ASYMPTOTIC DISTRIBUTON OF EIGENVALUES AND EIGENFUNCTIONS FOR GENERAL LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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The asymptotic distribution of eigenvalues and eigenfunctions of elliptic operators has been studied extensively by Weyl, Courant, Carleman, Pleijel and others. During the last decade, Gårding [9] and Browder solved the problem for an elliptic operator with infinitely differentiable coefficients and null Dirichlet boundary conditions. It is the purpose of this paper to consider the problem for a general class of elliptic boundary value problems investigated during the last few years by Agmon-Douglis-Nirenberg [2], Browder [3], [6] and Schechter [14].

Let (A,γ) with $\gamma = \{B_1,\cdots,B_m\}$ be a uniformly regularly elliptic boundary value problem on S. It is assumed that A is positively strongly elliptic and (A,γ) is formally positive. If A_γ is the realization of A as an operator on $L^2(S)$ with null boundary conditions γ, then the following results are obtained:

(1) When A_γ is self-adjoint:

$$N(t) = \sum_{\lambda_j \leqq t} 1 \sim (2\pi)^{-n} t^{n/2m} \int_S \int_{a(x,\xi)<1} d\xi dx \quad \text{as } t \to +\infty.$$

Let $e(x,y,t)$ be the spectral function of A_γ, then:

$$t^{-(n+|\alpha|+|\beta|)/2m} D_x^\alpha D_y^\beta e(x,y,t) = t^{-(n+|\alpha|+|\beta|)/2m} \sum_{\lambda_j \leqq t} D^\alpha \phi_j(x) D^\beta \phi_j(y) \to 0$$

as $t \to +\infty$; x, y in S and $x \neq y$.

$$D^{\alpha+\beta} e(x,x,t) \sim (2\pi)^{-n} t^{(n+|\alpha|+|\beta|)/2m} \frac{\Gamma(2p)}{\Gamma\left(1+\dfrac{n}{2m}\right)\Gamma\left(2p - \dfrac{n+|\alpha|+|\beta|}{2m}\right)} \cdot \int_{E^n} \xi^{\alpha+\beta}[a(x,\xi)+1]^{-2p} d\xi$$

for x in S as $t \to +\infty$ $(4mp > n + |\alpha| + |\beta|)$.

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λ_j, ϕ_j are the eigenvalues and eigenfunctions of A_γ.

(2) When A_γ is nonself-adjoint, then:

$$N(t) = \sum_{\mathrm{Re}\,\lambda_j \leqq t} 1 \sim (2\pi)^{-n} t^{n/2m} \int_S \int_{a(x,\xi)<1} d\xi dx \quad \text{as } t \to +\infty.$$

In §1, we give the notations and definitions; in §2, we state some known results; in §3 the Green's function of (A,γ) corresponding to the case of a half-space and constant coefficients is constructed. In §4, using the parametrix method, we construct the Green's function associated with (A,γ) when A and B_j are defined on S,Γ and have infinitely differentiable coefficients. Results for the self-adjoint case are given in §5 and for the nonself-adjoint case in §6.

1. **Notations and definitions.** Let E^n be the n-dimensional Euclidean space, S an open subset of E^n. The points of E^n will be denoted by $x=(x_1,\cdots,x_n)$ and integration with respect to dx over a subset of E^n denotes integration with respect to Lebesgue n-measure.

For $1 \leqq j \leqq n$, $D_j = i^{-1}\partial/\partial x_j$. If $\alpha=(\alpha_1,\cdots,\alpha_n)$ is any n-tuple of nonnegative integers, we set:

$$|\alpha| = \sum_{j=1}^{n} \alpha_j, \qquad D^\alpha = \prod_{j=1}^{n} D_j^{\alpha} .$$

The linear partial differential operator $A = \sum_{|\alpha|\leqq 2m} a_\alpha(x)D^\alpha$ with coefficients defined on S is said to be uniformly elliptic on S if there exists a constant $c>0$ such that:

$$|a(x,\xi)| \geqq c|\xi|^{2m} \quad \text{for every point } \xi \in R^n.$$

$a(x,\xi)$ is the characteristic form of A and is given by:

$$a(x,\xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha, \qquad \xi \text{ in } R^n.$$

We shall assume throughout for the sake of simplicity that S is bounded and that its boundary is locally a C^∞, $(n-1)$-dimensional manifold with C^∞ imbedding in E^n. In particular, S will satisfy the uniform regularity conditions (Browder [5]).

DEFINITION 1.1. *Let j be a nonnegative integer:*

$$W^{j,2}(S) = \{u : u \in L^2(S);\ D^\alpha u \in L^2(S),\ |\alpha| \leqq j\},$$

where $D^\alpha u$ denotes the derivative in the distribution sense.

$W^{j,2}(S)$ is a Hilbert space with respect to the inner product:

$$(u,v)_j = \sum_{|\alpha|\leqq j} (D^\alpha u, D^\alpha v), \quad u,v \text{ in } W^{j,2}(S),$$

$$(u,v) = \int_S u(x)\overline{v(x)}\,dx$$

is the usual inner product in $L^2(S)$.

The norm in $W^{j,2}(S)$ is given by:

$$\| u \|_{j,2} = \left(\sum_{|\alpha| \leqq j} \| D^\alpha u \|^2_{L^2(S)} \right)^{1/2}.$$

Let $\gamma = (B_1, \cdots, B_m)$ be a family of m differential operators with coefficients defined on S. We assume that the order r_j of each B_j is less than $2m$ and we write:

$$B_j = \sum_{|\beta| \leqq r_j} b_{j\beta}(x) D^\beta.$$

We also assume that the coefficients $b_{j\beta}$ lie in $C^\infty(\bar{S})$. For each j, the characteristic form is defined for $\zeta \in C^n$ by

$$b_j(x, \zeta) = \sum_{|\beta| = r_j} b_{j\beta}(x) \zeta^\beta.$$

DEFINITION 1.2. *A is said to be regularly elliptic on S if it is uniformly elliptic on S and if for each x of* Γ, *the polynomial* $a(x, \lambda N_x + T)$ *in* λ ($\lambda \in C; N_x$ *is the unit exterior normal vector to* Γ *at* x, T *any unit tangent vector to* Γ *at* x) *has exactly m roots (counting multiplicities) in the* λ *upper half plane.*

DEFINITION 1.3. *The boundary value problem* (A, γ) *is said to be uniformly regular if*:

(1) *A is uniformly regularly elliptic.*

(2) *For each unit tangent vector T to* Γ *at x, let* C_T *be a closed, Jordan rectifiable curve in the* λ *half plane which contains in its interior all the zeros of* $a(x, \lambda N_x + T)$ *with positive imaginary parts.*

For $1 \leqq j, k \leqq m$; *set*:

$$c_{jk}(x, T) = \int_{C_T} \lambda^{j-1} b_k(x, \lambda N_x + T) [a(x, \lambda N_x + T)]^{-1} d\lambda.$$

Then there exists $c > 0$ *such that*:

$$|\mathrm{Det}(c_{jk}(x, T))| \geqq c > 0$$

for all $x \in \Gamma$ *and all unit tangent vectors to* Γ *at x.*

2. DEFINITION 2.1. *Let* x_0 *be a point of* Γ; A_0 *the homogeneous differential operator of order 2m with constant coefficients*

$$A_0 = \sum_{|\alpha| = 2m} a_\alpha(x_0) D^\alpha.$$

Let $\gamma_0 = (B_{10}, \cdots, B_{m0})$ *where*:

$$B_{j0} = \sum_{|\beta| = r_j} b_{j\beta}(x_0) D^\beta.$$

Then (A, γ) *is said to be formally positive if the boundary value problem*:

$$(A_0 + tI)u = f \;\; on \;\; S, \qquad t > 0,$$
$$B_j u = 0 \;\; on \;\; \Gamma, \qquad j = 1, \cdots, m,$$

has a unique solution u in $C^\infty(S) \cap L^2(S)$ *for every f in* $C_c^\infty(E^n)|_S$ *and* $t > 0$, *such that*:

$$\| u \|_{L^2(S)} \leqq c \| f \|_{L^2(S)}$$

with the constant c independent of u,f.

We shall take the above definition as our basic assumption on (A,γ). Alternative assumptions which are equivalent to Definition 2.1 may be given in a more computational form; e.g. as given by Agmon [1]: the polynomial in the complex variable λ, $a(x_0, \lambda N_{x_0} + T) + t$ has no real roots; it has m roots with positive imaginary parts and:

$$\left| \operatorname{Det} \left(\int_{C_{t,T}} \lambda^{j-1} b_r(x_0, \lambda N_{x_0} + T) \, [a(x_0, \lambda N_{x_0} + T) + t]^{-1} d\lambda \right) \right| > 0,$$
$$r,j = 1, \cdots, m.$$

$C_{t,T}$ is a closed Jordan rectifiable curve surrounding all the roots with positive imaginary parts of $a(x_0, \lambda N_{x_0} + T) + t$ considered as a polynomial in λ.

DEFINITION 2.2. (1) *Let* A_γ *be the operator on* $L^2(S)$ *defined as follows*:

$$D(A_\gamma) = \{u : u \;\; in \;\; W^{2m,2}(S); \; B_j u = 0 \;\; on \;\; \Gamma; \; j = 1, \cdots, m\},$$
$$A_\gamma u = Au \quad if \;\; u \in D(A_\gamma).$$

(2) (A,γ) *is said to be formally self-adjoint if* A_γ *is a symmetric operator in* $L^2(S)$; *i.e.*

$$(A_\gamma u, v) = (u, A_\gamma v)$$

for all u,v in $D(A_\gamma)$.

THEOREM 2.1. *Let* (A,γ) *be a uniformly regularly elliptic boundary value problem as above, such that A is uniformly strongly elliptic and* (A,γ) *is formally positive.*

Then:

(1) *If* (A,γ) *is formally self-adjoint*; A_γ *is then a self-adjoint operator on* $L^2(S)$.

(2) *If* $t \geqq t_0 > 0$; $(A_\gamma + tI)^{-1}$ *exists on all of* $L^2(S)$ *and*:

$$\| (A_\gamma + tI)^{-1} \| \leqq ct^{-1}.$$

(3) *If p is a positive integer such that* $2mp > n$; $(A_\gamma + tI)^{-p}$ *is an operator of Hilbert-Schmidt type,*

$$(A_\gamma + tI)^{-p}f(x) = \int_S \mathscr{G}_{(p)}(x,y,t)f(y)dy \quad f \text{ in } L^2(S)$$

and $\mathscr{G}_{(p)}(x,y,t) \in L^2(S) \times L^2(S)$.

Proof. It has been proved by Agmon [1] that when (A,γ) is formally positive then $\|(A_\gamma + tI)^{-1}\| \leqq ct^{-1}$ and moreover if A_γ is formally self-adjoint, then A_γ is self-adjoint.

Since (A,γ) is a regular elliptic boundaryvalue problem; we have the a priori estimate:

$$\|u\|_{W^{2mp,2}} \leqq C\{\|u\|_{L^2} + \|(A_\gamma + tI)u\|^{W^{2m(p-1),2}}\},$$

$$\|u\|_{W^{2mp,2}} \leqq C\|(A_\gamma + tI)^p u\|_{L^2}.$$

Therefore $(A_\gamma + tI)^{-p}$ is a continuous mapping of $L^2(S)$ into $W^{2mp,2}(S)$. From the Sobolev imbedding theorem, the injection mapping:

$$W^{2mp,2}(S) \to L^\infty(S)$$

is continuous when $2mp > n$. Hence $(A_\gamma + tI)^{-p}$ considered as a mapping of $L^2(S)$ to $L^\infty(S)$ is continuous. By the Dunford-Pettis theorem, it follows that $(A_\gamma + tI)^{-p}$ is of Hilbert-Schmidt type:

$$(A_\gamma + tI)^{-p}f(x) = \int_S \mathscr{G}_{(p)}(x,y,t)f(y)dy, \quad f \text{ in } L^2(S).$$

3. Let A be a homogeneous linear elliptic differential operator on E^n with constant coefficients, of order $2m$.

Let $\gamma = \{B_j, j = 1,\cdots,m\}$ be a family of homogeneous linear differential operators defined on E^n with constant coefficients and of order $r_j < 2m-1$. Let t be a positive parameter such that (A,γ) is formally positive in the sense of Definition 2.1. In this section, we construct: (1) the Green's function $G(x,y,t)$ associated with $(A + tI,\gamma)$; (2) the iterates of G. Finally we study the asymptotic behavior of $G_{(p)}$ as $t \to +\infty$.

LEMMA 3.1. *Let A be a homogeneous, linear elliptic operator of order $2m$, on E^n with constant coefficients. Let t be a positive parameter such that $A + tI$ is positively strongly elliptic. Then $A + tI$ has a fundamental solution $E(x,y,t)$ given by:*

$$E(x,y,t) = (2\pi)^{-n}\int_{E^n} \exp(i\langle x-y,\xi\rangle)[a(\xi) + t]^{-1}d\xi$$

where the integral is taken as the Fourier transform of a tempered distribution if $2m < n$.

$E(x,y,t)$ is infinitely differentiable for $x \neq y$ and if $t = \tau^{2m}$ then:

$$D^\alpha E(x,y,t) = O(1)\tau^{-\varepsilon}|x-y|^{-n+2m-|\alpha|-\varepsilon}(1+|\tau(x-y)|^N)^{-1}$$

if $-n+2m-|\alpha| \leqq 0$ *and*

$$D^\alpha E(x,y,t) = O(1)\tau^{-\varepsilon}(1+|\tau(x-y)|^N)^{-1}$$

if $-n+2m-|\alpha|>0$, *where* $0<\varepsilon<1$; N *is an arbitrary positive number (Gårding* [9]).

THEOREM 3.1. *Let v be the solution of the boundary value problem*:

$$(A+tI)v = 0 \quad on\ E^n_+,$$
$$B_j v = B_j E \quad on\ E^{n-1}, \qquad j=1,\cdots,m,$$

where (A,γ) is a regular elliptic boundary value problem; A and B_j are homogeneous and have constant coefficients. Then:

(1) *$v(x,y,t)$ is infinitely differentiable for $x \neq y$ and is given by*:

$$v(x,y,t) = \sum_{r,j=1}^{m} (2\pi)^{-n}\int_{E^{n-1}} \exp(i\langle \hat{x}-\hat{y},\hat{\xi}\rangle)\, V_{rj}(\hat{\xi},x_1,y_1,t)\,d\hat{\xi}$$

with:

$$V_{rj}(\hat{\xi},x_1,y_1,t) = Q_{rj}(\hat{\xi},t)H_j(\hat{\xi},y_1,t) \int_{C_{\hat{\xi},t}} \zeta_1^{r-1}\exp(i\zeta_1 x_1)[a(\zeta_1,\hat{\xi})+t]^{-1}d\zeta_1$$

where $C_{\hat{\xi},t}$ is a closed, Jordan rectifiable curve in the ζ_1 upper half plane and surrounding all the m roots (counting multiplicities) of $a(\zeta_1,\hat{\xi})+t=0$ for fixed $\hat{\xi},t$.

$$H_j(\hat{\xi},y_1,t) = \int_{E^1} \exp(-iy_1\xi_1)b_j(\xi_1,\hat{\xi})[a(\xi_1,\hat{\xi})+t]^{-1}d\xi_1 .$$

Finally Q_{rj} are the elements of the transpose of the inverse of the matrix $(c_{rj}(\hat{\xi},t))$ with:

$$c_{rj}(\hat{\xi},t) = \int_{C_{\hat{\xi}_t,}} \zeta_1^{r-1}b_j(\zeta_1,\hat{\xi})[a(\zeta_1,\hat{\xi})+t]^{-1}d\zeta_1, \qquad r,j=1,\cdots,m.$$

(2) *Let $t=\tau^{2m}$ then $D^\alpha v(x,y,t) = O(1)\tau^{-\varepsilon}|\hat{x}-\hat{y}|^{-n+2m-|\alpha|-\varepsilon}(1+|\hat{x}-\hat{y}|^N)^{-1}$ if $-n+2m-|\alpha|\leqq 0$ and $D^\alpha v = O(1)\tau^{-\varepsilon}(1+|\hat{x}-\hat{y}|^N)^{-1}$ if $n+2m-|\alpha|>0$; $0<\varepsilon<1$; N is an arbitrary positive number.*

Proof. We write:

$$A = \sum_{q=0}^{2m} A_q(\hat{D})D_1^q, \qquad B_j = \sum_{q=0}^{r_j} B_{jq}(\hat{D})D_1^q,$$

where $\hat{D}$ denotes differential operators involving only $D_2,\cdots,D_n$. Taking the

Fourier transform with respect to the tangential variables $\hat{x} = (x_2, \cdots, x_n)$ we are reduced to the initial value problem:

$$\left(\sum_{q=0}^{2m} a_q(\hat{\xi}) D_1^q + tI \right) V(x_1, \hat{\xi}) = 0, \qquad x_1 > 0,$$

$$\sum_{q=0}^{r_j} b_{jq}(\hat{\xi}) D_1^q V(x_1, \hat{\xi}) = H_j(\hat{\xi}, t); \qquad x_1 = 0; \, j = 1, \cdots, m,$$

where:

$$V(x_1, \hat{\xi}) = (2\pi)^{-(n-1)/2} \int_{E^{n-1}} \exp(-i\langle \hat{x}, \hat{\xi} \rangle) v(\hat{x}_1, \hat{x}) d\hat{x}.$$

$H_j(\hat{\xi}, t)$ is similarly defined $(r_j < 2m - 1)$.

We consider a solution of the form

$$V(x_1, \hat{\xi}) = (2\pi)^{-1} \int_{C_{\hat{\xi},t}} \exp(ix_1\zeta_1) [a(\zeta_1, \hat{\xi}) + t]^{-1} p(\zeta_1) d\zeta_1,$$

where $p(\zeta_1)$ is a polynomial of degree less than or equal to $2m$ and $C_{\hat{\xi},t}$ is a closed, Jordan rectifiable curve in the upper half plane surrounding all the m roots (counting multiplicities) of $a(\zeta_1, \hat{\xi}) + t = 0$ for fixed $\hat{\xi}, t$.

By Cauchy's theorem, we may assume that $p(\zeta_1)$ is a polynomial of degree less than m:

$$p(\zeta_1) = \sum_{r=1}^{m} p_r \zeta_1^{r-1}.$$

We obtain a system of m equations with m unknowns.

$$(2\pi)^{-1} \sum_{r=1}^{m} p_r(\hat{\xi}) \int_{C_{\hat{\xi},t}} \zeta_1^{r-1} b_j(\zeta_1, \hat{\xi}) [a(\zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1 = H_j(\hat{\xi}, t); \qquad j = 1, \cdots, m.$$

We may solve it in a unique fashion. Indeed, we have:

$$\mathrm{Det}\left(\int_{\hat{\xi},t} \zeta_1^{r-1} b_j(\zeta_1, \hat{\xi}) [a(\zeta_1, \hat{\xi}') + t]^{-1} d\zeta_1 \right)$$

$$= |\hat{\xi}|^{\Sigma_j r_j + (m - m^2)/2} \mathrm{Det}\left(\int_{C_{\hat{\xi},t,|\hat{\xi}'|^{-m2}}} \zeta_1^{r-1} b_j(\zeta_1, \hat{\xi}') [a(\zeta_1, \hat{\xi}') + t |\hat{\xi}'|^{-2m}]^{-1} d\zeta_1 \right)$$

and the last expression is nonnull by hypothesis for $|\hat{\xi}| \neq 0$. We get:

$$p_r(\hat{\xi}) = \sum_{j=1}^{m} Q_{rj}(\hat{\xi}, t) H_j(\hat{\xi}, t),$$

$$V(x_1, \hat{\xi}, t) = \sum_{j=1}^{m} Q_{rj}(\hat{\xi}, t) H_j(\hat{\xi}, t) \int_{C_{\hat{\xi},t}} \zeta_1^{r-1} \exp(i\zeta_1 x_1) [a(\zeta_1, \hat{\xi}) + t]^{-1} d\zeta_1.$$

We want to take the inverse Fourier transform of V. To prove the summability of $V(x_1,\xi,t)$ with respect to ξ we establish the following lemmas.

LEMMA 3.2. *The following estimates hold uniformly for* $t \geqq t_0 > 0$:

$$f(x_1,\xi,t) = \int_{C_{\xi,t}} \zeta_1^{r-1}\exp(i\zeta_1 x_1)[a(\zeta_1,\xi)+t]^{-1}d\zeta_1$$
$$= O((1+|\xi|^{2m-r})^{-1})\exp(-d|\xi|x_1)$$

where $x_1 > 0$, $d > 0$ *independent of* t *and* $C_{\xi,t}$ *is a closed, Jordan rectifiable curve in thc* ζ_1 *upper half plane surrounding all the m roots of* $a(\zeta_1,\xi)+t$ *for fixed* ξ,t. $a(\zeta_1,\xi)$ *is a homogeneous polynomial of degree* $2m$.

Proof. Making the change of variables $\xi = t^{1/2m}\xi'$, $\zeta_1 = t^{1/2m}\zeta_1'$, we get:

$$f(x_1,\xi,t) = t^{r/2m-1}\int_{C_{\xi',1}} \zeta_1^{r-1}\exp(i\zeta_1 x_1 t^{1/2m})[a(\zeta_1,\xi)+1]^{-1}d\zeta_1$$
$$= t^{r/2m-1}f(x_1 t^{1/2m},\xi t^{-1/2m},1).$$

We consider:

$$f(x_1,\xi,1) = \int_{C_\xi} \zeta_1^{r-1}\exp(i\zeta_1 x_1)(a(\zeta_1,\xi)+1]^{-1}d\zeta_1.$$

(1) the equation $a(\zeta_1,\xi)+1=0$ may be written as $c\zeta_1^{2m}+P(\zeta_1,\xi)=0$ where $P(\zeta_1,\xi)$ is a polynomial of degree less than or equal to $2m-1$ in the ζ_1 variable and such that $P(\zeta_1,0)=1$; c is a constant. The roots of $a(\zeta_1,\xi)+1=0$ depend continuously on the parameter ξ and as ξ goes to zero they go to the roots of $c\zeta_1^{2m}+1=0$. Therefore, there is a closed curve C containing all the roots of $a(\zeta_1,\xi)+1=0$ for fixed and small ξ.

For $|\xi|<1$, we have:

$$f(x_1,\xi,1) = \int_C \zeta_1^{r-1}\exp(ix_1\zeta_1)[a(\zeta_1,\xi)+1]^{-1}d\zeta_1$$

so:

$$|f(x_1,\xi,1)| \leqq M,$$
$$|f(x_1,\xi,t)| \leqq Mt_0^{r/2m-1}.$$

(2) To study the behavior of $f(x_1,\xi,1)$ as $|\xi|$ goes to infinity, we make the change of variables $\xi=\tau\xi'$; $\zeta_1=\tau\zeta_1$ with $\tau=|\xi|$. Then

$$f(x_1,\xi,1) = |\xi|^{r-2m}\int_{C_{\xi,t^{-2m}}} \exp(i\zeta_1\tau x_1)[a(\zeta_1,\xi')+\tau^{-2m}]^{-1}d\zeta_1.$$

Consider the equation $a(\zeta_1,\xi')+\tau^{-2m}=0$ for large τ. The roots of the equation considered as a polynomial in ζ_1 depend continuously on τ^{-1} and so as τ goes to infinity they go to the roots of $a(\zeta_1,\xi')=0$; $|\xi'|=1$. Therefore there exists a closed curve $C_{\xi'}$ in the upper half plane surrounding all the roots of $a(\zeta_1,\xi') + \tau^{-2m} = 0$ for large τ

$$f(x_1,\xi,1) = \tau^{r-2m}\int_{C_{\xi'}} \zeta_1^{r-1}\exp(i\zeta_1\tau x_1)[a(\zeta_1,\xi') + \tau^{-2m}]^{-1}d\zeta_1.$$

On the curve $C_{\xi'}$, we have, $|a(\zeta_1,\xi)| \geqq c > 0$. On the other hand $|a(\zeta_1,\xi') + \tau^{-2m}| \geqq |a(\zeta_1,\xi')| - \tau^{-2m}$. For sufficiently large τ, $|a(\zeta_1,\xi') + \tau^{-2m}| \geqq c_1 > 0$. Therefore for large ξ

$$|f(x_1,\xi,1)| \leqq M\exp(-d|\xi|x_1)|\xi|^{r-2m}$$

where $d = \operatorname{Inf}_{\zeta_1\in C_{\xi'}}(\operatorname{Im}\zeta_1) > 0$, M is a constant and

$$|f(x_1,\xi,t)| \leqq M\exp(-d|\xi|x_1)|\xi|^{r-2m}, \qquad |\xi| > 1.$$

M is independent of x_1, t.

Combining (1), (2) we get the lemma.

Lemma 3.3. 1. *The following estimates hold uniformly for* $t \geqq t_0 > 0$

$$c_{rj}(\xi,t) = \int_{C_{\xi,t}} \zeta_1^{r-1}b_j(\zeta_1,\xi)[a(\zeta_1,\xi) + t]^{-1}d\zeta_1 = O((1 + |\xi|^{r+r_j})(1 + |\xi|^{2m})^{-1})$$

where $r = 1,\cdots,m$; $r_j < 2m$ *and* $C_{\xi,t}$ *is a closed Jordan rectifiable curve in the* ζ_1 *upper half plane surrounding all the* m *roots of* $a(\zeta_1,\xi) + t = 0$ *for fixed* ξ, t. $b_j(\zeta_1,\xi)$; $a(\zeta_1,\xi)$ *are homogeneous polynomials of degree* $r_j, 2m$ *respectively.*

2. *Let* $Q_{rj}(\xi,t)$ *be the elements of the transpose of the inverse of the matrix* $(c_{rj}(\xi,t))$ *then*:

$$Q_{rj}(\xi,t) = O((1 + |\xi|^{2m})(1 + |\xi|^{r+r_j})^{-1}).$$

Proof. The proof is similar to that of Lemma 3.2. We will not repeat it.

We return to the proof of Theorem 3.1. From Lemmas 3.2, 3.3 and noting that $H_j(\xi,t) = O(1 + |\xi|^{r_j})(1 + |\xi|^{2m})^{-1})$ we obtain:

$$V(x_1,\xi,t) = O((1 + |\xi|^{2m})^{-1})\exp(-d|\xi|x_1)$$

where $d > 0$; $x_1 > 0$; $t \geqq t_0 > 0$.

For $x_1 > 0$, $V(x_1,\xi,t)$ is integrable with respect to ξ we have:

$$(2\pi)^n v(x,t) =$$

$$\sum_{r,j=1}^{m}\int_{E^{n-1}} \exp(i\langle\hat{x},\xi\rangle)Q_{rj}(\xi,t)H_j(\xi,t)\int_{C_{\xi,t}} \zeta_1^{r-1}\exp(i\zeta_1 x_1)[a(\zeta_1,\xi) + t]^{-1}d\zeta_1 d\xi.$$

We study the regularity of $v(x,t)$. The results are stated in the following lemma:

Lemma 3.4. *Let* $v(x,t)$ *be given by the expression*:

$v(x,t) =$

$$\sum_{r,j=1}^{m}(2\pi)^{-}\int_{E^{n-1}}\exp(i\langle\hat{x},\hat{\xi}\rangle)Q_{rj}(\hat{\xi},t)H_j(\hat{\xi},t)\int_{C_{\hat{\xi},t}}\xi_1^{r-1}\exp(i\zeta_1x_1)[a(\zeta_1,\hat{\xi})+t]^{-1}d\zeta_1d\hat{\xi}$$

where $Q_{rj};H_j$ are defined in Theorem 3.1, then:

(1) *$v(x,t)$ is infinitely differentiable for $\hat{x}\neq 0$.*

(2) *$D^{\alpha}v(0,\hat{x},t)=O(1)t^{-\varepsilon/2m}|\hat{x}|^{-n-|\alpha|+2m-\varepsilon}(1+|\hat{x}|^{N})^{-1}$ if $-n-|\alpha|+2m\leqq 0$ and $D^{\alpha}v(0,\hat{x},t)=O(1)t^{-\varepsilon/2m}(1+|\hat{x}|^{N})^{-1}$ otherwise. $0<\varepsilon<1$ and N is an arbitrary positive integer.*

Proof. (1) First we consider the case when $x_1>0$. We may take the differentiation under the integral sign. Indeed:

$$K_{rj}(\hat{\xi},x_1,t,\alpha+\beta) = \hat{\xi}^{\alpha}Q_{rj}(\hat{\xi},t)H_j(\hat{\xi},t)\int_{C_{\hat{\xi},t}}\zeta_1^{r-1+\beta}\exp(i\zeta_1x_1)[a(\zeta_1,\hat{\xi})+t]^{-1}d\zeta_1,$$

$$K_{rj}(\hat{\xi},x_1,t,\alpha+\beta) = O(|\hat{\xi}|^{\alpha+\beta}(1+|\hat{\xi}|^{2m-1})^{-1}\exp(-d|\hat{\xi}|x_1)),$$

hence for $x_1>0$; $v(x,t)$ is infinitely differentiable.

(2) We now study the regularity of $v(x,t)$ as x_1 goes to zero.

For $x_1>0$, we have:

$$\hat{x}^{p}D^{\alpha+\beta}v(x_1,\hat{x},t)=(2\pi)^{-n}\sum_{r,j=1}^{m}\int_{E^{n-1}}\exp(i\langle\hat{x},\hat{\xi}\rangle)D_{\hat{\xi}}^{p}K_{rj}(\hat{\xi},x_1,\alpha+\beta,t)\,d\hat{\xi}.$$

Making the change of variables $\hat{\xi}=t^{1/2m}\hat{\xi}',\zeta_1=t^{1/2m}\zeta_1'$, we get:

$$(2\pi)^{n}\hat{x}^{p}D^{\alpha+\beta}v(x_1,\hat{x},t)$$

$$=\sum_{r,j=1}^{m}t^{(\alpha+\beta+n-p)/2m-1}\int_{E^{n-1}}\exp(i\langle\hat{x}t^{1/2m},\hat{\xi}\rangle)D_{\hat{\xi}}^{p}K_{rj}(\hat{\xi},x_1t^{1/2m},\alpha+\beta)d\hat{\xi}.$$

Consider the expression $D_{\hat{\xi}}^{p}K_{rj}(\hat{\xi},x_1t^{1/2m},\alpha+\beta)$. We have:

$$|D_{\hat{\xi}}^{p}K_{rj}(\hat{\xi},x_1t^{1/2m},\alpha+\beta)|\leqq M \quad \text{for } |\hat{\xi}|\leqq 1.$$

M is a constant independent of x_1,t.

Making the change of variables $\hat{\xi}=\tau\hat{\xi}'$; $\zeta_1=\tau\zeta_1'$ with $\tau=|\hat{\xi}|$; we get:

$$D_{\hat{\xi}}^{p}K_{rj}(\hat{\xi},x_1t^{1/2m},\alpha+\beta)=\tau^{\alpha+\beta+p-2m+1}D_{\hat{\xi}'}^{p}K_{rj}(\hat{\xi}',x_1t^{1/2m}\tau,\alpha+\beta)$$

so:

$$|D_{\hat{\xi}}^{p}K_{rj}(\hat{\xi},x_1t^{1/2m},\alpha+\beta)|\leqq M|\hat{\xi}|^{|\alpha|+|\beta|-2m+1-p} \quad \text{for } |\hat{\xi}|\geqq 1.$$

If $|\alpha|+|\beta|+n+1-2m\geqq 0$, we take $p=n+1+|\alpha|+|\beta|-2m$ and:

$$|D_{\hat{\xi}}^{p}K_{rj}(\hat{\xi},x_1t^{1/2m},\alpha+\beta)|\leqq M(1+|\hat{\xi}|^{n})^{-1} \quad \text{for all } \hat{\xi}.$$

Since $\hat{x}^{p}D^{\alpha+\beta}v(x_1,\hat{x},t)=0$ for $\hat{x}=0$ and $p=n+|\alpha|+|\beta|+1-2m\geqq 0$, we obtain:

$$D^{\alpha+\beta}v(x_1,\hat{x},t) = O(1)t^{-\varepsilon/2m}|\hat{x}|^{-n-|\alpha|-|\beta|+2m-\varepsilon}(1+|\hat{x}|^N)^{-1}; \qquad x_1 \geqq 0.$$

If $|\alpha| + |\beta| + 1 + n - 2m < 0$, we take $p = 0$, then:

$$|D_{\hat{\xi}}^p K_{rj}(\hat{\xi}, x_1 t^{1/2m}, \alpha+\beta)| \leqq M|\hat{\xi}|^{-n} \qquad \text{for } |\hat{\xi}| \geqq 1.$$

Therefore:

$$|D_{\hat{\xi}}^p K_{rj}(\hat{\xi}, x_1 t^{1/2m}, \alpha+\beta)| \leqq M(1+|\hat{\xi}|^n)^{-1} \qquad \text{for all } \hat{\xi}.$$

We get:

$$D^{\alpha+\beta}v(x_1,\hat{x},t) = O(1)t^{-\varepsilon/2m}(1+|\hat{x}|^N)^{-1} \qquad \text{for } x_1 \geqq 0.$$

The lemma is proved.

THEOREM 3.2. *Let A be a positively strongly elliptic operator on E^n, homogeneous, of order $2m$, with constant coefficients. Let $B_1, \cdots, B_m$ be m homogeneous, differential operators on E^n of order $r_j < 2m-1$ with constant coefficients. Let t be a positive parameter such that (A,γ) is formally positive. Then $G(x\ y,t)$ is given by the expression:*

$$G(x,y,t) = (2\pi)^{-n}\int_{E^n} \exp(i\langle x-y,\xi\rangle)[a(\xi)+t]^{-1}d\xi + v(x,y,t)$$

where the integral is taken as the Fourier transform of a tempered distribution if $2m < n$; $v(x,y,t)$ is given by Theorem 3.1.

(1) $G(x,y,t)$ is infinitely differentiable for $x \neq y$,

(2) $D^\alpha G(x,y,t) = O(1)t^{-\varepsilon/2m}|x-y|^{n+2m-|\alpha|-\varepsilon}(1+|x-y|^N)^{-1}$ if $2m-n \leqq |\alpha|$ and

$$D^\alpha G(x,y,t) = O(1)t^{-\varepsilon/2m}(1+|x-y|^N)^{-1}, \quad \text{if } 0 \leqq |\alpha| < 2m-n,$$

where $0 < \varepsilon < 1$ and N is an arbitrary positive number.

Proof. The theorem follows immediately from Lemma 3.1 and Theorem 3.1.

THEOREM 3.3. *Let $G(x,y,t)$ be the Green's function defined in Theorem 3.2 and $2m > n$; then:*

(1) $x \neq y$, $t^{1-n/2m}G(x,y,t) \to 0$ as $t \to +\infty$,

(2) $t^{1-n/2m}G(x,x,t) \sim (2\pi)^{-n}\int_{E^n}[a(\xi)+1]^{-1}d\xi + O(1)t^{-\varepsilon/2m}$ as $t \to +\infty$.

Proof. Let $t = \tau^{2m}$ and make the change of variables $\hat{\xi} = \tau\hat{\xi}'$; $\zeta_1 = \tau\zeta_1'$. We obtain:

$$G(x,y,t) = \tau^{n-2m}E(\tau x,\tau y,1) + v(x,y,t).$$

We consider the expression $v(x,y,t)$. From Lemma 3.4, we have:

$$v(x,y,t) = \sum_{r,j=1}^{m} \tau^{n-1-r_j}\int_{E^{n-1}} \exp(i\langle \tau\hat{x}-\tau\hat{y},\hat{\xi}\rangle)Q_{rj}(\hat{\xi},1)H_j(\hat{\xi},y_1,t)$$

$$\cdot \int_{C_{\hat{\xi}}} \zeta_1^{r-1}\exp(i\zeta_1\tau x_1)\,[a(\zeta_1,\hat{\xi})+1]^{-1}d\zeta_1 d\hat{\xi}.$$

On the other hand; $H_j(\hat{\xi},y_1,t) = O(1)\tau^{r_j+1-2m-\varepsilon}(1+|\hat{\xi}|^{2m-r_j})^{-1}$. When $x \neq y$ then by the Riemann-Lebesgue theorem, we have:

$$\tau^{2m-n}G(x,y,t)\to 0 \quad \text{as } t\to+\infty.$$

THEOREM 3.4. *Let $G_{(p)}(x,y,t)$ be the pth iterate of the Green's function G defined by Theorem 3.1. Let p be such that $2mp > n$; then:*

(1) *If $x\neq y$, $t^{p-n/2m}G_{(p)}(x,y,t)\to 0$ as $t\to+\infty$,*

(2) $t^{p-n/2m}G_{(p)}(x,x,t) = (2\pi)^{-n}\int_{E^n}[a(\xi)+1]^{-p}d\xi + O(1)t^{-\varepsilon/2m}$, $0<\varepsilon<1$, $t\to+\infty$.

Proof. We construct the iterates of G. We know that:

$$G_{(2)}(x,y,t) = \int_{E^n} G(x,z,t)\,G(z,y,t)dz, \qquad x\neq y.$$

The integral is well defined. We also have:

$$F_{\hat{x}}G(x,z,t) = (2\pi)^{-(n-1)/2}\exp(-i\langle\hat{z},\hat{\xi}\rangle)\int_{E'}\exp(i(x_1-z_1)\xi_1)[a(\xi_1,\hat{\xi})+t]^{-1}d\xi_1$$

$$+ \exp(-i\langle\hat{z},\hat{\xi}\rangle)\,V(x_1,z_1,\hat{\xi},t)$$

with:

$$V(x_1,z_1,\hat{\xi},t) = \sum_{r,j=1}^{m} Q_{rj}(\hat{\xi},t)H_j(\hat{\xi},z_1,t)\int_{C_{\hat{\xi},t}} \zeta_1^{r-1}\exp(i\zeta_1x_1)[a(\zeta_1,\hat{\xi})+t]^{-1}d\zeta_1.$$

Consider the Fourier transform of $G_{(2)}(x,y,t)$ with respect to the tangential variables $\hat{x}$.

$$F_{\hat{x}}G_{(2)}(x,y,t) = (2\pi)^{(1-n)/2}\int_{E^{n-1}}\exp(-i\langle\hat{x},\hat{\xi}\rangle)\,G_{(2)}(x,y,t)d\hat{x}, \qquad x\neq y,$$

$$= (2\pi)^{(1-n)/2}\int_{E^{n-1}}\int_{E^n}\exp(-i\langle\hat{x},\hat{\xi}\rangle)\,G(x,z,t)\,G(y,z,t)dzd\hat{x}.$$

By Fubini's theorem we may interchange the order of integration. We obtain:

$$F_{\hat{x}}G_{(2)}(x,y,t) = \int_{E^n} F_{\hat{x}}G(x,z,t)\,G(y,z,t)dz,$$

$$F_{\hat{x}}G_{(2)}(x,y,t)$$

$$=(2\pi)^{(1-n)/2}\int_{E^n}\int_{E^1}\exp(-i\langle\hat{z},\hat{\xi}\rangle)\exp(i\xi_1(x_1-z_1))[a(\xi_1,\hat{\xi})+t]^{-1}\,G(y,z,t)d\xi_1\,dz$$

$$+(2\pi)^{(1-n)/2}\int_{E^n}\exp(-i\langle\hat{z},\hat{\xi}\rangle)\,V(\hat{\xi},x_1,z_1,t)\,G(y,z,t)dz.$$

From Theorem 3.2; Lemmas 3.2; 3.3; it follows that all the integrals are well defined. The first integral of the above expression may be written as follows:

$$(2\pi)^{(1-n)/2}\int_{E^n} \exp(-i\langle \hat{z},\hat{\xi}\rangle)\, G(y,z,t)\int_{E^1} \exp[i\xi_1(x_1-z_1)]\,[a(\xi_1,\hat{\xi})+t]^{-1}\, d\xi_1 dz.$$

Denote by $k(z_1,x_1,t,\hat{\xi}) = \int_{E^1}\exp[i\xi_1(x_1-z_1)]\,[a(\xi_1,\hat{\xi})+t]^{-1}d\xi_1$. Then:

$$(2\pi)^{(1-n)/2}\int_{E^n} \exp(-i\langle \hat{z},\hat{\xi}\rangle)\, G(y,z,t)\, k(z_1,x_1,\hat{\xi},t)dz$$

$$= (2\pi)^{(1-n)/2}\int_{E^n} k(z_1,x_1,\hat{\xi},t)\int_{E^{n-1}} \exp(-i\langle \hat{z},\hat{\xi}\rangle)\, G(y,z,t)d\hat{z}dz_1$$

$$= \int_{E^1} k(z_1,x_1,\hat{\xi},t)\, F_2G(x,y,t)dz_1$$

$$=(2\pi)^{(1-n)/2}\int_{E^1} k(z_1-x_1,\hat{\xi},t)\exp(-i\langle\hat{y},\hat{\xi}\rangle)\int_{E^1}\exp[i\eta_1(z_1-y_1)][a(\eta_1,\hat{\xi})+t]^{-1}d\eta_1 dz_1$$

$$+\ (2\pi)^{(1-n)/2}\int_{E^1} k(z_1-x_1,\hat{\xi},t)\exp(-i\langle\hat{y},\hat{\xi}\rangle)\, V(\hat{\xi},y_1,z_1,t)dz_1.$$

Consider the first term. It is easy to see that $k(u,\hat{\xi},t)$ is integrable with respect to u. Applying Fubini's theorem, we get:

$$(2\pi)^{-n/2}\exp(-i\langle\hat{y},\hat{\xi}\rangle)\int_{E^1}\exp[i\xi_1(x_1-y_1)]\,[a(\xi_1,\hat{\xi})+t]^{-2}d\xi_1$$

since

$$\int_{E^1}\exp(i\eta_1 z_1)k(z_1-x_1,\hat{\xi},t)dz_1 = (2\pi)^{-1/2}\exp(i\eta_1 x_1)\,[a(\eta_1,\hat{\xi})+t]^{-1}.$$

We consider:

$$(2\pi)^{(1-n)/2}\int_{E^n}\exp(-i\langle\hat{z},\hat{\xi}\rangle)\, V(\hat{\xi},x_1,z_1,t)\, G(y,z,t)dz.$$

As before we may write it as

$$\int_{E^1} V(\hat{\xi},x_1,z_1,t)\left\{(2\pi)^{(1-n)/2}\int_{E^{n-1}}\exp(-i\langle\hat{z},\hat{\xi}\rangle)G(y,z,t)d\hat{z}\right\}dz_1.$$

The integral in the bracket is the Fourier transform of G with respect to the tangential variable $\hat{z}$. We obtain:

$$\int_{E^1}\int_{E^1}\exp(-i\langle\hat{y},\hat{\xi}\rangle)\exp[i\xi_1(z_1-y_1)]\, V(\hat{\xi},y_1,z_1,t)\,[a(\xi_1,\hat{\xi})+t]^{-1}d\xi_1 dz_1$$

$$+\int_{E^1}\exp(-i\langle\hat{y},\hat{\xi}\rangle)\, V(\hat{\xi},t,y_1,z_1)\, V(\hat{\xi},t,y_1,x_1)dz_1.$$

Denote the first integral by $h_1(\hat{\xi}, x_1, y_1, t)$ and the second one by $h_2(\hat{\xi}, x_1, y_1, t)$; then:

$$F_{\hat{x}}G_{(2)}(x,y,t) = (2\pi)^{(1-n)/2}\exp(-i\langle\hat{y},\hat{\xi}\rangle)\int_{E^1}\exp[i\xi_1(x_1-y_1)][a(\xi_1,\hat{\xi})+t]^{-2}d\xi_1$$

$$+ \; h_1(\hat{\xi},x_1,y_1,t) + h_2(\hat{\xi},x_1,y_1,t) + g(\hat{\xi},x_1,y_1,t)$$

with

$$g(\hat{\xi},x_1,y_1,t) = \int_{E^1}\exp(-i\langle\hat{y},\hat{\xi}\rangle)k(z_1-x_1,\hat{\xi},t)V(\hat{\xi},z_1,y_1,t)dz_1.$$

If $4m > n$, we want to take the inverse Fourier transform of $F_{\hat{x}}G_{(2)}$. First we establish the following lemma:

LEMMA 3.5. *Let* $g(\hat{\xi},x_1,y_1,t)$; $h_1(\hat{\xi},x_1,y_1,t)$, $h_2(\hat{\xi},x_1,y_1,t)$ *be as above; then the following estimates hold uniformly for* $t \geqq t_0 > 0$

(1) $g(\hat{\xi},x_1,y_1,t) = O((1+|\hat{\xi}|^{4m})^{-1})$,
(2) $h_1(\hat{\xi},x_1,y_1,t) = O((1+|\hat{\xi}|^{4m})^{-1})$,
(3) $h_2(\hat{\xi},x_1,y_1,t) = O((1+|\hat{\xi}|^{4m})^{-1})$.

Proof. Making the change of variable $\hat{\xi} = t^{1/2m}\hat{\xi}'$ and taking into account the results of Lemmas 3.2, 3.3, we get the above estimates immediately. We return to the proof of the theorem. If $4m > n$, we may take the inverse Fourier transform of $F_{\hat{x}}G_{(2)}$ with respect to $\hat{\xi}$. We obtain:

$$G_{(2)}(x,y,t) = (2\pi)^{-n}\int_{E^n}\exp(-i\langle x-y,\xi\rangle)[a(\xi)+t]^{-2}d\xi$$

$$+ \; (2\pi)^{(1-n)/2}\int_{E^{n-1}}\exp(i\langle\hat{x},\hat{\xi}\rangle)\{h_1(\hat{\xi},x_1,y_1,t)+h_2(\hat{\xi},x_1,y_1,t)\}d\hat{\xi}.$$

If $4m < n$, we construct $F_{\hat{x}}G_{(3)}(x,y,t)$ etc. step by step. We take only the first term and we have to make an estimate of the error involved in terms of the parameter t, for large t.

From the proof of Theorem 3.3, we have:

$g(\hat{\xi},x_1,y_1,t) = O(1)\tau^{-4m+1-\varepsilon}(1+|\hat{\xi}|^{4m-1})^{-1}$,
$h_1(\hat{\xi},x_1,y_1,t) = O(1)\tau^{-4m+1-\varepsilon}(1+|\hat{\xi}|^{4m-1})^{-1}$,
$h_2(\hat{\xi},x_1,y_1,t) = O(1)\tau^{-4m+1-\varepsilon}(1+|\hat{\xi}|^{4m-1})^{-1}$.

Therefore if $4m > n$, we have:

$$G_{(2)}(x,y,t) = (2\pi)^{-n}\int_{E^n}\exp(i\langle x-y,\xi\rangle)[a(\xi)+t]^{-2}d\xi + O(1)t^{-2+(n-\varepsilon)/2m}.$$

More generally if $2mp > n$,

$$G_{(p)}(x,y,t) = (2\pi)^{-n}\int_{E^n}\exp(i\langle x-y,\xi\rangle)[a(\xi)+t]^{-p}d\xi + O(1)t^{-p+(n-\varepsilon)/2m}.$$

The conclusion of the theorem follows immediately.

4. In this section the Green's function $\mathscr{G}(x,y,t)$ corresponding to the elliptic boundary value problem $\{A+tI, B_j,\ j=1,\cdots,m\}$ where A and B_j are defined respectively on S and on Γ and have infinitely differentiable coefficients is constructed. We will:

(1) Construct the Green's function G associated with the elliptic boundary value problem $\{A+tI; B_j; j=1,\cdots,m\}$ where A and B_j are defined on E^n_+, E^{n-1} respectively, with infinitely differentiable coefficients.

(2) Seek an integral representation of a function $u(x)$, infinitely differentiable function with compact support in $E^n_+ \cup E^{n-1}$ in terms of $(A+tI)u, B_ju$.

(3) Get the function $\mathscr{G}$ using (1) and (2).

LEMMA 4.1. *Let $H_{jy}(x_1,\hat{x}-\hat{y},t)$ be given by the expression:*

$$H_{j\hat{y}}(x_1,\hat{x}-\hat{y},t) = \sum_{r=1}^{m}\int_{E^{n-1}} \exp(i\langle\hat{x}-\hat{y},\hat{\xi}\rangle)Q_{rj}(\hat{y},\hat{\xi},t)\int_{C_{\hat{\xi},t}} \zeta_1^{r-1}\exp(i\zeta_1 x_1)[a(\hat{y},\zeta_1,\hat{\xi})+t]^{-1}\,d\zeta_1 d\hat{\xi}$$

where $a(\hat{y},\zeta_1,\hat{\xi})$ is the characteristic form of the homogeneous regularly elliptic operator $A_{\hat{y}}$ with coefficients evaluated at $\hat{y}$; $C_{\hat{\xi},t}$ is a closed curve in the ζ_1 upper half plane surrounding the roots of $a(\hat{y},\zeta_1,\hat{\xi})+t=0$ for fixed $\hat{\xi},t$.

$Q_{rj}(\hat{y},\hat{\xi},t)$ are the elements of the transpose of the inverse of the matrix:

$$\left(c_{rj}(\hat{y},\hat{\xi},t) = \int_{C_{\hat{\xi},t}} \zeta_1^{r-1} b_j(\hat{y},\zeta_1,\hat{\xi})[a(\hat{y},\zeta_1,\hat{\xi})+t]^{-1}\,d\zeta_1\right).$$

$b_j(\hat{y},\xi)$ is the characteristic form of the differential operator $B_{j\hat{y}}$ of order r_j and with coefficients evaluated at $\hat{y}$.

$H_{j\hat{y}}(x_1,\hat{x}-\hat{y},t)$ is infinitely differentiable for $\hat{x}\neq\hat{y}$ and:

$$(B_k - B_{k\hat{y}})H_{j\hat{y}}(0,\hat{x}-\hat{y},t) = O(1)t^{-\varepsilon/2m}\,|\hat{x}-\hat{y}|^{-n+2-\varepsilon},\quad k,j=1,\cdots,m,$$

$$AH_{j\hat{y}}(x_1,\hat{x}-\hat{y},t) = O(1)t^{-\varepsilon/2m}\,|\hat{x}-\hat{y}|^{-n+2-\varepsilon},\qquad x_1>0.$$

Proof. $H_{j\hat{y}}(x_1,\hat{x}-\hat{y},t)$ is well defined and for $x_1>0$ is infinitely differentiable (Lemmas 3.2, 3.3). We study the case when $x_1\to 0$. Let p be a positive integer and consider:

$$(\hat{x}-\hat{y})^p\hat{D}^\alpha D_1^\beta H_{j\hat{y}}(x_1,\hat{x}-\hat{y},t) = \sum_{r=1}^{m}\int_{E^{n-1}} \exp(i\langle\hat{x}-\hat{y},\hat{\xi}\rangle) \cdot D_{\hat{\xi}}^p\left\{\hat{\xi}^\alpha Q_{rj}(\hat{\xi},\hat{y},t)\int_{C_{\hat{\xi},t}} \zeta_1^{r-1+\beta}\exp(i\zeta_1 x_1)[a(\hat{y},\zeta_1,\hat{\xi})+t]^{-1}\,d\zeta_1\right\}d\hat{\xi}.$$

The expression $D_{\hat\xi}^p\{\ \}$ is integrable at the origin when $x_1 \to 0$. We have only to consider it at infinity.

$$D_{\hat\xi}^p\left\{\hat\xi^\alpha Q_{rj}(\hat y,\hat\xi,t)\int_{C_{\hat\xi,t}} \zeta_1^{\beta+r-1}[a(\hat y,\zeta_1,\hat\xi)+t]^{-1}d\zeta_1\right\}$$

$$= t^{(|\alpha|+|\beta|-r_j-p)/2m}D_{\hat\xi}^p\left\{\hat\xi^\alpha Q_{rj}(\hat y,\hat\xi,1)\int_{C_{\hat\xi,1}} \zeta_1^{r-1+\beta}[a(\hat y,\zeta_1,\hat\xi)+1]^{-1}d\zeta_1\right\}.$$

Making the change of variables $\hat\xi = \tau\hat\xi'$, $\zeta_1 = \tau\zeta_1'$ with $\tau = |\hat\xi|$; we get:

$$D_{\hat\xi}^p\left\{\hat\xi^\alpha Q_{rj}(\hat y,\hat\xi,1)\int_{C_{\hat\xi,1}} \zeta_1^{r-1+\beta}[a(\hat y,\zeta_1,\hat\xi)+1]^{-1}d\zeta_1\right\}$$

$$\tau^{\alpha+\beta-r_j-p}D_{\hat\xi'}^p\left\{\hat\xi'^\alpha Q_{rj}(\hat y,\hat\xi',\tau^{-2m})\int_{C_{\hat\xi'}} \zeta_1^{r-1+\beta}[a(\hat y,\zeta_1,\hat\xi')+\tau^{-2m}]^{-1}d\zeta_1\right\}.$$

So:

$$D_{\hat\xi}^p\left\{\hat\xi^\alpha Q_{rj}(\hat y,\hat\xi,t)\int_{C_{\hat\xi,t}} \zeta_1^{r-1+\beta}[a(\hat y,\zeta_1,\hat\xi)+t]^{-1}d\zeta_1\right.$$

$$= t^{(\alpha+\beta-r_j-p)/2m}O(1)|\hat\xi|^{-r_j-p+|\alpha|+|\beta|} \quad \text{for large } |\hat\xi|.$$

If we take $p = -r_j + n + |\alpha| + |\beta| \geqq 0$, the expression $(\hat x-\hat y)^p\hat D^\alpha D_1^\beta H_{j\hat y}$ is well defined. When $\hat x=\hat y$, it is equal to zero. We may replace $\exp(i\langle\hat x-\hat y,\hat\xi\rangle)$ by $\exp(i\langle\hat x-\hat y,\hat\xi\rangle)-1$ which is less than $|\hat x-\hat y|^{1-\varepsilon}|\hat\xi|^{-\varepsilon+1}$. We get:

$$D^\alpha H_{j\hat y}(0,\hat x-\hat y,t) = O(1)t^{-\varepsilon/2m}|\hat x-\hat y|^{r_j-n-|\alpha|-|\beta|+1-\varepsilon}.$$

It follows that if $r_k \leqq r_j$:

$$(B_k - B_{k\hat y})H_{j\hat y}(0,\hat x-\hat y,t) = O(1)t^{-\varepsilon/2m}|\hat x-\hat y|^{-n+2-\varepsilon}.$$

When $r_k > r_j$; consider

$$B_kH_{j\hat y}(x_1,\hat x-\hat y,t) = \sum_{r=1}^m \int_{E^{n-1}} \exp(i\langle\hat x-\hat y,\hat\xi\rangle)Q_{rj}(\hat y,\hat\xi,t)$$

$$\cdot\int_{C_{\hat\xi,t}} \zeta_1^{r-1}\exp(i\zeta_1x_1)b_k(\hat x,\zeta_1,\hat\xi)[a(\hat y,\zeta_1,\hat\xi)+t]^{-1}d\zeta_1d\hat\xi.$$

It is well defined for $x_1>0$, has a discontinuity at $(x_1,\hat x-\hat y)=0$ and:

$$B_kH_{j\hat y}(x_1,0,t)\Big|_{x_1=0} = 0.$$

The integrand is nonnull for $\hat x=\hat y$; $x_1>0$.

Consider: $B_kH_{j\hat y}(|\hat x-\hat y|,\hat x-\hat y,t)$. We have:

$$B_kH_{j\hat y}(|\hat x-\hat y|,\hat x-\hat y,t)\Big|_{\hat x=\hat y} = 0.$$

$b_k(\hat{y},\zeta_1,\hat{\xi})$ is infinitely differentiable; taking the Taylor's development of b_k in powers of $\hat{x}-\hat{y}$ for $\hat{x}$ near $\hat{y}$, up to the order $n+r_k+2$ and putting in the above integral, we obtain:

$$B_kH_{j\hat{y}}(x_1,\hat{x}-\hat{y},t) = B_{k\hat{y}}H_{j\hat{y}}(x_1,\hat{x}-\hat{y},t) + \int_{E^{n-1}} \exp(i\langle\hat{x}-\hat{y},\hat{\xi}\rangle)P^1_{jk}(\hat{\xi},\hat{y},x_1,t)d\hat{\xi}$$
$$+ \int_{E^{n-1}} \exp(i\langle\hat{x}-\hat{y},\hat{\xi}\rangle)P^2_{jk}(\hat{\xi},x,t,\hat{y})d\hat{\xi}.$$

The integrals are well defined for $x_1>0$ and the last expression is majorized by $Mt^{-\varepsilon/2m}|\hat{x}-\hat{y}|^{-\varepsilon+r_j+1}$ for $x_1 \geqq 0$. The expression $P^1_{jk}(\hat{\xi},x_1,\hat{y},t)$ is not integrable when $x_1=0$. Since $B_kH_{j\hat{\xi}}(|\hat{x}-\hat{y}|,\hat{x}-\hat{y},t)|_{\hat{x}=\hat{y}}=0$ we must have:

$$P^1_{jk}(\hat{\xi},0,\hat{y},t) = 0.$$

It follows then that:

$$(B_k - B_{k\hat{y}})H_{j\hat{y}}(x_1,\hat{x}-\hat{y},t) = O(1)t^{-\varepsilon/2m}|\hat{x}-\hat{y}|^{-n+2-\varepsilon}$$

for all j,k.

We note that $(\hat{x}-\hat{y})^{n-2}AH_{j\hat{y}}(x_1,\hat{x}-\hat{y},t)$ is uniformly continuous in $\hat{x}-\hat{y}$ for $x_1>0$ and is equal to zero for $\hat{x}=\hat{y}$. So for large t when $|\hat{x}-\hat{y}| \leqq t^{-\varepsilon/2(2m+2)}$, we have:

$$|(\hat{x}-\hat{y})^{n-2}AH_{j\hat{y}}(x_1,\hat{x}-\hat{y},t)| \leqq Mt^{-s}$$

for some positive number s.

On the other hand, we have:

$$|(\hat{x}-\hat{y})^{n+2m-1+\varepsilon}AH_{j\hat{y}}(x_1,\hat{x}-\hat{y},t)| \leqq Mt^{-\varepsilon/2m}.$$

So for $|\hat{x}-\hat{y}| > t^{-\varepsilon/2(2m+2)}$, we get:

$$|(\hat{x}-\hat{y})^{n-2}AH_{j\hat{y}}(x_1,\hat{x}-\hat{y},t)| \leqq Mt^{-\varepsilon/4m}.$$

Hence for large t and $x_1>0$, we obtain:

$$(\hat{x}-\hat{y})^{n-2}AH_{j\hat{y}}(x_1,\hat{x}-\hat{y},t) = O(1)t^{-\nu/2m}, \qquad 0<\nu<1.$$

LEMMA 4.2. *Let $H_{j\hat{y}}$ be as above, then*:

$$B_{k\hat{y}}H_{j\hat{y}}(0,\hat{x}-\hat{y},t) = 0 \quad \textit{if } k \neq j,$$
$$B_{j\hat{y}}H_{j\hat{y}}(0,\hat{x}-\hat{y},t) = \delta_{\hat{y}}.$$

Proof. It follows immediately from the definition of H_j.

LEMMA 4.3. *Let $\{A; B_j; j=1,\cdots,m\}$ be a uniformly regularly elliptic boundary value problem where A and B_j are defined on E^n_+, E^{n-1}, have infinitely differentiable coefficients and are homogeneous of orders $2m, r_j$ re-*

spectively. Let A_z, B_{jz} be the operators obtained from A; B_j by taking the values of the coefficients at the point z. Let $G_{(z)}(x,z,t)$ be the Green's function associated with $\{A_z + tI;\ B_{jz};\ j = 1,\cdots,m\}$ constructed in §3, *(Theorem* 3.2).

Let

$$\begin{aligned} \alpha_0(x,z,t) &= (A - A_z)G_{(z)}(x,z,t), \qquad x \neq z;\ x,z \text{ in } B^+, \\ \alpha_j(x,z,t) &= (B_j - B_{jz})G_{(z)}(x,z,t), \qquad j = 1,\cdots,m, \\ \alpha(x,z,t) &= (\alpha_0,\cdots,\alpha_m), \end{aligned} \tag{1}$$

$$\begin{aligned} \beta_{0k}(x,\hat{y},t) &= (A - A_{\hat{y}})H_{k\hat{y}}(x,\hat{y},t), \\ \beta_{jk}(\hat{x},\hat{y},t) &= (B_j - B_{j\hat{y}})H_{k\hat{y}}(\hat{x},\hat{y},t), \qquad j = 1,\cdots,m, \\ \beta_k(x,\hat{y},t) &= (\beta_{0k},\cdots,\beta_{mk}), \end{aligned} \tag{2}$$

$H_{k\hat{y}}$ is given by Theorem 4.1.

$$w(x,z,t) = (v(x,z),\ h(\hat{x},\hat{z}),\cdots,h_m(\hat{x},\hat{z})). \tag{3}$$

Define the linear transformations:

$$T_0 w(x,z,t) = \int_{B^+} \alpha(x,y,t)\,v(y,z)\,dy,$$

$$T_k w(x,z,t) = \int_{\Gamma_0} \beta_k(x,\hat{y},t)\,h_k(\hat{y},\hat{z})\,d\hat{y},$$

with:

$$Tw = T_0 w + \sum_{k=1}^{m} (T_k w),$$

$$B^+ = \{x : |x| < 1,\ x_1 > 0\}; \quad \Gamma_0 = \{x : |x| < 1,\ x_1 = 0\}.$$

Then the integral equation:

$$w(x,z,t) + Tw(x,z,t) + \alpha(x,z,t) = 0$$

may be solved by the Neumann series for sufficiently large t and:

$$\begin{aligned} v(x,z,t) &= O(1)t^{-\varepsilon/2m}|x-z|^{1-n-\varepsilon}(1 + |t^{1/2m}(x-z)|^N)^{-1}, \\ h_j(\hat{x},\hat{z},t) &= O(1)t^{-\varepsilon/2m}|\hat{x}-\hat{z}|^{-n+2-\varepsilon}(1 + |t^{1/2m}(\hat{x}-\hat{z})|^N)^{-1}, \end{aligned}$$

$t = \tau^{2m}$ $< \varepsilon < 1$; N any positive integer.

Proof. We have from Theorem 3.2:

$$\begin{aligned} \alpha_0(x,z,t) &= (A - A_z)G_{(z)}(x,z,t) = O(1)\tau^{-\varepsilon}|x-z|^{1-n-\varepsilon}(1 + |(x-z)|^N)^{-1}, \\ \alpha_j(\hat{x},z,t) &= (B_j - B_{jz})G_{(z)}(\hat{x},z,t) = O(1)\tau^{-\varepsilon}|\hat{x}-\hat{z}|^{-n+2-\varepsilon}(1 + |(\hat{x}-\hat{z})|^N)^{-1}. \end{aligned}$$

Finally from Lemma 4.1:

$$\beta_{jk}(\hat{x},\hat{y},t) = O(1)\tau^{-\varepsilon}|\hat{x}-\hat{y}|^{-n+2-\varepsilon}.$$

Consider the series:

$$w(z,x,t) = \alpha(x,z,t) + T\alpha + T^2\alpha + \cdots.$$

It may be written as:

$$v(x,z,t) = \alpha_0(x,z,t) + \int_{B^+} \alpha_0(x,y,t)\alpha_0(y,z,t)dy$$

$$+ \sum_{k=1}^{m} \int_{\Gamma_0} \beta_{0k}(x,\hat{y},t)\alpha_k(\hat{y},\hat{z})h_j(\hat{x},z,t) = \alpha_j(\hat{x},\hat{z},t) + \int_{B^+} \alpha_0(\hat{x},y,t)\alpha_j(y,\hat{z},t)dy$$

$$+ \sum_{k=1}^{m} \int_{\Gamma_0} \beta_{jk}(\hat{x},\hat{y},t)\alpha_k(\hat{y},\hat{z})d\hat{y} + \cdots, \qquad j = 1,\cdots,m.$$

The first series is majorized by:

$$O(1)t^{-\varepsilon/2m}|x-z|^{1-n-\varepsilon} + O(1)t^{-\varepsilon/2m}\int_{B^+}|x-y|^{1-n-\varepsilon}\ |y-z|^{1-n-\varepsilon}\,dy$$

$$+\ O(1)t^{-\varepsilon/m}\int_{\Gamma_0} |\hat{x}-\hat{y}|^{-n+2-\varepsilon}|\hat{z}-\hat{y}|^{-n+2-\varepsilon}d\hat{y} + \cdots,$$

which is uniformly convergent for large t; x,z in B^+. The second series is majorized by:

$$O(1)t^{-\varepsilon/2m}|\hat{x}-\hat{z}|^{2-n-\varepsilon} + O(1)t^{-\varepsilon/m}\int_{B^+} |x-y|^{1-n-\varepsilon}|y-z|^{1-n-\varepsilon}dy$$

$$+\ O(1)t^{-\varepsilon/m}\int_{\Gamma_0} |\hat{x}-\hat{y}|^{-n+2-\varepsilon}|\hat{z}-\hat{y}|^{2-n-\varepsilon}d\hat{y} + \cdots,$$

which is also uniformly convergent for large t.

The proof of the theorem is completed.

THEOREM 4.1. *Let $\{A;B_j;\ j=1,\cdots,m\}$ be a uniformly regularly elliptic boundary value problem where A,B_j are defined on E^n_+, E^{n-1} and have infinitely differentiable coefficients. If $G_{(z)}(x,z,t)$ is the Green's function associated with the constant coefficients problem $\{A_z + tI;B_{jz};j=1,\cdots,m\}$ constructed in Theorem 3.2; then:*

$$G(x,z,t) = G_{(z)}(x,z,t) + \int_{B^+} G_{(y)}(x,y,t)v(y,z,t)dy + \sum_{k=1}^{m}\int_{\Gamma_0} H_{k\hat{y}}(x_1,\hat{y},t)h_k(\hat{y},z)$$

is the Green's function associated with the elliptic boundary value problem $\{A + tI;\ B_j;j=1,\cdots,m\}$. $H_{k\hat{y}}$ are the kernels defined by Lemmas 4.1, v, h_k are the solutions of the system of integral equations of Lemma 4.3.

$$B^+ = \{x : x_1 > 0;\ |x| < 1\};\quad \Gamma_0 = \{x : x_1 = 0; |x| < 1\}.$$

Proof. We verify that

$$(A + tI)\, G(x,z,t) = \delta_z,$$
$$B_j G(x,z,t) = 0, \qquad x_1 = 0;\ j = 1,\cdots,m.$$

(1) Consider $(A+tI)\,G(x,z,t)$. We have:

$$\begin{aligned}(A+tI)G(x,z,t) &= (A_z + tI)G_{(z)}(x,z,t) + (A - A_z)G_z(x,z,t)\\ &\quad + \sum_{k=1}^{m} \int_{\Gamma_0} (A - A_{\hat{y}})H_{k\hat{y}}(x,\hat{y},t)h_k(\hat{y},t,z)d\hat{y}\\ &\quad + (A+tI)\left(\int_{B^+} G_y(x,y,t)v(y,z,t)dy\right).\end{aligned}$$

(2) Let $\phi(x)$ be an infinitely differentiable function with compact support in B^+. We have:

$$\begin{aligned}\int_{B^+}(A+tI)G(x,z,t)\phi(x)dx &= \phi(z) + \int_{B^+} (A-A_z)G_z(x,t,z)\,\phi(x)dx\\ &\quad + \sum_{k=1}^{m}\int_{B^+} (A-A_{\hat{y}})H_{k\hat{y}}(x,\hat{y},t)h_k(\hat{y},z,t)\phi(x)d\hat{y}dx\\ &\quad + \int_{B^+}(A+tI)\left(\int_{B^+} G_y(x,y,t)\,v(y,z,t)dy\right)\phi(x)dx.\end{aligned}$$

(3) Consider the last integral. Since $\phi(x) \in C_c^\infty(B^+)$ we may write it as:

$$\begin{aligned}&\sum_{|\alpha|=2m} \int_{B^+}\int_{B^+} G_y(x,t,y)\,v(y,z,t)\,[t\phi + D_x^\alpha(\bar{a}_\alpha(x)\phi(x))]dydx\\ &= \sum_{|\alpha|=2m}\int_{B^+}\int_{B^+}[t\phi(x) + D_x^\alpha(\bar{a}_\alpha(x)\phi(x))]G_y(x,y,t)\,v(y,z,t)dxdy\end{aligned}$$

by Fubini's theorem. Integrating by parts, we obtain:

$$\begin{aligned}&\int_{B^+}\int_{B^+}(A+tI)G_y(x,y,t)\,v(y,z,t)\phi(x)dxdy\\ &= \int_{B^+}\int_{B^+} (A_y + tI)G_y(x,y,t)\phi(x)v(y,z,t)dxdy\\ &\qquad + \int_{B^+}\int_{B^+}(A - A_y)G_y(x,y,t)v(y,z)\phi(x)dxdy\\ &= \int_{B^+}\phi(y)v(y,z,t)dy + \int_{B^+}\int_{B^+}(A - A_y)G_y(x,y,t)\,v(y,z,t)\phi(x)dydx\end{aligned}$$

by Fubini's theorem.

(4) But v, h_j satisfy the equation:

$$0 = v(x,z,t) + (A - A_z)G_z(x,z,t) + \int_{B^+}(A-A_y)G_y(x,y,t)v(y,z)dy$$
$$+ \sum_{k=1}^{m} \int_{\Gamma_0} (A - A_{\hat{y}})H_{k\hat{y}}(x,\hat{y},t)h_k(\hat{y},z,t)d\hat{y}.$$

Hence: $(A + tI)G(x,z,t) = \delta_z$.

(5) We show that $B_jG(x,z,t) = 0$ if $x_1 = 0$; $j = 1,\cdots,m$. We have with x,z in B^+:

$$B_jG(x,z,t) = B_{jz}G_z(x,z,t) + (B_j - B_{jz})G_z(x,z,t) + \int_{B^+} B_jG_y(x,y,t)v(y,z)dy$$
$$+ \sum_{k=1}^{m} \int_{\Gamma_0} B_jH_{k\hat{y}}(x,\hat{y},t)h_k(\hat{y},z,t)d\hat{y}.$$

The differentiations under the integral sign are valid (Theorems 3.2, 4.1). The last integral may be written as:

$$\int_{\Gamma_0}(B_j - B_{j\hat{y}})H_{k\hat{y}}(x,\hat{y},t)h_k(\hat{y},z,t)d\hat{y} + \int_{\Gamma_0} B_{j\hat{y}}H_{k\hat{y}}(x,\hat{y},t)h_k(\hat{y},z,t)d\hat{y}.$$

Let $\phi(\hat{x}) \in C_c^\infty(\Gamma_0)$. Consider:

$$\int_{\Gamma_0}\int_{\Gamma_0} B_{j\hat{y}}H_{k\hat{y}}(x,\hat{y},t)h_k(\hat{y},z,t)\phi(\hat{x})d\hat{x}d\hat{y}.$$

By Fubini's theorem, we have:

$$\int_{\Gamma_0}\int_{\Gamma_0} B_{j\hat{y}}H_{k\hat{y}}(x,\hat{y},t)h_k(\hat{y},z,t)\phi(\hat{x})d\hat{x}d\hat{y}.$$

We know that $B_{j\hat{y}}H_{k\hat{y}}(\hat{x},\hat{y}) = \delta_{jk}\delta_{\hat{y}}$ so as $x_1 \to 0$, we get:

$$\delta_{jk}\int_{\Gamma_0} h_k(\hat{y},z,t)\phi(\hat{y})d\hat{y}.$$

(6) On the other hand, v and h_k satisfy the equation:

$$(B_j - B_{j\hat{z}})G_z(\hat{x},\hat{z},t) + \int_{B^+}(B_j - B_{j\hat{y}})G_y(\hat{x},\hat{y},t)v(\hat{y},z,t)d\hat{y}$$
$$+ \sum_{k=1}^{m} \int_{\Gamma_0} B_{j\hat{y}}H_{k\hat{y}}(\hat{x},\hat{y},t)h_k(\hat{y},z,t)d\hat{y} + h_j(z,x,t) = 0; \qquad j = 1,\cdots,m.$$

So $B_jG(\hat{x},z,t) = 0$, $j = 1,\cdots,m$.

LEMMA 4.4. *Let* $\{A;B_j;j=1,\cdots,m\}$ *be a uniformly regularly elliptic boundary value problem where A and* B_j *are defined on* E^n_+, E^{n-1} *and have infinitely differentiable coefficients. A and* B_j *are homogeneous differential operators. Let* $G_z(x,z,t)$ *be the Green's function corresponding to the constant coefficients problem* $\{A_z+tI;\ B_{jz};\ j=1,\cdots,m\}$. G_z *is given by Theorem* 3.2.

Let $H_{j\hat{y}}(x,\hat{y})$ *be the kernels given in Lemma* 4.1. *Set*:

$$\begin{aligned}\alpha_{0j}(x,\hat{y}) &= (A-A_{\hat{y}})H_{j\hat{y}}(x,\hat{y}),\\ \alpha_{rj}(\hat{x},\hat{y}) &= (B_r-B_{r\hat{y}})H_{j\hat{y}}(\hat{x},\hat{y}),\\ \alpha_j(x,\hat{y}) &= (\alpha_{0j},\cdots,\alpha_{mj}),\end{aligned} \tag{1}$$

$$\begin{aligned}\beta_0(x,z) &= (A-A_z)G_z(x,z),\\ \beta_r(\hat{x},z) &= (B_r-B_{rz})G_z(x,z),\\ \beta(x,z) &= (\beta_0,\cdots,\beta_m),\end{aligned} \tag{2}$$

$$w_j(x,\hat{y}) = (v_j,h_j,\cdots,h_{mj}). \tag{3}$$

Define the transformations:

$$Tw_j(x,\hat{y}) = \int_{B^+}\beta(x,z)v_j(z,\hat{y})dz,$$

$$T_kw_j(x,\hat{y}) = \int_{\Gamma_0}\alpha_k(x,z)h_{jk}(z,\hat{y})dz,$$

$$Tw_j+\sum_{k=1}^{m}T_kw_j = \mathscr{I}w_j(x,\hat{y}).$$

Then the integral equation:

$$w_j(x,y)+\mathscr{I}w_j(x,y)+\alpha_j(x,y)=0, \qquad j=1,\cdots,m,$$

may be solved by the Neumann series for large t and for x,y in $B^+\cup\Gamma=\{x:|x|<1;\ x_1\geqq 0\}$.

Moreover:

$$\begin{aligned}v_j(x,\hat{y}) &= O(1)t^{-\varepsilon/2m}|\hat{x}-\hat{y}|^{-n+2-\varepsilon}(1+|(x-\hat{y})|^N)^{-1},\\ h_{jk}(\hat{x},\hat{y}) &= O(1)t^{-\varepsilon/2m}|\hat{x}-\hat{y}|^{-n+2-\varepsilon}(1+|(\hat{x}-\hat{y})|^N)^{-1}\quad,\end{aligned}$$

$0<\varepsilon<1$; *N is a positive integer.*

Proof. From Lemma 4.1, we have:

$$\begin{aligned}\alpha_{0j}(x,\hat{y}) &= (A-A_{\hat{y}})H_{j\hat{y}}(x,\hat{y}) = O(1)t^{-\varepsilon/2m}|x-\hat{y}|^{-n+2-\varepsilon},\\ \alpha_{rj}(\hat{x},\hat{y}) &= (B_r-B_{r\hat{y}})H_{j\hat{y}}(\hat{x},\hat{y}) = O(1)t^{-\varepsilon/2m}|\hat{x}-\hat{y}|^{-n+2-\varepsilon}.\end{aligned}$$

From Theorem 3.2, we get:

$$\beta_0(x,z) = (A - A_z)G_z(x,z) = O(1)t^{-\varepsilon/2m}|x-z|^{-n+1-\varepsilon},$$
$$\beta_r(\hat{x},z) = (B_r - B_{rz})G_z(\hat{x},z) = O(1)t^{-\varepsilon/2m}|\hat{x}-\hat{z}|^{-n+2-\varepsilon}.$$

Consider the series:

$$\alpha_j(x,\hat{y}) + \mathscr{I}\alpha_j(x,\hat{y}) + \mathscr{I}\alpha_j + \cdots.$$

It may be written as:

$$\alpha_{0j}(x,\hat{y}) + \int_{B^+} \beta_0(x,z,t)\alpha_{0j}(z,\hat{y})dz + \sum_{k=1}^{m} \int_{\Gamma_0} \alpha_{0k}(x,\hat{z})\alpha_{jk}(\hat{z},\hat{y})d\hat{z} + \cdots$$

and:

$$\alpha_{rj}(\hat{x},\hat{y}) + \int_{B^+} \beta_r(\hat{x},z,t)\alpha_{0j}(z,\hat{y})\,dz + \sum_{k=1}^{m} \int_{\Gamma_0} \alpha_{rk}(\hat{x},\hat{z})\alpha_{jk}(\hat{z},\hat{y})d\hat{z} + \cdots.$$

They are majorized by the series:

$$O(1)t^{-\varepsilon/2m}|\hat{x}-\hat{y}|^{-n+2-\varepsilon} + O(1)t^{-\varepsilon/m}\int_{B^+}|\hat{x}-\hat{z}|^{-n+2-\varepsilon}|\hat{y}-\hat{z}|^{-n+2-\varepsilon}dz$$
$$+ O(1)t^{-\varepsilon/m}\int_{\Gamma_0}|\hat{x}-\hat{z}|^{-n+2-\varepsilon}|\hat{z}-\hat{y}|^{-n+2-\varepsilon}d\hat{z} + \cdots$$

which is uniformly convergent for large t.

The lemma is proved.

LEMMA 4.5. *Let $H_{j\hat{z}}(x,\hat{z})$ be the kernels constructed in Lemma* 4.1 *for the constant coefficients problem $\{A_{\hat{z}} + tI;\ B_{j\hat{z}};\ j = 1,\cdots,m\}$ Let $G_z(x,z,t)$ be the Green's function associated with the elliptic boundary value problem $\{A_z + tI;\ B_{jz};\ j = 1,\cdots,m\}$. The differential operators A, B_j are homogeneous and have infinitely differentiable coefficients.*

Let:

$$H_j(x,\hat{y}) = H_{j\hat{y}}(x,\hat{y}) + \int_{B^+} G_z(x,z,t)v_j(z,y)dz + \sum_{k=1}^{m}\int_{\Gamma_0} H_{k\hat{z}}(x,\hat{z})h_{kj}(\hat{z},\hat{y})d\hat{z}$$

where v_j and h_{kj} satisfy the system of integral equations of Lemma 4.4. *Then*:

$$(A + tI)H_j(x,\hat{y}) = 0, \quad x \text{ in } B^+ = \{x: x_1 > 0,\ |x| < 1\},$$
$$B_rH_r(0,\hat{x},\hat{y}) = \delta_{\hat{y}},$$
$$B_rH_j(0,\hat{x},\hat{y}) = 0 \quad \text{if } r \neq j.$$

Proof. The proof is long but easy and is similar to that of Theorem 4.2.

THEOREM 4.2. *Let $u(x)$ be an infinitely differentiable function with compact support in $E^n_+ \cup E^{n-1}$. Then u has the following integral representation*:

$$u(x,t) = \int_{E^n_+} G(x,y,t)(A + tI)u(y)dy + \sum_{j=1}^{m} \int_{E^{n-1}} H_j(x-\hat{y},t)B_j u(0,\hat{y})d\hat{y}.$$

$\{A, B_j; j = 1,\cdots,m\}$ is a uniformly regularly elliptic boundary value problem; A and B_j are defined on E^n_+, E^{n-1} with infinitely differentiable coefficients. $G(x,y,t)$ is the Green's function associated with $\{A+tI;\ B_j; j=1,\cdots,m\}$ and is given by Theorem 4.1. *The kernels H_j are given by Lemma* 4.5.

Proof. We consider the boundary value problem:

$$(A + tI)u(x) = f(x) \quad \text{on } E^n_+,$$

$$B_j u(\hat{x}) = g_j(\hat{x}) \quad \text{on } E^{n-1}, \quad j = 1,\cdots,m.$$

Since u is infinitely differentiable and has compact support in $E^n_+ \cup E^{n-1}$; f and g_i also have compact supports.

We may write $u(x)$ as $u(x) = v(x) + w(x)$ where $v(x)$ is the solution of $(A + tI)v(x) = f(x)$ on E^n_+, $B_j v(\hat{x}) = 0$ on E^{n-1}, $j = 1,\cdots,m$, and $w(x)$ is the solution of the boundary value problem: $(A + tI)w(x) = 0$ on E^n_+, $B_j w(x) = g_j$ on $E^{n-1}, j = 1,\cdots,m$. Let $G(x,y,t)$ be the Green's function associated with the elliptic boundary value problem $\{A + tI;\ B_j; j = 1,\cdots,m\}$ given by Theorem 4.2. We get:

$$v(x) = \int_{E^n_+} G(x,y,t)f(y)dy.$$

Now we construct w. Let $H_j(x,y)$ be the kernels given by Lemma 4.5; then w is given by the expression:

$$w(x) = \sum_{j=1}^{m} \int_{E^{n-1}} h_j(\hat{y})H_j(x-\hat{y},t)d\hat{y}.$$

The conclusion of the theorem follows immediately.

THEOREM 4.3. *Let $\mathcal{G}(x,y,t)$ be the Green's function associated with the uniformly regularly elliptic boundary value problem (A,γ) where A is defined on a bounded open subset S of E^n with infinitely differentiable coefficients; $\gamma = (B_1,\cdots,B_m)$ is a family of differential operators defined on the boundary Γ of S with infinitely differentiable coefficients. A and B_j are homogeneous differential operators.*

(A,γ) is assumed to be formally positive in the sense of Definition 2.1. *Let $G(x,y,t)$ be the Green's function of Theorem* 4.1 (*i.e. corresponding to the case of a half space*). *Then*:

$$\mathcal{G}(x,y,t) = G(x,y,t) - u(x,y,t), \qquad y \text{ in } S,$$

$$u(x,y,t) = \sum_k \tilde{u}(\phi^{-1}(x), \phi_k^{-1}(y), t),$$

ϕ_k are the diffeomorphisms corresponding to the uniform regularity of S and:

$$\tilde{u}(x,y,t) = \sum_{j=1}^{m} \int_{\Gamma_0} H_j(x,\hat{z},t) B_j G(y,\hat{z},t) d\hat{z}$$

$H_j(x,\hat{z},t)$ is given by Lemma 4.5

$$\Gamma_0 = \{z: \ z_1 = 0; \ |z| < 1\}.$$

Proof. There is no loss of generality in assuming that $y = 0$ is in S. Let $G(x,t)$ be as in Theorem 4.1. Then $G(x,t)$ is a fundamental solution of the elliptic operator $A + tI$.

If u is the solution of the boundary value problem:

$$(A + tI)u(x) = 0 \quad \text{on } S,$$
$$B_j(x)u = B_j G(x,t) \quad \text{on } \Gamma; \ j = 1,\cdots,m,$$

then $\mathcal{G}(x,t) = G(x,t) - u(x,t)$.

S is a bounded domain which is uniformly regular. It may be covered by a finite number of open sets N_k and there exists a family of infinitely differentiable unctions η_k with compact supports in N_k, and such that:

$$\sum_k \eta_k^2(x) = 1, \qquad x \text{ in } S.$$

We have:

$$(A + tI)(u(x)\eta_k^2(x)) = \eta_k^2(x)(A + tI)u(x) + \sum_{|\alpha|+|\beta|=2m; |\alpha|<2m} a_{\alpha\beta}(x) D^\alpha u D^\beta \eta_k^2(x).$$

Similarly for $B_j(u\eta_k^2)$.

We consider the boundary value problem:

$$(A + tI)u\eta_k^2 = f_k \quad \text{on } N_k \cap S,$$
$$B_j(u\eta_k^2) = g_{jk} + h_{jk} \quad \text{on } N_k \cap \Gamma; \ j = 1,\cdots,m,$$

where:

$$f_k(x) = \sum_{|\alpha|<2m; |\alpha|+|\beta|=2m} a_{\alpha\beta}(x) D^\alpha u D^\beta \eta_j^2(x),$$
$$h_{jk}(x) = \sum_{|\alpha|<r_j; |\alpha|+|\beta|=r_j} b_{j\beta k}(x) D^\alpha u D^\beta \eta_k^2(x).$$

Using the diffeomorphisms $\phi_k(x)$ we map N_k into the positive half ball. Set: $\tilde{u}_k(x) = (\eta_k^2(u))\,(\phi_k(x))$; $\tilde{f}_k(x)$; $\tilde{g}_{jk}(x)$; $\tilde{h}_{jk}$ are similarly defined.

Using the same notations for the transplanted operators, we get:

$$(A + tI)\tilde{u}_k(x) = \tilde{f}_k(x) \quad \text{on } B^+ = \{x : x_1 > 0; \ |x| < 1\},$$
$$B_j \tilde{u}(0,\hat{x}) = \tilde{g}_{jk}(0,\hat{x}) + \tilde{h}_{jk}(0,\hat{x}), \quad \text{on } \Gamma; \ j = 1,\cdots,m.$$

$\tilde{f}_k$ is an infinitely differentiable function with compact support in $B^+ \cup \Gamma_0$. Applying Theorem 4.2; we obtain:

$$\tilde{u}_k(x) = \int_{B^+} \tilde{f}_k(y)G(x,y,t)dy + \sum_{j=1}^{m} \int_{\Gamma_0} H_j(x-\hat{y},t)\{\tilde{g}_{jk}(\hat{y}) + \tilde{h}_{jk}(\hat{y})\}d\hat{y}.$$

Since $\sum_k \eta_k^2(x) = 1$, we have:

$$\tilde{u}(x) = \sum_k \tilde{u}_k(x) = \sum_{j=1}^{m} \int_{\Gamma_0} H_j(x-\hat{y},t)B_jG(0,\hat{y},t)d\hat{y}.$$

The theorem is proved.

5. Theorem 5.1. *Let $\mathcal{G}_{2p}$ be the 2pth iterate of the Green's function $\mathcal{G}$ defined in Theorem 4.3. Let A_γ be the realization of A under null boundary conditions γ as an operator on $L^2(S)$. If (A,γ) is formally self-adjoint and λ_j, ϕ_j are respectively the eigenvalues and eigenfunctions of A_γ; then: for $2mp > n$:*

$$D_x^\alpha D_y^\beta \mathcal{G}_{2p}(x,y,t) = \sum_j D^\alpha\phi_j(x)D^\beta\phi_j(y)(\lambda_j+t)^{-2p}; \qquad |\alpha|,|\beta| \leqq 2m.$$

Proof. Let $2mp > n$, then $(A_\gamma + tI)^{-p}$ is of Hilbert-Schmidt type. Since (A,γ) is formally self-adjoint, it follows from Theorem 2.1 that A_γ is self-adjoint; $\lambda_j + t > 0$, we have a complete orthonormal system of eigenfunctions ϕ_j.

Consider:

$$((A_\gamma + tI)^{-p}f,\phi_j) = \int_S \int_S \mathcal{G}_{(p)}(x,y,t)f(y)\phi_j(z)dydz, \quad f \text{ in } L^2(S).$$

We get:

$$(\lambda_j + t)^{-p}\phi_j(y) = \int_S \mathcal{G}_{(p)}(z,y,t)\phi_j(z)dz.$$

Using Parseval's formula, we obtain:

$$\sum_j (\lambda_j+t)^{-2p}\phi_j(x)\phi_j(y) = \int_S \mathcal{G}_{(p)}(z,x,t)\mathcal{G}_{(p)}(z,y,t)dz$$
$$= \mathcal{G}_{(2p)}(x,y,t).$$

Let $\mathcal{G}_{2p,k}(x,y,t) = \sum_{j=1}^{k}(\lambda_j+t)^{-2p}\phi_j(x)\phi_j(y)$ then:

$$\|\mathcal{G}_{2p,k} - \mathcal{G}_{2p}\|_{W^{2m,2}\times W^{2m,2}} \leq \sum_k^l (\lambda_j+t)^{-2p}\|\phi_j\|_{W^{2m,2}},$$

$$\leqq \sum_k^l (\lambda_j+t)^{-2p+2} \to 0 \quad \text{as } l,\ k \to \infty,$$

$$\mathcal{G}_{2p,k}(x,y,t) \to \mathcal{G}_{(2p)}(x,y,t) \quad \text{in } W^{2m,2}\times W^{2m,2}.$$

In particular; $D_x^\alpha D_y^\beta \mathscr{G}_{2p,k}(x,y,t) \to D_x^\alpha D_y^\beta \mathscr{G}_{2p}(x,y,t)$ in $L^2 \times L^2$ and we get:

$$D_x^\alpha D_y^\beta \mathscr{G}_{2p}(x,y,t) = \sum_j (\lambda_j + t)^{-2p} D^\alpha \phi_j(x) D^\beta \phi_j(y).$$

LEMMA 5.1. *Let $\mathscr{G}_p(x,y,t)$ be the pth iterate of the Green's function defined in Theorem 4.3. Then if $2mp > n + |\alpha| + |\beta|$,*

$$x \neq y,\ t^{p-(n+|\alpha|+|\beta|)/2m} D_x^\alpha D_y^\beta \mathscr{G}_p(x,y,t) \to 0 \quad \text{as } t \to +\infty,$$

$$t^{p-(n+|\alpha|+|\beta|)/2m} D^{\alpha+\beta} \mathscr{G}_p(x,x,t) = (2\pi)^{-n} \int_{E^n} \xi^{\alpha+\beta} [a(x,\xi)+1]^{-p} d\xi$$

as $t \to +\infty$; for x, y in S.

Proof. We prove the lemma for $|\alpha| = |\beta| = 0$; the general case may be treated in the same fashion. Let G be the Green's function associated with the elliptic boundary value problem $\{A + tI; \gamma\}$ on a half space with infinitely differentiable coefficients. From Theorem 4.2, we have

$$G(x,y,t) = G_y(x,y,t) + \int_{B^+} G_z(x,z,t) v(z,y,t) dz + \sum_{j=1}^m \int_{\Gamma_0} H_{j\hat{z}}(x-\hat{z},t) h_j(\hat{z},\hat{y}) d\hat{z}$$

x, y in B^+.

G_y is the Green's function associated with the constant coefficients problem $\{A_y + tI;\ B_{jy};\ j = 1,\cdots,m\}$ on a half space.

We show that:

$$\lim_{t\to+\infty} t^{p-n/2m} G_p(x,y,t) = \lim_{t\to+\infty} t^{p-n/2m} G_{p,y}(x,y,t).$$

First consider the case $2m > n$. With $t = \tau^{2m}$, we have:

$$\tau^{2m-n} \int_{B^+} G_z(x,z,t) v(y,z,t) dz = O(1)\tau^{-\varepsilon} \int_{B^+} |y-z|^{1-n-\varepsilon} (1 + |\tau(x-y)|^N)^{-1} dz,$$

$$\tau^{2m-n} \int_{\Gamma_0} H_{j\hat{z}}(x-\hat{z},t) h_j(\hat{z},y,t) d\hat{z} = O(1)\tau^{-\varepsilon} |x_1|^{-n} \int_{\Gamma_0} |\hat{z}-\hat{y}|^{-n+2-\varepsilon} (1 + |\tau(\hat{z}-\hat{y})|^N)^{-1} d\hat{z}.$$

So:

$$\lim_{t\to+\infty} t^{1-n/2m} G(x,y,t) = \lim_{t\to+\infty} t^{1-n/2m} G_{(y)}(x,y,t).$$

Now if $2m < n$, from Theorem 4.1; we have:

$$G(x,y,t) = G_{(y)}(x,y,t) \quad (1 + O(1)\tau^{-\varepsilon})$$

so

$$\lim_{t\to+\infty} t^{p-n/2m} G_{(p)}(x,y,t) = \lim_{t\to+\infty} t^{p-n/2m} G_{(p),y}(x,y,t).$$

On the other hand;

$$\mathscr{G}(x,y,t) = G(x,y,t) + \sum_k \tilde{v}(\phi_k^{-1}(x),\phi_k^{-1}(y),t)$$

with

$$\tilde{v}(x,y,t) = \sum_{j=1}^{m} \int_{\Gamma_0} H_j(x-\hat{z},t)B_jG(\hat{z},y,t)d\hat{z}.$$

An argument as above gives:

$$\lim_{t\to+\infty} t^{p-n/2m}\mathscr{G}_{(p)}(x,y,t) = \lim_{t\to+\infty} t^{p-n/2m}G(x,y,t).$$

The conclusion of the lemma follows from Theorem 3.4.

THEOREM 5.2. *Let A_γ be the realization of the positively strongly elliptic operator A under null boundary conditions $\gamma=(B_1,\cdots,B_m)$ as an operator on $L^2(S)$. The operators A and B_j are defined on a bounded open set S and on the boundary Γ respectively and have infinitely differentiable coefficients.*

(A,γ) is assumed to be uniformly regularly elliptic, formally self-adjoint andformally positive in the sense of Definition 2.1. Let λ_j,ϕ_j be the eigenvalues and eigenfunctions of A_γ. Then:

(1) $$N(t) = \sum_{\lambda_j\leqq t} 1 \sim (2\pi)^{-n}t^{n/2m}\int_S\int_{a(x,\xi)<1} d\xi dx \quad as\ t\to+\infty.$$

(2) $$t^{-(n+|\alpha|+|\beta|)/2m}D_x^\alpha D_y^\beta e(x,y,t) = t^{-(n+|\alpha|+|\beta|)}\sum_{\lambda_j\leqq t} D^\alpha\phi_j(x)D^\beta\phi_j(y)\to 0$$

as $t\to\infty$ for x,y in S and $x\neq y$.

(3) $$D^{\alpha+\beta}e(x,x,t)\sim(2\pi)^{-n}t^{(n+|\alpha|+|\beta|)/2m}K(n,m,p,\alpha,\beta)\int_{E^n}\xi^{\alpha+\beta}[a(x,\xi)+1]^{-2p}d\xi$$

as $t\to\infty$ for x in S and $4mp>n+|\alpha|+|\beta|$.

$$K(n,m,p,\alpha,\beta) = \frac{\Gamma(2p)}{\Gamma\left(1+\frac{n}{2m}\right)\Gamma\left(2p-\frac{n+|\alpha|+|\beta|}{2m}\right)}.$$

Proof. One can show easily that:

$$\sum_j (\lambda_j+t)^{-2p} = \int_S \mathscr{G}_{(2p)}(x,x,t)dx.$$

Consider the sequence of integrable functions $t^{2p-n/2m}\mathscr{G}_{(2p)}(x,x,t)$. For large t, we have from the previous lemmas: $t^{2p-n/2m}|\mathscr{G}_{(2p)}(x,x,t)|\leqq M$ for all x in S and M is a constant independent of x and t. We apply the Lebesgue dominated convergence theorem and we get:

$$t^{2p-n/2m}\int_S \mathscr{G}_{(2p)}(x,x,t)dx \sim (2\pi)^{-n}\int_S\int_{E^n}[a(x,\xi)+1]^{-2p}d\xi dx$$

as $t \to +\infty$.

Applying the Tauberian theorem of Hardy-Littlewood [10], we get the results for $N(t)$.

We may write:

$$\mathscr{G}_{(2p)}(x,y,t) = \sum_j (\lambda_j + t)^{-2p}\phi_j(x)\phi_j(y) = \int_0^\infty (\lambda + t)^{-2p} de(x,y,\lambda)$$

where $e(x,y,t)$ is the spectral function. Taking into account the results of Lemma 5.1 and applying the Tauberian theorem of Hardy-Littlewood again, we get the results stated in the theorem.

6. The case of a nonself-adjoint regular elliptic boundary value problem is considered. The study of the asymptotic distribution of eigenvalues for the nonself-adjoint case has been carried out by Carleman [8] and Keldych [11] for second order elliptic equations.

THEOREM 6.1. *Let $\{A;\ B_j;\ j=1,\cdots,m\}$ be a uniformly regularly elliptic boundary value problem where A and B_j are defined on a bounded open subset S of E^n and on the boundary Γ with infinitely differentiable coefficients. (A,γ) with $\gamma=(B_1,\cdots,B_m)$ is assumed to be formally positive in the sense of Definition 2.1. Let A_γ be the realization of A under null boundary conditions γ as an operator $L^2(S)$. If $2mp > n$ where $2m$ is the order of A, the operator $(A_\gamma + tI)^{-2p}$ is of trace class. Let λ_j be the eigenvalues of A, then:*

$$\operatorname{tr}(A+tI)^{-2p} = \sum_j (\lambda_j + t)^{-2p} = \int_S \mathscr{G}_{(2p)}(x,x,t)dx.$$

$\mathscr{G}_{(2p)}(x,y,t)$ is the $2p$th iterate of the Green's function associated with $A_\gamma + tI$ on S.

Proof. With the above hypothesis, it has been proved in §2 that $(A_\gamma + tI)^{-p}$ is of Hilbert-Schmidt type, so $(A_\gamma + tI)^{-2p}$ is of trace class. Let ϕ_j be the generalized eigenfunctions of A_γ. They form an orthonormal basis in $L^2(S)$. Denote by P_j the orthogonal projection of $L^2(S)$ onto the subspace of $L^2(S)$ spanned by $\{\phi_1,\cdots,\phi_j\}$; consider the operator: $T_j = P_j(A+tI)^{-2p}P_j$.

It takes the subspace spanned by $\{\phi_1,\cdots,\phi_j\}$ into itself. The subspace is of finite dimension and we have:

$$\operatorname{tr}(T_j) = \sum_{k=1}^{j} (\lambda_k + t)^{-2p},$$

$$\operatorname{tr}(T_n - T_m) = \operatorname{tr}(T_n) - \operatorname{tr}(T_m) = \sum_{j=m}^{n} (\lambda_j + t)^{-2p}.$$

Denote by $|||T|||$ the trace norm of an operator of finite rank. (Ruston [13].) Then:

$$|||T_n - T_m||| \leqq \sum_m^n |\lambda_j + t|^{-2p}.$$

Since $(A_\gamma + tI)^{-p}$ is of Hilbert-Schmidt type: $\Sigma_j|\lambda_j + t|^{-2p} < \infty$. It follows that $T \to \mathscr{C}$ in the trace norm topology and:

$$\mathrm{tr}(\mathscr{C}) \sum_j (\lambda_j + t)^{-2p}.$$

We now show that $\mathscr{C} = (A_\gamma + tI)^{-2p}$. We know that $(A_\gamma + tI)^{-2p}$ is a compact operator and $T_j \to (A_\gamma + tI)^{-2p}$ in the operator norm topology. Since $T_j \to \mathscr{I}$ in the trace norm, it converges to τ in the operator norm; hence $\mathscr{C} = (A_\gamma + tI)^{-2p}$ and:

$$\mathrm{tr}(A_\gamma + tI)^{-2p} = \sum_j (\lambda_j + t)^{-2p}.$$

We may write $\mathscr{C} = \tau_R + i\mathscr{C}_I$ where $\mathscr{C}_R = (\mathscr{C}^* - \mathscr{C})/2$, $\mathscr{C}_I = (\mathscr{C} - \mathscr{C}^*)/2i$, Since $\mathscr{C}_R, \mathscr{C}_I$ are self-adjoint, we may apply results of §5 to get

$$\mathrm{tr}(A_\gamma + tI)^{-2p} = \sum_j (\lambda_j + t)^{-2p} = \int_S \mathscr{G}_{(2p)}(x, x, t)dx.$$

THEOREM 6.2. *Let (A,γ) be a uniformly regularly elliptic boundary value problem with infinitely differentiable coefficients and formally positive in the sense of Definition 2.1. If A_γ is the realization of the positively strongly elliptic operator A as an operator on $L^2(S)$ under null boundary conditions γ and λ_j are the eigenvalues of A_γ then*:

$$N(t) = \sum_{\mathrm{Re}\lambda_j \leqq t} 1 \sim (2\pi)^{-n} t^{n/2m} w_a(S) \quad as\ t \to +\infty$$

where $w_a(S) = \int_S w_a(x)dx$ *and* $w_a(x) = \int_{a(x,\xi)<1} d\xi$.

Proof. Set $\lambda_j = \alpha_j + i\beta_j$; $f(t) = \Sigma_j(\alpha_j + t)^{-2p}$ and $g(t) = \Sigma_j(\lambda_j + t)^{-2p}$. Let $h(t) = f(t) - g(t) = \Sigma_j\{(\alpha_j + t)^{-2p} - (\lambda_j + t)^{-2p}\}$.

It has been proved by Browder [4] that the spectrum of A_γ is contained inside an algebraic curve $|\mathrm{Im}\,\zeta| \leqq c(\mathrm{Re}\,\zeta)^\mu$ with $\mu = (2m-1)/2m$, we get:

$$|h(t)| \leqq \sum_j (\alpha_j + t)^{-2p-1}|\alpha_j|^\mu.$$

The eigenvalues have an accumulation point at infinity, hence there exists a number N such that:

$$|\alpha_N| < t^\delta \leqq \alpha_{N+1}, \qquad 0 < \delta < 1.$$

We have:

$$\sum_{N+1} |\alpha_j|^\mu(\alpha_j + t)^{-2p-1} \leqq \sum_{N+1} t^{(\mu-1)\delta}(\alpha_j + t)^{-2p}, \quad |h(t)| \leqq ct^{(\mu-1)\delta} f(t).$$

It follows that: $\lim t^{2p-n/2m} \sum_j (\alpha_j + t)^{-2p} = \lim t^{2p-n/2m} \sum_j (\lambda_j + t)^{-2p}$.

By an argument as in Theorem 5.2 and applying the Tauberian theorem of Hardy-Littlewood we get:

$$N(t) = \sum_{\mathrm{Re}\lambda \leq t} 1 \sim t^{n/2m} w_a(S) \cdot (2\pi)^{-n} \quad \text{as } t \to +\infty .$$

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(2) Added in proof.