(C) The Kolmogorov 4/5 - law

We have focussed so far on the absolute structure functions, that were used to obtain bounds on the energy flux Π_ℓ. However, there are other types of structure functions of interest, some of them more directly related to energy flux and which, in fact, provide alternate definitions of it. Rather than define "large-scale energy" by

$$\bar{e}_\ell(\mathbf{x},t) = \tfrac{1}{2}|\bar{\mathbf{v}}_\ell(\mathbf{x},t)|^2$$

one can instead make an alternate definition by filtering just one factor

$$\begin{aligned} e_\ell(\mathbf{x},t) &= \frac{1}{2}\mathbf{v}(\mathbf{x},t)\cdot\bar{\mathbf{v}}_\ell(\mathbf{x},t) \\ &= \int d^d\mathbf{r}\ G_\ell(\mathbf{r})\frac{1}{2}\mathbf{v}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x}+\mathbf{r},t) \end{aligned}$$

where

$$e_{\mathbf{r}}(\mathbf{x},t) = \tfrac{1}{2}\mathbf{v}(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x}+\mathbf{r},t)$$

is the so-called point-split kinetic energy density. The quantities $e_\ell(\mathbf{x},t)$ and $e_{\mathbf{r}}(\mathbf{x},t)$ are well-defined whenever $\mathbf{v}$ has finite mean energy:

$$\frac{1}{T}\int_0^T dt\int_V d^d\mathbf{x}\ \frac{1}{2}|\mathbf{v}(\mathbf{x},t)|^2 < +\infty.$$

We now derive a balance equation for the point-split kinetic energy density of a Navier-Stokes solution, as follows

$$\begin{aligned} \partial_t(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}') &+ \boldsymbol{\nabla}\cdot\left[(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}')\mathbf{v} + \frac{1}{2}(p\mathbf{v}'+p'\mathbf{v}) + \frac{1}{4}|\mathbf{v}'|^2\delta\mathbf{v} - \nu\boldsymbol{\nabla}(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}')\right] \\ &= \frac{1}{4}\boldsymbol{\nabla}_{\mathbf{r}}\cdot[\delta\mathbf{v}\,|\delta\mathbf{v}|^2] - \nu\boldsymbol{\nabla}\mathbf{v}:\boldsymbol{\nabla}\mathbf{v}' + \frac{1}{2}(\mathbf{f}\cdot\mathbf{v}'+\mathbf{f}'\cdot\mathbf{v}) \end{aligned}$$

with the notations

$$\begin{aligned} \mathbf{v} &= \mathbf{v}(\mathbf{x},t),\ \ p = p(\mathbf{x},t) \\ \mathbf{v}' &= \mathbf{v}(\mathbf{x}+\mathbf{r},t),\ \ p' = p(\mathbf{x}+\mathbf{r},t) \\ \delta\mathbf{v} &= \mathbf{v}'-\mathbf{v} = \mathbf{v}(\mathbf{x}+\mathbf{r},t) - \mathbf{v}(\mathbf{x},t) \end{aligned}$$

Remarks:

#1. This identity was first derived by L. Onsager in the 1940's in a smoothed and space-integrated form. The result was communicated to C. C. Lin in a letter in 1945, but never formally published by Onsager. The space-local form was derived by J. Duchan & R. Robert, Nonlinearity **13** 249-255(2000) in a smoothed version, discussed a bit later.

#2. The relation is analogous to the energy balance equation that we derived in the filtering approach:

$$\partial_t(\tfrac{1}{2}|\bar{\mathbf{v}}_\ell|^2) + \boldsymbol{\nabla}\cdot\left[\tfrac{1}{2}|\bar{\mathbf{v}}_\ell|^2\bar{\mathbf{v}}_\ell + \bar{p}_\ell\bar{\mathbf{v}}_\ell - \boldsymbol{\tau}_\ell\cdot\bar{\mathbf{v}}_\ell - \nu\boldsymbol{\nabla}(\tfrac{1}{2}|\bar{\mathbf{v}}_\ell|^2)\right] = \boldsymbol{\nabla}\bar{\mathbf{v}}_\ell : \boldsymbol{\tau}_\ell - \nu|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell|^2 + \mathbf{f}_\ell\cdot\bar{\mathbf{v}}_\ell$$

Proof of the identity: Take

$$\begin{aligned}\partial_t\mathbf{v} + \boldsymbol{\nabla}\cdot(\mathbf{v}\mathbf{v}) - \nu\triangle\mathbf{v} + \boldsymbol{\nabla}p &= \mathbf{f},\ \ \boldsymbol{\nabla}\cdot\mathbf{v} = 0\\ \partial_t\mathbf{v}' + \boldsymbol{\nabla}\cdot(\mathbf{v}'\mathbf{v}') - \nu\triangle\mathbf{v}' + \boldsymbol{\nabla}p' &= \mathbf{f}',\ \ \boldsymbol{\nabla}\cdot\mathbf{v}' = 0\end{aligned}$$

Dot the first by $\mathbf{v}'$ and the second by $\mathbf{v}$ and add together, to obtain

$$\begin{aligned}\partial_t(\mathbf{v}\cdot\mathbf{v}') &+ \mathbf{v}\cdot[\boldsymbol{\nabla}\cdot(\mathbf{v}'\mathbf{v}') - \nu\triangle\mathbf{v}'] + \mathbf{v}'\cdot[\boldsymbol{\nabla}\cdot(\mathbf{v}\mathbf{v}) - \nu\triangle\mathbf{v}]\\ &+ \boldsymbol{\nabla}\cdot(p'\mathbf{v} + p\mathbf{v}') = \mathbf{f}'\cdot\mathbf{v} + \mathbf{f}\cdot\mathbf{v}'\end{aligned}$$

Now the viscous term is reorganized as

$$\begin{aligned}\mathbf{v}\cdot\triangle\mathbf{v}' + \mathbf{v}'\cdot\triangle\mathbf{v} &= \boldsymbol{\nabla}\cdot[v_i\boldsymbol{\nabla}v_i' + v_i'\boldsymbol{\nabla}v_i] - 2\boldsymbol{\nabla}v_i\cdot\boldsymbol{\nabla}v_i'\\ &= \boldsymbol{\nabla}\cdot[\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{v}')] - 2\boldsymbol{\nabla}\mathbf{v} : \boldsymbol{\nabla}\mathbf{v}'\end{aligned}$$

Lastly we discuss the crucial nonlinear term. Note first that

$$\begin{aligned}\mathbf{v}\cdot[\boldsymbol{\nabla}\cdot(\mathbf{v}'\mathbf{v}')] + \mathbf{v}'\cdot[\boldsymbol{\nabla}\cdot(\mathbf{v}\mathbf{v})] &= v_i\partial_j(v_i'v_j') + v_i'\partial_j(v_iv_j)\\ &= v_i\partial_j(v_i'v_j') + \partial_j(v_i'v_iv_j) - v_iv_j(\partial_jv_i')\\ &= \triangle + \boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\mathbf{v}')\mathbf{v}]\end{aligned}$$

with

$$\triangle \equiv v_i\partial_j(v_i'v_j') - v_iv_j(\partial_jv_i')$$

.

By incompressibility,

$$\begin{aligned} \triangle &= v_i v_j'(\partial_j v_i') - v_i v_j(\partial_j v_i') \\ &= v_i(v_j' - v_j)(\partial_j v_i') \\ &= v_i \delta v_j(\partial_j v_i') \end{aligned}$$

Also by incompressibility $\boldsymbol{\nabla}_{\mathbf{r}} \cdot (\delta\mathbf{v}) = \boldsymbol{\nabla}_{\mathbf{r}} \cdot \mathbf{v}' = 0$, so that

$$\begin{aligned} \boldsymbol{\nabla}_{\mathbf{r}} \cdot [\delta\mathbf{v}\,|\delta\mathbf{v}|^2] &= (\delta\mathbf{v} \cdot \boldsymbol{\nabla}')|\delta\mathbf{v}|^2 \\ &= 2(\delta v_j \partial_j) v_i' \cdot (v_i' - v_i) \\ &= 2\delta v_j \cdot v_i' \partial_j v_i' - 2 v_i \delta v_j (\partial_j v_i') \\ &= \delta\mathbf{v} \cdot \boldsymbol{\nabla}(|\mathbf{v}'|^2) - 2\triangle \\ &= \boldsymbol{\nabla} \cdot [|\mathbf{v}'|^2 \delta\mathbf{v}] - 2\triangle \end{aligned}$$

Thus,

$$\begin{aligned} \partial_t(\mathbf{v}\cdot\mathbf{v}') \quad &+ \quad \boldsymbol{\nabla}\cdot\left[(\mathbf{v}\cdot\mathbf{v}')\mathbf{v} + (p\mathbf{v}' + p'\mathbf{v}) + \frac{1}{2}|\mathbf{v}'|^2\delta\mathbf{v} - \nu\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{v}')\right] \\ &- \quad \frac{1}{2}\boldsymbol{\nabla}_{\mathbf{r}}\cdot[\delta\mathbf{v}\,|\delta\mathbf{v}|^2] + 2\nu\boldsymbol{\nabla}\mathbf{v} : \boldsymbol{\nabla}\mathbf{v}' = (\mathbf{f}\cdot\mathbf{v}' + \mathbf{f}'\cdot\mathbf{v}) \end{aligned}$$

QED!

Multiplying the point-split identity through by $G_\ell(\mathbf{r})$ and integrating over $\mathbf{r}$ gives a corresponding balance equation for the regularized energy density $\frac{1}{2}\mathbf{v}\cdot\bar{\mathbf{v}}_\ell$:

$$\begin{aligned} \partial_t(\frac{1}{2}\mathbf{v}\cdot\bar{\mathbf{v}}_\ell) \quad &+ \quad \boldsymbol{\nabla}\cdot\left[(\frac{1}{2}\mathbf{v}\cdot\bar{\mathbf{v}}_\ell)\mathbf{v} + \frac{1}{2}(p\bar{\mathbf{v}}_\ell + \bar{p}_\ell\mathbf{v}) + \frac{1}{4}\overline{(|\mathbf{v}|^2\mathbf{v})_\ell} - \frac{1}{4}\overline{(|\mathbf{v}|^2)_\ell}\mathbf{v} - \nu\boldsymbol{\nabla}(\frac{1}{2}\mathbf{v}\cdot\bar{\mathbf{v}}_\ell)\right] \\ &= \quad -\frac{1}{4\ell}\int d^d\mathbf{r}\,(\boldsymbol{\nabla}G)_\ell(\mathbf{r})\cdot\delta\mathbf{v}(\mathbf{r})\,|\delta\mathbf{v}(\mathbf{r})|^2 - \nu\boldsymbol{\nabla}\mathbf{v} : \boldsymbol{\nabla}\bar{\mathbf{v}}_\ell + \frac{1}{2}(\mathbf{f}\cdot\bar{\mathbf{v}}_\ell + \bar{\mathbf{f}}_\ell\cdot\mathbf{v}) \end{aligned}$$

The above equation can be shown to be valid even for singular Leray solutions of INS, if the space-time derivatives are interpreted in the sense of distributions. (See Appendix.)

We now consider the limit of vanishing viscosity $\nu \to 0$. If the Navier-Stokes solution $\mathbf{v}^\nu \to \mathbf{v}$ as $\nu \to 0$ in the space-time L^2-sense, i.e.

$$\|\mathbf{v}^\nu - \mathbf{v}\|^2_{L^2_{\rm spacetime}} = \int dt \int d^d\mathbf{x}\ |\mathbf{v}^\nu(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)|^2 \to 0, \text{ as } \nu \to 0$$

then it is not hard to show that the limiting velocity $\mathbf{v}$ is a solution of the incompressible Euler equations

$$\partial_t \mathbf{v} + \boldsymbol{\nabla}\cdot(\mathbf{v}\mathbf{v}) = -\boldsymbol{\nabla} p + \mathbf{f},\ \boldsymbol{\nabla}\cdot\mathbf{v} = 0$$

in the sense of space-time distributions. The relevant simple estimate is that

$$\begin{aligned} \|\mathbf{v}^\nu\mathbf{v}^\nu - \mathbf{v}\mathbf{v}\|_{L^1} &= \|\mathbf{v}^\nu(\mathbf{v}^\nu - \mathbf{v}) + (\mathbf{v}^\nu - \mathbf{v})\mathbf{v}\|_{L^1} \\ &\leq \|\mathbf{v}^\nu\|_{L^2}\|\mathbf{v}^\nu - \mathbf{v}\|_{L^2} + \|\mathbf{v}^\nu - \mathbf{v}\|_{L^2}\|\mathbf{v}\|_{L^2} \end{aligned}$$

The key assumption of convergence of the velocity as $\nu \to 0$ in the strong L^2-sense has never been proved *a priori*, but it is consistent with empirical observations, following an argument of

> P. Isett, "Nonuniqueness and existence of continuous, globally dissipative Euler flows,"' (2017); `https://arxiv.org/abs/1710.11186`

as further elaborated in these works:

> T. D. Drivas and G. L. Eyink, "An Onsager singularity theorem for Leray solutions of incompressible NavierStokes," Nonlinearity **32** 4465 (2019)
>
> T. D. Drivas and H. Q. Nguyen. "Remarks on the emergence of weak Euler solutions in the vanishing viscosity limit," J. Nonlin. Sci. **29** 709–721 (2019).

For example, assume that the 2nd-order structure function $S_2^\nu(\mathbf{r},t) = \langle|\delta\mathbf{v}^\nu(\mathbf{r})|^2\rangle$ defined by an instantaneous space-average satisfies a scaling relation of the form

$$S_2^\nu(\mathbf{r},t) \sim \begin{cases} C_2(t) u_{rms}^2 (r/L)^{\zeta_2}, & \eta_2 < r < L \\ C_2(t) Re^{\gamma(2-\zeta_2)} u_{rms}^2 (r/L)^2 & r < \eta_2 \end{cases} \qquad (i)$$

with $\eta_2 \sim L(Re)^{-\gamma}$ for some $\gamma > 0$ and with ν-independent constant $C_2(t)$ so that

$$\frac{1}{T}\int_0^T dt\, C_2(t) < \infty. \qquad (ii)$$

It then follows that the "Besov semi-norm" $|||\mathbf{v}^\nu(t)|||_{B_2^{\sigma_2,\infty}}$ with $\sigma_2 = \zeta_2/2$ is essentially given by the constant $C_2(t)$:

$$|||\mathbf{v}^\nu(t)|||^2_{B_2^{\sigma_2}} := \sup_{|\mathbf{r}|<L} \frac{\langle|\delta\mathbf{v}^\nu(\mathbf{r})|^2\rangle}{(r/L)^{2\sigma_2}} = C_2(t)u^2_{\rm rms}$$

and condition (*ii*) with an energy bound independent of ν gives $\mathbf{v}^\nu \in L^2([0,T], B_2^{\sigma_2,\infty}(\Omega))$ uniformly in the viscosity. Because $B_2^{\sigma_2,\infty}(\Omega)$ is compactly embedded in $L^2(\Omega)$ according to a theorem of Kolmogorov & Riesz, a strong limit $\mathbf{v}^\nu \to \mathbf{v}$ exists in $L^2_{\rm spacetime}$ as $\nu \to 0$ (at least along a subsequence) by a result known as the Aubin-Lions-Simon lemma. For example, see:

> F. Boyer & P. Fabrie, *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*, (Springer, New York, 2013), section 5.3.

This same argument implies also convergence strong in $L^p_{\rm spacetime}$ for $p > 2$ with an analogous hypothesis on the absolute structure-function $S_p^\nu(\mathbf{r}, t)$.

If it is furthermore true that $\int dt \int d^d\mathbf{x}\ |\mathbf{v}(\mathbf{x}, t)|^3 < +\infty$, then the point-split balance holds also for the Euler solution in the distribution sense:

$$\begin{aligned}\partial_t(\frac{1}{2}\mathbf{v}\cdot\bar{\mathbf{v}}_\ell) + \boldsymbol{\nabla}\cdot\left[(\frac{1}{2}\mathbf{v}\cdot\bar{\mathbf{v}}_\ell)\mathbf{v} + \frac{1}{2}(p\bar{\mathbf{v}}_\ell + \bar{p}_\ell\mathbf{v}) + \frac{1}{4}\overline{(|\mathbf{v}|^2\mathbf{v})_\ell} - \frac{1}{4}\overline{(|\mathbf{v}|^2)_\ell}\mathbf{v}\right] \\ = -\frac{1}{4\ell}\int d^d\mathbf{r}\,(\boldsymbol{\nabla}G)_\ell(\mathbf{r})\cdot\delta\mathbf{v}(\mathbf{r})\,|\delta\mathbf{v}(\mathbf{r})|^2 \qquad (\star)\end{aligned}$$

For simplicity, we consider the case with no external force, $\mathbf{f} = 0$. We now consider the limit $\ell \to 0$. Under the same basic assumption, that $\int dt \int d^d\mathbf{x}|\mathbf{v}(\mathbf{x}, t)|^3 < +\infty$, it is not hard to show that the LHS of equation $(\star)$

$$\begin{aligned}\partial_t(\frac{1}{2}\mathbf{v}\cdot\bar{\mathbf{v}}_\ell) \quad &+ \quad \boldsymbol{\nabla}\cdot\left[(\frac{1}{2}\mathbf{v}\cdot\bar{\mathbf{v}}_\ell)\mathbf{v} + \frac{1}{2}(p\bar{\mathbf{v}}_\ell + \bar{p}_\ell\mathbf{v}) + \frac{1}{4}\overline{(|\mathbf{v}|^2\mathbf{v})_\ell} - \frac{1}{4}\overline{(|\mathbf{v}|^2)_\ell}\mathbf{v}\right] \\ &\longrightarrow \ \partial_t(\frac{1}{2}|\mathbf{v}|^2) + \boldsymbol{\nabla}\cdot\left[(\frac{1}{2}|\mathbf{v}|^2 + p)\mathbf{v}\right] \equiv -D(\mathbf{v}), \quad \text{as } \ell \to 0\end{aligned}$$

in the sense of distributions. Just to consider one typical term,

$$\begin{aligned}&\left|\int dt \int d^d\mathbf{x}\ \boldsymbol{\nabla}\varphi(\mathbf{x},t)\cdot(\frac{1}{2}\mathbf{v}(\mathbf{x},t)\cdot\bar{\mathbf{v}}_\ell(\mathbf{x},t))\mathbf{v}(\mathbf{x},t) - \int dt \int d^d\mathbf{x}\ \boldsymbol{\nabla}\varphi(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t)\frac{1}{2}|\mathbf{v}(\mathbf{x},t)|^2\right| \\ \leq\ &\sup|\boldsymbol{\nabla}\varphi|\cdot\|(\frac{1}{2}\mathbf{v}\cdot\bar{\mathbf{v}}_\ell)\mathbf{v} - (\frac{1}{2}|\mathbf{v}|^2)\mathbf{v}\|_{L^1_{\rm spacetime}}\end{aligned}$$

Using $\frac{1}{2}(\mathbf{v}\cdot\bar{\mathbf{v}}_\ell)\mathbf{v} - \frac{1}{2}|\mathbf{v}|^2\mathbf{v} = \frac{1}{2}[\mathbf{v}\cdot(\bar{\mathbf{v}}_\ell - \mathbf{v})]\mathbf{v}$ and the Hölder inequality

$$\|[\mathbf{v}\cdot(\bar{\mathbf{v}}_\ell - \mathbf{v})]\mathbf{v}\|_{L^1} \le \|\mathbf{v}\|_{L^3}^2 \|\bar{\mathbf{v}}_\ell - \mathbf{v}\|_{L^3}$$

One can then show that the upper bound $\to 0$ as $\ell \to 0$. This implies that

$$\boldsymbol{\nabla}\cdot\left[(\tfrac{1}{2}\mathbf{v}\cdot\bar{\mathbf{v}}_\ell)\mathbf{v}\right] \longrightarrow \boldsymbol{\nabla}\cdot\left[\tfrac{1}{2}|\mathbf{v}|^2\mathbf{v}\right]$$

in the sense of distributions. The other terms are treated in a similar fashion. But since the LHS of equation $(\star)$ converges to $-D(\mathbf{v})$ in the sense of the distributions, so does the RHS! That is,

$$D(\mathbf{v}) = \lim_{\ell\to 0} \frac{1}{4\ell}\int d^d\mathbf{r}\ (\boldsymbol{\nabla}G)_\ell(\mathbf{r})\cdot\delta\mathbf{v}(\mathbf{r})|\delta\mathbf{v}(\mathbf{r})|^2 \tag{1}$$

in the sense of distributions. To summarize, we obtain the above formula for the anomalous dissipation $D(\mathbf{v})$ that appears in the energy balance relation

$$\partial_t(\frac{1}{2}|\mathbf{v}|^2) + \boldsymbol{\nabla}\cdot\left[(\frac{1}{2}|\mathbf{v}|^2 + p)\mathbf{v}\right] = -D(\mathbf{v}). \tag{2}$$

for the singular Euler solution $\mathbf{v}(\mathbf{x},t)$. This result is quite interesting in its own right and not just as a step in the proof of the Kolmogorov 4/5-law. It is a precise mathematical formulation of Onsager's idea that Euler solutions which arises in the zero-viscosity limit of turbulent flow may not conserve energy. We could derive the same balance equation (2) by starting from the balance equation for $\frac{1}{2}|\bar{\mathbf{v}}_\ell|^2$ and taking the limit $\ell \to 0$. This would give us another valid expression

$$D(\mathbf{v}) = \lim_{\ell\to 0}\Pi_\ell \ \ \text{(in the distribution sense)}$$

for the anomalous dissipation $D(\mathbf{v})$. In fact, the RHS of equation $(\star)$

$$D_\ell(\mathbf{v}) = \frac{1}{4\ell}\int d^d\mathbf{r}\ (\boldsymbol{\nabla}G)_\ell(\mathbf{r})\cdot\delta\mathbf{v}(\mathbf{r})|\delta\mathbf{v}(\mathbf{r})|^2 \tag{3}$$

is another way of measuring energy flux to small scales, alternative to Π_ℓ. We can get from equation (1) for $D(\mathbf{v})$ Onsager's bound on energy flux to small scales. For example, if $\mathbf{v}$ has Hölder exponent h, then it follows from equation (3) that

$$D_\ell(\mathbf{v}) = O(\ell^{3h-1})$$

the same bound derived earlier for Π_ℓ. These bounds imply the assertion of Onsager in his 1949 paper that Euler solutions must conserve energy if the velocity has Hölder exponent $h > 1/3$. Using L_p norms, one can easily show also that energy is conserved if $\sigma_p > \frac{1}{3}$ (equivalently, $\zeta_p > p/3$) for $p \geq 3$. Under any of these regularity assumptions, $D(\mathbf{v}) \equiv 0$!

Let us now return to our derivation of the 4/5-law, by obtaining a simplified expression for $D(\mathbf{v})$ for the case of a spherically symmetric filter kernel G that depends upon only the magnitude $r = |\mathbf{r}|$:

$$G(\mathbf{r}) = G(r)$$

so that

$$\boldsymbol{\nabla} G(\mathbf{r}) = \hat{\mathbf{r}} G'(r).$$

In that case, one can go to spherical coordinates in d-dimensions

$$D_\ell(\mathbf{v}) = \tfrac{1}{4\ell} \int_0^\infty r^{d-1} dr \ \int_{S^{d-1}} d\omega(\hat{\mathbf{r}}) \hat{\mathbf{r}} \cdot \delta\mathbf{v}(\mathbf{r}) |\delta\mathbf{v}(\mathbf{r})|^2 (G')_\ell(r)$$

where S^{d-1} is the unit sphere in d-dimensions and $d\omega$ is the measure on solid angles. Now introduce

$$\delta v_L(\mathbf{r}) = \hat{\mathbf{r}} \cdot \delta\mathbf{v}(\mathbf{r}) = \underline{\text{longitudinal velocity increment}}$$

and

$$\begin{aligned} \langle \delta v_L(\mathbf{r}) |\delta\mathbf{v}(\mathbf{r})|^2 \rangle_{ang} &= \frac{1}{\Omega_{d-1}} \int_{S^{d-1}} d\omega(\hat{\mathbf{r}}) \delta v_L(\mathbf{r}) |\delta\mathbf{v}(\mathbf{r})|^2 \\ &= \text{angular average of } \ \delta v_L |\delta\mathbf{v}|^2 \end{aligned}$$

where Ω_{d-1} is the $(d-1)$ -dimension volume of S^{d-1}. We thus find that

$$\begin{aligned} D_\ell(\mathbf{v}) &= \frac{1}{4\ell} \Omega_{d-1} \int_0^\infty r^{d-1} dr \ (G')_\ell(r) \langle \delta v_L(\mathbf{r}) |\delta\mathbf{v}(\mathbf{r})|^2 \rangle_{ang} \\ &= \Omega_{d-1} \int_0^\infty \rho^d d\rho \ G'(\rho) \ \left. \frac{\langle \delta v_L(\mathbf{r}) |\delta\mathbf{v}(\mathbf{r})|^2 \rangle_{ang}}{4r} \right|_{\mathbf{r}=\ell\boldsymbol{\rho}} \end{aligned}$$

where $\boldsymbol{\rho} = \mathbf{r}/\ell$. We know that the limit of the LHS exists as $\ell \to 0$ in the sense of distributions and gives $D(\mathbf{v})$. Taking the limit on the RHS, we see that

$$\tfrac{\langle \delta v_L(\mathbf{r})|\delta \mathbf{v}(\mathbf{r})|^2\rangle_{ang}}{4r} \longrightarrow D^*(\mathbf{v}) \ \text{, as} \quad r \to 0$$

with

$$\begin{aligned} D(\mathbf{v}) &= D^*(\mathbf{v}) \cdot \Omega_{d-1} \int_0^\infty \rho^d d\rho G'(\rho) \\ &= D^*(\mathbf{v}) \cdot (-d \cdot \Omega_{d-1} \int_0^\infty \rho^{d-1} d\rho G(\rho)) \quad \text{by integration by parts} \\ &= -d \cdot D^*(\mathbf{v}) \ \ \text{since} \ \ \Omega_{d-1} \int_0^\infty \rho^{d-1} d\rho G(\rho) = 1 \end{aligned}$$

We conclude finally that

$$\boxed{\lim_{r\to 0} \frac{\langle \delta v_L(\mathbf{r})|\delta \mathbf{v}(\mathbf{r})|^2\rangle_{ang}}{r} = -\frac{4}{d} D(\mathbf{v})}$$

The results in the above form were given in the paper

J. Duchon & R. Robert, "Inertial energy dissipation for weak solution of incompressible Euler and Navier-Stokes equations," Nonlinearity, **13** 249-255(2000).

It is possible, by an elaboration of these arguments, to derive expressions for $D(\mathbf{v})$ that involve only $\delta v_L(\mathbf{r})$, or mixed expressions that involve $\delta v_L(\mathbf{r})$ and the transverse velocity increment

$$\delta \mathbf{v}_T(\mathbf{r}) = \delta \mathbf{v}(\mathbf{r}) - \delta v_L(\mathbf{r})\hat{\mathbf{r}}$$

which satisfies $\hat{\mathbf{r}} \cdot \delta \mathbf{v}_T(\mathbf{r}) = 0$, or

$$\delta v_T^2(\mathbf{r}) = |\delta \mathbf{v}_T(\mathbf{r})|^2/(d-1),$$

the magnitude of the transverse velocity increment per component. These are, in d-dimensions,

$$\boxed{\begin{aligned} \lim_{r\to 0} \frac{\langle \delta u_L^3(\mathbf{r})\rangle_{ang}}{r} &= -\frac{12}{d(d+2)} D(\mathbf{v}) \\ \lim_{r\to 0} \frac{\langle \delta v_L(\mathbf{r}) \delta v_T^2(\mathbf{r})\rangle_{ang}}{r} &= -\frac{4}{d(d+2)} D(\mathbf{v}) \end{aligned}}$$

For the derivation, see G. L. Eyink, Nonlinearity **16**, 137-145(2003).

Example: Burgers Equation

The above discussion has been a bit abstract, so that it is useful to consider a concrete example. All the previous results have exact analogies for singular/distributional solutions of the inviscid Burgers equation, which can be shown to satisfy the energy balance equation

$$\partial_t(\tfrac{1}{2}u^2) + \partial_x(\tfrac{1}{3}u^3) = -D(u)$$

with $D(u) = \lim_{\ell\to 0} \frac{1}{12\ell}\int_{-\infty}^{+\infty} dr (G')_\ell(r)\delta u^3(r)$ in the sense of distributions. Alternately,

$$\lim_{r\to 0} \frac{\langle \delta u_L^3(r)\rangle_{ang}}{|r|} = -12D(u)$$

where $\delta u_L(r) := \text{sign}(r)\delta u(r)$ and $\langle \delta u_L^3(r)\rangle_{ang} = \frac{1}{2}[\delta u^3(+|r|) - \delta u^3(-|r|)]$. For the Khokhlov sawtooth solution in the limit $\nu \to 0$ it is straightforward to calculate explicitly that, with $r > 0$

$$\begin{aligned} \langle \delta u_L^3(r)\rangle_{ang} \quad = \quad & \frac{1}{2}\Big[(r/t - \triangle u)^3 + (r/t)^3\Big]\chi_{[-r,0]}(x) \\ & +\frac{1}{2}\Big[(r/t)^3 - (-r/t + \triangle u)^3\Big]\chi_{[0,r]}(x) \\ & +\Big(\frac{r}{t}\Big)^3 \chi_{[-r,r]^c}(x) \end{aligned}$$

where $\Delta u := 2L/t > 0$, so that

$$\frac{\langle \delta u_L^3(r)\rangle_{ang}}{r} \longrightarrow -[\tfrac{1}{2}(\triangle u)^3 + \tfrac{1}{2}(\triangle u)^3]\delta(x) = -(\triangle u)^3\delta(x) \;\; , \text{as} \quad r \to 0.$$

Notice that this is equal to $-12\varepsilon(x)$, where $\varepsilon(x) = \lim_{r\to 0}\nu(\partial_x u^\nu)^2$ is the distributional limit of the viscous dissipation in $u^\nu(x,t)$ as $\nu \to 0$.

A similar result can be obtained for the $\nu \to 0$ limit of Leray solutions of the Navier-Stokes equation. These satisfy a local energy balance of the form

$$\partial_t(\frac{1}{2}|\mathbf{v}^\nu|^2) + \boldsymbol{\nabla}\cdot\left[(\frac{1}{2}|\mathbf{v}^\nu|^2 + p^\nu)\mathbf{v}^\nu - \nu\boldsymbol{\nabla}(\frac{1}{2}|\mathbf{v}^\nu|^2)\right] = -\nu|\boldsymbol{\nabla}\mathbf{v}^\nu|^2 \quad (\text{or, } \le -\nu|\boldsymbol{\nabla}\mathbf{v}^\nu|^2), \qquad (\star)$$

For simplicity, we shall only consider the case where "=" holds above rather than "≤". (For the general case, see Appendix.) Let us assume that $\mathbf{v}^\nu \to \mathbf{v}$ as $\nu \to 0$ in the L^3-sense in spacetime:

$$\int dt \int d^dx \; |\mathbf{v}^\nu(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)|^3 \longrightarrow 0.$$

This is stronger than the L^2-convergence assumed earlier, so that, again, the limiting velocity

$\mathbf{v}$ is an Euler solution in distribution sense. Furthermore, it is now possible to check that the LHS of equation ($\star$) above has the limit

$$\lim_{\nu\to 0} \partial_t(\tfrac{1}{2}|\mathbf{v}^\nu|^2) + \boldsymbol{\nabla}\cdot\left[(\tfrac{1}{2}|\mathbf{v}^\nu|^2 + p^\nu)\mathbf{v}^\nu - \nu\boldsymbol{\nabla}(\tfrac{1}{2}|\mathbf{v}^\nu|^2)\right] = \partial_t(\tfrac{1}{2}|\mathbf{v}|^2) + \boldsymbol{\nabla}\cdot\left[(\tfrac{1}{2}|\mathbf{v}|^2 + p)\mathbf{v}\right]$$

in distribution sense. The argument is very similar to that which we gave earlier for the limit $\ell \to 0$. Furthermore, the limit is exactly the same, i.e. $-D(\mathbf{v})$! Since the limits of the LHS and the RHS of equation ($\star$) must be the same, we obtain

$$D(\mathbf{v}) = \lim_{\nu\to 0} \nu|\boldsymbol{\nabla}\mathbf{v}^\nu|^2 = \lim_{\nu\to 0} \varepsilon^\nu \quad \text{(Duchon \& Robert, 2000)}$$

in the sense of distributions. Notice the RHS of the above expression is non-negative, so that its limit also must be:

$$D(\mathbf{v}) \geq 0$$

More precisely, $D(\mathbf{v})$ is a nonnegative distribution, which satisfies $\int d^d\mathbf{x}\, dt\, \varphi(\mathbf{x},t) D(\mathbf{v})(\mathbf{x},t) \geq 0$ for every nonnegative test function φ (C^∞ with compact support). It is known that every nonnegative distribution is given by a measure, i.e.

$$\int d^d\mathbf{x} \int dt\, \varphi(\mathbf{x},t) D(\mathbf{v})(\mathbf{x},t) = \int\int \mu(d\mathbf{x}, dt)\, \varphi(\mathbf{x},t)$$

This "dissipation measure" has been much studied experimentally and observed to have multi-fractal scaling properties, as we discuss a bit later! If φ is nonnegative and also normalized

$$\int dt \int d^d\mathbf{x}\, \varphi(\mathbf{x},t) = 1,$$

then we can interpret

$$\int dt \int d^d\mathbf{x}\, \varphi(\mathbf{x},t)\, \nu|\boldsymbol{\nabla}\mathbf{v}^\nu(\mathbf{x},t)|^2 \equiv \langle \nu|\boldsymbol{\nabla}\mathbf{v}^\nu|^2\rangle_\varphi$$

as an average in spacetime over the compact support of φ, weighted by φ. The above result then says that

$$\lim_{\nu\to 0}\langle\varepsilon^\nu\rangle_\varphi = \langle D(\mathbf{v})\rangle_\varphi$$

Our earlier results can be stated in a similar fashion, e.g.

$$\lim_{r\to 0}\lim_{\nu\to 0} \frac{\langle[\delta v_L^\nu(\mathbf{r})]^3\rangle_{\varphi,ang}}{r} = -\frac{12}{d(d+2)}\langle D(\mathbf{v})\rangle_\varphi.$$

We thus see that, taking first $\nu \to 0$,

$$\langle \delta u_L^3(\mathbf{r})\rangle_{\varphi,ang} \sim -\tfrac{12}{d(d+2)}\langle\varepsilon\rangle_\varphi r$$

This is the famous Kolmogorov 4/5-law (since the coefficient $\frac{12}{d(d+2)} = \frac{4}{5}$ for $d = 3$), derived by Kolmogorov in the third of his celebrated 1941 papers on turbulence. The related results

$$\begin{aligned}
\langle \delta v_L(\mathbf{r})\delta u_T^2(\mathbf{r})\rangle_{\varphi,ang} &\sim -\frac{4}{d(d+2)}\langle\varepsilon\rangle_\varphi r \\
\langle \delta v_L(\mathbf{r})|\delta \mathbf{v}(\mathbf{r})|^2\rangle_{\varphi,ang} &\sim -\frac{4}{d}\langle\epsilon\rangle_\varphi r
\end{aligned}$$

are called the Kolmogorov 4/15- and 4/3-laws, respectively. These were derived by Kolmogorov in the statistical sense, averaging over an ensemble of solutions assuming statistical homogeneity and isotropy. He employed in his derivation an equation derived earlier for the 2-point velocity correlation $\langle v_i(\mathbf{x},t)v_j(\mathbf{x}+\mathbf{r},t)\rangle$ by van Kármán and Howarth (1938), so that this is sometimes called the Kolmogorov-Kármán-Howarth relation. The result presented here is much stronger, because there is no average over ensembles and no assumption of homogeneity and/or isotropy. It seems to have been Onsager in the 1940's who realized that such relations should hold for individual realizations, without averaging. He derived the formula

$$D_\ell(\mathbf{v}) = \tfrac{1}{4\ell}\int d^dr\, (\boldsymbol{\nabla} G)_\ell(\mathbf{r})\cdot\delta\mathbf{v}(\mathbf{r})|\delta\mathbf{v}(\mathbf{r})|^2$$

and discussed its limit for $\ell \to 0$. In the statistical framework, the corresponding result

$$\boldsymbol{\nabla}_{\mathbf{r}}\cdot\langle\delta\mathbf{v}(\mathbf{r})|\delta\mathbf{v}(\mathbf{r})|^2\rangle \sim -4\langle\varepsilon\rangle \quad \text{, as} \quad r\to 0$$

was derived by A. S. Monin (1959) and is sometimes called the Kolmogorov-Monin relation. It does not assume isotropy. There is another derivation of the 4/5-law by Nie & Tanveer (1999) without statistical averaging. It uses also space-time averaging and angle-averaging. It is stronger than the result presented here in that it includes viscous corrections, but it is weaker than the presented local results, since it requires a global spacetime average.

Ensemble Approach to the 4/5th Law

We have focused on the deterministic version of the 4/5th law, but the traditional approach using ensemble averages has some advantages. For one thing, it provides a simple framework within which to study the effects of finite Reynolds number. See:

R. A. Antonia et al. et al., "Finite Reynolds number effect and the 4/5 law," PRF, **4**, 084602 (2019)

which extensively reviews this aspect (although we disagree with a great many theoretical claims in this work!)

Another important contribution of the statistical approach is in the paper:

J. Bedrossian et al. "A sufficient condition for the Kolmogorov 4/5 Law for stationary martingale solutions to the 3D Navier-Stokes equations," Commun. Math. Phys. **367** 1045-1075 (2019)

This work involves some (unphysical) mathematical complications because of the assumption of a continuum Navier-Stokes description, for which only weak solutions are known to exist (see Appendix on Leray's theory). However, the paper makes an important contribution by attempting to derive the the 4/5th-law under the weakest possible hypotheses. The authors consider the situation with turbulence driven by an external body force which is spatially homogeneous and white-in-time:

$$\langle f_i(\mathbf{x},t) f_j(\mathbf{x}',t')\rangle = 2F_{ij}(\mathbf{r}/L)\delta(t-t'), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}'$$

as first discussed here:

E. A. Novikov, "Functionals and the random-force method in turbulence theory," Sov. Phys. JETP **20** 1290-1294 (1965)

The useful result with forcing white-in-time obtained by Novikov (and derived rigorously by Bedrossian et al. for a notion of weak solutions) is that the power input by the force is fixed as $Q = \mathrm{tr}(\mathbf{F}(\mathbf{0}))$. Therefore, in the long-time statistical steady-state $Q = \varepsilon$, or

$$\mathrm{tr}(\mathbf{F}(\mathbf{0})) = \nu\langle|\boldsymbol{\nabla}\mathbf{v}|^2\rangle.$$

Thus, energy dissipation rate is trivially independent of Reynolds number! This does not, however, mean that there is "dissipative anomaly," which is the requirement that

$$D(Re) := \varepsilon/(u_{rms}^3/L) \to D_* > 0, \text{ as } Re \to \infty.$$

Hence, there is no dissipative anomaly if

$$u_{rms}^2 = \langle|\mathbf{v}|^2\rangle \to \infty \text{ as } Re \to \infty!$$

Bedrossian et al. introduce the notion of a "weak anomaly", which occurs when

$$Re \cdot D(Re) \to \infty \text{ as } Re \to \infty, \qquad (*)$$

or, in other words, $D(Re)$ may vanish as Re increases but more slowly than $D(Re) \propto 1/Re$. Equivalently, this means that the *Taylor microscale* $\lambda \propto \nu u_{rms}^2/\varepsilon$ satisfies $\lambda/L \to 0$ as $Re \to \infty$. The interesting result obtained by Bedrossian et al. under the hypothesis (*) is that, for any lengths ℓ_i, ℓ_d satisfying

$$\ell_i/L \to 0 \text{ and } \ell_d/\lambda \to \infty \text{ as } Re \to \infty$$

then with $\langle\langle\cdot\rangle\rangle$ denoting both ensemble-averaging and angle-averaging

$$\lim_{\ell_i/L\to 0} \limsup_{Re\to\infty} \sup_{\ell\in[\ell_d,\ell_i]} \left| \frac{1}{\ell}\langle\langle \delta u_L^3(\boldsymbol{\ell})\rangle\rangle + \frac{4}{5}\varepsilon \right| = 0.$$

Hence, it follows that the 4/5th-law holds to any desired degree of accuracy over the interval $[\ell_d, \ell_i]$ for $Re \gg 1$. An important implication of this result is that <u>the validity of the 4/5th-law cannot be taken as evidence for a (strict) dissipative anomaly.</u>

Experiments and Simulations

⋆ R. A. Antonia et al. et al., PRF, **4**, 084602 (2019)

This paper presents data on the 4/5th-law from a compilation of laboratory experiments.

FINITE REYNOLDS NUMBER EFFECT AND THE 4/5 LAW

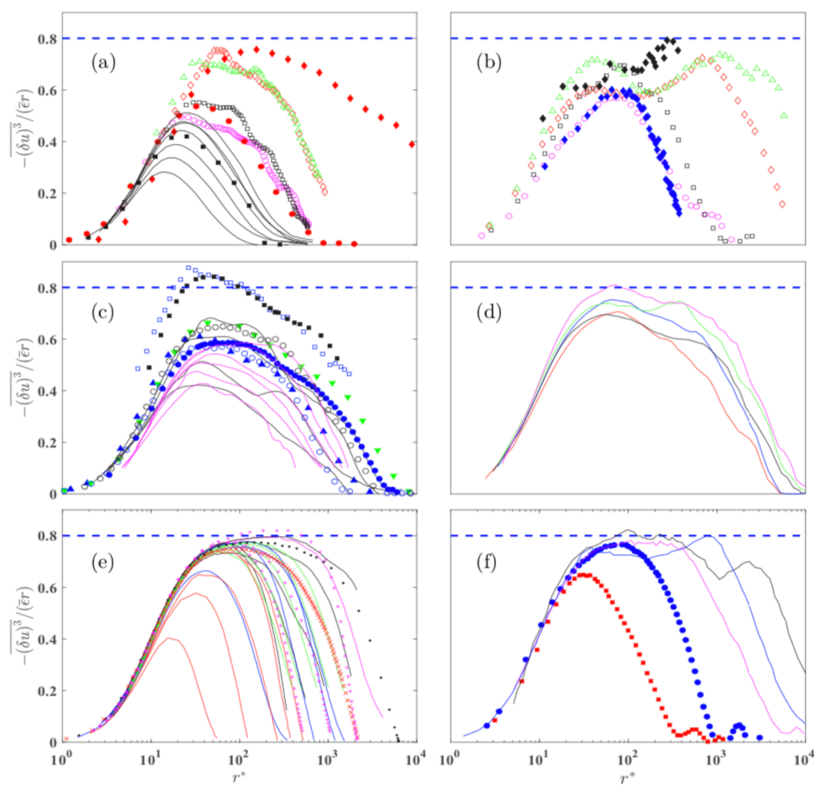

FIG. 1. Distributions of $\overline{(\delta u^*)^3}/r^*$ in various flows. (a) Shearless grid turbulence: black curves, $Re_\lambda = 26$–99 [40]; ■, $Re_\lambda = 72$ [46]; ●, $Re_\lambda = 144$ [46]; ○, $Re_\lambda = 99$ [47]; □, $Re_\lambda = 134$ [47]; △, $Re_\lambda = 319$ [47]; and ◇, $Re_\lambda = 448$ [47]. Also shown are the data along the axis of the ONERA wind tunnel (similar to grid turbulence) at $Re_\lambda = 2260$, ◆ [46]. (b) Sheared grid turbulence: ◆, $Re_\lambda = 170$ [48]; ◆, $Re_\lambda = 660$ [48]; ○, $Re_\lambda = 196$ [49]; □, $Re_\lambda = 254$ [49]; ◇, $Re_\lambda = 875$ [49]; and △, $Re_\lambda = 938$ [49]. (c) Along the axis of a circular jet: ■, $Re_\lambda = 835$ [50]; □, $Re_\lambda = 966$ [51]; black curves, $Re_\lambda = 235$–545 [52]; ●, $Re_\lambda = 485$ [41]; pink curves, $Re_\lambda = 122$–310 [53]; ▲, $Re_\lambda = 350$ [46]; ▼, $Re_\lambda = 695$ [46]; ○, $Re_\lambda = 200$ [54]; and ○, $Re_\lambda = 430$ [54]. (d) Along the axis of a plane jet: $Re_\lambda = 550$–1067 [44]. (e) SFPBT: red curves, $Re_\lambda = 38$–240 [43]; green curves, $Re_\lambda = 177$–435 [36]; blue curves, $Re_\lambda = 70$–460 [42]; black curves, $Re_\lambda = 167$–1131 [22]; pink curve, $Re_\lambda = 805$ [55]; •, $Re_\lambda = 1300$ [56]; ×, $Re_\lambda = 650$ [57]; and +, $Re_\lambda = 240$–700 [58]. (f) In a cylindrical container between counterrotating disks (■, $Re_\lambda = 120$; ●, $Re_\lambda = 300$; and pink curve, $Re_\lambda = 1170$) [59] and at $y/\delta \approx 0.5$ in a high Reynolds number boundary layer: $Re_\lambda = 600$ (blue curve) and $Re_\lambda = 1450$ (black curve) [26]. The dashed horizontal line in each plot corresponds to the value of 4/5.

⋆ K. R. Sreenivasan & B. Dhruva, Prog. Theor. Phys. Suppl. **130**, 103–120 (1998)

This paper presents data from hot-wire measurements in the atmospheric boundary layer, with Re_λ in the range $10,000 - 20,000$. The plot shows the "Kolmogorov function" $K(r) = \langle \delta u_L^3(r) \rangle / \varepsilon r$ and its "local slope" $d \log K(r)/d(\log r)$.

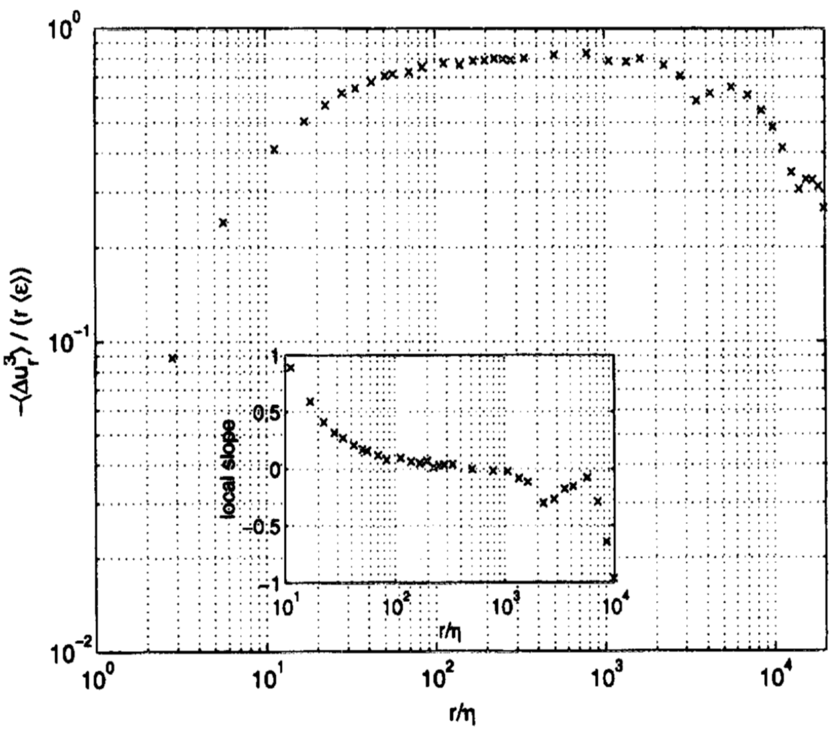

Fig. 3. The Kolmogorov function K for the present atmospheric data (for the first data set in Table I). It appears that the scaling range is quite extensive (although substantially smaller than in Fig. 2). The inset indicates the local slope, obtained by finite difference approximation of K. Interpreted blindly, the inset suggests that the scaling range is no more than half-an-order of magnitude in extent, corresponding to the region where the local slope is close to zero.

⋆ K. P. Iyer, K. R. Sreenivasan, and P. K. Yeung, PRF **5** 054605 (2020)

This paper presents data from a $16,384^3$ simulation of forced turbulence in a periodic domain, with $Re_\lambda = 1300$. The quantity $\langle \delta u_L^3(r) \rangle$ was calculated by averaging over space, time, and orientation angle of the displacement vector $\mathbf{r}$.

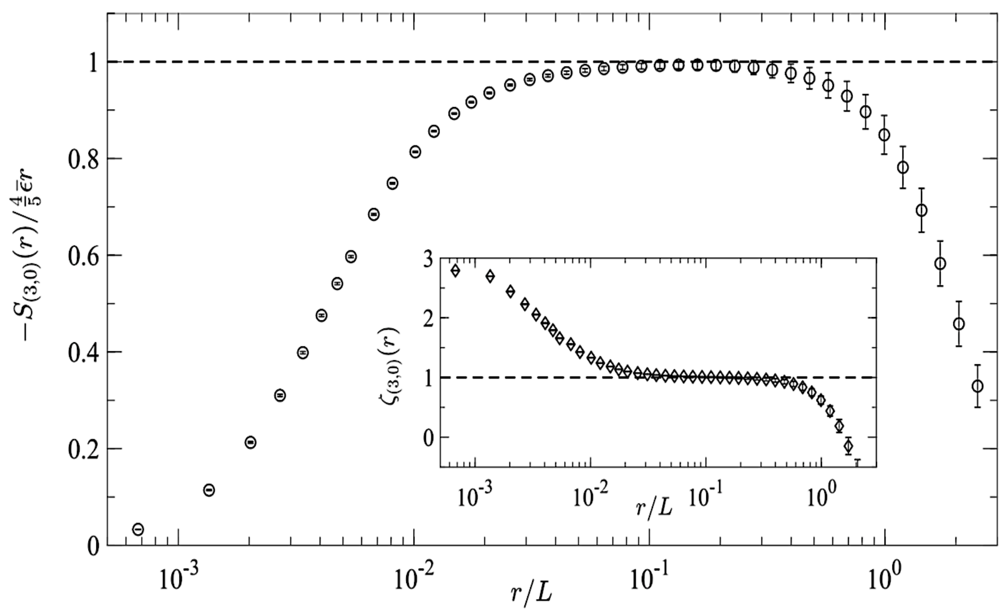

FIG. 1. Compensated third-order isotropic longitudinal structure function versus scale r on log-linear scales. The maximum value of $[-S_{(3,0)}(r)/(\frac{4}{5})\bar{\epsilon}r] = 0.99 \pm 0.01$. Inset shows the logarithmic local slope $\zeta_{(3,0)}(r) = d[\log|S_{(3,0)}(r)|]/d[\log r]$. Dashed line at unity, in both the main figure and the inset, is the exact result of Kolmogorov [see Eq. (3)].

⋆ M. Taylor et al. Phys. Rev. E **68**, 026310 (2003)

Another study with a 512^3 DNS of homogenous forced turbulence, with $Re_\lambda \cong 249 - 263$. This paper shows the importance of angle-averaging, obtaining results with such averaging comparable to those at nearly twice the Reynolds number.

TAYLOR, KURIEN, AND EYINK PHYSICAL REVIEW E **68**, 026310 (2003)

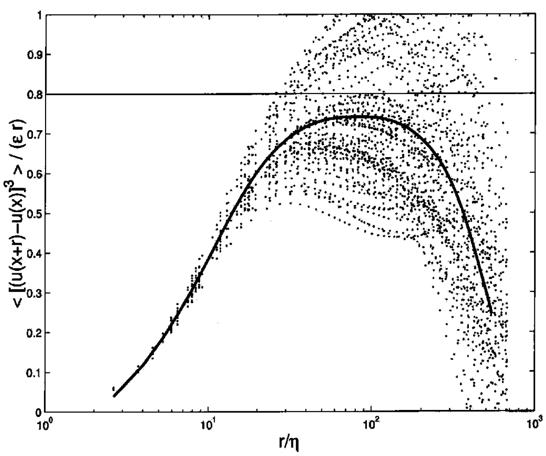

FIG. 6. The nondimensional third-order longitudinal structure function, computed from a single snapshot of the stochastic dataset, vs the nondimensional scale r/η. The dots indicate the values of the structure function computed at various $\ell \mathbf{r}_j$. The thick curve is the angle average. The horizontal line indicates the 4/5 mark.

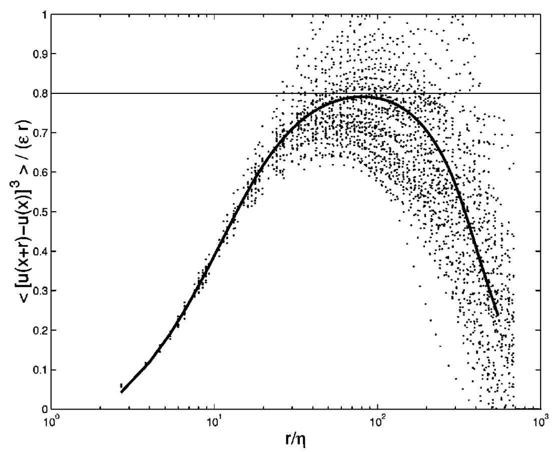

FIG. 7. The nondimensional third-order longitudinal structure function computed from a single snapshot of the deterministic dataset vs the nondimensional scale r/η. The various symbols and lines mean the same as in Fig. 6.

One last remark: The derivation that we have given applies even if $D(\mathbf{v}) \equiv 0$, i.e. vanishes everywhere. For example, this holds in a smooth solution of the Euler equations, for which $\langle \delta u_L^3(\mathbf{r})\rangle_{ang} \sim \langle \delta v_L(\mathbf{r}) \delta \mathbf{v}_T^2(\mathbf{r})\rangle \sim O(r^3)$, so that $D_\ell(\mathbf{v}) = O(\ell^2) \to 0$ as $\ell \to 0$. Another example is 2D Euler solutions where, under very general assumptions, $D(\mathbf{v}) \equiv 0$ and there is no energy cascade to small scales. E.g. see Proposition 6 in Duchon & Robert (2000). There is a nontrivial extension of the $\frac{4}{5}$-law to 2D turbulence, but with $\langle \delta u_L^3(\mathbf{r})\rangle_{ang}$ positive, corresponding to inverse energy cascade. E.g. see D. Bernard (1999).

Additional References:

T. von Kármán & L. Howarth, "On the statistical theory of isotropic turbulence," Proc. Roy. Soc. Lond. A **164**, 192-215 (1938).

A. N. Kolmogorov, "Dissipation of energy in locally isotropic turbulence," Dokl. Akad. Nauk. SSR **32**, 16-18 (1941).

A. S. Monin, "Theory of locally isotropic turbulence," Dokl. Akad. Nauk. SSSR **125** 515-518(1959); see also, A.S. Monin & A. M. Yaglom, Statistical Fluid Mechanics, vol.2 (MIT, 1975), p.403.

Papers on Deterministic Versions of $\frac{4}{5}$-law

Q. Nie & S. Tanveer, "A note on the third-order struncture functions in turbulence," Proc. R. Soc. A **455**, 1615-1635(1999).

J. Duchon & Robert, "Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes," Nonlinearity, **13** 249-255(2000)

G. L. Eyink, "Local $\frac{4}{5}$-law and energy dissipation anomaly in turbulence," Nonlinearity, **16** 137-145(2003).

G. L. Eyink, "Onsager and the theory of hydrodynamic turbulence," Rev. Mod. Phys. **78** 87-135(2006), Section IV, B.
Onsager's unpublished notes: `https://ntnu.tind.io/record/121183#`

2D Analogues of the $\frac{4}{5}$-law

D. Bernard, "Three-point velocity correlation functions in two-dimensional forced turbulence," Phys. Rev. E **60** 6184-6187(1993).

A. M. Polyakov, "The theory of turbulence in two dimensions," Nucl. Phys. B **396**, 367-385(1993). This paper, in particular, discusses the analogy of the $\frac{4}{5}$-law with conservation-law anomalies in quantum field theories.

The "Onsager Conjecture" and the h-Principle

What is now called the "Onsager conjecture" goes back to the following remark that Onsager made at the very end of his 1949 paper on fluid turbulence:

> "It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity feld does not remain differentiable! In fact it is possible to show that the velocity field in such 'ideal' turbulence cannot obey any LIPSCHITZ condition of the form
>
> $$|\vec{v}(\vec{r'}+\vec{r})-\vec{v}(\vec{r'})| < (const.)r^n \qquad (26)$$
>
> for any order n greater than 1/3; otherwise the energy is conserved. Of course, under the circumstances, the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description; for example, the formulation (15) in terms of FOURIER series will do. The detailed conservation of energy (17) does not imply conservation of the total energy if the number of steps in the cascade is infinite, as expected and the double sum of $Q(\vec{k},\vec{k'})$ converges only conditionally."

First, Onsager claims here that "it is possible to show that" that energy is conserved by ideal (Euler) fluid equations if the Hölder exponent of the velocity is greater than 1/3 and he gives a brief sketch of a proof using Fourier series. Second, Onsager remarks, after discussing the K41 theory in the preceding paragraphs, that "in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity", so that he clearly believed that Euler solutions with Hölder exponents 1/3 (or smaller) could dissipate kinetic energy.

The first published proof of Onsager's claim of conservation of kinetic energy for exponents $> 1/3$ was given in the paper:

> G. L. Eyink, "Energy dissipation without viscosity in ideal hydrodynamics, I. Fourier analysis and local energy transfer," Physica D **78** 222–240 (1994)

which made rigorous Onsager's heuristic argument using Fourier series but which required a more restrictive notion than standard Hölder continuity. The paper

> P. Constantin, W. E., and E.S. Titi, "Onsager's conjecture on the energy conservation for solutions of Euler's equation", Commun. Math. Phys. **165** 207–209 (1994)

very concisely proved Onsager's original statement on energy conservation for Euler solutions with velocities in Hölder spaces but also proved conservation for velocities only in Besov spaces. Their argument employed as a regularization a spatial mollification/low-pass filtering/coarse-graining operation, which is a very general and powerful technique that we have exploited throughout these lectures. Another important paper was

> J. Duchon and R. Robert, "Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations," Nonlinearity **13** 249–255 (2000)

which proved the Besov-space result using a point-splitting regularization that makes a connection with the Kolmogorov 4/5th law. Remarkably, Onsager had performed calculations identical to those of Duchon-Robert in his unpublished notes:

`https://ntnu.tind.io/record/121183`

see pp.14-18 in that folder. These results were never formally published but only communicated in a private letter to T. von Kármán and C. C. Lin in 1945, which is reproduced and discussed further here:

K. R. Sreenivasan, "Onsager and the theory of hydrodynamic turbulence," Rev. Mod. Phys. **78** 87–135 (2006)

The second remark of Onsager, that energy dissipation should be possible for Euler solutions whose velocities have Hölder regularity $\leq 1/3$, was almost certainly not proved by him in any mathematical way but instead presumably suggested by the empirical evidence. The modern form of Onsager's conjecture in terms of "weak solutions" was stated in Eyink (1994). That paper also constructed a simple example of a time-independent velocity field which showed that the paper's own proof of conservation (and also that of Constantin-E-Titi) could not be extended to regularity exponents $\leq 1/3$. Eyink (1994) then noted that

> "It must not, of course, be concluded that, simply because our argument fails when $h \leq 1/3$, that non-conservation is actually possible for $h \leq 1/3$. We emphasize that to demonstrate this it is necessary to construct an appropriate solution $\mathbf{v}(.,.)$ with $\mathbf{v}(.,t) \in C^h$, $0 < h < 1/3$ for $t \in [0,T]$, for which the energy indeed decreases or increases in the interval."

This issue then lay dormant until 13 years later when preprints appeared by Camillo De Lellis and László Székelyhidi Jr

C. De Lellis and L. Székelyhidi Jr, "The Euler equations as a differential inclusion," arXiv:math/0702079 (2007) [published in Annals of Mathematics, **170** 1417–1436 (2009)]

C. De Lellis and L. Székelyhidi Jr, "On admissibility criteria for weak solutions of the Euler equations", arXiv 0712.3288 (2007) [published in Arch. Ration. Mech. Anal. **195** 225–260 (2010)]

which initiated a long effort that culminated in full proofs in 2016-2017 that dissipative Euler solutions exist with any Hölder exponent $h < 1/3$:

P. Isett, "A proof of Onsager's conjecture," arXiv:1608.08301 (2016) [published in Annals Math. **188** 1–93 (2018)]

T. Buckmaster, C. De Lellis, L. Szkelyhidi Jr. and V. Vicol, "Onsager's conjecture for admissible weak solutions," arXiv:1701.08678 (2017) [Commun. Pure Appl. Math. **72** 229-274 (2019)]

Remarkably, these developments are very closely connected with famous work of the mathematician John Nash on C^1 isometric embeddings:

J. Nash, "C^1 isometric imbeddings", Ann. of Math. **60** 383–396 (1954),

a connection which is very lucidly discussed in the following review papers:

C. De Lellis and L. Székelyhidi, Jr., "Continuous Dissipative Euler Flows and a Conjecture of Onsager," in Proceedings of the European Congress of Mathematics, Kraków, 2012, edited by R. Latała, A. Ruciński, P. Strzelecki, J. Światkowski, and D. Wrzosek (European Mathematical Society, Zurich, 2013), pp. 13–30.
`http://cvgmt.sns.it/media/doc/paper/2187/de_lellis_proc_ECM_4.pdf`

C. De Lellis and L. Szkelyhidi, Jr., "On turbulence and geometry: from Nash to Onsager," Notices Amer. Math. Soci. **5** 677-685 (2019)

We give here a very succinct review, following the discussions in the previous references.

The paper of Nash (1954) addressed a classical problem of differential geometry, whether a smooth manifold M of dimension $n \geq 2$ with Riemannian metric g may be isometrically imbedded in m-dimensional Euclidean space $\mathbb{R}^m$, that is, whether a C^1 embedding map $\mathbf{u} : M \to \mathbb{R}^m$ exists so that the Riemannian metric induced by the embedding agrees with g, or

$$\partial_i \mathbf{u} \cdot \partial_j \mathbf{u} = g_{ij}. \qquad (*)$$

To answer this question, Nash considered a more general problem of *short embeddings* which do not preserve lengths of curves on M but can only *decrease* lengths, so that

$$\partial_i \overline{\mathbf{u}} \cdot \partial_j \overline{\mathbf{u}} \leq g_{ij}$$

in the matrix sense. The startling result obtained by Nash, with some improvement due to

N. H. Kuiper, On C^1-isometric imbeddings. I, II, Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17 (1955), 545–556, 683–689.

is the following:

Nash-Kuiper Theorem: *Let (M, g) be a smooth closed n-dimensional Riemannian manifold, and let $\overline{\mathbf{u}} : M \to \mathbb{R}^m$ be a C^∞ short embedding with $m \geq n+1$. For any $\epsilon > 0$ there exists a C^1 isometric embedding $\mathbf{u} : M \to \mathbb{R}^m$ with $|\mathbf{u} - \overline{\mathbf{u}}|_{C^0} < \epsilon$.*

This result is surprising for two reasons. First, the condition (*) is a set of $n(n+1)/2$ equations in m unknowns. A reasonable guess would be that the system is solvable, at least locally, when $m \geq n(n+1)/2$ and this indeed was a classical conjecture of Schläfli (1871). However, for $n \geq 3$ and $m = n+1$, the system (*) is overdetermined! It is not obvious that there should be any solutions at all, but the Nash-Kuiper Theorem shows that there exists a huge (C^0-dense) set of solutions in C^1. Moreover, for $n = 2$ one can compare with classical rigidity results of Herglotz and Cohn-Vossen for the so-called Weyl problem: if (S, g) is a compact Riemannian surface with positive Gauss curvature and if $\mathbf{u} \in C^2$ is an isometric embedding of S into $\mathbb{R}^3$, then $\mathbf{u}$ is uniquely determined up to a rigid motion! Thus it is clear that isometric embeddings have very different qualitative behavior at low and high regularity (i.e. C^1 versus C^2). This type of wild non-uniqueness at low regularity is a central aspect of the h-principle introduced by mathematician Mikhail Gromov:

M. Gromov, "A topological technique for the construction of solutions of differential equations and inequalities," Intern.Congr.Math.(Nice 1970) **2** 221-225 (1971)

M. Gromov, *Partial Differential Relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 9 (Springer-Verlag, Berlin, 1986).

with the isometric embedding problem as a primary example. We shall not discuss here the details of Nash's proof of his remarkable result, but just remark that his construction of the isometry $\mathbf{u}$ was in a series of stages, by adding at each stage a new small, high-frequency perturbation. A brief popular account is here:

`http://www.abelprize.no/c63466/binfil/download.php?tid=63580`

written on the occasion of the award to Nash of the 2015 Abel Prize in mathematics.

The fundamental contribution of De Lellis and Székelyhidi Jr was to realize that there is a very deep mathematical analogy between the problem of isometrically embedding a smooth manifold by a map of low regularity and the problem of solving the Cauchy initial-value problem for incompressible Euler equations by a velocity field of low regularity and that Nash's method of construction can be carried over to the latter. The analog of a "short mapping" for the Euler system is what De Lellis and Székelyhidi Jr call a smooth <u>subsolution</u>, i.e. a smooth triple $(\overline{\mathbf{v}}, \overline{p}, \overline{\boldsymbol{\tau}})$ with $\overline{\boldsymbol{\tau}}$ a symmetric, positive-definite tensor such that

$$\partial_t \overline{\mathbf{v}} + \boldsymbol{\nabla} \cdot (\overline{\mathbf{v}}\ \overline{\mathbf{v}} + \overline{\boldsymbol{\tau}}) = -\boldsymbol{\nabla}\overline{p}, \quad \boldsymbol{\nabla} \cdot \overline{\mathbf{v}} = 0.$$

This exactly corresponds to the incompressible Euler equations with the addition of a positive-definite "turbulent stress" tensor $\overline{\boldsymbol{\tau}}$! The fundamental theorem of Buckmaster et al. (2018) is then the following:

<u>Theorem</u>: *Let $(\overline{\mathbf{v}}, \overline{p}, \overline{\boldsymbol{\tau}})$ be any smooth, strict subsolution of the Euler equations on $\mathbb{T}^3 \times [0, T]$ and let $h < 1/3$. Then there exists a sequence $(\mathbf{v}_k, p_k)$ of weak Euler solutions such that $\mathbf{v}_k \in C^h(\mathbb{T}^3 \times [0, T])$ satisfy, as $k \to \infty$,*

$$\int_{\mathbb{T}^3} d^3x\ f\ \mathbf{v}_k \to \int_{\mathbb{T}^3} d^3x\ f\overline{\mathbf{v}}, \quad \int_{\mathbb{T}^3} d^3x\ f\ \mathbf{v}_k\mathbf{v}_k \to \int_{\mathbb{T}^3} d^3x\ f(\overline{\mathbf{v}}\ \overline{\mathbf{v}} + \overline{\boldsymbol{\tau}})$$

for all $f \in L^1(\mathbb{T}^3)$ uniformly in time, and furthermore for all $t \in [0,T]$ and all k

$$\int_{\mathbb{T}^3} d^3x\ \frac{1}{2}|\mathbf{v}_k|^2 = \int_{\mathbb{T}^3} d^3x\ \frac{1}{2}(|\overline{\mathbf{v}}|^2 + \mathrm{Tr}\ \overline{\boldsymbol{\tau}}).$$

If we take $f(\mathbf{r}) = \widetilde{G}_\ell(\mathbf{x}+\mathbf{r})$, then the convergence property can be restated as

$$\widetilde{\mathbf{v}}_{k,\ell} \to \widetilde{\overline{\mathbf{v}}}_\ell, \quad \widetilde{\tau}_\ell(\mathbf{v}_k, \mathbf{v}_k) \to \widetilde{\tau}_\ell(\overline{\mathbf{v}}, \overline{\mathbf{v}}) + \widetilde{\overline{\boldsymbol{\tau}}}_\ell \quad \text{(cf. Germano's identity!)}$$

This result is an h-principle for weak Euler solutions with Hölder regularity $h < 1/3$. Clearly, there is a huge number of subsolutions, since one may add any positive definite tensor $\overline{\boldsymbol{\tau}}$ whatsoever. An immediate consquence is therefore

Corollary: *Let $e : [0,T] \to \mathbb{R}^+$ be any strictly positive, smooth function. Then for any $0 < h < 1/3$ there exists a weak Euler solution $\mathbf{v} \in C^h(\mathbb{T}^3 \times [0,T])$ such that*

$$\int_{\mathbb{T}^3} d^3x\ \frac{1}{2}|\mathbf{v}|^2 = e(t).$$

In particular, one may take $e(t)$ to be any function strictly decreasing in time and then the Euler solution of the Corollary (globally) dissipates kinetic energy.

The same "convex integration methods" have also implications for non-uniqueness of the Cauchy initial-value problem, as discussed already by C. De Lellis and L. Székelyhidi Jr (2010). The Theorem 1 from that paper shook many previous expectations:

Theorem: *Let $d \geq 2$. There exist compactly-supported divergence-free vector fields $\mathbf{v}_0 \in L^\infty$ for which there are infinitely many weak Euler solutions with that initial data, satisfying both the strong energy equality*

$$\int_{\mathbb{R}^d} |\mathbf{v}(\mathbf{x},t)|^2\, d^n x = \int_{\mathbb{R}^d} |\mathbf{v}(\mathbf{x},s)|^2\, d^d x, \quad \textit{for all } t > s$$

and the local energy equality

$$\partial_t \left(\frac{1}{2}|\mathbf{v}|^2\right) + \boldsymbol{\nabla} \cdot \left[\left(\frac{1}{2}|\mathbf{v}|^2 + p\right)\mathbf{v}\right] = 0.$$

Furthermore, there are infinitely many weak Euler solutions with that initial data satisfying the strong energy inequality:

$$\int_{\mathbb{R}^d} |\mathbf{v}(\mathbf{x},t)|^2\, d^dx < \int_{\mathbb{R}^n} |\mathbf{v}(\mathbf{x},s)|^2\, d^dx, \quad \textit{for all } t > s$$

This result showed that one cannot add a local energy inequality

$$\partial_t \left(\frac{1}{2}|\mathbf{v}|^2\right) + \boldsymbol{\nabla} \cdot \left[\left(\frac{1}{2}|\mathbf{v}|^2 + p\right)\mathbf{u}\right] \leq 0,$$

(or even an equality) and obtain a unique weak solution for the Euler equations for certain L^∞ initial data. Furthermore, non-uniqueness occurs even if total energy is strictly decreasing.

Note that such initial data with non-unique solutions cannot be smooth (say, $C^{1,\epsilon}$ or higher smoothness) because this would violate the following important type of result:

<u>Theorem</u>: *Let* $\mathbf{v} \in L^\infty((0,T); L^2(\mathbb{T}^d))$ *be a weak Euler solution,* $\mathbf{V} \in C^1(\mathbb{T}^d \times [0,T])$ *a strong solution, and assume that* $\mathbf{v}$ *and* $\mathbf{V}$ *share the same initial datum* $\mathbf{v}_0$*. Assume moreover that*

$$\int_{\mathbb{T}^d} |\mathbf{v}(\mathbf{x},t)|^2\, d^dx \leq \int_{\mathbb{T}^d} |\mathbf{v}_0(\mathbf{x})|^2\, d^dx \qquad (\#)$$

for almost every $t \in (0,T)$*. Then* $\mathbf{v}(\mathbf{x},t) = \mathbf{V}(\mathbf{x},t)$ *for almost every* $(\mathbf{x},t)$*.*

Results of this type go by the name of <u>strong-weak uniqueness</u>. For an excellent review, see:

E. Wiedemann, "Weak-strong uniqueness in fluid dynamics," in: *Partial Differential Equations in Fluid Mechanics*, London Mathematical Society Lecture Note Series, vol. 452, Eds. Fefferman, C. L., Robinson, J. C., Rodrigo, J. L., & Diez, J. L. R. (Cambridge University Press, 2018). `https://arxiv.org/abs/1705.04220`

The conclusion of such results is that any "admissable" weak Euler solution satisfying the weak energy inequality (#) must coincide with a classical Euler solution, as long as that exists. Note that strong-weak uniqueness applies also to "measure-valued weak Euler solutions" such as constructed by DiPerna-Majda (1987): see the Wiedemann (2018) review.

These non-uniqueness results have since been considerably extended and are still currently under active investigation. Some very important results are contained in this paper:

S. Daneri, E. Runa, and L. Székelyhidi Jr, "Non-uniqueness for the Euler equations up to Onsager's critical exponent," Annals of PDE **7** 8 (2021)

who prove:

Theorem: *For any $h \in (0, 1/3)$, there is a set of divergence-free vector fields $\mathbf{v}_0 \in C^h(\mathbb{T}^3)$ which is a dense subset of the divergence-free vector fields in $L^2(\mathbb{T}^3)$ such that infinitely many Euler solutions exist with that initial data for which $\mathbf{v}(t) \in C^{h'}(\mathbb{T}^3)$ for all $h' < h$, $t \in (0, T)$ and for which the weak energy inequality (#) holds.*

This theorem shows that the non-uniqueness holds right up to the critical Onsager 1/3 exponent and for a dense set of initial data. Thus, uniqueness is in some sense "typical". To our knowledge, it is not yet known if the same result still holds if solutions are not merely "admissable" but instead satisfy a local energy inequality. A partial result in this direction was proved by

P. Isett, "Nonuniqueness and existence of continuous, globally dissipative Euler flows," arXiv:1710.11186v2.

whose Theorem 6 states that:

Theorem: *For any $h < 1/15$, the set of divergence-free initial data $\mathbf{v}_0 \in C(\mathbb{T}^3)$ that admit infinitely many incompressible Euler solutions of class C^h that obey local energy conservation*

$$\partial_t \left(\frac{1}{2}|\mathbf{v}|^2\right) + \boldsymbol{\nabla} \cdot \left[\left(\frac{1}{2}|\mathbf{v}|^2 + p\right) \mathbf{v}\right] = 0$$

on a common interval of time containing 0 is C^0 dense.

This situation must be contrasted with that for somewhat smoother initial data, for example $\mathbf{v}_0 \in C^h$ with $h > 1$, where unique solutions of the incompressible Euler equations exist (at least locally in time) that satisfy local kinetic energy conservation and possess the same regularity as the initial data. The non-uniqueness of weak solutions to the Cauchy problem with lower regularity is a "Nash non-rigidity" phenomenon, implying an essential unpredictability.

The constructions of such weak Euler solutions follow a strategy similar to that of Nash, by a sequence of stages $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$, ... At stage n one has, after coarse-graining, a subsolution

$$\partial_t \mathbf{v}_n + \boldsymbol{\nabla} \cdot (\mathbf{v}_n \mathbf{v}_n + \boldsymbol{\tau}_n) = -\boldsymbol{\nabla} p_n$$

which is supported on wavenumbers $< 2^n$. By adding a small-scale carefully chosen perturbation one can (together with other operations, such as evolving under smooth Euler dynamics locally in time and "gluing" the different time-segments) succeed to cancel a large part of the stress $\boldsymbol{\tau}_n$ so that, in the limit, $\boldsymbol{\tau}_n \to \mathbf{0}$ weakly and one obtains a weak limit $\mathbf{v}$ which is a distributional Euler solution. This is therefore a kind of "inverse renormalization group" procedure.

These Euler solutions, one must stress, are <u>not</u> obtained in the physically relevant manner by taking zero-viscosity limits of smooth solutions of the incompressible Navier-Stokes equations. One of the outstanding issues is to show that similar weak Euler solutions are obtained as viscosity tends to 0, at least along suitably selected subsequences of viscosities. If such inviscid-limit solutions do exist, then another open problem is to formulate suitable "admissibility/selection conditions" that such limits must possess and that uniquely characterize those solutions (or classes of such solutions). Some relatively simple proposals are ruled out by the previous results. For example, the local dissipation condition

$$\partial_t \left(\frac{1}{2} u^2 \right) + \partial_x \left(\frac{1}{2} u^3 \right) \leq 0$$

selects a unique weak solution of inviscid Burgers equation, under very modest regularity assumptions. This is not true (even with $= 0$!) for incompressible Euler solutions in C^h, $h < 1/15$.