity of the forward recursion for $x \leq 0$ but some improvement is possible for $x > 0$ because the continued fraction approximants are slowly convergent for x close to zero. The results developed above are exploited to get accurate values of r_n so that the ratios

$$(21) \quad r_{k-1} = \frac{.5}{x + kr_k}, \quad k = n, n - 1, \dots, 1, \quad x > 0,$$

can be computed for the relation

$$(22) \quad y_k = \frac{2e^{-x^2}}{\pi^{1/2}} \prod_{i=0}^k r_i, \quad k = 0, 1, 2, \dots, n, \quad x > 0.$$

The method which has proved successful in computing r_n is based upon a restatement of (21) with k replaced by $k + 1$,

$$[2(k + 1)(r_{k+1}/r_k)r_k + 2x]r_k = 1,$$

in the form

$$(23a) \quad r_k = \frac{1}{x + (x^2 + 2(k + 1)R_{k+1})^{1/2}}, \quad R_{k+1} = \frac{r_{k+1}}{r_k}.$$

Then, with D_k of (20b) as an initial approximation to r_k for $k \geq n$ (see Table 4), the algorithm becomes

$$(23b) \quad r_k^0 = \frac{1}{x + (x^2 + [2(k + 1)r_{k+1}(0)]^2)^{1/2}}, \quad k = n, n + 1, \dots, n + \nu, x \geq 0,$$

where $r_k(0)$ is defined in (19), and

$$(23c) \quad R_{k+1}^m = r_{k+1}^m / r_k^m, \quad k = n, n + 1, \dots, n + \nu - m - 1,$$

$$r_k^{m+1} = \frac{1}{x + (x^2 + 2(k + 1)R_{k+1}^m)^{1/2}}, \quad m = 0, 1, 2, \dots, \nu - 1.$$

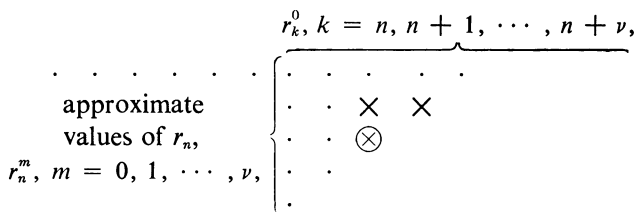


FIGURE 1

The iteration diagram is shown in Fig. 1. r_n^m is the approximate value of r_n , and only two successive diagonals need be stored.

Although the convergence proof below does not show that r_k^m decreases monotonically to r_k on the columns of Fig. 1, the numerical results were all of this character. If this were true in general, it would follow, on using (21), that

$$c_n^\nu = \frac{1}{2x + 2(n + 1)r_{n+1}^{\nu-1}} \leq r_n \leq r_n^\nu,$$

and the error criterion

$$|r_n^\nu - r_n|/r_n \leq |r_n^\nu - c_n^\nu|/c_n^\nu \leq \epsilon$$

could be used to terminate the process to guarantee a relative error ϵ . Even if the monotonicity is violated, this is a sensible method of termination since $r_{n+1}^{\nu-1}$, and hence c_n^ν , can be expected to be slightly less accurate than r_n^ν . The construction of rigorous bounds using (23a) is shown in the convergence proof of the algorithm.

Some experimentation shows that if $n < 25$ (say) and $x < 2$, n should be increased to 25 before the iteration is started to increase the rate of convergence. Upon completion, the backward recursive step in (21) is applied followed by (22). Notice that for $x = 0$, $r_k^1 = r_k(0)$ and $\nu = 1$. With a relative error $\epsilon = 5 \times 10^{-9}$, extensive evaluation of this procedure showed that $\nu \leq 5$ (a maximum of 6 applications of (23b) and 15 of (23c)) held for $x \geq 0$ and $0 \leq n \leq 100$. For $n \geq 50$ the maximum value of ν was 4, but the number of steps in (21) to reduce the index when the starting index is small diminishes this advantage somewhat.

Straight backward recursion with (21) starting with C_{200} or D_{200} of (9) gave only 4 significant digits in $\text{erfc}(0.1)$. The corresponding computation with C_{200} or D_{200} of (20a) or (20b), which are accurate at $x = 0$, gave 6 significant figures. This amounts to iteration of these bounds according to (6) or (21). It is common to avoid underflow problems by scaling y_k by e^{x^2} in (22).

The forward recursive loop for $x < 0$ is started with $y_{-1}(x)$ and $y_0(x)$ where $y_0(x) = 2 - \text{erfc}(|x|)$ for $x < 0$. The scheme for $x \geq 0$ is used to compute $\text{erfc}(|x|)$. If $x < X_0$ ($X_0 = -6$ for a CDC 6600 computer), $y_0(x) = 2$ to the word length of the machine and the $\text{erfc}(|x|)$ computation can be avoided.

The methods exploited here have concentrated on recursion, primarily for sake of computation. However, the differential inequalities developed in the Appendix can be integrated for other types of approximations.

Convergence of the Algorithm.

THEOREM. *If $x \geq 0$, the sequence r_k^m generated by (23c) converges to r_k as $m \rightarrow \infty$ for each $k \geq n \geq 0$.*

The proof consists of constructing monotone sequences of upper and lower bounds on r_k^m which converge to r_k . Let $D_k^0 = r_k^0$. Using (20b) and (21), we have

$$(24) \quad C_k^0 = \frac{1}{2x + 2(k + 1)D_{k+1}^0} \leq r_k \leq r_k^0 = D_k^0.$$

Substitution of these bounds into the expressions r_{k+1}/r_k and r_{k+1}^0/r_k^0 yield

$$(25) \quad \frac{C_{k+1}^0}{D_k^0} \leq \frac{r_{k+1}}{r_k} \leq \frac{D_{k+1}^0}{C_k^0} \quad \text{and} \quad \frac{C_{k+1}^0}{D_k^0} \leq \frac{r_{k+1}^0}{r_k^0} \leq \frac{D_{k+1}^0}{C_k^0}.$$

Another substitution of the bounds in (25) into the denominators of (23a) and (23c) for $m = 0$ give new bounds D_k^1 and C_k^1 ,

$$D_k^1 \geq r_k \geq C_k^1, \quad D_k^1 \geq r_k^1 \geq C_k^1$$

where

(26)

$$D_k^1 = \frac{1}{x + (x^2 + 2(k + 1)C_{k+1}^0/D_k^0)^{1/2}} \quad \text{and} \quad C_k^1 = \frac{1}{x + (x^2 + 2(k + 1)D_{k+1}^0/C_k^0)^{1/2}}.$$

Notice that equality holds throughout for $x = 0$. Continuing in this way, we can inductively construct sequences of bounds D_k^m and C_k^m on r_k and r_k^m . However, convergence is obtained by showing monotonicity of each sequence,

$$D_k^{m+1} \leq D_k^m \quad \text{and} \quad C_k^{m+1} \geq C_k^m, \quad m = 0, 1, 2, \dots$$

Thus, for $m = 0$, we need to establish $D_k^0 \geq D_k^1$ and $C_k^1 \geq C_k^0$ before going on to the induction. $D_k^0 \geq D_k^1$ follows by showing that

$$g(x) = \frac{C_{k+1}^0}{D_k^0} = \frac{R_{k+3}(0)(x + (x^2 + 2(k + 1)R_{k+1}(0))^{1/2})}{\left[2R_{k+3}(0) - \frac{k + 2}{k + 3} \right]x + \frac{k + 2}{k + 3}(x^2 + 2(k + 3)R_{k+3}(0))^{1/2}}$$

in the denominator of D_k^1 is greater than

$$(27) \quad R_{k+1}(0) = r_{k+1}(0)/r_k(0) = 2(k + 1)r_{k+1}^2(0)$$

in the denominator of D_k^0 for $x > 0$. (The last expression in (27) is obtained from (6) with $x = 0$ and C_{k+1}^0 is obtained from (24) by rationalizing the denominator of D_{k+2}^0 .) This inequality, $g(x) \geq R_{k+1}(0)$, can be proved by showing monotone increasing behavior of $g(x)$ together with $g(0) = R_{k+1}(0)$. $g(0) = R_{k+1}(0)$ follows from (25) and (27) and the fact that equality holds for $x = 0$ in (25). The positivity of $g'(x)$,

$$g'(x) = \frac{R_{k+3}(0)}{(x^2 + a)^{1/2}(x^2 + c)^{1/2}} \cdot \left[\frac{(ab - cd)(ab + cd)x^2 + ac(ab^2 - cd^2) + bx(a - c)A(x)}{A(x)(dx + b(x^2 + a)^{1/2})^2} \right],$$

where

$$\begin{aligned} a &= 2(k + 3)R_{k+3}(0), & c &= 2(k + 1)R_{k+1}(0), \\ b &= (k + 2)/(k + 3), & d &= 2R_{k+3}(0) - b, \\ A(x) &= ab(x^2 + c)^{1/2} + cd(x^2 + a)^{1/2}, \end{aligned}$$

will follow if the quantities $a - c$, d , $ab - cd$, $ab^2 - cd^2$ are shown to be positive.

A direct application of (33) and (39) for $x = 0$ gives $a > c$ since

$$R_k(x) = 1 + r'_k(x), \quad r'_{k+1}(x) > r'_k(x) \Rightarrow R_{k+1}(x) > R_k(x).$$

On the other hand, $d > 0$ follows from (33), (35) and (7), with (9), for $x = 0$ since

$$R_{k+3}(0) = 1 + r'_{k+3}(0) \geq 1 - 2r_{k+3}^2(0) > 1 - \frac{2}{2(k + 3)} = \frac{k + 2}{k + 3} = b$$

implies $d > b > 0$.

In order to deduce the signs of $ab - cd$ and $ab^2 - cd^2$, we first express $R_{k+3}(0)$ in terms of $r_{k+1}(0)$ by means of (27) and (6),

$$(28) \quad R_{k+3}(0) = 2(k + 3)r_{k+3}^2(0), \quad r_{k+3}(0) = \frac{k + 2}{k + 3} r_{k+1}(0).$$

Then, with (27) and (28),

$$\begin{aligned} ab - cd &= 4br_{k+1}^2(0)[(k+2)^2 + (k+1)^2 - 4(k+1)^2(k+2)r_{k+1}^2(0)], \\ ab^2 - cd^2 &= 4b^2r_{k+1}^2(0)[(2k+3) - 4(k+1)(k+2)r_{k+1}^2(0)] \\ &\quad \cdot [1 + 4(k+1)(k+2)r_{k+1}^2(0)]. \end{aligned}$$

If we use (19) for $r_{k+1}(0)$ together with $\Gamma(z+1) = z\Gamma(z)$ in the denominator, we get

$$r_{k+1}^2(0) = \frac{1}{(k+1)^2} \frac{\Gamma^2(k/2+1)}{\Gamma^2((k+1)/2)},$$

and the upper bound of [18],

$$n^{1-s} \leq \frac{\Gamma(n+1)}{\Gamma(n+s)} \leq (n+s)^{1-s}, \quad n > 0, 0 \leq s \leq 1,$$

with $n = k/2$, $s = 1/2$, suffices to establish the sign of $ab - cd$,

$$ab - cd \geq 4br_{k+1}^2(0) \left[2k^2 + 6k + 5 - 4(k+2) \left(\frac{k+1}{2} \right) \right] = 4br_{k+1}^2(0) > 0.$$

However, sharper results are needed to show $ab^2 - cd^2 > 0$ for $k \geq 0$. The results stated in (45) can be applied, for $k \geq 4$,

$$\frac{\Gamma^2(k/2+1)}{\Gamma^2((k+1)/2)} < \frac{k}{2} + \frac{1}{4} + \frac{1}{16k} - \frac{1}{32k^2} + \frac{48}{5k^3}, \quad k \geq 4,$$

and this yields

$$ab^2 - cd^2 > \frac{4b^2r_{k+1}^2(0)}{k+1} \left[\frac{3}{4} - \frac{3}{8k} - \frac{763}{20k^2} - \frac{384}{5k^3} \right] [1 + 4(k+1)(k+2)r_{k+1}^2(0)] > 0$$

for $k \geq 9$. Direct substitution was used to verify $ab^2 - cd^2 > 0$ for $k = 0$ through $k = 9$. Thus, $D_k^0 \geq D_k^1$ with strict inequality for $x > 0$.

For C_k^1 , we take the defining equation (26) and substitute (24) for C_k^0 to get

$$C_k^1 = \frac{1}{x + (x^2 + 2(k+1)D_{k+1}^0(2x + 2(k+1)D_{k+1}^0))^{1/2}} = \frac{1}{2x + 2(k+1)D_{k+1}^0} = C_k^0.$$

To summarize the situation for $m = 0$, we have

$$(29) \quad D_k^0 \geq D_k^1 \geq r_k \geq C_k^1 = C_k^0 \quad \text{and} \quad D_k^0 \geq D_k^1 \geq r_k^1 \geq C_k^1 = C_k^0.$$

Now, we repeat the induction steps (24) through (26) for $m = 1$. Thus, (29) applied to r_{k+1}/r_k and r_{k+1}^1/r_k^1 yields

$$\frac{C_{k+1}^1}{D_k^1} \leq \frac{r_{k+1}}{r_k} \leq \frac{D_{k+1}^1}{C_k^1} \quad \text{and} \quad \frac{C_{k+1}^1}{D_k^1} \leq \frac{r_{k+1}^1}{r_k^1} \leq \frac{D_{k+1}^1}{C_k^1}.$$

These expressions with (23a) and (23c) for $m = 1$ give new bounds D_k^2 and C_k^2 ,

$$D_k^2 \geq r_k \geq C_k^2, \quad D_k^2 \geq r_k^2 \geq C_k^2,$$

where

$$D_k^2 = \frac{1}{x + (x^2 + 2(k+1)C_{k+1}^1/D_k^1)^{1/2}} \quad \text{and} \quad C_k^2 = \frac{1}{x + (x^2 + 2(k+1)D_{k+1}^1/C_k^1)^{1/2}}.$$

Then, by (29),

$$\frac{C_{k+1}^1}{D_k^1} \geq \frac{C_{k+1}^0}{D_k^0} \Rightarrow D_k^1 \geq D_k^2 \quad \text{and} \quad \frac{D_{k+1}^1}{C_k^1} \leq \frac{D_{k+1}^0}{C_k^0} \Rightarrow C_k^2 \geq C_k^1.$$

Thus, for $m = 1$ we have

$$D_k^0 \geq D_k^1 \geq D_k^2 \geq r_k \geq C_k^2 \geq C_k^1 = C_k^0 \quad \text{and} \quad D_k^0 \geq D_k^1 \geq D_k^2 \geq r_k^2 \geq C_k^2 \geq C_k^1 = C_k^0$$

with strict inequality in $C_k^2 \geq C_k^1$ for $x > 0$ because $D_k^0 > D_k^1$ for $x > 0$.

Continuing in this way, we compute inductively a sequence D_k^m which is bounded and monotone decreasing while C_k^m is bounded and monotone increasing with r_k^m and r_k between these bounds. Each sequence therefore has a limit D_k and C_k such that

$$D_k = \frac{1}{x + (x^2 + 2(k + 1)C_{k+1}/D_k)^{1/2}}, \quad C_k = \frac{1}{x + (x^2 + 2(k + 1)D_{k+1}/C_k)^{1/2}}.$$

Solving for each of these radicals and squaring gives

$$(30) \quad D_k = \frac{1}{2x + 2(k + 1)C_{k+1}}, \quad C_k = \frac{1}{2x + 2(k + 1)D_{k+1}}$$

and combining the relations in (30) produces

$$D_k = \frac{1}{2x + \frac{2(k + 1)}{2x + 2(k + 2)D_{k+2}}}, \quad C_k = \frac{1}{2x + \frac{2(k + 1)}{2x + 2(k + 2)C_{k+2}}}.$$

Each of these lead to the continued fraction for r_k which can be developed similarly by repeated application of (6). Therefore, $D_k = C_k = r_k$ and r_k^m converges to r_k .

Acknowledgements. The author wishes to acknowledge the assistance of the referee for his part in clarifying the presentation and offering suggestions on several tedious details. The assistance of Mrs. Sharon L. Daniel is also appreciated for the coding of the numerical experiments for a CDC 6600 computer.

Appendix. Some Relations Involving r_n, r'_n and r''_n . We start with the self-joint differential equation

$$\frac{d}{dt} (e^{t^2} y'_n) = 2ne^{t^2} y_n, \quad n \geq -1,$$

and apply Green's theorem to the relation

$$\int_x^\infty \left[y_{n-1} \frac{d}{dt} (e^{t^2} y'_n) - y_n \frac{d}{dt} (e^{t^2} y'_{n-1}) \right] dt = 2 \int_x^\infty e^{t^2} y_n(t) y_{n-1}(t) dt$$

to get

$$y_{n-1} y'_n - y_n y'_{n-1} = -2e^{-x^2} \int_x^\infty e^{t^2} y_n(t) y_{n-1}(t) dt, \quad n \geq 1,$$

or with (11),

$$(31) \quad y_n(x) y_{n-2}(x) - y_{n-1}^2(x) = -2e^{-x^2} \int_x^\infty e^{t^2} y_n(t) y_{n-1}(t) dt < 0, \quad n \geq 1.$$

Therefore,

$$(32) \quad r'_n(x) = \frac{y_n y_{n-2}}{y_{n-1}^2} - 1 = -\frac{2e^{-x^2}}{y_{n-1}^2(x)} \int_x^\infty e^{t^2} y_n(t) y_{n-1}(t) dt < 0,$$

$$(33) \quad r'_n(x) = \frac{r_n}{r_{n-1}} - 1 < 0 \quad \text{and} \quad 0 < \frac{r_n}{r_{n-1}} < 1, \quad n \geq 0.$$

The recursion relation (6), reciprocated and differentiated, provides the recursion relation for r'_n ,

$$(34) \quad r'_{n-1} = -2r_{n-1}^2(1 + nr'_n),$$

which shows that $r'_n > -2r_n^2$. On the other hand, a direct estimate of (32) using

$$\begin{aligned} \exp\{t^2 - x^2\} &\geq 1 && \text{for } t \geq x \geq 0, \\ &\geq e^{-x^2} && \text{for } t \geq x \quad \text{and} \quad x < 0, \end{aligned}$$

and the differential form of (11) gives

$$(35) \quad \begin{aligned} -2r_n^2 < r'_n < -r_n^2, && x \geq 0, \\ &< -r_n^2 \exp\{-x^2\}, && x < 0. \end{aligned}$$

(34) in the form

$$-r'_{n-1}/(2r_{n-1}^2) = 1 + nr'_n > 0$$

also shows that

$$(36) \quad r'_n > -1/n,$$

which is better for large negative x than (35). The recursion relation for r''_n follows from (34) by differentiation,

$$(37) \quad r_{n-1}r''_{n-1} + 2nr_{n-1}^3r''_n = 2(r'_{n-1})^2.$$

The positivity of r''_n can be obtained from (32) as follows:

$$\begin{aligned} y_{n-1}^3 r''_n &= y_{n-2}[y_n y_{n-2} - y_{n-1}^2] - y_n[y_{n-1} y_{n-3} - y_{n-2}^2] \\ &= -2y_{n-2}(x)e^{-x^2} \int_x^\infty e^{t^2} y_n(t) y_{n-1}(t) dt + 2y_n(x)e^{-x^2} \int_x^\infty e^{t^2} y_{n-1}(t) y_{n-2}(t) dt \\ &= 2e^{-x^2} \int_x^\infty e^{t^2} y_{n-1}(t)[y_{n-2}(t)y_n(x) - y_{n-2}(x)y_n(t)] dt. \end{aligned}$$

This gives, after factorization,

$$(38) \quad y_{n-1}^3 r''_n = 2e^{-x^2} \int_x^\infty e^{t^2} y_{n-1}(t) y_n(t) y_n(x) \left[\frac{1}{r_n(t)r_{n-1}(t)} - \frac{1}{r_n(x)r_{n-1}(x)} \right] dt > 0.$$

This not only implies monotone increasing behavior of r'_n in x , but differentiation of (33) establishes monotone behavior in n as well,

$$r_{n-1}r''_n + r'_{n-1}r'_n = r'_n - r'_{n-1} > 0$$

or

(39) $0 > r'_n > r'_{n-1}.$

Upper bounds on r''_n can be obtained from (37) using $r''_n > 0$. $r''_n > 0$ also establishes

(40) $r'_n/r_n > r'_{n-1}/r_{n-1}$

through differentiation of (33),

$$r''_n = \frac{r_n}{r_{n-1}} \left(\frac{r'_n}{r_n} - \frac{r'_{n-1}}{r_{n-1}} \right)$$

and with (33) again,

$$1/r_{n-1} - 1/r_n > 1/r_{n-2} - 1/r_{n-1}.$$

That is, second differences $\delta^2(1/r_{n-1})$ are negative,

(41) $\delta^2(1/r_{n-1}) < 0, \quad n = 2, 3, \dots$

(39) together with (33) also establishes the monotone decreasing behavior of the differences $r_{n-1} - r_n$,

$$0 < r_{n-1} - r_n < (r_{n-2} - r_{n-1})r_{n-1}/r_{n-2} < (r_{n-2} - r_{n-1})$$

and hence

(42) $\delta^2(r_{n-1}) > 0, \quad n = 2, 3, \dots$

The expression in (31) is the numerator of

(43a) $\frac{d^2}{dx^2} \ln y_n(x) = \frac{y_n y_{n-2} - y_{n-1}^2}{y_n^2} < 0, \quad n \geq 1.$

For $n = 0$ this works out to be

(43b) $\frac{d^2}{dx^2} \ln y_0(x) = -\frac{2y_{-1}(x)y_1(x)}{y_0^2(x)} < 0.$

Thus we have also established that $w = \ln i^n \operatorname{erfc}(x)$ is concave down for all x and all $n \geq 0$.

A Sharp Upper Bound on a Gamma Ratio. We start with the asymptotic expansion [1, p. 257]

$$\ln \Gamma(z) = \left(z - \frac{1}{2} \right) \ln z - z + \frac{1}{2} \ln 2\pi + \frac{1}{12z} - \frac{1}{360z^3} + R$$

for $z > 0$ and estimate R by the next term $|R| \leq 1/(1260z^5)$. The application of the final results dictated this accuracy. This expression can be applied for $z = x + 1$ and $z = x + \frac{1}{2}$ to yield

(44)
$$\begin{aligned} \ln \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} &= \frac{1}{2} \ln x + \left(x + \frac{1}{2} \right) \ln \left(1 + \frac{1}{x} \right) - x \ln \left(1 + \frac{1}{2x} \right) - \frac{1}{2} \\ &+ \frac{1}{12x} \left[\left(1 + \frac{1}{x} \right)^{-1} - \left(1 + \frac{1}{2x} \right)^{-1} \right] \\ &- \frac{1}{360x^3} \left[\left(1 + \frac{1}{x} \right)^{-3} - \left(1 + \frac{1}{2x} \right)^{-3} \right] + R_1 - R_2 \end{aligned}$$

where

$$|R_1| \leq 1/(1260x^5), \quad |R_2| \leq 1/(1260x^5).$$

Now, the following alternating series for $x \geq 2$ can be used for terms up to and including x^{-4} in (44),

$$\begin{aligned} \ln\left(1 + \frac{1}{x}\right) &= \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} + R_4\left(\frac{1}{x}\right), & \left|R_4\left(\frac{1}{x}\right)\right| &\leq \frac{1}{4x^4}, \\ &= \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + R_5\left(\frac{1}{x}\right), & \left|R_5\left(\frac{1}{x}\right)\right| &\leq \frac{1}{5x^5}, \\ \left(1 + \frac{1}{x}\right)^{-1} &= 1 - \frac{1}{x} + \frac{1}{x^2} + \bar{R}_3\left(\frac{1}{x}\right) & \left|\bar{R}_3\left(\frac{1}{x}\right)\right| &\leq \frac{1}{x^3}, \\ \left(1 + \frac{1}{x}\right)^{-3} &= 1 + \hat{R}_1\left(\frac{1}{x}\right), & \left|\hat{R}_1\left(\frac{1}{x}\right)\right| &\leq \frac{3}{x}. \end{aligned}$$

Then

$$\ln \frac{\Gamma^2(x+1)}{\Gamma^2(x+\frac{1}{2})} = \ln x + \frac{1}{4x} - \frac{1}{96x^3} + E$$

where

$$\begin{aligned} E &= R_4\left(\frac{1}{x}\right) + 2xR_5\left(\frac{1}{x}\right) - 2xR_5\left(\frac{1}{2x}\right) + \frac{1}{6x} \left[\bar{R}_3\left(\frac{1}{x}\right) - \bar{R}_3\left(\frac{1}{2x}\right) \right] \\ &\quad - \frac{1}{180x^3} \left[\hat{R}_1\left(\frac{1}{x}\right) - \hat{R}_1\left(\frac{1}{2x}\right) \right] + 2R_1 - 2R_2 \end{aligned}$$

and

$$|E| < 1/x^4, \quad x \geq 2.$$

For consistency, terms of degree three or less are carried accurately in estimating the exponential

$$\begin{aligned} \frac{\Gamma^2(x+1)}{\Gamma^2(x+\frac{1}{2})} &= xe^\alpha = x \left(1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} + \sum_{i=4}^{\infty} \frac{\alpha^i}{i!} \right) \\ &< x \left(1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} + \frac{1}{4!} \sum_{i=4}^{\infty} \alpha^i \right) \\ &< x \left(1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} + \frac{\alpha^4}{4!(1-\alpha)} \right) \end{aligned}$$

where

$$\alpha = 1/(4x) - 1/(96x^3) + E, \quad 0 < \alpha < 3/16, \quad x \geq 2.$$

Now we expand the powers of α and bound terms of higher order,

$$x^{-j} \leq x^{-4}/2^{j-4}, \quad j \geq 5, \quad x \geq 2,$$

to obtain

$$(45) \quad \frac{\Gamma^2(x+1)}{\Gamma^2(x+\frac{1}{2})} < x \left(1 + \frac{1}{4x} + \frac{1}{32x^2} - \frac{1}{128x^3} + \frac{6}{5x^4} \right), \quad x \geq 2.$$

This expression is asymptotically correct in all terms except the last.

Applied Mathematics Division
Sandia Laboratories
Albuquerque, New Mexico 87115

1. M. ABRAMOWITZ & I. A. STEGUN (Editors), *Handbook of Mathematical Functions, With Formulas, Graphs and Mathematical Tables*, Nat. Bur. Standards Appl. Math. Series, 55, Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964. MR 29 #4914.
2. D. F. BARROW & A. C. COHEN, JR., "On some functions involving Mill's ratio," *Ann. Math. Statist.*, v. 25, 1954, pp. 405-408. MR 15, 807.
3. Z. W. BIRNBAUM, "An inequality for Mill's ratio," *Ann. Math. Statist.*, v. 13, 1942, pp. 245-246. MR 4, 19.
4. A. V. BOYD, "Inequalities for Mills' ratio," *Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs.*, v. 6, 1959, pp. 44-46. MR 22 #9625.
5. W. GAUTSCHI, "Computational aspects of three-term recurrence relations," *SIAM Rev.*, v. 9, 1967, pp. 24-82. MR 35 #3927.
6. W. GAUTSCHI, "Recursive computation of the repeated integrals of the error function," *Math. Comp.*, v. 15, 1961, pp. 227-232. MR 24 #B2113.
7. R. D. GORDON, "Values of Mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument," *Ann. Math. Statist.*, v. 12, 1941, pp. 364-366. MR 3, 171.
8. Y. KOMATU, "Elementary inequalities for Mills' ratio," *Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs.*, v. 4, 1955, pp. 33-34.
9. K. B. OLDHAM, "Approximations for the $x \exp x^2 \operatorname{erfc} x$ function," *Math. Comp.*, v. 22, 1968, p. 454.
10. H. O. POLLAK, "A remark on 'Elementary inequalities for Mills' ratio,' by Y. Komatu," *Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs.*, 4, 1956, p. 40. MR 18, 722.
11. W. D. RAY & A. E. N. T. PITMAN, "Chebyshev polynomial and other new approximations to Mills' ratio," *Ann. Math. Statist.*, v. 34, 1963, pp. 892-902. MR 27 #3070.
12. H. RUBEN, "A convergent asymptotic expansion for Mills' ratio and the normal probability integral in terms of rational functions," *Math. Ann.*, v. 151, 1963, pp. 355-364. MR 27 #5308.
13. H. RUBEN, "A new asymptotic expansion for the normal probability integral and Mills' ratio," *J. Roy. Statist. Soc. Ser. B*, v. 24, 1962, pp. 177-179. MR 25 #2662.
14. H. RUBEN, "Irrational fraction approximations to Mills' ratio," *Biometrika*, v. 51, 1964, pp. 339-345. MR 30 #3519.
15. M. R. SAMPFORD, "Some inequalities on Mills' ratio and related functions," *Ann. Math. Statist.*, v. 24, 1953, pp. 130-132. MR 14, 995.
16. L. R. SHENTON, "Inequalities for the normal integral including a new continued fraction," *Biometrika*, v. 41, 1954, pp. 177-189. MR 15, 884.
17. R. F. TATE, "On a double inequality of the normal distribution," *Ann. Math. Statist.*, v. 24, 1953, pp. 132-134. MR 14, 995.
18. V. R. R. UPPULURI, "A stronger version of Gautschi's inequality satisfied by the gamma function," *Skand. Aktuarietidskr.* v. 1-2, 1964, pp. 51-52. MR 31 #4934.