FATIGUE INITIATION – SOME NOTES

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INTRODUCTION AND PURPOSE

This report is part of the graduate course FATIGUE AND FRACTURE MECHANICS. This is a "literature course" during summer of -97. The contents of the course should be equivalent to 6 weeks of full time work.

The literature in the course is chosen in order to put the emphasis on multi-axial fatigue and fracture mechanics. In this fatigue initiation part, several papers on fatigue initiation, mainly with a continuum mechanics approach, are used.

Note that the comments reflect my thoughts on the paper. I may well have misunderstood some of the contents etc. Also, I have included my own associations and comments in the notes below (not always explicitly stated). So, read the following with a "suspicious mind".

1. EQUIVALENT STRESS CRITERION

1.1. INTRODUCTION (OWN COMMENTS)

This is an attempt to compare and relate some common equivalent stress criteria.

Consider a general multiaxial state of stress in a material. The state of stress in a material point is defined by the stress tensor.

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{1}$$

If this matrix is assumed to be symmetric (i.e. $\sigma_{ij} = \sigma_{ji}$), there are six components that together defines the current state of stress.

Direction Invariant Equivalent Stress Measures

In order to compare this state of stress with experimental data, we need a scalar measure, that completely defines the stress field. This measure should be chosen in such a way that it reflects the fatigue behavior of the material. As a first assumption, could be to use von Mises equivalent stress, which is widely used in plasticity.

$$\sigma_{eq} = \sqrt{\frac{3}{2}\sigma_{ij}^{d}\sigma_{ij}^{d}} \quad \text{where } \sigma_{ij}^{d} = \sigma_{ij} - \frac{\sigma_{kk}}{3} \tag{2}$$

here, σ_{ij}^{d} is the deviatoric stress tensor. It is stated that fatigue initiation will occur if σ_{eq} exceeds a certain threshold value (which is found experimentally and considered to be a material parameter).

Influence of a Superposed Constant Shear Stress

Experimental data shows that there are no influence of a superposed static shear stress for fatigue initiation. In order to define what a “static” stress is, the concept of a stress cycle has to be introduced. That there is no influence of a static shear stress implies that the mid value of the stress should be eliminated in some way.

It is then convenient to express von Mises equivalent stress in terms of principal stresses. If the direction of the principal stresses is constant and the loading is in-phase (i.e. the principal stresses are varying with the same frequency and phase angle), the amplitude of the principal stresses could be used in the von Mises equivalent stress in order to eliminate the influence of a static shear stress.A modified von Mises equivalent stress could then be expressed as

$$\sigma_{eq} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{1,a} - \sigma_{2,a})^2 + (\sigma_{2,a} - \sigma_{3,a})^2 + (\sigma_{3,a} - \sigma_{1,a})^2} \tag{3}$$

where index a, denotes maximum deviation from mid-value during a stress cycle.

If the conditions described above are not fulfilled, the mid-value of the stress tensor has to be found and eliminated. The modified von Mises equivalent stress could then be expressed as

$$\sigma_{eq}(t) = \sqrt{\frac{3}{2}\sigma^{d}_{ij,\,a}\,\sigma^{d}_{ij,\,a}} \quad \text{where} \quad \sigma^{d}_{ij,\,a} = \sigma^{d}_{ij} - \sigma^{d}_{ij,\,m} \tag{4}$$

and $\sigma^{d}_{ij,\,m}$ is the mid value of the deviatoric stress tensor during a stress cycle. Note that once the mid-value has been eliminated, von Mises equivalent stress is no longer given as a single value for a load cycle, but is defined for every instant of time. (Or put in other words, it is no longer obvious when the maximum equivalent stress will occur).

INFLUENCE OF A SUPERPOSED CONSTANT HYDROSTATIC STRESS

From experiments, it has also been found that a static hydrostatic stress do have an influence on the fatigue behavior. In order to account for this influence, von Mises equivalent stress has to be modified

$$\sigma_{eq}(t) = \sqrt{\frac{3}{2}\sigma^{d}_{ij,\,a}\,\sigma^{d}_{ij,\,a}} + f(\sigma_h) \quad \text{where} \quad \sigma_h = \frac{\sigma_{kk}}{3} \tag{5}$$

σ_h is the hydrostatic stress and f is a function. Typically, f can be expressed as

$$f = a\sigma_h(t) \quad \text{or} \quad f = b\sigma_{h,\,mean} \quad \text{or} \quad f = c\sigma_{h,\,max} \tag{6}$$

where a, b and c are material constants and index “mean” and “max” denotes mean and max-values in a stress cycle.

SHEAR STRESS BASED EQUIVALENT STRESS MEASURES

Fatigue initiation is confined to shear in specific directions (i.e. acting on intergranular slip bands [and possibly grain interfaces]) in the material. Due to this, and also in order to achieve a possibility to account for anisotropy, it can be suitable to use an equivalent stress measure based on the shear strain acting in a specific direction. The shear stress vector can be computed as

$$\tau_k = \sigma_{ki}\mathbf{n}_i - (\sigma_{ji}\mathbf{n}_i\mathbf{n}_j)\mathbf{n}_k \tag{7}$$

where n is the normal vector for the shear plane studied. The shear stress vector can be applied in (5) as a measure of the influence of the shear stress. This leads to a shear stress based equivalent stress[1]

[1] The influence of a superposed hydrostatic stress can, in such a criterion, also be represented by the normal stress acting on the considered shear plane. However, this would predict an influence of a superposed static shear stress, which is in contradiction to observed fatigue behavior.

$$\sigma_{eq}(t) = \tau_a(t) + f(\sigma_h) \quad (8)$$

Where index a denotes the deviation from the mid-value of the shear stress vector during a load cycle.

The mid value of the shear stress vector for a load cycle can be defined as origo for the smallest circle that circumscribes the stress path that the tip of the shear stress vector follows during a stress cycle, see FIG. 1. In the same manner, the mid value of the second invariant of the deviatoric stress tensor, that is used in (5) can be found by identifying the smallest hypersphere that encloses its 5D-path [1]. However, if the stresses are symmetric during the stress cycle, the mid value during a stress cycle will be this point of symmetry [1].

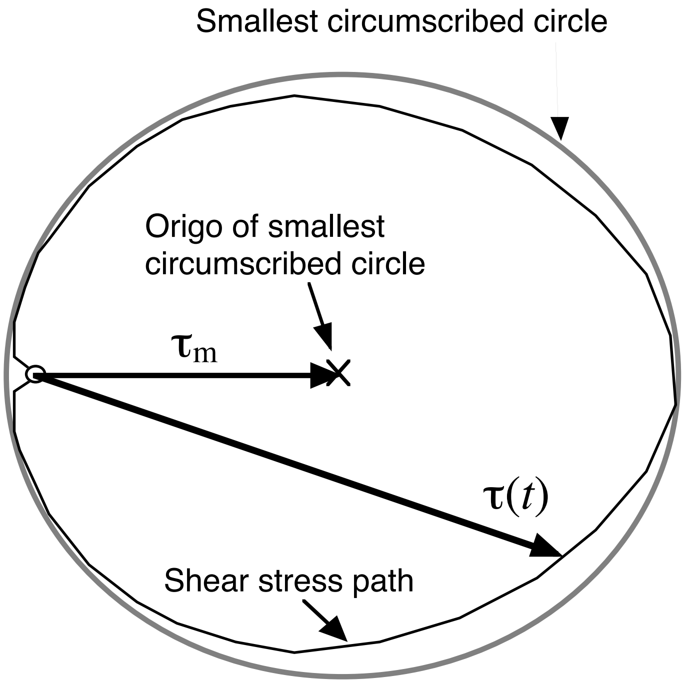

Fig. 1 Definition of mid shear stress τ_m for a stress cycle.

Note that (8) defines an equivalent stress measure for every shear plane (i.e. every possible n). This can be used to take anisotropy into account since material parameters can be expressed as functions of n. Equation (5), on the other hand, is based on two stress invariants, namely the second invariant of the deviatoric stress tensor and the hydrostatic stress (which is the first invariant of the stress tensor). Thus, this criterion is, in some sense, based on energy assumptions.

GRADIENT DEPENDENT EQUIVALENT STRESS MEASURES

An other experimental finding, is that the fatigue thresholds for cyclic bending and cyclic tension/compression differs. This can, to some extent be explained by volumetric effects, but some of the effect is due to the influence of stress gradients, see [2] and [3]. The equivalent stress criterion can then be expressed as

$$\sigma_{eq}(t) = \tau(t) + f_1(\sigma_h) + f_2(\sigma_h) \cdot (\mathbf{G}) \quad (9)$$

where f_1 and f_2 are functions, and

$$\mathbf{G} = \begin{bmatrix} \frac{\partial\sigma_h}{\partial x} & \frac{\partial\sigma_h}{\partial y} & \frac{\partial\sigma_h}{\partial z} \end{bmatrix} \quad (10)$$

Note that f_2 should be a function of the hydrostatic stress such that it assures that the gradient does not have an influence when there is no acting hydrostatic stress at the material point considered.

2. PHYSICAL JUSTIFICATION FOR EQUIVALENT STRESS CRITERIA

Mainly from [4]. In the following section, capital greek letters are used to denote macroscopic stresses. In all other sections, small capital greek letters denote macroscopic stresses.

2.1. MESOSCOPIC AND MACROSCOPIC STRESSES

Consider stresses acting on a mesoscopic scale under macroscopically elastic conditions. The macroscopic stresses are then defined by Hooke's law as $\Sigma = C : E$, where Σ is the stress tensor, C is the stiffness tensor and E is the strain tensor. On the mesoscopic, scale, we have the similar relation $\sigma = \mathbf{c} : \varepsilon^e$.

Even though the material is macroscopically elastic, we can have plastic strains on the mesoscopic scale, i.e. $\varepsilon = \varepsilon^e + \varepsilon^p$. Assuming the macroscopic strain is the average of the mesoscopic strain gives $\mathbf{E} = \varepsilon^e + \varepsilon^p$. Thus, we have the macroscopic stress as

$$\Sigma = \mathbf{C} : \varepsilon^e + \mathbf{C} : \varepsilon^p = \mathbf{C} : \mathbf{c}^{-1} : \sigma + 2\mu\varepsilon^p = \sigma + 2\mu\varepsilon^p \quad (11)$$

Where Hooke's law on the mesoscopic scale has been used, together with the fact that the mesoscopic plastic strain is a deviatoric tensor (has isotropic material been assumed???). Also, it has been assumed that $Cc^{-1}=I$, where I is the fourth-order unit-tensor.

Now, the projected shear stress acting in a specified crystal slip direction, m, on the shear plane can be written as, see (7).

$$\mathbf{T} = (\mathbf{m} \bullet t)\mathbf{m} = (\mathbf{m} \bullet \Sigma \bullet \mathbf{n} - (\mathbf{n} \bullet \Sigma \bullet \mathbf{n})\mathbf{n} \bullet \mathbf{m})\mathbf{m} = (\Sigma : \mathbf{a})\mathbf{m} \tag{12}$$

where $a_{ij} = \frac{n_i m_j + n_j m_i}{2}$ and it has been used that $\mathbf{m} \bullet \Sigma \bullet \mathbf{n} = \Sigma : \mathbf{a}$ due to symmetry of Σ, ($\mathbf{m} \bullet \Sigma \bullet \mathbf{n} = \Sigma_{ij} m_i n_j = [\text{sym}] = \frac{1}{2}(\Sigma_{ij} m_i n_j + \Sigma_{ij} m_j n_i) = \Sigma_{ij} a_{ij}$). Also, $\mathbf{n} \bullet \mathbf{m} = 0$, since these vectors are orthogonal.

Assuming one active glide system per crystal gives $\varepsilon^p = g^p \mathbf{a}$ (since a defines the orientation of the glide plane and direction). Here, g is the magnitude of plastic shear strain. Introducing this in (11) gives the mesoscopic stress as

$$\sigma = \Sigma - 2\mu\gamma^p \mathbf{a} \tag{13}$$

The projection of the mesoscopic stress vector ($\mathbf{n} \bullet \sigma$) on the normal (n) of the slip plane is

$$\begin{aligned} \nu &= (\mathbf{n} \bullet \sigma \bullet \mathbf{n})\mathbf{n} = (\mathbf{n} \bullet (\Sigma - 2\mu g^p \mathbf{a}) \bullet \mathbf{n})\mathbf{n} \\ &= (\mathbf{n} \bullet \Sigma \bullet \mathbf{n})\mathbf{n} - 2\mu g^p (\mathbf{n} \bullet \mathbf{a} \bullet \mathbf{n})\mathbf{n} \end{aligned} \tag{14}$$

But $\mathbf{n} \bullet \mathbf{a} \bullet \mathbf{n} = \mathbf{n} \bullet \mathbf{nm} \bullet \mathbf{n} = 0$, since m and n are orthogonal. Thus, we have

$$\nu = (\mathbf{n} \bullet \Sigma \bullet \mathbf{n})\mathbf{n} \tag{15}$$

which is equal to the macroscopic normal stress, N, acting on the slip plane.

The mesoscopic shear stress is the projection of the mesoscopic stress vector ($\mathbf{n} \bullet \sigma$) on the direction (m) of the slip plane

$$\tau = (\mathbf{n} \bullet \sigma \bullet \mathbf{m})\mathbf{m} = (\sigma : \mathbf{a})\mathbf{m} \tag{16}$$

because $n \bullet \sigma \bullet m = \frac{1}{2}\sigma_{ij} n_i m_j + \frac{1}{2}\sigma_{ij} n_i m_j = \sigma_{ij}\left(\frac{n_i m_j + n_j m_i}{2}\right) = \sigma_{ij} a_{ij}$ since σ_{ij} is symmetric.

Introducing (13) in (16) gives

$$\begin{aligned} \tau &= ([\Sigma - 2\mu g^p \mathbf{a}] : \mathbf{a})\mathbf{m} = [\Sigma : \mathbf{a} - 2\mu g^p (\mathbf{a} : \mathbf{a})]\mathbf{m} \\ &= (\Sigma : \mathbf{a})\mathbf{m} - \mu g^p \mathbf{m} = (\Sigma : \mathbf{a})\mathbf{m} - \mu\gamma^p \end{aligned} \tag{17}$$

where it was used that

$$\begin{aligned} a : a &= a_{ij} a_{ij} = \left(\frac{n_i m_j + n_j m_i}{2}\right)\left(\frac{n_i m_j + n_j m_i}{2}\right) \\ &= \frac{n_i n_i m_j m_j + n_i m_i n_j m_j + n_j n_j m_i m_i + n_i m_i n_j m_j}{4} \\ &= \frac{1 + 0 + 1 + 0}{4} = \frac{1}{2} \end{aligned} \tag{18}$$

since n and m are orthogonal unit tensors. Also, $\gamma^{p} = g^{p}\mathbf{m}$. Inserting (12) in (17) gives

$$\tau = \mathbf{T} - \mu\gamma^{p}\mathbf{m} \tag{19}$$

From (19) and (15), it can be seen that, on a slip line, the mesoscopic normal stress equals the macroscopic normal stress, whereas the mesoscopic shear stress is the macroscopic shear stress plus a correction.

2.2. Crystal Plasticity

Schmid's law states that a crystal, is deforming plastically if

$$(\tau - \mathbf{b}) \bullet (\tau - \mathbf{b}) = \tau_y^2 \text{ and } \frac{\partial[(\tau - \mathbf{b}) \bullet (\tau - \mathbf{b}) - \tau_y^2]}{\partial\tau} \bullet \dot{\tau} > 0 \tag{20}$$

where b is the kinematically hardening parameter and

$$\frac{\partial[(\tau - \mathbf{b}) \bullet (\tau - \mathbf{b}) - \tau_y^2]}{\partial\tau} = 2(\tau - \mathbf{b}) \tag{21}$$

The plastic shear strain rate is given by

$$\dot{\gamma}^{p} = \dot{\lambda}^{p} \cdot \frac{\partial[(\tau - \mathbf{b}) \bullet (\tau - \mathbf{b}) - \tau_y^2]}{\partial\tau} = 2\dot{\lambda}^{p}(\tau - \mathbf{b}) \tag{22}$$

Now, introduce isotropic and kinematical hardening rules

$$\dot{\tau}_y = g\sqrt{\dot{\gamma}^{p} \cdot \dot{\gamma}^{p}} \text{ and } \dot{\mathbf{b}} = c\dot{\gamma}^{p} \tag{23}$$

where g and c are positive material constants.

The consistency condition is

$$\begin{aligned} \frac{\partial[(\tau - \mathbf{b}) \bullet (\tau - \mathbf{b}) - \tau_y^2]}{\partial t} &= \frac{\partial[(\tau - \mathbf{b}) \bullet (\tau - \mathbf{b}) - \tau_y^2]}{\partial\tau}\dot{\tau} \\ &+ \frac{\partial[(\tau - \mathbf{b}) \bullet (\tau - \mathbf{b}) - \tau_y^2]}{\partial\mathbf{b}}\dot{\mathbf{b}} + \frac{\partial[(\tau - \mathbf{b}) \bullet (\tau - \mathbf{b}) - \tau_y^2]}{\partial\tau_y}\dot{\tau}_y \\ &= 2(\tau - \mathbf{b})\dot{\tau} - 2(\tau - \mathbf{b})\dot{\mathbf{b}} - 2\tau_y\dot{\tau}_y \\ &= 2(\tau - \mathbf{b})\dot{\tau} - 2(\tau - \mathbf{b})c\dot{\gamma}^{p} - 2\tau_y g\sqrt{\dot{\gamma}^{p} \cdot \dot{\gamma}^{p}} \\ &= 2(\tau - \mathbf{b})(\dot{\tau} - c\dot{\gamma}^{p}) - 2\tau_y g\sqrt{\dot{\gamma}^{p} \cdot \dot{\gamma}^{p}} = 0 \end{aligned} \tag{24}$$

which, with (20), gives

$$\dot{\gamma}^p = \frac{2(\tau - \mathbf{b})\tau}{2(c(\tau - \mathbf{b}) + \tau_y g)} = \frac{\tau}{(c + g)} \qquad (25)$$

Apply (19) in rate form as $\dot{\tau} = \dot{\mathbf{T}} - \mu\dot{\gamma}^p\mathbf{m}$, and insert in (25). This gives

$$\dot{\gamma}^p = \frac{\dot{\mathbf{T}} - \mu\dot{\gamma}^p}{(c + g)} \Leftrightarrow \dot{\gamma}^p = \frac{\dot{\mathbf{T}}}{(c + g + \mu)} \qquad (26)$$

Thus, the mesoscopic plastic shear strain rate is proportional to the rate of the macroscopic shear stress acting on the slip line.

Introducing (19) in the yield criterion for the crystal (20) and applying (23), gives

$$[\mathbf{T} - (\mu + c)\gamma^p][\mathbf{T} - (\mu + c)\gamma^p] - \tau_y^2 \le 0 \qquad (27)$$

Both T and γ^p acts along m. Thus, (27) can be rewritten as

$$\mathbf{T} - (\mu + c)\gamma^p = \pm\tau_y \qquad (28)$$

which is a segment of this line, centered at $(\mu + c)\gamma^p$ and with length $2\tau_y$. At cyclic loading, the shear stress vector T will act along m. The plastic mesostrain can be expressed as

$$\dot{\Gamma} = \sqrt{\dot{\gamma}^p\dot{\gamma}^p} = \frac{\sqrt{\dot{\mathbf{T}}\dot{\mathbf{T}}}}{c + g + \mu} \qquad (29)$$

where (26) has been applied.

When T acts along a path $\overrightarrow{O\mathbf{T}^A}$, where O denotes a relaxed state of stress, the resulting plastic strain will consequently be

$$\Gamma_{O \to A} = \frac{\left\|\mathbf{T}^A - \tau_y\mathbf{m}\right\|}{c + g + \mu} \qquad (30)$$

where $\left\|\mathbf{T}^A - \tau_y\mathbf{m}\right\|$ denotes the length of the vector T^A, exceeding $\tau_y\mathbf{m}$. Due to isotropic hardening, according to (23), the yield limit when the shear stress reverses its direction, at T^A, will be

$$\tau_y^A = \tau_y + g\Gamma = \tau_y + \frac{g\left\|\mathbf{T}^A\right\| - \tau_y}{c + g + \mu} \qquad (31)$$

Now, we start a stress cycle by loading the shear stress to its first peak value at T^B. The accumulated plastic strain for $\overrightarrow{\mathbf{T}^A\mathbf{T}^B}$ is (note that the behavior is plastic in the region $\overrightarrow{\mathbf{T}^{A}2\tau_y^{A}}$)

$$\Gamma_{A\to B} = \frac{1}{(c+g+\mu)}\left\|\mathbf{T}^{B} - \mathbf{T}^{A} + 2\tau_y^{A}\mathbf{m}\right\| \\ = \frac{1}{(c+g+\mu)}(\|\Delta\mathbf{T}\| - 2\tau_y^{A}) \tag{32}$$

where $\|\Delta\mathbf{T}\| = \left\|\mathbf{T}^{A} - \mathbf{T}^{B}\right\|$.

The new yield limit will be

$$\tau_y^{B} = \tau_y^{A} + g\Gamma_{A\to B} = \tau_y^{A} + \frac{g}{(c+g+\mu)}(\|\Delta\mathbf{T}\| - 2\tau_y^{A}) \tag{33}$$

Completing the stress cycle, by loading the shear stress from the first peak value at T^{B} to its starting peak value at T^{A}, results in a additional induced plastic strain

$$\Gamma_{B\to A} = \frac{1}{(c+g+\mu)}(\|\Delta\mathbf{T}\| - 2\tau_y^{B}) \tag{34}$$

thus, the increment of plastic strain over the first cycle ($A \to B \to A$)is

$$\delta\Gamma^{1} = \frac{1}{(c+g+\mu)}(\|\Delta\mathbf{T}\| - 2\tau_y^{A} + \|\Delta\mathbf{T}\| - 2\tau_y^{B}) \tag{35}$$

$$= \frac{2}{(c+g+\mu)}\left(\|\Delta\mathbf{T}\| - \tau_y^{A} - \tau_y^{A} - \frac{g\|\Delta\mathbf{T}\|}{(c+g+\mu)} - \frac{2g\tau_y^{A}}{(c+g+\mu)}\right)$$

$$= \frac{2}{(c+g+\mu)^2}(\|\Delta\mathbf{T}\|[(c+g+\mu)-g] - 2\tau_y^{A}[(c+g+\mu)-g])$$

$$= \frac{2(c+\mu)}{(c+g+\mu)^2}(\|\Delta\mathbf{T}\| - 2\tau_y^{A})$$

where τ_y^{B} according to (33) has been used.

By repeating the calculations above, the plastic strain increment for the N:th cycle can be expressed as

$$\delta\Gamma^{N} = \left(\frac{\mu+c-g}{\mu+c+g}\right)^{N}\delta\Gamma^{1} \tag{36}$$

And the accumulated plastic strain from the cyclic loading is given by an adding all increments using (36) and (35), i.e.

$$\Gamma_\infty = \sum_{i=1}^{\infty}\left(\frac{\mu + c - g}{\mu + c + g}\right)^{i-1}\delta\Gamma^1 = \tag{37}$$

$$\frac{1}{1 - \frac{(\mu + c - g)^2}{(\mu + c + g)^2}} \cdot \frac{2(c + \mu)}{(c + g + \mu)^2} \cdot (\|\Delta \mathbf{T}\| - 2\tau_y^A) =$$

$$\frac{1}{2g}(\|\Delta \mathbf{T}\| - 2\tau_y^A)$$

And finally adding the plastic strain from the first loading $\overrightarrow{O\mathbf{T}^A}$, from (30), and τ_y^A from (31),gives the total accumulated strain as

$$\begin{aligned}
\Gamma_\infty &= \frac{1}{2g}(\|\Delta \mathbf{T}\| - 2\tau_y^A) + \frac{\|\mathbf{T}^A\| - \tau_y}{c + g + \mu} \\
&= \frac{\|\Delta \mathbf{T}\|}{2g} - \frac{\tau_y^A}{g} + \frac{\|\mathbf{T}^A\|}{c + g + \mu} - \frac{\tau_y}{c + g + \mu} \\
&= \frac{\|\Delta \mathbf{T}\|}{2g} - \frac{\tau_y}{g} - \frac{\|\mathbf{T}^A\|}{c + g + \mu} + \frac{\|\mathbf{T}^A\|}{c + g + \mu} + \frac{\tau_y}{c + g + \mu} - \frac{\tau_y}{c + g + \mu} \\
&= \frac{1}{g}\left(\frac{\|\Delta \mathbf{T}\|}{2} - \tau_y\right) = \frac{1}{g}(T_a - \tau_y)
\end{aligned} \tag{38}$$

Where T_a is the amplitude of the macroscopic shear strain acting on the slip line. τ_y can often be neglected, since it is the yield stress of the weakest crystal plane and thus very small.

It has thus been shown that the amplitude of the macroscopic shear stress can be connected to plastic flow of the crystals. The hydrostatic stress in the equivalent stress expressions is the average of the macroscopic normal stress acting at a material point (which is equal to the average of the mesoscopic stresses acting on the same point, see (15)). The hydrostatic stress term is used to account for the fact that no true “initiation” can be distinguished. In practice, initiation is a mix of this true “initiation” (which is only due to applied shear stresses according to the reasoning above) and the propagation of microcracks, which are influenced by a hydrostatic stress.

3. Equivalent Stress Criteria for Prediction of Fatigue Initiation

Comment: The criteria described below are primarily aimed at predicting fatigue initiation, i.e. the load levels at which fatigue could be expected. However, their limits of use can be extended by introducing the calculated equivalent stresses in a Wöhler relation and thus predict a limited fatigue life. However, since the criteria are developed for fatigue threshold values, this approach should only be defendable if the stresses in the component studied are close to these threshold values.

3.1. THE DANG VAN CRITERION

The Dang van criterion [5] states that fatigue damage will occur when, for a shear plane, the following inequality is fulfilled

$$\tau_a + a_{DV} > \tau_e \tag{39}$$

where τ_e is the fatigue treshold in pure shear and a_{DV} is a material parameter that can be evaluated, for instance, from the expression

$$a_{DV} = \frac{\tau_e - \sigma_e/2}{\sigma_e/3} = \frac{3}{2} \cdot \frac{2\tau_e - \sigma_e}{\sigma_e} \tag{40}$$

where σ_e is the fatigue treshold amplitude in alternating bending (or in tension/compression). τ_a is the shear stress "amplitude", which in this case is the deviation from the mid value during a stress cycle of the shear stress vector acting in a plane specified by the normal vector n, see FIG. 2.

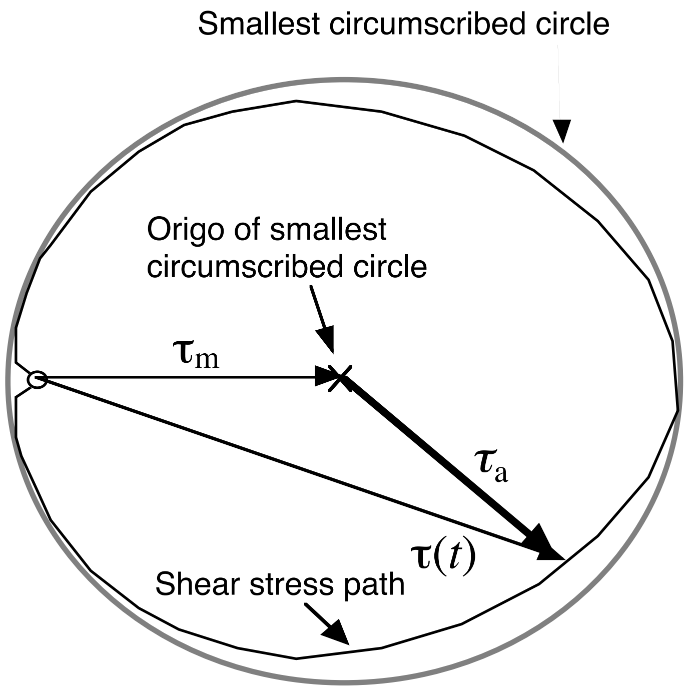

Fig. 2 Definition of shear stress "amplitude" in the Dang Van criterion

Comment: The Dang Van-criterion does not take gradient effects into account and is based on a "Tresca-type" condition. I.e. the fatigue limit is dependent on the most stressed slip direction. No stress redistributions are assumed to take place.

3.2. Papadopoulos Criterion

The Papadopoulos criterion states that fatigue occurs if the following inequality holds

$$\sqrt{\langle T_a^2 \rangle} + \alpha\sigma_{H,\max} > \tau_e \quad (41)$$

Where $\sqrt{\langle T_a^2 \rangle}$ is defined as

$$\sqrt{\langle T_a^2 \rangle} = \sqrt{5}\sqrt{\frac{1}{8\pi^2}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{\chi=0}^{2\pi}(T_a(\varphi,\theta,\chi))^2 d\chi \sin\theta d\theta d\varphi} \quad (42)$$

and is the root mean square value (for all possible slip lines in a material point) of the amplitude of the shear stress acting along the slip line, i.e. $T = (\mathbf{m} \bullet \Sigma \bullet \mathbf{n} - (\mathbf{n} \bullet \Sigma \bullet \mathbf{n})\mathbf{n} \bullet \mathbf{m})_a$.

$\sigma_{H,\max}$ is the maximum (algebraic) value of the hydrostatic stress (positive in tension) during a stress cycle, i.e.

$$\sigma_{H,\max} = \max_{t \in T}\left(\frac{\sigma_{xx}(t) + \sigma_{yy}(t) + \sigma_{zz}(t)}{3}\right) \quad (43)$$

α is a material parameters, that can be evaluated from fatigue tests as (for example)

$$\alpha = \frac{\tau_e - \sigma_e/\sqrt{3}}{\sigma_e/3} = \sqrt{3} \cdot \frac{\sqrt{3}\tau_e - \sigma_e}{\sigma_e} \quad (44)$$

In order to achieve a beneficial effect of a compressive hydrostatic stress, this criterion can only be applied to materials for which the inequality $\frac{\sigma_e}{\tau_e} \leq \sqrt{3}$ holds (this is fulfilled for "hard metals").

Comment: Papadopoulos criterion is aimed at predicting fatigue initiation. It seems reasonable to adopt an approach where the accumulated plastic strain in a material point (regarding all slip directions) should not exceed a limiting value. The results obtained seem to fit experimental data very well (see [1]). If a criterion is aimed at predicting initiation/propagation however, it would perhaps be more suitable to apply a direction dependent criterion, since the propagation phase is very sensitive to direction of loading.

3.3. Gradient Dependent Criterion (Papadopoulos and Panoskaltis)

Direction Dependent Formulation

This criterion has been expressed in two different ways. In [2], the criterion is expressed as

$$T_a + \alpha N_{max} - \beta\sqrt{G}\langle N_{max}\rangle > \tau_e \tag{45}$$

Where N_{max} is the maximum value of normal stress acting on the shear plane studied. T_a is the shear stress amplitude on this shear plane. <x> denotes that values ≤ 0 should be replaced by zero. Also,

$$G = \sqrt{\left(\frac{\partial N_{max}}{\partial x}\right)^2 + \left(\frac{\partial N_{max}}{\partial y}\right)^2 + \left(\frac{\partial N_{max}}{\partial z}\right)^2} \tag{46}$$

is the gradient of the normal stress acting on the shear plane.

The material parameters can be evaluated as

$$\alpha = \frac{\tau_e}{\sigma_e} - 1 \quad \text{and} \quad \beta = \sqrt{R}\left(\frac{\tau_e}{\sigma_e} - \frac{\tau_e}{f_e}\right) \tag{47}$$

Here, σ_e is the fatigue threshold amplitude in fully reversed tension/ compression and f_e the fatigue threshold amplitude in fully reversed, constant moment, bending. R is half the height of the component.

Comment: In [2], the fatigue limits have been defined as the stress range, i.e. σ_e (in this paper) = s/2 (in [2]). Also, the use of N_{max} (instead of a hydrostatic stress) should imply a dependence on the fatigue treshold of a superposed static shear stress. However, the critical plane is chosen as the plane of the maximum shear stress amplitude. If it is taken as the plane of maximum equivalent stress according to (45), this dependence would (probably) have (see [1] and compare the Findley and the Matake-criteria).

Invariant Based Formulation

In [3] an invariant formulation of a gradient dependent criterion is proposed. The criterion is then written as

$$\sqrt{J_{2,a}} + \alpha P_{max}\left(1 - \beta\left\langle\frac{G}{P_{max}}\right\rangle^n\right) \le \tau_e \tag{48}$$

where J2,a is the amplitude of the second invariant and P_{max} is the maximum value of the hydrostatic stress. The material parameters can be calculated as

$$\alpha = \frac{3\tau_e}{\sigma_e} - \sqrt{3} \quad \text{and} \quad \beta = R^n \frac{1 - \dfrac{\sigma_e}{f_e}}{1 - \dfrac{\sigma_e}{\sqrt{3}\tau_e}} \tag{49}$$

The value of n is not given in [3], since the applications studied lead to expressions where n could be omitted. However, in the general case, this criterion involves four material constants.

The criterion is based on the Crossland criterion (which is equals with β=0).

4. Criteria for Prediction of Limited Fatigue Life

4.1. Socies Shear Stress/Strain Based Criterion

In [7], Socie presents an equivalent stress criteria for limited fatigue life. The criteria is aimed at fatigue where the majority of fatigue life is spent in initiating cracks (or actually growing cracks from some 0.01 to 1.0 mm). This process is mainly shear based, but there is an influence of the stresses and strains acting perpendicular to the crack. According to Socie, this influence is mainly due to the ability of these strains (stresses) to prevent mechanical interlocking, see Fig. 3.

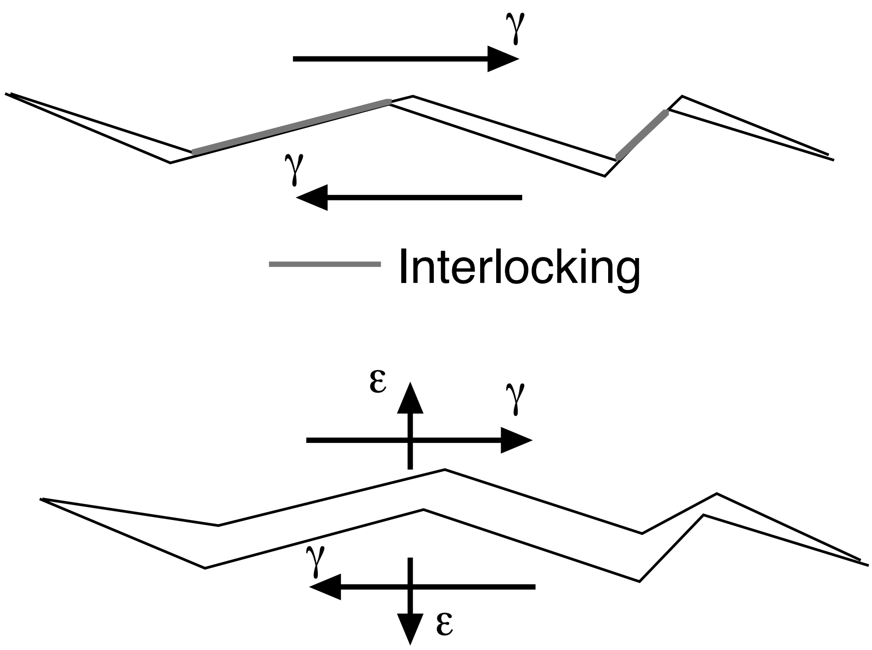

Fig. 3 Interlocking in shear crack growth and the influence of a normal strain (stress)

Since the fatigue life model is intended for both low cycle fatigue (LCF) and high cycle fatigue (HCF), which are best described by strain- and stress-components respectivly, the criterion is a mix of this measures and can be expressed as

$$\hat{\gamma} + \hat{\varepsilon}_n + \frac{\hat{\sigma}_{n0}}{E} = \gamma_f'(2N)^c + \frac{\tau_f'}{G}(2N)^b \tag{50}$$

where $\hat{\gamma}$ is the maximum shear strain amplitude, $\hat{\varepsilon}_n$ is the tensile strain perpendicular to the maximum shear strain amplitude, $\hat{\sigma}_{n0}$ is the mean stress perpendicular to the maximum shear strain amplitude. As for material parameters, E is Young's modulus, G is the shear modulus, c is a fatigue ductility exponent, b is a fatigue strength exponent, $\hat{\gamma}_f'$ is a shear fatigue ductility coefficient, $\hat{\tau}_f'$ is a shear fatigue strength coefficient. Finally, N is the number of load cycles (i.e. 2N is the number of load reversals) needed to initiate a 1.0 mm surface crack. Note that there are three fatigue material parameters in this criterion. These parameters are found by curve-fitting the outcome of several limited fatigue life experiments.

For a specific loading (and given material parameters), the fatigue life N can be evaluated from (50). The criterion has been used for fatigue lifes of $N=10^3$ - 10^6.

4.2. SMITH-WATSON-TOPPER CRITERION

For fatigue conditions, where fatigue cracks tend to occur on planes of maximum principal strain, Socie [7] proposed the use of the Smith-Watson-Topper (SWT) criterion. Since the criterion should be applicable for both LCF and HCF conditions, both stresses and strains are included. The criterion can be expressed as

$$\sigma_1^{max}\frac{\Delta\varepsilon_1}{2} = \sigma_f'\varepsilon_f'(2N)^{b+c} + \frac{\sigma_f'^2}{E}(2N)^{2b} \qquad (51)$$

where $\frac{\Delta\varepsilon_1}{2}$ is the maximum principal strain amplitude, σ_1^{max} is the maximum principal stress on the maximum principal strain plane. As for material parameters, E is Young’s modulus, c is a fatigue ductility exponent, b is a fatigue strength exponent, ε_f' is a tensile fatigue ductility coefficient, σ_f' is a tensile fatigue strength coefficient. Finally,N is the number of load cycles needed to initiate a 1.0 mm surface crack. Note that there are three fatigue material parameters in this criterion.

The criterion has been used for fatigue lifes of $N=10^3$ - 10^6.

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APPENDIX

MATLAB code used to plot diagrams in the text above.