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ORIGINAL PAPER

# Equivariant Euler Characteristics of Symplectic Buildings

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**Abstract**
We compute the equivariant Euler characteristics of the buildings for the symplectic groups over finite fields.

**Keywords** Equivariant Euler characteristic · Totally isotropic subspace · Symplectic group over a finite field · Generating function · Irreducible polynomial

**Mathematics Subject Classification** 05E18 · 06A07

## 1 Introduction

Let $G$ be a finite group, $\Pi$ a finite $G$-poset, and $r \geq 1$ a natural number. Atiyah and Segal [2] defined the *rth equivariant reduced Euler characteristic* of the $G$-poset $\Pi$ as the normalised sum

$$\tilde{\chi}_r(\Pi, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)} \tilde{\chi}(C_\Pi(X(\mathbf{Z}^r)))$$

of the reduced Euler characteristics of the $X(\mathbf{Z}^r)$-fixed $\Pi$-subposets, $C_\Pi(X(\mathbf{Z}^r))$, with $X$ ranging over all homomorphisms of $\mathbf{Z}^r$ to $G$. (See Appendix A for more information on equivariant Euler characteristics.) Here are two examples of equivariant Euler characteristics:

1. The general linear group $\mathrm{GL}_n^+(\mathbf{F}_q)$ acts on the poset $\mathrm{L}_n^+(\mathbf{F}_q)^*$ of non-extreme subspaces of the $n$-dimensional vector space over the field $\mathbf{F}_q$ of prime power order $q$. The generating function for the $(r+1)$th equivariant reduced Euler characteristics of the $\mathrm{GL}_n^+(\mathbf{F}_q)$-posets $\mathrm{L}_n^+(\mathbf{F}_q)^*$ is

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$$1+\sum_{n\geq 0}\tilde{\chi}_{r+1}(\mathrm{L}_n^+(\mathbf{F}_q)^*,\mathrm{GL}_n^+(\mathbf{F}_q))x^n=\prod_{0\leq j\leq r}(1-q^jx)^{(-1)^{r-j}\binom{r}{j}}$$

according to [19, Theorem 1.4].

2. The general unitary group $\mathrm{GL}_n^-(\mathbf{F}_q)$ acts on the poset $\mathrm{L}_n^-(\mathbf{F}_q)^*$ of non-extreme totally isotropic subspaces of the $n$-dimensional unitary geometry over the field $\mathbf{F}_{q^2}$ of prime power order $q^2$. The generating function for (minus) the $(r+1)$th equivariant reduced Euler characteristics of the $\mathrm{GL}_n^-(\mathbf{F}_q)$-posets $\mathrm{L}_n^-(\mathbf{F}_q)^*$ is

$$1-\sum_{n\geq 0}\tilde{\chi}_{r+1}(\mathrm{L}_n^-(\mathbf{F}_q)^*,\mathrm{GL}_n^-(\mathbf{F}_q))x^n=\prod_{0\leq j\leq r}(1+(-1)^{r-j}q^jx)^{(-1)^{r-j}\binom{r}{j}}$$

according to [20, Theorem 1.4].

In this paper we consider the symplectic case. For a prime power $q$, $\mathrm{Sp}_{2n}(\mathbf{F}_q)$, the isometry group of the symplectic $2n$-geometry, acts on the poset $\mathrm{L}_{2n}^*(\mathbf{F}_q)=\{0\subsetneq U\subsetneq \mathbf{F}_q^{2n} \mid U\subseteq U^\perp\}$ of nonzero totally isotropic subspaces. The general definition of equivariant Euler characteristics (Definition A.2) takes in this special case the following form.

**Definition 1.1** [2] The $r$th, $r\geq 1$, equivariant reduced Euler characteristic of the $\mathrm{Sp}_{2n}(\mathbf{F}_q)$-poset $\mathrm{L}_{2n}^*(\mathbf{F}_q)$ is the normalised sum

$$\tilde{\chi}_r(\mathrm{Sp}_{2n}(\mathbf{F}_q))=\frac{1}{|\mathrm{Sp}_{2n}(\mathbf{F}_q)|}\sum_{X\in\mathrm{Hom}(\mathbf{Z}^r,\mathrm{Sp}_{2n}(\mathbf{F}_q))}\tilde{\chi}(C_{\mathrm{L}_{2n}^*(\mathbf{F}_q)}(X(\mathbf{Z}^r)))$$

of the reduced Euler characteristics of the induced subposets $C_{\mathrm{L}_{2n}^*(\mathbf{F}_q)}(X(\mathbf{Z}^r))$ of $X(\mathbf{Z}^r)$-invariant subspaces as $X$ ranges over all homomorphisms of the free abelian group $\mathbf{Z}^r$ on $r$ generators into the symplectic group.

We now state the main results about equivariant Euler characteristics in the symplectic case.

The generating function for the *negative* of the $r$th equivariant reduced Euler characteristics of the sequence $(\mathrm{L}_{2n}^*(\mathbf{F}_q),\mathrm{Sp}_{2n}(\mathbf{F}_q))_{n\geq 1}$ is the power series

$$\mathrm{FSp}_r(q,x)=1-\sum_{n\geq 1}\tilde{\chi}_r(\mathrm{Sp}_{2n}(\mathbf{F}_q))x^n \tag{1.2}$$

with coefficients in the ring $\mathbf{Z}[q]$ of integral polynomials in $q$.

**Theorem 1.3** $\mathrm{FSp}_1(q,x)=1$ *and* $\mathrm{FSp}_{r+1}(q,x)=\prod_{\substack{0\leq j\leq r\\ j\not\equiv r \bmod 2}}(1-q^jx)^{-\binom{r}{j}}$ *for all* $r\geq 1$.

The first generating functions $\mathrm{FSp}_{r+1}(q,x)$ for $0 \le r \le 5$ are

$$1, \quad \frac{1}{1-x}, \quad \frac{1}{(1-qx)^2}. \quad \frac{1}{(1-x)(1-q^2x)^3},$$
$$\frac{1}{(1-qx)^4(1-q^3x)^4}, \quad \frac{1}{(1-x)(1-q^2x)^{10}(1-q^4x)^5}$$

The generating function can also be expressed in the following alternative way.

**Corollary 1.4** $\mathrm{FSp}_{r+1}(q,x) = \exp\left(\sum_{n \ge 1} \frac{1}{2}((q^n+1)^r - (q^n-1)^r)\frac{x^n}{n}\right)$ *for all* $r \ge 0$.

We study also the $p$-primary equivariant reduced Euler characteristics $\tilde{\chi}_r(p, \mathrm{Sp}_{2n}(\mathbf{F}_q))$ of $(\mathrm{L}^*_{2n}(\mathbf{F}_q), \mathrm{Sp}_{2n}(\mathbf{F}_q))$ for a fixed prime $p$ (Definition 5.1). The $r$th $p$-primary generating function, $\mathrm{FSp}_r(p,q,x)$, is defined as in (1.2) but with $\tilde{\chi}_r(\mathrm{Sp}_{2n}(\mathbf{F}_q))$ replaced by $\tilde{\chi}_r(p, \mathrm{Sp}_{2n}(\mathbf{F}_q))$ (5.3).

**Theorem 1.5** $\mathrm{FSp}_{r+1}(p,q,x) = \exp\left(\sum_{n \ge 1} \frac{1}{2}((q^n+1)^r_p - (q^n-1)^r_p)\frac{x^n}{n}\right)$ *for all* $r \ge 0$.

The infinite product expansions of the generating functions

$$\mathrm{FSp}_{r+1}(q,x) = \prod_{n \ge 1}(1-x^n)^{c_{r+1}(q,n)}$$
$$c_{r+1}(q,n) = \frac{1}{2n}\sum_{d|n}\mu(n/d)((q^d-1)^r - (q^d+1)^r)$$
$$\mathrm{FSp}_{r+1}(p,q,x) = \prod_{n \ge 1}(1-x^n)^{c_{r+1}(p,q,n)}$$
$$c_{r+1}(p,q,n) = \frac{1}{2n}\sum_{d|n}\mu(n/d)((q^d-1)^r_p - (q^d+1)^r_p)$$

follow immediately from the elementary [19, Lemma 3.7]. Here, $\mu$ is the number theoretic Möbius function.

The ($p$-primary) equivariant reduced Euler characteristics are directly linked to the structure of the symplectic group $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ as a finite group of Lie type.

**Theorem 1.6** *For all* $n \ge 1$ *and all* $r \ge 0$,

$$-\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q)) = \frac{(-1)^n}{|W(C_n)|}\sum_{w \in W(C_n)} \det(w)\det(q-w)^r$$
$$-\tilde{\chi}_{r+1}(p, \mathrm{Sp}_{2n}(\mathbf{F}_q)) = \frac{(-1)^n}{|W(C_n)|}\sum_{w \in W(C_n)} \det(w)\det(q-w)^r_p$$

*where* $W(C_n)$ *is the Weyl group representation for the algebraic group* $\mathrm{Sp}_{2n}(\overline{\mathbf{F}}_s)$, $s = \mathrm{char}(\mathbf{F}_q)$.

The paper is organised as follows. In Sect. 2 we briefly recall the definition of the symplectic group as the isometry group of an even dimensional alternating bilinear form over $\mathbf{F}_q$. All symplectic automorphisms have self-reciprocal characterisic polynomials (Proposition 3.2) and Sect. 3 deals with the number $\mathrm{SRIM}_n^-(q)$ of self-reciprocal irreducible monic polynomials of even degree $n$ over $\mathbf{F}_q$ (Definition 3.6). Section 4 contains the proof of Theorem 1.3 based upon the vanishing result of Lemma 4.2 and the recursive relation (4.7) which is the specific manifestation of the general recurrence of Lemma A.3. Theorem 1.3 with $r = 1$ says that $-\tilde{\chi}_2(\mathrm{Sp}_{2n}(\mathbf{F}_q)) = 1$ for all $n \geq 1$ and all prime powers $q$ confirming the non-block-wise form of the Knörr–Robinson conjecture for $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ relative to the defining characteristic (Remark 4.13). The $r$th $p$-primary equivariant Euler characteristic is Euler characteristic computed in Morava $K(r)$-theory. Section 5 is the $p$-primary version of Sect. 4. The proof of Theorem 1.5 consists in solving recurrence (5.12) which is the $p$-primary version of (4.7). We observe that the $p$-primary equivariant Euler characteristic $\tilde{\chi}_r(p, \mathrm{Sp}_{2n}(\mathbf{F}_q))$ for $p \nmid q$ depends only on the closure $\overline{\langle q \rangle}$ of the subgroup generated by $q$ in the unit group $\mathbf{Z}_p^\times$ of the $p$-adic integers. In Section 6, the equivariant Euler characteristics $\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q))$ and $\tilde{\chi}_{r+1}(p, \mathrm{Sp}_{2n}(\mathbf{F}_q))$ are expressed directly in terms of integer partitions (Corollary 6.2) or in terms of determinants of Weyl group elements (Theorem 1.6). We also consider the reciprocal power series $\mathrm{FSp}_{r+1}(q,x)^{-1}$ and $\mathrm{FSp}_{r+1}(p,q,x)^{-1}$ (Corollary 6.7) and the generating functions $\sum_{r \geq 0} -\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q))x^r$ with fixed parameter $n$ (Corollary 6.4). Example 6.8 offers several concrete examples of the identities established in this section. In the short Section 7, we formulate the symplectic analogs of Thévenaz' polynomial identities [27, Theorems A–B]. The paper closes with two appendices. Appendix A is a review of basic properties of equivariant Euler characteristics and Appendix B recalls facts, helpful for concrete calculations of equivariant Euler characteristics, about Hall's eulerian functions of groups [9].

The following notation will be used in this paper:

| | |
|---|---|
| $p$ | is a prime number |
| $v_p(n)$ | is the $p$-adic valuation of $n$ |
| $n_p$ | is the $p$-part of the natural number $n$ ($n_p = p^{v_p(n)}$) |
| $\mathbf{Z}_p$ | is the ring of $p$-adic integers |
| $q$ | is a prime power |
| $\mathbf{F}_q$ | is the finite field with $q$ elements |
| $\mu(d)$ | is the value of the number theoretic Möbiusfunction at the natural number $d$ [24, Example 3.8.4] |

## 2 The Symplectic Group $\mathrm{Sp}_{2n}(\mathbf{F_q})$

Let $q$ be a prime power and $n \geq 1$ a natural number. The symplectic $2n$-geometry is the vector space $V_{2n}(\mathbf{F}_q) = \mathbf{F}_q^{2n}$ of dimension $2n$ over the field $\mathbf{F}_q$ equipped with the non-degenerate [1, Definition 3.1] alternating ($\langle u, v \rangle = -\langle v, u \rangle$) bilinear form given by

$$\langle u, v \rangle = uJv^t = \sum u_i v_{-i} - \sum u_{-i} v_i, \qquad J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \qquad u, v \in V_{2n}(\mathbf{F}_q) \tag{2.1}$$

for all $u = (u_1, \ldots, u_n, u_{-1}, \ldots, u_{-n}), v = (v_1, \ldots, v_n, v_{-1}, \ldots, v_{-n}) \in V_{2n}(\mathbf{F}_q)$. The symplectic $2n$-geometry is the orthogonal direct sum, $\langle e_1, e_{-1} \rangle \perp \cdots \perp \langle e_n, e_{-n} \rangle$, of the $n$ hyperbolic planes $\langle e_i, e_{-i} \rangle$, $1 \leq i \leq n$ [1, Theorem 3.7]. The symplectic group $\mathrm{Sp}_{2n}(\mathbf{F}_q) = \{g \in \mathrm{GL}_n^+(\mathbf{F}_q) | gJg^t = J\}$ is the group of all automorphisms of the symplectic $2n$-geometry [8, §2.7]. Its order is $|\mathrm{Sp}_{2n}(\mathbf{F}_q)| = q^{2n} \prod_{1 \leq i \leq n} (q^{2i} - 1)$ and its center is trivial if $q$ is even and of order 2 if $q$ is odd [1, Theorem 5.2].

A subspace $U$ of the symplectic geometry $(V_{2n}(\mathbf{F}_q), \langle \cdot, \cdot \rangle)$ is totally isotropic if $\langle U, U \rangle = 0$. The symplectic group acts on the poset $\mathrm{L}_{2n}^*(\mathbf{F}_q)$ of all nontrivial totally isotropic subspaces. Since all vectors are isotropic, $\langle u, u \rangle = 0$, all 1-dimensional subspaces are in $\mathrm{L}_{2n}^*(\mathbf{F}_q)$ (and $\mathrm{L}_2^*(\mathbf{F}_q)$ is simply the set of 1-dimensional subspaces of $V_2(\mathbf{F}_q)$).

When the prime power $q = 2^e$ is even, $\mathrm{Sp}_{2n}(\mathbf{F}_q) \cong \mathrm{SO}_{2n+1}(\mathbf{F}_q)$ [8, Theorem 2.2.10].

## 3 Characteristic Polynomials of Symplectic Automorphisms

**Definition 3.1** [14, Definition 3.12] The reciprocal of a degree $n$ polynomial $p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ over $\mathbf{F}_q$ with nonzero constant term is the degree $n$ polynomial $p^*(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n = x^n p(x^{-1})$. The polynomial $p$ is self-reciprocal if $p^* = p$.

The operation $p(x) \to p^*(x)$ is involutory, multiplicative, and divisibility respecting ($p^{**}(x) = p(x)$, $(p_1(x) p_2(x))^* = p_1^*(x) p_2^*(x)$, $p_1 | p_2 \Rightarrow p_1^* | p_2^*$) on the set of polynomials $p(x) \in \mathbf{F}_q[x]$ with $p(0) \neq 0$. The multisets of roots for a polynomial and its reciprocal correspond under the inversion map $\bar{\mathbf{F}}_q^\times \to \bar{\mathbf{F}}_q^\times : \alpha \to \alpha^{-1}$. The polynomial $p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ is self-reciprocal if and only if it has a palindromic coefficient sequence, $a_i = a_{n-i}$, $0 \leq i \leq n$.

**Proposition 3.2** *The characteristic polynomial of any symplectic automorphism* $g \in \mathrm{Sp}_{2n}(\mathbf{F}_q)$ *is a self-reciprocal monic polynomial.*

***Proof*** Taking for given that $\det(g) = +1$ [5], we get that the characteristic polynomial, $c_g$, of $g$ satisfies

$$
\begin{aligned}
c_g(\lambda) &= c_{g^t}(\lambda) = \det(g^t - \lambda I) \\
&= \det(Jg^{-1}J^{-1} - \lambda I) \overset{J=-J^{-1}}{=} \det(-Jg^{-1}J + J(\lambda I)J) \overset{\det(J)=1}{=} \\
&\quad \det(-g^{-1} + \lambda I) \overset{\det(g)=1}{=} \det(\lambda g - I) \\
&= \lambda^{2n} \det(g - \lambda^{-1} I) = \lambda^{2n} c_g(\lambda^{-1})
\end{aligned}
$$

where $I$ is the identity matrix. □

Conversely, any self-reciprocal polynomial is the characteristic polynomial for a symplectic automorphism [23, Theorem A.1].

**Proposition 3.3** *The number of self-reciprocal monic polynomials of even degree $2n$ is $q^n$.*

***Proof*** Self-reciprocal monic polynomials of degree $2n$ have palindromic coefficients. □

**Lemma 3.4** *The transformation $r(x) \to r^*(x)/r(0)$ is an involution on the set of irreducible monic polynomials $r(x) \in \mathbf{F}_q[x]$ with $r(0) \neq 0$.*

*The irreducible monic polynomial $r(x)$ with $r(0) \neq 0$ is fixed under this involution, when $\deg r = 1$, if and only if $r(x) = x \pm 1$, and when $\deg r > 1$, if and only if $r(x)$ has even degree, $r(0) = 1$, and $r(x)$ is self-reciprocal.*

***Proof*** If $r(x)$ has degree at least 2, the degree of $r$ must be even, since the set of roots in the algebraic closure is invariant under inversion $\bar{\mathbf{F}}_q^\times \to \bar{\mathbf{F}}_q^\times : \alpha \to \alpha^{-1}$ and the fixed points, $\pm 1$, are not roots of $r(x)$. The relation $r(0)r(x) = r^*(x) = x^{\deg r} r(1/x)$ evaluated at $x = -1$ gives $r(0) = 1$. Thus $r(x) = r^*(x)$ and $r(x)$ is self-reciprocal. □

**Proposition 3.5** *Let $p(x)$ be a self-reciprocal monic polynomial. The canonical factorisation [14, Theorem 1.59] of $p(x)$ has the form*

$$
p(x) = \begin{cases} (x-1)^{a_-} \times (x+1)^{a_+} \times \prod_i r_i^-(x)^{m_i^-} \times \prod_j (s_j(x) s_j^*(x)/s_j(0))^{m_j^+} & q \text{ odd} \\ (x+1)^{a_+} \times \prod_i r_i(x)^{m_i^-} \times \prod_j (s_j(x) s_j^*(x)/s_j(0))^{m_j^+} & q \text{ even} \end{cases}
$$

*where*

$$
\deg p = \begin{cases} a_- + a_+ + \sum_i m_i^- \deg r_i + 2\sum_j m_j^+ \deg s_j & q \text{ odd} \\ a_+ + \sum_i m_i^- \deg r_i + 2\sum_j m_j^+ \deg s_j & q \text{ even} \end{cases}
$$

*and $a_-, a_+, m_i^-, m_j^+ \geq 0$, $a_-$ is even, the $r_i^-(x)$ are self-reciprocal irreducible monic polynomials of even degree at least 2, and the $s_j(x)$ are non-self-reciprocal irreducible monic polynomials distinct from $x - 1$. Conversely, any polynomial with a canonical factorisation of this form is a self-reciprocal monic polynomial.*

***Proof*** Let $p(x) = \prod r_i(x)^{e_i}$ be the canonical factorisation. Since $p(x)$ is monic and self-reciprocal, $p(0) = 1$, and $p(x) = p^*(x) = p^*(x)/p(0) = \prod (r_i^*(x)/r_i(0))^{e_i}$ where $r_i^*(x)/r_i(0)$ are irreducible monic polynomials [21, Remark 2.1.49]. Thus the multiset of the irreducible factors of $p(x)$ is invariant under the involution $r(x) \longleftrightarrow r^*(x)/r(0)$. Group the irreducible factors into those fixed by this involution and pairs interchanged by it. An irreducible factor of degree $\geq 2$ is fixed by the involution if and only if it is self-reciprocal according to Lemma 3.4. Any irreducible linear factor, which has the form $x - \alpha$ for some $\alpha \in \mathbf{F}_q^\times$, is fixed by the involution if and only if $\alpha = \pm 1$ ($\alpha = 1$ when $q$ is even). Thus $p(x)$ has a canonical factorisation of the form shown in the proposition. When $q$ is odd, the multiplicity, $a_-$, of the factor $x - 1$ is even because $1 = p(0) = (-1)^{a_-}$.

Conversely, if $p(x)$ has a factorisation as in the proposition, then $(x-1)^{a_-}$ is self-reciprocal as $a_-$ is even, and as also the other factors, $x + 1$, $r_i^-(x)$, $s_j(x)s_j^*(x)/s_j(0)$, are self-reciprocal, the polynomial $p(x)$ is self-reciprocal. □

All factors on the right hand side of the formula of Proposition 3.5 are self-reciprocal. The exponent $a_-$ is even while $a_+$ has the same parity as the degree of $p$.

**Definition 3.6** [21, Definition 2.1.23, Remark 3.1.19] For every integer $n \geq 1$

- $\mathrm{IM}_n(q)$ is the number of Irreducible Monic polynomials $p(x)$ of degree $n$ over $\mathbf{F}_q$ with $p(0) \neq 0$
- $\mathrm{SRIM}_n^-(q)$ is the number of Self-Reciprocal Irreducible Monic polynomials $p(x)$ of even degree $2n$ over $\mathbf{F}_q$
- $\mathrm{SRIM}_n^+(q)$ is the number of unordered pairs $\{p(x), p^*(x)/p(0)\}$ of irreducible monic polynomials $p(x)$ of degree $n$ over $\mathbf{F}_q$ with $p(0) \neq 0$ and $p(x) \neq p^*(x)/p(0)$

For any $n \geq 1$ [21, Theorem 2.1.24, Theorem 3.1.20] [17, Theorem 3]

$$\mathrm{IM}_n(q) = \frac{1}{n} \sum_{d|n} \mu(d)(q^{n/d} - 1)$$

$$\mathrm{SRIM}_n^-(q) = \begin{cases} \dfrac{1}{2n} \displaystyle\sum_{\substack{d|n \\ d \equiv 1 \bmod 2}} \mu(d)(q^{n/d} - 1) & q \quad \text{odd} \\ \dfrac{1}{2n} \displaystyle\sum_{\substack{d|n \\ d \equiv 1 \bmod 2}} \mu(d) q^{n/d} & q \quad \text{even} \end{cases} \tag{3.7}$$

and we have

$$\mathrm{IM}_n(q) = \begin{cases} 2\mathrm{SRIM}_n^+(q) & n > 1 \quad \text{odd} \\ 2\mathrm{SRIM}_n^+(q) + \mathrm{SRIM}_{n/2}^-(q) & n > 1 \quad \text{even} \end{cases} \tag{3.8}$$

In degree $n = 1$, in particular, $\mathrm{IM}_1(q) = q - 1$ and

$$\mathrm{SRIM}_1^+(q) = \begin{cases} \frac{1}{2}(q-3) & q \text{ odd} \\ \frac{1}{2}(q-2) & q \textit{ even} \end{cases}$$

For odd $q$, the $\frac{1}{2}(q-3)$ unordered pairs are the pairs $\{x-\alpha, x-\alpha^{-1}\}$ with $\alpha \in \mathbf{F}_q^* - \{-1, +1\}$. For even $q$, the $\frac{1}{2}(q-2)$ unordered pairs are the pairs $\{x-\alpha, x-\alpha^{-1}\}$ with $\alpha \in \mathbf{F}_q^* - \{1\}$.

**Lemma 3.9** *Let $m \geq 1$ be and $k \geq 0$. Then $2^k\mathrm{SRIM}_{2^km}^-(q) = \mathrm{SRIM}_m^-(q^{2^k})$ for all prime powers $q$. When $m$ is odd,*

$$2^{k+1}\mathrm{SRIM}_{2^km}^-(q) = \begin{cases} \mathrm{IM}_1(q^{2^k}) + 1 & q \textit{ even and } m = 1 \\ \mathrm{IM}_m(q^{2^k}) & \textit{otherwise} \end{cases}$$

***Proof*** Since the odd divisors of $2^km$ or of $m$ are the divisors of $m_{2'}$, the odd part of $m$, we have

$$2^k\mathrm{SRIM}_{2^km}^-(q) = \begin{cases} \frac{1}{2m}\sum_{d|m_{2'}}(q^{2^km/d} - 1) = \mathrm{SRIM}_m^-(q^{2^k}) & q \text{ odd} \\ \frac{1}{2m}\sum_{d|m_{2'}} q^{2^km/d} = \mathrm{SRIM}_m^-(q^{2^k}) & q \text{ even} \end{cases}$$

by (3.7).

Assume now that $m$ is odd. If $q$ is odd, $2^{k+1}\mathrm{SRIM}_{2^km}^-(q) = 2\mathrm{SRIM}_m^-(q^{2^k}) = \mathrm{IM}_m(q^{2^k})$ since $2\mathrm{SRIM}_m^-(q) = \mathrm{IM}_m(q)$. If $q$ is even and $m > 1$, $\sum_{d|m}\mu(d) = 0$ and $2\mathrm{SRIM}_m^-(q) = \frac{1}{m}\sum_{d|m} q^{m/d} = \frac{1}{m}\sum_{d|m}(q^{m/d} - 1) = \mathrm{IM}_m(q)$ again. If $q$ is even and $m = 1$, $2^{k+1}\mathrm{SRIM}_{2^k}^-(q) = q^{2^k} = \mathrm{IM}_1(q^{2^k}) + 1$. □

**Lemma 3.10** *For all $n \geq 1$,*

$$\mathrm{IM}_n(q) = \begin{cases} 2\mathrm{SRIM}_n^-(q) - \mathrm{SRIM}_{n/2}^-(q) & n \textit{ even} \\ 2\mathrm{SRIM}_n^-(q) & n \textit{ odd and } n > 1 \textit{ if } q \textit{ even} \\ 2\mathrm{SRIM}_1^-(q) - 1 & n = 1 \textit{ and } q \textit{ even} \end{cases}$$

***Proof*** Assume first that $n \geq 2$ is even. Write $n = 2^km$ for some $k \geq 1$ and odd $m \geq 1$. If $m = 1$ and $q$ is even,

$$\begin{aligned}\mathrm{IM}_n(q) = \mathrm{IM}_{2^k}(q) &= \frac{1}{2^k}\sum_{d|2^k}\mu(d)(q^{2^k/d}-1) = \frac{1}{2^k}((q^{2^k}-1)-(q^{2^{k-1}}-1))\\ &= \frac{1}{2^k}(q^{2^k}-q^{2^{k-1}}) = 2\mathrm{SRIM}^-_{2^k}(q) - \mathrm{SRIM}^-_{2^{k-1}}(q)\\ &= 2\mathrm{SRIM}^-_n(q) - \mathrm{SRIM}^-_{n/2}(q)\end{aligned}$$

as $\mu(1)=1$, $\mu(2)=-1$, and $\mu(2^j)=0$ for $j>1$. Otherwise, $q$ is odd in case $m=1$, and then

$$\begin{aligned}\mathrm{IM}_n(q) = \mathrm{IM}_{2^k m}(q) &= \frac{1}{2^k m}\sum_{d|2^k m}\mu(d)(q^{2^k m/d}-1)\\ &= \frac{1}{2^k m}\left(\sum_{d|m}\mu(d)(q^{2^k m/d}-1) - \sum_{d|m}\mu(d)(q^{2^{k-1}m/d}-1)\right)\\ &= \frac{1}{2^k}(\mathrm{IM}_m(q^{2^k}) - \mathrm{IM}_m(q^{2^{k-1}})) \overset{\text{Lemma 3.9}}{=} 2\mathrm{SRIM}^-_{2^k m}(q) - \mathrm{SRIM}^-_{2^{k-1}m}(q)\\ &= 2\mathrm{SRIM}^-_n(q) - \mathrm{SRIM}^-_{n/2}(q)\end{aligned}$$

Here we used that the divisors of $2^k m$ are $2^j d$, $0 \le j \le k$, where $d$ is a divisor of $m$, and that $\mu(2d) = -\mu(d)$ and $\mu(2^j d) = 0$ for $j>1$ as $d$ is odd.

Next, assume that $n \ge 1$ is odd. If $n=1$ then

$$2\mathrm{SRIM}^-_1(q) = \begin{cases} q-1 = \mathrm{IM}_1(q) & q \text{ odd}\\ q = \mathrm{IM}_1(q)+1 & q \text{ even}\end{cases}$$

which proves the lemma in this case. If $n>1$, then

$$2n\mathrm{SRIM}^-_n(q) = \begin{cases} \sum_{d|n}\mu(d)(q^{n/d}-1) = n\mathrm{IM}_n(q) & q \text{ odd}\\ \sum_{d|n}\mu(d)q^{n/d} = \sum_{d|n}\mu(d)(q^{n/d}-1) = n\mathrm{IM}_n(q) & q \text{ even}\end{cases}$$

because $\sum_{d|n}\mu(d)=0$. Thus $\mathrm{IM}_n(q) = 2\mathrm{SRIM}^-_n(q)$ for all $q$ when $n>1$ is odd (Fig. 1). □

**Lemma 3.11** *For all* $n \ge 1$,

$$\mathrm{SRIM}^-_n(q) + \mathrm{SRIM}^+_n(q) = \begin{cases} \mathrm{IM}_1(q) - 1 & n=1 \textit{ and } q \textit{ odd}\\ \mathrm{IM}_n(q) & \textit{otherwise}\end{cases}$$

| | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ | $n=7$ |
|---|---|---|---|---|---|---|---|
| $\mathrm{SRIM}_n^-(q)$ | $\frac{1}{2}(q-1)$ | $\frac{1}{4}(q^2-1)$ | $\frac{1}{6}(q^3-q)$ | $\frac{1}{8}(q^4-1)$ | $\frac{1}{10}(q^5-q)$ | $\frac{1}{12}(q^6-q^2)$ | $\frac{1}{14}(q^7-q)$ |
| $\mathrm{SRIM}_n^+(q)$ | $\frac{1}{2}(q-3)$ | $\frac{1}{4}(q^2-2q+1)$ | $\frac{1}{6}(q^3-q)$ | $\frac{1}{8}(q^4-2q^2+1)$ | $\frac{1}{10}(q^5-q)$ | $\frac{1}{12}(q^6-2q^3-q^2+2q)$ | $\frac{1}{14}(q^7-q)$ |

**Fig. 1** The polynomials $\mathrm{SRIM}_n^{\pm}(q)$ for odd $q$

***Proof*** Assume the prime power $q$ is odd. If $n=1$ then $\mathrm{IM}_1(q)=q-1$, $\mathrm{SRIM}_1^-(q)=\frac{1}{2}(q-1)$, and $\mathrm{SRIM}_1^+(q)=\frac{1}{2}(q-3)$ so that indeed $\mathrm{SRIM}_1^-(q)+\mathrm{SRIM}_1^+(q)=q-2=\mathrm{IM}_1(q)-1$. For odd $m>1$, $\mathrm{SRIM}_m^-(q)=\frac{1}{2}\mathrm{IM}_m(q)\overset{(3.8)}{=}\mathrm{SRIM}_m^+(q)$ so that clearly $\mathrm{SRIM}_m^-(q)+\mathrm{SRIM}_m^+(q)=\mathrm{IM}_m(q)$. For odd $m\geq 1$ and $k\geq 1$,

$$\begin{aligned}
&\mathrm{SRIM}_{2^km}^-(q)+\mathrm{SRIM}_{2^km}^+(q)\overset{(3.8)}{=}\mathrm{SRIM}_{2^km}^-(q)+\frac{1}{2}\left(\mathrm{IM}_{2^km}(q)-\mathrm{SRIM}_{2^{k-1}m}^-(q)\right)\\
&=\frac{1}{2}\left(2\mathrm{SRIM}_{2^km}^-(q)-\mathrm{SRIM}_{2^{k-1}m}^-(q)\right)+\frac{1}{2}\mathrm{IM}_{2^km}(q)\overset{\text{Lemma 3.10}}{=}\frac{1}{2}\mathrm{IM}_{2^km}(q)\\
&+\frac{1}{2}\mathrm{IM}_{2^km}(q)=\mathrm{IM}_{2^km}(q)
\end{aligned}$$

This finishes the proof for odd $q$.

Assume that $q$ is a power of 2. In degree 1, $\mathrm{SRIM}_1^-(q)+\mathrm{SRIM}_1^+(q)=\frac{1}{2}q+\frac{1}{2}q-1=q-1=\mathrm{IM}_1(q)$. For odd $m>1$, $\mathrm{SRIM}_m^-(q)=\frac{1}{2}\mathrm{IM}_m(q)=\mathrm{SRIM}_m^+(q)$ by Lemma 3.9 and (3.8) so $\mathrm{SRIM}_m^-(q)+\mathrm{SRIM}_m^+(q)=\mathrm{IM}_m(q)$. For odd $m\geq 1$ and any $k\geq 1$, we can again use Lemma 3.10 and it follows, as for odd $q$, that $\mathrm{SRIM}_{2^km}^-(q)+\mathrm{SRIM}_{2^km}^-(q)=\mathrm{IM}_{2^km}(q)$. □

## 4 Proofs of Theorem 1.3 and Corollary 1.4

We recursively compute the generating functions $\mathrm{FSp}_r(q,x)$, $r\geq 1$, of Theorem 1.3.

**Lemma 4.1** $-\tilde{\chi}_1(\mathrm{Sp}_{2n}(\mathbf{F}_q))=0$ *for all* $n\geq 1$.

***Proof*** The first equivariant reduced Euler characteristic is the usual reduced Euler characteristic of the orbit space $|\mathrm{L}_{2n}^*(\mathbf{F}_q)|/\mathrm{Sp}_{2n}(\mathbf{F}_q)$ for the action of $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ on its building (Lemma A.4). Webb's theorem [33, Proposition 8.2.(i)] says that the reduced Euler characteristic of this orbit space equals 0. □

**Lemma 4.2** [22] *If $A$ is an abelian subgroup of* $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ *and* $\gcd(|A|,q)>1$, *then* $\tilde{\chi}_r(C_{\mathrm{L}_{2n}^*(\mathbf{F}_q)}(A),C_{\mathrm{Sp}_{2n}(\mathbf{F}_q)}(A))=0$ *for all* $r\geq 1$.

***Proof*** It suffices to show that $\tilde{\chi}(C_{\mathrm{L}^*_{2n}(\mathbf{F}_q)}(A)) = 0$ for any abelian subgroup $A$ of $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ with $O_s(A) \neq 1$ where $s$ is the characteristic of the field $\mathbf{F}_q$. We may replace the poset $\mathrm{L}^*_{2n}(\mathbf{F}_q)$ by the poset $\mathcal{S}^{s+*}_{\mathrm{Sp}_{2n}(\mathbf{F}_q)}$ of non-trivial $s$-subgroups of $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ [22, Theorem 3.1]. The fixed poset $C_{\mathcal{S}^{s+*}_{\mathrm{Sp}_{2n}(\mathbf{F}_q)}}(A)$ admits the conical contraction $B \leq BO_s(A) \geq O_s(A)$ defined for all $B \in C_{\mathcal{S}^{s+*}_{\mathrm{Sp}_{2n}(\mathbf{F}_q)}}(A)$. □

**Lemma 4.3** *For $n \geq 1$ and $r \geq 1$, the $(r+1)$th equivariant Euler characteristicof the $\mathrm{Sp}_{2n}(\mathbf{F}_q)$-poset $\mathrm{L}^*_{2n}(\mathbf{F}_q)$ is*

$$\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q)) = \sum_{\substack{X \in \mathrm{Hom}(\mathbf{Z}, \mathrm{Sp}_{2n}(\mathbf{F}_q))/\mathrm{Sp}_{2n}(\mathbf{F}_q) \\ \gcd(q, |X(\mathbf{Z})|) = 1}} \tilde{\chi}_r(C_{\mathrm{L}^*_{2n}(\mathbf{F}_q)}(X), C_{\mathrm{Sp}_{2n}(\mathbf{F}_q)}(X))$$

*where the sum ranges over semisimple conjugacy classes in $\mathrm{Sp}_{2n}(\mathbf{F}_q)$.*

***Proof*** This is a special case of the general formula from Lemma A.3. By Lemma 4.2, we need only the conjugacy classes of order prime to $q$ (semisimple classes). □

The centraliser of the semisimple element $g$ of $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ with characteristic polynomial as in Proposition 3.5 is [6] [29, (3.3)]

$$C_{\mathrm{Sp}_{2n}(\mathbf{F}_q)}(g) = \mathrm{Sp}_{a_-}(\mathbf{F}_q) \times \mathrm{Sp}_{a_+}(\mathbf{F}_q) \times \prod_i \mathrm{GL}^-_{m_i^-}(\mathbf{F}_{q^{\frac{1}{2}d_i^-}}) \times \prod_j \mathrm{GL}^+_{m_j^+}(\mathbf{F}_{q^{d_j}}) \tag{4.4}$$

and the contribution, $-\tilde{\chi}_r(C_{\mathrm{L}_{2n}(\mathbf{F}_q)^*}(g), C_{\mathrm{Sp}_{2n}(\mathbf{F}_q)}(g))$, to the sum $-\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q))$ of Lemma 4.3 from $g$ is

$$-\tilde{\chi}_r(\mathrm{Sp}_{a^-}(\mathbf{F}_q)) \times -\tilde{\chi}_r(\mathrm{Sp}_{a^+}(\mathbf{F}_q)) \times \prod_i -\tilde{\chi}_r(\mathrm{GL}^-_{m_i^-}(\mathbf{F}_{q^{\frac{1}{2}d_i^-}})) \times \prod_j \tilde{\chi}_r(\mathrm{GL}^+_{m_j^+}(\mathbf{F}_{q^{d_j}}))$$

with a sign change similar to that of [20, Lemma 4.3]. The characteristic polynomial induces a bijection between the set of semisimple classes in $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ and the set of self-reciprocal polynomials of degree $2n$ [29, §3.1] [31]. We conclude from these facts that $-\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q))$ equals

$$\sum_{(a^-,a^+,\lambda^-,\lambda^+)} (-\tilde{\chi}_r(\mathrm{Sp}_{2a^-}(\mathbf{F}_q)))(-\tilde{\chi}_r(\mathrm{Sp}_{2a^+}(\mathbf{F}_q)))$$
$$\times \prod_{\substack{d^- \\ \exists m^- : (m^-,d^-)^{E(m^-,d^-)} \in \lambda^-}} \binom{\mathrm{SRIM}^-_{d^-}(q)}{[E(m^-,d^-) : (m^-,d^-)^{E(m^-,d^-)} \in \lambda^-]}$$
$$\times \prod_{(m^-,d^-)^{E(m^-,d^-)} \in \lambda^-} -\tilde{\chi}_r(\mathrm{GL}^-_{m^-}(\mathbf{F}_{q^{\frac{1}{2}d^-}}))^{E(m^-,d^-)}$$
$$\times \prod_{\substack{d^+ \\ \exists m^+ : (m^+,d^+)^{E(m^+,d^+)} \in \lambda^+}} \binom{\mathrm{SRIM}^+_{d^+}(q)}{[E(m^+,d^+) : (m^+,d^+)^{E(m^+,d^+)} \in \lambda^+]}$$
$$\times \prod_{(m^+,d^+)^{E(m^+,d^+)} \in \lambda^+} \tilde{\chi}_r(\mathrm{GL}^+_{m^+}(\mathbf{F}_{q^{d^+}}))^{E(m^+,d^+)} \tag{4.5}$$

where the sum runs over all $(a^-, a^+, \lambda^-, \lambda^+)$ where $a^\pm$ are positive integers, $\lambda^\pm = \{(m_i^\pm, d_i^\pm)^{E_i^\pm}\}$ are multisets of pairs of positive integers such that $a^- + a^+ + \sum m_i^- d_i^- E_i^- + \sum m_j^+ d_j^+ E_j^+ = n$ and the $d_i^-$ are even.

We are here using multinomial coefficients as defined below.

**Definition 4.6** For a rational polynomial $m \in \mathbf{Q}[q]$ and $k_1, \ldots, k_s \geq 0$ a finite sequence of nonnegative integers, define the multinomial coefficients to be

$$\binom{m}{k_1,\ldots,k_s} = \frac{m(m-1)\cdots(m+1-\sum k_i)}{k_1!\cdots k_s!} = \begin{cases} \dfrac{m!}{k_1!\cdots k_s!\cdot(m-\sum k_i)!} & \sum k_i \leq m \\ 0 & \sum k_i > m \end{cases}$$
$$\binom{m}{-k_1,\ldots,-k_s} = (-1)^{k_1+\cdots+k_s}\binom{m}{k_1,\ldots,k_s}$$
$$\binom{-m}{-k_1,\ldots,-k_s} = \frac{m(m+1)\cdots(m-1+\sum k_i)}{k_1!\cdots k_s!} = \binom{m-1+\sum k_i}{k_1,\ldots,k_s}$$

Using the concept of $T_S$-transforms from [19, §3.2] we may express Lemma 4.3 or (4.5) by the recurrence

$$\mathrm{FSp}_{r+1}(x) = \begin{cases} \mathrm{FSp}_r(x)^2 T_{\mathrm{SRIM}^-(q)}(\mathrm{FGL}^-_r(x)) T_{\mathrm{SRIM}^+(q)}(\mathrm{FGL}^+_r(x)) & q \text{ odd} \\ \mathrm{FSp}_r(x) T_{\mathrm{SRIM}^-(q)}(\mathrm{FGL}^-_r(x)) T_{\mathrm{SRIM}^+(q)}(\mathrm{FGL}^+_r(x)) & q \text{ even} \end{cases} \quad (r \geq 1) \tag{4.7}$$

where the generating functions $\mathrm{FGL}^\pm_r(x)$ of [19, (1.3)] and [20, (1.2)] have been transformed relative to the polynomial sequences $(\mathrm{SRIM}^\pm_d(q))_{d\geq 1}$. We now start the computation, facilitated by the multiplicative property [19, (3.2)], of the product of these two transformed generating functions. The first lemma is well-known [32, p 258].

**Lemma 4.8** $T_{\mathrm{IM}(q)}(1-x)=\dfrac{1-qx}{1-x}$.

***Proof*** The $T_{\mathrm{IM}(q)}$-transform of $1-x$ is

$$T_{\mathrm{IM}(q)}(1-x)=\prod_{n\geq 1}(1-x^n)^{\mathrm{IM}_n(q)}=\exp\left(\sum_{n\geq 1}(1-q^n)x^n/n\right)=\frac{1-qx}{1-x}$$

where the second equality is justified by [19, Lemma 3.7] and the identity $q^n-1=\sum_{d|n}d\mathrm{IM}_d(q)$, the Möbius inverse of the left equation of (3.7). □

**Lemma 4.9** $T_{\mathrm{SRIM}^-(q)}(1-x)T_{\mathrm{SRIM}^+(q)}(1-x)=\begin{cases}\dfrac{1-qx}{(1-x)^2} & q \text{ odd}\\ \dfrac{1-qx}{1-x} & q \text{ even}\end{cases}$ *and* $T_{\mathrm{SRIM}^-(q)}(1+x)T_{\mathrm{SRIM}^+(q)}(1-x)=\dfrac{1}{1-x}$.

***Proof*** Thanks to the identity of Lemma 4.8 it is easy to determine

$$\begin{aligned}T_{\mathrm{SRIM}^-(q)}(1-x)T_{\mathrm{SRIM}^+(q)}(1-x)&=T_{\mathrm{SRIM}^-(q)+\mathrm{SRIM}^+(q)}(1-x)\\ &\overset{\text{Lemma 3.11}}{=}\begin{cases}(1-x)^{-1}T_{\mathrm{IM}(q)}(1-x)=\dfrac{1-qx}{(1-x)^2} & q\text{ odd}\\ T_{\mathrm{IM}(q)}(1-x)=\dfrac{1-qx}{1-x} & q\text{ even}\end{cases}\end{aligned}$$

Observe that

$$T_{\mathrm{SRIM}^-(q)}\left(\frac{1-x}{1+x}\right)=\begin{cases}\dfrac{1-qx}{1-x} & 2\nmid q\\ 1-qx & 2|q\end{cases}\tag{4.10}$$

because, as $1+x=\frac{1-x^2}{1-x}$,

$$\begin{aligned}T_{\mathrm{SRIM}^-(q)}\left(\frac{1-x}{1+x}\right)&=\frac{T_{\mathrm{SRIM}^-(q)}(1-x)^2}{T_{\mathrm{SRIM}^-(q)}(1-x^2)}=\frac{\prod_m(1-x^m)^{2\mathrm{SRIM}_m^-(q)}}{\prod_m(1-x^{2m})^{\mathrm{SRIM}_m^-(q)}}\\ &=\prod_{2\nmid m}(1-x^m)^{2\mathrm{SRIM}_m^-(q)}\times\prod_{2|m}(1-x^m)^{2\mathrm{SRIM}_m^-(q)-\mathrm{SRIM}_{m/2}^-(q)}\\ &\overset{\text{Lemma 3.10}}{=}\begin{cases}T_{\mathrm{IM}(q)}(1-x)=(1-x)^{-1}(1-qx) & q\text{ odd}\\ (1-x)T_{\mathrm{IM}(q)}(1-x)=1-qx & q\text{ even}\end{cases}\end{aligned}$$

Therefore

$$\begin{aligned}&T_{\mathrm{SRIM}^-(q)}(1+x)T_{\mathrm{SRIM}^+(q)}(1-x)\\&\quad=T_{\mathrm{SRIM}^-(q)}\left(\frac{1+x}{1-x}\right)T_{\mathrm{SRIM}^-(q)}(1-x)T_{\mathrm{SRIM}^+(q)}(1-x)=\frac{1}{1-x}\end{aligned}$$

for all prime powers $q$. □

From recurrence (4.7) and Lemma 4.9, we get that the second generating function is

$$\begin{aligned}\mathrm{FSp}_2(x)&=T_{\mathrm{SRIM}^-(q)}(\mathrm{FGL}_1^-(x))T_{\mathrm{SRIM}^+(q)}(\mathrm{FGL}_1^+(x))\\&=T_{\mathrm{SRIM}^-(q)}(1+x)T_{\mathrm{SRIM}^+(q)}(1-x)=\frac{1}{1-x}\end{aligned}$$

as the first generating function is $\mathrm{FSp}_1(x)=1$ by Lemma 4.1.

**Lemma 4.11** *For all* $r\geq 1$

$$T_{\mathrm{SRIM}^-(q)}(\mathrm{FGL}_{r+1}^-(x))T_{\mathrm{SRIM}^+(q)}(\mathrm{FGL}_{r+1}^+(x))\prod_{\substack{0\leq j\leq r+1\\ j\equiv r \bmod 2}}(1-q^jx)^{\binom{r+1}{j}}$$

$$=\begin{cases}\displaystyle\prod_{\substack{0\leq j\leq r\\ j\not\equiv r \bmod 2}}(1-q^jx)^{2\binom{r}{j}} & q\text{ odd}\\ \displaystyle\prod_{\substack{0\leq j\leq r\\ j\not\equiv r \bmod 2}}(1-q^jx)^{\binom{r}{j}} & q\ \text{ even}\end{cases}$$

***Proof*** With the formulas for $\mathrm{FGL}_{r+1}^{\pm}(x)$ from [19, Theorem 1.4] and [20, Theorem 1.3] as input we compute for odd $q$ that

$$T_{\mathrm{SRIM}^-(q)}(\mathrm{FGL}^-_{r+1}(x))T_{\mathrm{SRIM}^+(q)}(\mathrm{FGL}^+_{r+1}(x))$$

$$= \frac{\prod\limits_{\substack{0\le j\le r\\ j\equiv r \bmod 2}} T_{\mathrm{SRIM}^-(q)}(1+q^jx)^{\binom{r}{j}} \prod\limits_{\substack{0\le j\le r\\ j\equiv r \bmod 2}} T_{\mathrm{SRIM}^+(q)}(1-q^jx)^{\binom{r}{j}}}{\prod\limits_{\substack{0\le j\le r\\ j\not\equiv r \bmod 2}} T_{\mathrm{SRIM}^-(q)}(1-q^jx)^{\binom{r}{j}} \prod\limits_{\substack{0\le j\le r\\ j\not\equiv r \bmod 2}} T_{\mathrm{SRIM}^+(q)}(1-q^jx)^{\binom{r}{j}}}$$

$$= \frac{\prod\limits_{\substack{0\le j\le r\\ j\not\equiv r \bmod 2}} (1-q^jx)^{2\binom{r}{j}}}{\prod\limits_{\substack{0\le j\le r\\ j\not\equiv r \bmod 2}} (1-q^{j+1}x)^{\binom{r}{j}} \prod\limits_{\substack{c0\le j\le r\\ j\equiv r \bmod 2}} (1-q^jx)^{\binom{r}{j}}}$$

$$= \frac{\prod\limits_{\substack{0\le j\le r\\ j\not\equiv r \bmod 2}} (1-q^jx)^{2\binom{r}{j}}}{\prod\limits_{\substack{0\le j\le r+1\\ j\equiv r \bmod 2}} (1-q^jx)^{\binom{r+1}{j}}}$$

by using properties of the $T_{\mathrm{SRIM}^\pm(q)}$-transform [19, Chp 3] and Lemma 4.9. When $q$ is even, the computations are essentially identical. □

***Proof of Theorem 1.3*** The formula of Theorem 1.3 is the solution to the recurrence (4.7) given the result of Lemma 4.11. □

***Proof of Corollary 1.4*** The logarithm of the $(r+1)$th generating function $\mathrm{FSp}_{r+1}(q,x)$ is

$$\sum_{\substack{\leq j \leq r \\ j \equiv r \bmod 2}} -\binom{r}{j} \log(1-q^j x) = \sum_{\substack{0 \leq j \leq r \\ j \equiv r \bmod 2}} \binom{r}{j} \sum_{n \geq 1} \frac{(q^j x)^n}{n}$$

$$= \sum_{n \geq 1} \sum_{\substack{0 \leq j \leq r \\ j \equiv r \bmod 2}} 2\binom{r}{j} q^{nj} \frac{x^n}{2n}$$

$$= \sum_{n \geq 1} ((q^n+1)^r - (q^n-1)^r) \frac{x^n}{2n}$$

□

The binomial formula applied to right hand side of Theorem 1.3 gives the more direct expression

$$-\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q)) = \sum_{\substack{n_0+\cdots+n_r=n \\ j \equiv r \bmod 2 \Rightarrow n_j = 0}} \prod_{0 \leq j \leq r} (-1)^{n_j} \binom{-\binom{r}{j}}{n_j} q^{jn_j} \qquad (r \geq 1) \tag{4.12}$$

where the sum ranges over all the $\binom{n+\lfloor \frac{1}{2}(r-1) \rfloor}{n}$ weak compositions $n_0 + \cdots + n_r$ of $n$ into $r+1$ parts [24, p 15] with $n_j = 0$ for all $j \equiv r \bmod 2$.

Elementary properties of the binomial coefficients imply that the generating functions satisfy the recurrence $\mathrm{FSp}_1(q,x) = 1$ and

$$\mathrm{FSp}_{r+1}(q,x) = \frac{\mathrm{FSp}_r(q,qx)}{1-q^{r-1}x} \prod_{1 \leq j \leq r-1} \mathrm{FSp}_{r-j}(q, q^{j-1}x)$$

for all $r \geq 1$.

***Remark 4.13*** The (non-block-wise form of the) Knörr–Robinson conjecture [13] [28, §3] for the group $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ relative to the characteristic $s$ of $\mathbf{F}_q$ states that

$$-\tilde{\chi}_2(\mathrm{Sp}_{2n}(\mathbf{F}_q)) = z_s(\mathrm{Sp}_{2n}(\mathbf{F}_q))$$

where $z_s(\mathrm{Sp}_{2n}(\mathbf{F}_q)) = |\{\chi \in \mathrm{Irr}_{\mathbf{C}}(\mathrm{Sp}_{2n}(\mathbf{F}_q)) | |\mathrm{Sp}_{2n}(\mathbf{F}_q)|_s = \chi(1)_s\}|$ is the number of irreducible complex representations of $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ of $s$-defect 0. As $\mathrm{FSp}_2(q,x) = (1-x)^{-1} = 1 + x + x^2 + \cdots$, the left side is 1 and so is the right side [11, Remark p 69]. This confirms the Knörr–Robinson conjecture for $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ relative to the defining characteristic.

## 5 Proof of Theorem 1.5

Let $p$ be a prime and, as in the previous sections, $q$ a prime power. (The prime $p$ may or may not divide the prime power $q$ although it will soon emerge that $p\nmid q$ is the most interesting case.) In this section we discuss Tamanoi's $p$-primary equivariant reduced Euler characteristics of the $\mathrm{Sp}_{2n}(\mathbf{F}_q)$-poset $\mathrm{L}^*_{2n}(\mathbf{F}_q)$ of nonzero totally isotropic subspaces.

$\mathbf{Z}_p$ denotes the ring of $p$-adic integers and $Z_p^r$ the product $\mathbf{Z}\times\mathbf{Z}_p^{r-1}$ of one copy of the integers with $r-1$ copies of the $p$-adic integers.

**Definition 5.1** [25, (1–5)] The $r$th, $r\geq 1$, $p$-primary equivariant reduced Euler characteristic of the $\mathrm{Sp}_{2n}(\mathbf{F}_q)$-poset $\mathrm{L}^*_{2n}(\mathbf{F}_q)$ is the normalised sum

$$\tilde{\chi}_r(p,\mathrm{Sp}_{2n}(\mathbf{F}_q))=\frac{1}{|\mathrm{Sp}_{2n}(\mathbf{F}_q)|}\sum_{X\in\mathrm{Hom}(Z_p^r,\mathrm{Sp}_{2n}(\mathbf{F}_q))}\tilde{\chi}(C_{\mathrm{L}^*_{2n}(q)}(X(Z_p^r)))$$

of reduced Euler characteristics.

In this definition, the sum ranges over all commuting $r$-tuples $(X_1,X_2,\ldots,X_r)$ of elements of $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ such that the elements $X_2,\ldots,X_r$ have $p$-power order. The first $p$-primary equivariant reduced Euler characteristic is independent of $p$ and agrees with the first equivariant reduced Euler characteristic.

The $r$th $p$-primary equivariant *unreduced* Euler characteristic $\chi_r(p,\mathrm{Sp}_{2n}(\mathbf{F}_q))$ agrees with the Euler characteristic of the homotopy orbit space $B\mathrm{L}^*_{2n}(\mathbf{F}_q)_{h\mathrm{Sp}_{2n}(\mathbf{F}_q)}$ computed in Morava $K(r)$-theory at $p$ [10] [18, Remark 7.2] [25, 2-3, 5-1].

For $r=1$, the $p$-primary equivariant reduced Euler characteristic and the equivariant reduced Euler characteristic agree, $\tilde{\chi}_1(p,\mathrm{Sp}_{2n}(\mathbf{F}_q))=\tilde{\chi}_1(\mathrm{Sp}_{2n}(\mathbf{F}_q))$, and for $r\geq 1$,

$$\begin{aligned}&\tilde{\chi}_{r+1}(p,\mathrm{Sp}_{2n}(\mathbf{F}_q))\\&\quad=\sum_{X\in\mathrm{Hom}(\mathbf{Z}_p,\mathrm{Sp}_{2n}(\mathbf{F}_q))/\mathrm{Sp}_{2n}(\mathbf{F}_q)}\tilde{\chi}_r(p,C_{\mathrm{L}^*_{2n}(\mathbf{F}_q)}(X),C_{\mathrm{Sp}_{2n}(\mathbf{F}_q)}(X))\\&\qquad(n\geq 1)\end{aligned}\tag{5.2}$$

where the sum runs over conjugacy classes of $p$-elements in the symplectic group (Lemma A.3).

The $r$th $p$-primary generating function at $q$ is the integral power series

$$\mathrm{FSp}_r(p,q,x)=1-\sum_{n\geq 1}\tilde{\chi}_r(p,\mathrm{Sp}_{2n}(\mathbf{F}_q))x^n\in\mathbf{Z}[[x]]\tag{5.3}$$

associated to the sequence $(-\tilde{\chi}_r(p,\mathrm{Sp}_{2n}(\mathbf{F}_q)))_{n\geq 1}$ of the *negative* of the $p$-primary equivariant reduced Euler characteristics. We have $\mathrm{FSp}_1(p,q,x)=\mathrm{FSp}_1(q,x)=1$ and, directly from the definition and Lemma 4.2, $\mathrm{FSp}_r(p,q,x)=1$ for all $r\geq 1$ when $p|q$. Thus we now restrict to the case where $p$ does not divide $q$.

**Definition 5.4** [14, Definition 3.2] [12, Definition, Chp 4, §1] Let $f\in\mathbf{F}_q[x]$ be a polynomial with $f(0)\neq 0$. The order of $f$, $\mathrm{ord}(f)$, is the least positive integer $e$ for which $f(x)|x^e - 1$.

Let $a$ and $n$ be relatively prime integers. The multiplicative order of $a$ modulo $n$, $\mathrm{ord}_n(a)$, is the order of $a$ in the unit group $(\mathbf{Z}/n\mathbf{Z})^\times$ of the modulo $n$ residue ring $\mathbf{Z}/n\mathbf{Z}$.

**Definition 5.5** For every integer $n \geq 1$, prime number $p$, and prime power $q$,

- $\mathrm{IM}_n(p,q)$ is the number of Irreducible Monic $p$-power order polynomials $p(x)$ of degree $n$ over $\mathbf{F}_q$ with $p(0) \neq 0$
- $\mathrm{SRIM}_n^-(p,q)$ is the number of Self-Reciprocal Irreducible Monic $p$-power order polynomials $p(x)$ of even degree $2n$ over $\mathbf{F}_q$
- $\mathrm{SRIM}_n^+(p,q)$ is the number of unordered pairs $\{p(x), p^*(x)/p(0)\}$ of irreducible monic $p$-power order polynomials $p(x)$ of degree $n$ over $\mathbf{F}_q$ with $p(0) \neq 0$ and $p(x) \neq p^*(x)/p(0)$

In degree $n = 1$, in particular, $\mathrm{IM}_1(p,q) = (q-1)_p$, represented by the polynomials $x - \alpha$ with $\alpha$ in the Sylow $p$-subgroup of the unit group $\mathbf{F}_q^\times$, and

$$2\mathrm{SRIM}_1^+(p,q) = \begin{cases} (q-1)_2 - 2 & p = 2 \\ (q-1)_p - 1 & p > 2 \end{cases} \tag{5.6}$$

as $x - \alpha$ is fixed if and only if $\alpha^2 = 1$. By the $p$-version [19, (4,7)] of a classical identity [14, Theorem 3.25] and by the definition of $\mathrm{SRIM}_n^\pm(p,q)$,

$$\mathrm{IM}_n(p,q) = \frac{1}{n}\sum_{d|n} \mu(d)(q^{d|n} - 1)_p$$
$$\mathrm{IM}_n(p,q) = \begin{cases} 2\mathrm{SRIM}_1^+(p,q) + \varepsilon & n = 1 \\ 2\mathrm{SRIM}_n^+(p,q) & n > 1 \text{ odd} \\ 2\mathrm{SRIM}_n^+(p,q) + 2\mathrm{SRIM}_{n/2}^-(p,q) & n > 1 \text{ even} \end{cases} \tag{5.7}$$

where $\varepsilon = 2$ if $p = 2$ and $\varepsilon = 1$ if $p > 2$.

**Lemma 5.8** *Assume $p \nmid q$. Let $D = \mathrm{ord}_p(q^2)$ and let $f \in \mathbf{F}_q[x]$ be a self-reciprocal irreducible monic p-power order polynomial of degree $2d$ for some $d \geq 1$. Then*

1. $q^d \equiv -1 \bmod p^j$ for some $j \geq 1$
2. $D|d$
3. $f(x)|(x^{(q^{2d}-1)_p} - 1)$ and $f(x)|(x^{q^d+1} - 1)$
4. $f(x)|(x^{(q^d+1)_p} - 1)$

***Proof*** Let $f \in \mathbf{F}_q[x]$ be a self-reciprocal irreducible monic $p$-power order polynomial of degree $2d$, $d \geq 1$. Then $p|\mathrm{ord}(f)|q^{2d} - 1$ by [14, Corollary 3.4]. In other words, $q^{2d} \equiv 1 \bmod \mathrm{ord}(f)$, $q^{2d} \equiv 1 \bmod p$, and thus $d$ is a multiple of $D$. Moreover,

$f(x)|(x^{(q^{2d}-1)_p}-1)$ by [14, Lemma 3.6] as $\operatorname{ord}(f)|(q^{2d}-1)_p$, and $f(x)|(x^{q^d+1}-1)$ by [17, Theorem 1.(i)]. But then $f(x)|(x^{(q^d+1)_p}-1)$ by [14, Corollary 3.7] as $f(x)$ is irreducible and $\gcd(q^{2d}-1)_p, q^d+1 = \gcd(q^{2d}-1)_p, (q^d+1)_p = \gcd(q^d-1)_p(q^d+1)_p, (q^d+1)_p = (q^d+1)_p$.

The irreducible factors of $x^{(q^d+1)_p}-1$ of degree $\geq 2$ are the irreducible factors of the cyclotomic polynomials $\Phi_{p^j}(x)$ where $j\geq 1$ and $p^j|q^d+1$. Thus $q^d \equiv -1 \bmod p^j$ for some $j\geq 1$. □

**Lemma 5.9** *Assume $p \nmid q$ and $d\geq 1$. Each irreducible monic factor $f\in \mathbf{F}_q[x]$ of $x^{(q^n+1)_p}-1$, $n\geq 1$, of degree $2d\geq 2$ is self-reciprocal, has p-power order, and $d|n$ with n/d odd where* $\deg(f)=2d$.

***Proof*** Suppose $f(x)|(x^{(q^n+1)_p}-1)$. Then $f(0)\neq 0$ and $f(x)$ has $p$-power order by [14, Lemma 3.6]. Since $f$ is irreducible of degree $\deg(f)\geq 2$ and $f(x)|x^{(q^n+1)_p}-1|x^{q^n+1}-1$, $f(x)$ is self-reciprocal and $d=\frac{1}{2}\deg(f)$ divides $n$ with odd quotient $n/d$ by [17, Theorem 1.(ii)]. □

**Lemma 5.10** *Assume $p\nmid q$. For any $n\geq 1$,*

$$2n\mathrm{SRIM}_n^-(p,q) = \begin{cases} \sum_{d|\frac{n}{n_2}} \mu(d)(q^{n/d}+1)_p & n\neq n_2 \\ (q^n+1)_p - \varepsilon & n = n_2 \end{cases}$$

where $\varepsilon = 2$ if $p=2$ and $\varepsilon=1$ if $p>2$. For any *odd* $n\geq 1$, $\mathrm{SRIM}^-_{2^k n}(p,q) = 2^{-k}\mathrm{SRIM}_n^-(p,q^{2^k})$ for all $k\geq 0$.

***Proof*** Recall that the irreducible factors of the polynomial $x^{q^n+1}-1$ are distinct and that there is one linear factor, $x+1$, of order 1, when $q$ is even and two, $x+1$, $x-1$, of order 1 and 2, if $q$ is odd. The polynomial $x^{(q^n+1)_p}-1$ thus has $\varepsilon$ linear factors of $p$-power order where $\varepsilon=2$ if $q$ is odd and $p=2$ and $\varepsilon=1$ in all other cases. Lemma 5.9 and Möbius inversion thus imply that

$$(q^n+1)_p = \varepsilon + \sum_{\substack{d|n \\ n/d \text{ odd}}} 2d\mathrm{SRIM}_d^-(p,q), \qquad 2n\mathrm{SRIM}_n^-(p,q) = \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d)((q^{n/d}+1)_p - \varepsilon)$$

The first part of the lemma follows because $\sum_{d|\frac{n}{n_2}}\mu(d)$ is 1 if $n=n_2$ is a power of 2 and 0 otherwise. If $n=2^k m$, $k\geq 0$, $m\geq 1$ odd, then $2n\mathrm{SRIM}_n^-(p,q) = \sum_{d|m}\mu(d)(q^{2^k m/d}+1)_p = 2m\mathrm{SRIM}_m^-(p,q^{2^k})$. □

**Lemma 5.11** *Assume $p\nmid q$. For every odd $n\geq 1$ and $k\geq 0$,*

$$\mathrm{IM}_{2^k n}(p,q) = 2^{-k}\mathrm{IM}_n(p,q^{2^k}) - 2^{-k}\mathrm{IM}_n(p,q^{2^{k-1}})$$
$$\mathrm{SRIM}^+_{2^k n}(p,q) = 2^{-k}\mathrm{SRIM}^+_n(p,q^{2^k}) - 2^{-k}\mathrm{SRIM}^+_n(p,q^{2^{k-1}}) - 2^{-k}\mathrm{SRIM}^-_n(p,q^{2^{k-1}})$$

***Proof*** Since $\mu(2^j n) = 0$ for $j \geq 2$,

$$\begin{aligned} n2^k\mathrm{IM}_{n2^k}(p,q) &= \sum_{d|n}\mu(n/d)(q^{2^k d}-1)_p + \sum_{d|n}\mu(2n/d)(q^{2^{k-1}d}-1)_p \\ &= \sum_{d|n}\mu(n/d)(q^{2^k d}-1)_p - \sum_{d|n}\mu(2n/d)(q^{2^{k-1}d}-1)_p \\ &= n\mathrm{IM}_n(p,q^{2^k}) - n\mathrm{IM}_n(p,q^{2^{k-1}}) \end{aligned}$$

for all $k \geq 0$. This proves the first assertion. Now,

$$\begin{aligned} \mathrm{SRIM}^+_{2^k n}(p,q) &\overset{(5.7)}{=} 2^{-1}\mathrm{IM}_{2^k n}(p,q) - 2^{-1}\mathrm{SRIM}^-_{2^{k-1}n}(p,q) \\ &= 2^{-k}(2^{-1}\mathrm{IM}_n(p,q^{2^k}) - 2^{-1}\mathrm{IM}_n(p,q^{2^{k-1}})) - 2^{-1}\mathrm{SRIM}^-_{2^{k-1}n}(p,q) \\ &\overset{(5.7),L.5.10}{=} 2^{-k}\mathrm{SRIM}^+_n(p,q^{2^k}) - 2^{-k}\mathrm{SRIM}^+_n(p,q^{2^{k-1}}) \\ &- 2^{-k}\mathrm{SRIM}^-_n(p,q^{2^{k-1}}) \end{aligned}$$

proves the second assertion. □

For all pairs ($p$, $q$), where $p$ is a prime, $q$ is a prime power, and $p \nmid q$, and for all $r \geq 1$, the $p$-primary analogue of (4.7),

$$\mathrm{FSp}_{r+1}(p,q)(x) = \begin{cases} \mathrm{FSp}_r(p,q)(x)^2 T_{\mathrm{SRIM}^-(p,q)}(\mathrm{FGL}^-_r(p,q,x))T_{\mathrm{SRIM}^+(p,q)}(\mathrm{FGL}^+_r(p,q,x)) & p=2 \\ \mathrm{FSp}_r(p,q)(x)T_{\mathrm{SRIM}^-(p,q)}(\mathrm{FGL}^-_r(p,q,x))T_{\mathrm{SRIM}^+(p,q)}(\mathrm{FGL}^+_r(p,q,x)) & p>2 \end{cases} \tag{5.12}$$

is a consequence of recurrence (5.2). Note here that a semisimple $g \in \mathrm{Sp}_{2n}(\mathbf{F}_q)$ has $p$-power order if and only if all the irreducible factors in its characteristic polynomial, described in Proposition 3.5, have $p$-power order. This is because multiplication by $x$ in the $\mathbf{F}_q[x]$-module $\mathbf{F}_q[x]/(r(x))$, where $r(x)$ is irreducible with $r(0) \neq 0$, has $p$-power order if and only if $r(x)$ has $p$-power order by [14, Lemma 3.5]. Also note that in Proposition 3.5 with odd $q$, the polynomial $(x+1)$ has even order and is therefore allowed only when $p = 2$.

The proof of Theorem 1.5 consists in verifying that the solution to recurrence (5.12) satisfies the infinite product expansion

$$\begin{aligned} \mathrm{FSp}_r(p,q,x) &= \prod_{n\geq 1}(1-x^n)^{c_r(p,q,n)} \\ c_r(p,q,n) &= \frac{1}{2n}\sum_{d|n}\mu(n/d)((q^d-1)_p^{r-1}-(q^d+1)_p^{r-1}) \end{aligned}$$

Since the infinite product expansions of $\mathrm{FGL}_r^{\pm}(p,q,x)$ are [19, §1] [20, §1]

$$\begin{aligned} \mathrm{FGL}_r^{+}(p,q,x) &= \prod_{n\geq 1}(1-x^n)^{a_r^{+}(p,q,n)} \\ a_r^{+}(p,q,n) &= \frac{1}{n}\sum_{d|n}\mu(n/d)(q^d-1)_p^{r-1} \\ \mathrm{FGL}_r^{-}(p,q,x) &= \prod_{n\geq 1}(1-x^n)^{a_r^{-}(p,q,n)} \\ a_r^{-}(p,q,n) &= \frac{1}{n}\sum_{d|n}(-1)^d\mu(n/d)(q^d-(-1)^d)_p^{r-1} \end{aligned}$$

and [19, (3.2)]

$$\begin{aligned} T_{\mathrm{SRIM}^{\pm}(p,q)}\mathrm{FGL}_r^{\pm}(p,q,x) &= \prod_{d\geq 1}\mathrm{FGL}_r^{\pm}(p,q^d,x^d)^{\mathrm{SRIM}_d^{\pm}(p,q)} \\ &= \prod_{n,d\geq 1}(1-x^{nd})^{a_r^{\pm}(p,q^d,n)\mathrm{SRIM}_d^{\pm}(p,q)} \end{aligned}$$

we must show that

$$\begin{aligned} &c_{r+1}(p,q,N) \\ &= \begin{cases} 2c_r(p,q,N)+\sum_{d|N}a_r^{-}(p,q^d,N/d)\mathrm{SRIM}_d^{-}(p,q)+\sum_{d|N}a_r^{+}(p,q^d,N/d)\mathrm{SRIM}_d^{+}(p,q) \\ \qquad p=2 \\ c_r(p,q,N)+\sum_{d|N}a_r^{-}(p,q^d,N/d)\mathrm{SRIM}_d^{-}(p,q)+\sum_{d|N}a_r^{+}(p,q^d,N/d)\mathrm{SRIM}_d^{+}(p,q) \\ \qquad p>2 \end{cases} \end{aligned} \tag{5.13}$$

for all $N\geq 1$ and all $r\geq 0$.

Theorem 1.5 will here be proved only for odd primes $p$. The below proof can easily be modified to cover $p=2$.

***Proof*** Assume $p>2$ and let $N\geq 1$ be an odd integer. Induction shows that

$$2^k a_r^{\pm}(p,q,2^kN) = \begin{cases} a_r^{+}(p,q^2,N)-a_r^{\pm}(p,q,N) & k=1 \\ a_r^{+}(p,q^{2^k},N)-a_r^{+}(p,q^{2^{k-1}},N) & k>1 \end{cases} \tag{5.14}$$

and

$$2^k c_r(p,q,2^kN) = \begin{cases} a_r^-(p,q,N) + a_r^+(p,q,N) & k=0 \\ c_r(p,q^{2^k},N) - c_r(p,q^{2^{k-1}},N) & k>0 \end{cases} \tag{5.15}$$

Indeed, $a_r^{\pm}(p,q,2N) = \frac{1}{2}a_r^+(p,q^2,N) - \frac{1}{2}a_r^{\pm}(p,q,N)$ because

$$\begin{aligned} 2Na_r^-(p,q,2N) &= \sum_{d|N} -\mu(2N/d)(q^d+1)_p^{r-1} + \sum_{d|N}\mu(N/d)(q^{2d}-1)_p^{r-1} \\ &= -\sum_{d|N} -\mu(N/d)(q^d+1)_p^{r-1} + \sum_{d|N}\mu(N/d)(q^{2d}-1)_p^{r-1} \\ &= -Na_r^-(p,q,N) + Na_r^+(p,q^2,N) \end{aligned}$$

$$\begin{aligned} 2Na_r^+(p,q,2N) &= \sum_{d|N}\mu(2N/d)(q^d-1)_p^{r-1} + \sum_{d|N}\mu(N/d)(q^{2d}-1)_p^{r-1} \\ &= -Na_r^+(p,q,N) + Na_r^+(p,q^2,N) \end{aligned}$$

Since $\mu(2^jN) = 0$ for $j \geq 2$,

$$\begin{aligned} 2^kNa_r^{\pm}(p,q,2^kN) &= \sum_{d|N}\mu(N/d)(q^{2^kd}-1)_p^{r-1} + \sum_{d|N}\mu(2N/d)(q^{2^{k-1}d}-1)_p^{r-1} \\ &= Na_r^+(p,q^{2^k},N) - Na_r^+(p,q^{2^{k-1}},N) \end{aligned}$$

for all $k>1$. Since $N$ is odd, $c_r(p,q,N) = \frac{1}{2}a_r^-(p,q,N) + \frac{1}{2}a_r^+(p,q,N)$, and, as $\mu(2^jN)=0$ for $j\geq 2$, $2^kNc_r(p,q^{2^k},N) = Nc_r(p,q^{2^k},N) - Nc_r(p,q^{2^{k-1}},N)$ when $k>0$.

The first equality of the below display holds (at $d=1$) because $\mathrm{SRIM}_1^-(p,q) + \frac{1}{2} = \frac{1}{2}(q+1)_p$ by Lemma 5.10. The next to last equality holds because

$$\sum_{\{d_1 : f|d_1|d_2\}} \mu(d_1/f) = \begin{cases} 1 & f = d_2 \\ 0 & f < d_2 \end{cases}$$

contributes only when $f = d_2$. Remembering these observations we find that

$$\begin{aligned}
&\sum_{d|N} a_r^-(p,q^d,N/d)\mathrm{SRIM}_d^-(p,q)+\frac{1}{2}a_r^-(p,q,N)\\
&\quad=\sum_{d|N} a_r^-(p,q^d,N/d)\frac{1}{2d}\sum_{f|d}\mu(f)(q^{d/f}+1)_p\\
&\quad=\sum_{d|N} a_r^-(p,q^d,N/d)\frac{1}{2d}\sum_{f|d}\mu(d/f)(q^f+1)_p\\
&\quad=-\sum_{d|N}\frac{d}{N}\sum_{e|\frac{N}{d}}\mu(N/de)(q^{de}+1)_p^{r-1}\frac{1}{2d}\sum_{f|d}\mu(d/f)(q^f+1)_p\\
&\quad=-\frac{1}{2N}\sum_{f|d_1|d_2|N}\mu(N/d_2)(q^{d_2}+1)_p^{r-1}(q^f+1)_p\mu(d_1/f)\\
&\quad=-\frac{1}{2N}\sum_{d|N}\mu(N/d)(q^d+1)_p^r=\frac{1}{2}a_{r+1}^-(p,q,N)
\end{aligned}\tag{5.16}$$

The first equality of the below display holds (at $d=1$) because $\mathrm{SRIM}_1^+(p,q)+\frac{1}{2}=\frac{1}{2}(q-1)_p=\frac{1}{2}\mathrm{IM}_1(p,q)$ by (5.6) and (5.7). For all odd $d>1$, $\mathrm{SRIM}_d^+(p,q)=\frac{1}{2}\mathrm{IM}_d(p,q)$ by (5.7). Remembering these observations we find that

$$\begin{aligned}
&\sum_{d|N} a_r^+(p,q^d,N/d)\mathrm{SRIM}_d^+(p,q)+\frac{1}{2}a_r^+(p,q,N)\\
&\quad=\sum_{d|N} a_r^+(p,q^d,N/d)\frac{1}{2}\mathrm{IM}_d(p,q)\\
&\quad=\sum_{d|N} a_r^+(p,q^d,N/d)\frac{1}{2d}\sum_{f|d}\mu(f)(q^{d/f}-1)_p\\
&\quad=\sum_{d|N} a_r^+(p,q^d,N/d)\frac{1}{2d}\sum_{f|d}\mu(d/f)(q^f-1)_p\\
&\quad=\sum_{d|N}\frac{d}{N}\sum_{e|N/d}\mu(N/de)(q^{de}-1)_p^{r-1}\frac{1}{2d}\sum_{f|d}\mu(d/f)(q^f-1)_p\\
&\quad=\frac{1}{2N}\sum_{f|d_1|d_2|N}\mu(N/d_2)(q^{d_2}-1)_p^{r-1}(q^f-1)_p\mu(d_1/f)\\
&\quad=\frac{1}{2N}\sum_{d|N}\mu(N/d)(q^d-1)_p^r=\frac{1}{2}a_{r+1}^+(p,q,N)
\end{aligned}\tag{5.17}$$

By adding (5.16) and (5.17) we get

$$\begin{aligned}
&\sum_{d|N} a_r^-(p,q^d,N/d)\mathrm{SRIM}_d^-(p,q) + \sum_{d|N} a_r^+(p,q^d,N/d)\mathrm{SRIM}_d^+(p,q) \\
&\quad = \frac{1}{2}a_{r+1}^-(p,q,N) - \frac{1}{2}a_r^-(p,q,N) + \frac{1}{2}a_{r+1}^+(p,q,N) - \frac{1}{2}a_r^+(p,q,N) \qquad (5.18)\\
&\quad \overset{(5.15)}{=} c_{r+1}(p,q,N) - c_r(p,q,N)
\end{aligned}$$

proving (5.13) for all odd $N$.

Next consider $2N$, $N$ odd. The expression $\sum_{d|2N} a_r^-(p,q^d,2N/d)\mathrm{SRIM}_d^-(p,q) + \sum_{d|2N} a_r^-(p,q^d,2N/d)\mathrm{SRIM}_d^+(p,q)$ is the sum of the four terms

$$\begin{aligned}
&\sum_{d|N} a_r^-(p,q^d,2N/d)\mathrm{SRIM}_d^-(p,q) \\
&\quad \overset{(5.14)}{=} \frac{1}{2}\sum_{d|N}(a_r^+(p,q^{2d},N/d) - a_r^-(p,q^d,N/d))\mathrm{SRIM}_d^-(p,q) \\
&\sum_{d|N} a_r^-(p,q^{2d},N/d)\mathrm{SRIM}_{2d}^-(p,q) \\
&\quad \overset{\mathrm{L.}\,5.11}{=} \frac{1}{2}\sum_{d|N} a_r^-(p,q^{2d},N/d)\mathrm{SRIM}_d^-(p,q^2) \\
&\sum_{d|N} a_r^+(p,q^d,2N/d)\mathrm{SRIM}_d^+(p,q) \\
&\quad \overset{(5.14)}{=} \frac{1}{2}\sum_{d|N}(a_r^+(p,q^{2d},N/d) - a_r^+(p,q^d,N/d))\mathrm{SRIM}_d^+(p,q) \\
&\sum_{d|N} a_r^+(p,q^{2d},N/d)\mathrm{SRIM}_{2d}^+(p,q) \\
&\quad \overset{\mathrm{L.}\,5.11}{=} \frac{1}{2}\sum_{d|N} a_r^+(p,q^{2d},N/d)(\mathrm{SRIM}_d^+(p,q^2) - \mathrm{SRIM}_d^+(p,q) - \mathrm{SRIM}_d^-(p,q))
\end{aligned}$$

which is

$$2^{-1}\left(\sum_{d|N} a_r^-(p,q^{2d},N/d)\mathrm{SRIM}_d^-(p,q^2)+\sum_{d|N} a_r^+(p,q^{2d},N/d)\mathrm{SRIM}_d^+(p,q^2)\right)$$
$$-2^{-1}\left(\sum_{d|N} a_r^-(p,q^{d},N/d)\mathrm{SRIM}_d^-(p,q)+\sum_{d|N} a_r^+(p,q^{d},N/d)\mathrm{SRIM}_d^+(p,q)\right)$$
$$\overset{(5.18)}{=}-2^{-1}(c_{r+1}(p,q,N)-c_r(p,q,N))\quad+2^{-1}(c_{r+1}(p,q^2,N)-c_r(p,q^2,N))$$
$$=2^{-1}(c_{r+1}(p,q^2,N)-c_{r+1}(p,q,N))-2^{-1}(c_r(p,q^2,N)-c_r(p,q,N))$$
$$\overset{(5.15)}{=}c_{r+1}(p,q,2N)-c_r(p,q,2N)$$

This proves (5.13) for $2N$, $N$ odd.

Finally, we consider $2^kN$, $N$ odd, $k>1$. We shall evaluate the sum

$$\sum_{d|2^kN} a_r^-(p,q^d,2^kN/d)\mathrm{SRIM}_d^-(p,q)+\sum_{d|2^kN} a_r^+(p,q^d,2^kN/d)\mathrm{SRIM}_d^+(p,q)$$
$$=\sum_{0\le j\le k}\sum_{d|N} a_r^-(p,q^{2^jd},2^{k-j}N/d)\mathrm{SRIM}_{2^jd}^-(p,q)$$
$$+\sum_{0\le j\le k}\sum_{d|N} a_r^+(p,q^{2^jd},2^{k-j}N/d)\mathrm{SRIM}_{2^jd}^+(p,q)$$

which occurs on the right hand side of (5.13). For $j=0$ we get

$$\sum_{d|N} a_r^\pm(p,q^d,2^kN/d)\mathrm{SRIM}_d^\pm(p,q)\overset{(5.14)}{=}2^{-k}\Bigg(\sum_{d|N}(a_r^+(p,q^{2^kd},N/d)$$
$$-a_r^+(p,q^{2^{k-1}d},N/d))\mathrm{SRIM}_d^\pm(p,q)\Bigg)$$

For $0<j<k$ we get

$$\sum_{d|N} a_r^-(p,q^{2^jd},2^{k-j}N/d)\mathrm{SRIM}_{2^jd}^-(p,q)$$
$$\overset{\mathrm{L.}\,5.10}{=}2^{-j}\sum_{d|N} a_r^-(p,q^{2^jd},2^{k-j}N/d)\mathrm{SRIM}_d^-(p,q^{2^j})$$
$$\overset{(5.14)}{=}2^{-k}\sum_{d|N}(a_r^+(p,q^{2^kd},N/d)-a_r^-(p,q^{2^{k-1}d},N/d))\mathrm{SRIM}_d^-(p,q^{2^j})$$

$$\sum_{d|N} a_r^+(p, q^{2^j d}, 2^{k-j}N/d)\mathrm{SRIM}^+_{2^j d}(p,q)$$

$$\overset{\text{L.5.11}}{=} 2^{-j}\sum_{d|N} a_r^+(p, q^{2^j d}, 2^{k-j}N/d)(\mathrm{SRIM}^+_d(p, q^{2^j}) - \mathrm{SRIM}^+_d(p, q^{2^{j-1}})$$
$$- \mathrm{SRIM}^-_d(p, q^{2^{j-1}}))$$
$$\overset{(5.14)}{=} 2^{-k}\sum_{d|N}(a_r^+(p, q^{2^k d}, N/d) - a_r^+(p, q^{2^{k-1}d}, N/d))(\mathrm{SRIM}^+_d(p, q^{2^j})$$
$$- \mathrm{SRIM}^+_d(p, q^{2^{j-1}}) - \mathrm{SRIM}^-_d(p, q^{2^{j-1}}))$$

For $j = k$ we get

$$\sum_{d|N} a_r^-(p, q^{2^k d}, N/d)\mathrm{SRIM}^-_{2^k d}(p,q) \overset{\text{L.5.10}}{=} 2^{-k}\sum_{d|N} a_r^-(p, q^{2^k d}, N/d)\mathrm{SRIM}^-_d(p, q^{2^k})$$
$$\sum_{d|N} a_r^+(p, q^{2^k d}, N/d)\mathrm{SRIM}^+_{2^k d}(p,q) \overset{\text{L.5.11}}{=} 2^{-k}\sum_{d|N} a_r^+(p, q^{2^k d}, N/d)(\mathrm{SRIM}^+_d(p, q^{2^k})$$
$$- \mathrm{SRIM}^+_d(p, q^{2^{k-1}}) - \mathrm{SRIM}^-_d(p, q^{2^{k-1}}))$$

The sum of these $2(k+1)$ terms is

$$2^{-k}\sum_{d|N} a_r^-(p, q^{2^k d}, N/d)\mathrm{SRIM}^-_d(p, q^{2^k}) + 2^{-k}\sum_{d|N} a_r^+(p, q^{2^k d}, N/d)\mathrm{SRIM}^+_d(p, q^{2^k})$$
$$- 2^{-k}\left(\sum_{d|N} a_r^-(p, q^{2^{k-1}d}, N/d)\mathrm{SRIM}^-_d(p, q^{2^{k-1}}) + \sum_{d|N} a_r^+(p, q^{2^{k-1}d}, N/d)\mathrm{SRIM}^+_d(p, q^{2^{k-1}})\right)$$
$$\overset{(5.18)}{=} 2^{-k}(c_{r+1}(p, q^{2^k}, N) - c_r(p, q^{2^k}, N)) - 2^{-k}(c_{r+1}(p, q^{2^{k-1}}, N) - c_r(p, q^{2^{k-1}}, N))$$
$$= 2^{-k}(c_{r+1}(p, q^{2^k}, N) - c_{r+1}(p, q^{2^{k-1}}, N)) - 2^{-k}(c_r(p, q^{2^k}, N) - c_r(p, q^{2^{k-1}}, N))$$
$$\overset{(5.15)}{=} c_{r+1}(p, q, 2^k N) - c_r(p, q, 2^k N)$$

This proves (5.13) for $2^k N$, $N$ odd , $k > 1$.

We can now conclude that (5.13) holds for all $N$ when the prime $p$ is odd. □

Two prime powers prime to $p$ are declared to be $p$-equivalent if the generate the same closed subgroup of the topological unit group $\mathbf{Z}_p^\times$ in $\mathbf{Z}_p$. More concretely, if we let

$$O(p,q)=\begin{cases}(q \bmod 8, v_2(q^2-1)) & p=2\\ (\mathrm{ord}_p(q), v_p(q^{\mathrm{ord}_p(q)}-1)) & p>2\end{cases}$$

then the prime powers $q_1$ and $q_2$ are $p$-equivalent, $\overline{\langle q_1\rangle}=\overline{\langle q_2\rangle}\leq \mathbf{Z}_p^\times$, if and only if $O(p,q_1)=O(p,q_2)$ [4, §3]. The sequences $(\mathrm{SRIM}_d^{\pm}(p,q))_{d\geq 1}$ and hence the power series $\mathrm{FSp}_r(p,q,x)$ depend only on the $p$-class of $q$ when $p\nmid q$.

For example, the 2-classes are represented by the 2-adic units $\pm 3^{2^e}$, $e\geq 0$, and the 3-classes by the prime powers $2^{3^e}$ and $4^{3^e}$, $e\geq 0$ [4, Lemma 1.11.(a)].

***Example 5.19*** For all $r\geq 1$ and $e\geq 0$

$$\mathrm{FSp}_{r+1}(2,3^{2^e},x)=\begin{cases}\dfrac{\prod_{n\geq 0}Q(x^{2^{n+1}})^{2^{(n+3)(r-1)}}}{Q(x)^{2^{2(r-1)}}(1+x)^{2^{r-1}}} & e=0\\[2ex] \dfrac{\prod_{n\geq 0}Q(x^{2^n})^{2^{(n+2)(r-1)+re}}}{(1-x)^{2^{r-1}}} & e>0\end{cases}$$

$$\mathrm{FSp}_{r+1}(2,-3^{2^e},x)=\begin{cases}\dfrac{\prod_{n\geq -1}Q(x^{2^{n+1}})^{2^{(n+3)(r-1)}}}{(1-x)^{2^{r-1}}} & e=0\\[2ex] \dfrac{\prod_{n\geq 0}Q(x^{2^{n+1}})^{2^{(n+3)(r-1)+re}}}{Q(x)^{2(r-1)+re}(1+x)^{2^{r-1}}} & e>0\end{cases}$$

where $Q(x)=\frac{1-x}{1+x}$. This follows from Theorem 1.5 after some power series manipulations as $\exp(-\sum_{n\geq 1}(2n)_2^r\frac{x^n}{n})=\prod_{n\geq 0}Q(x^{2^n})^{2^{(n+1)(r-1)}}$, $\exp(-2\sum_{n\geq 0}\frac{x^{2n+1}}{2n+1})=Q(x)$, and (in the case of $+3^{2^e}$)

$$(3^n+1)_2=\begin{cases}4 & 2\nmid n\\ 2 & 2\mid n\end{cases}\qquad (3^n-1)_2=\begin{cases}2 & 2\nmid n\\ 4n_2 & 2\mid n\end{cases}$$

$$((3^{2^e})^n-1)_2=\begin{cases}2^{2+e} & 2\nmid n\\ 2^{2+e}n_2 & 2\mid n\end{cases}\qquad ((3^{2^e})^n+1)_2=2$$

for all $e>0$.

## 6 Proof of Theorem 1.6

As already used several times in this paper, a multiset $\lambda$ is a base set $B(\lambda)$ with a multiplicity function assigning an integer $E(\lambda,b)\geq 0$ to every element $b$ of the base set. The sum $|\lambda|=\sum_{b\in B(\lambda)}E(\lambda,b)$ of the multiplicities is the *cardinality* of the multiset $\lambda$. A *partition of* $n$, in symbols $\lambda\vdash n$, is a multiset $\lambda$ on the set $\mathbf{N}$ of natural numbers such that $\sum_{b\in\mathbf{N}}bE(\lambda,b)=n$. The *multiset sum*, $\lambda_1+\lambda_2$, is the multiset with multiplicity function $E(\lambda_1+\lambda_2,b)=E(\lambda_1,b)+E(\lambda_2,b)$. A *partition of* $n$ *into parts of two kinds* is a pair $(\lambda_-,\lambda_+)$ of partitions, $\lambda^-$ and $\lambda^+$, such that the multiset sum $\lambda_-+\lambda_+$ partitions $n$, in symbols $(\lambda_-,\lambda_+)\vdash n$.

**Lemma 6.1** *Let $A(q) \in \mathbf{Q}[q]$ be a rational polynomial in the indeterminate q. The polynomial sequence $(B_n(q))_{n\geq 0}$ with $B_0(q) = 1$ and*

$$B_n(q) = \sum_{\lambda \vdash n} (-1)^{|\lambda|} \prod_{d \in B(\lambda)} \frac{A(q^d)^{E(\lambda,d)}}{|C_d \wr \Sigma_{E(\lambda,d)}|}, \qquad n \geq 1,$$

*satisfies the recurrence $B_0(q) = 1$ and $nB_n(q) + \sum_{1\leq j\leq n} A(q^j)B_{n-j}(q) = 0$ for $n \geq 1$.*

***Proof*** Writing $\Sigma_m$ for the symmetric group of degree $m$ and $C_m$ for the cyclic group of order $m$, the claim is that

$$n \sum_{\lambda \vdash n} (-1)^{|\lambda|} \prod_{d \in B(\lambda)} \frac{A(q^d)^{E(\lambda,d)}}{|C_d \wr \Sigma_{E(\lambda,d)}|} + \sum_{1\leq j\leq n} A(q^j) \sum_{\mu \vdash n-j} (-1)^{|\mu|} \prod_{d \in B(\mu)} \frac{A(q^d)^{E(\lambda,d)}}{|C_d \wr \Sigma_{E(\lambda,d)}|} = 0$$

for all $n \geq 1$. Since, for all $d \in B(\lambda)$,

$$A(q^d) \prod_{f \in B(\lambda - \{d\})} \frac{A(q^f)^{E(\lambda-\{d\})}}{|C_f \wr \Sigma_{E(\lambda-\{d\})}|} = dE(\lambda, d) \prod_{d \in B(\lambda)} \frac{A(q^d)^{E(\lambda,d)}}{|C_d \wr \Sigma_{E(\lambda,d)}|}$$

it suffices to show that

$$(-1)^{|\lambda|} n + \sum_{d \in B(\lambda)} (-1)^{|\lambda - \{d\}|} dE(\lambda, d) = 0$$

But this is obvious since $|\lambda - \{d\}| = |\lambda| - 1$ and $\sum_{d \in B(\lambda)} dE(\lambda, d) = n$ as $\lambda$ partitions $n$. □

**Corollary 6.2** *For all $n \geq 1$ and $r \geq 0$,*

$$-\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q)) = \sum_{\lambda \vdash n} (-1)^{|\lambda|} \prod_{d \in B(\lambda)} \frac{((q^d-1)^r/2 - (q^d+1)^r/2)^{E(\lambda,d)}}{|C_d \wr \Sigma_{E(\lambda,d)}|}$$

$$-\tilde{\chi}_{r+1}(p, \mathrm{Sp}_{2n}(\mathbf{F}_q)) = \sum_{\lambda \vdash n} (-1)^{|\lambda|} \prod_{d \in B(\lambda)} \frac{((q^d-1)_p^r/2 - (q^d+1)_p^r/2)^{E(\lambda,d)}}{|C_d \wr \Sigma_{E(\lambda,d)}|}$$

***Proof*** Let $H(r,q) = \frac{1}{2}(q-1)^r - \frac{1}{2}(q+1)^r$ ($H(r,q) = \frac{1}{2}(q-1)_p^r - \frac{1}{2}(q+1)_p^r$ in the $p$-primary case). By Theorem 1.4, $\sum_{n\geq 0} -\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q))x^n = \exp(-\sum_{n\geq 1} H(r,q^n)\frac{x^n}{n})$ (with the convention that $-\tilde{\chi}_{r+1}(\mathrm{Sp}_0(\mathbf{F}_q)) = 1$ for all $r \geq 0$), so the sequence $-\tilde{\chi}_{r+1}(\mathrm{Sp}_n(\mathbf{F}_q))$, $n \geq 1$, satisfies the recurrence

$$n(-\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q))) + \sum_{1\leq j\leq n} H(r,q^j)(-\tilde{\chi}_{r+1}(\mathrm{Sp}_{2(n-j)}(\mathbf{F}_q)) = 0, \qquad n \geq 1,$$

according to [19, Lemma 3.7]. We can now apply Lemma 6.1. □

Let $F_q$ denote the standard Frobenius endomorphism of the symplectic algebraic group $\mathrm{Sp}_{2n}(\overline{\mathbf{F}}_s)$, $s = \mathrm{char}(\mathbf{F}_q)$, with fixed points $\mathrm{Sp}_{2n}(\overline{\mathbf{F}}_s)^{F_q} = \mathrm{Sp}_{2n}(\mathbf{F}_q)$. The standard maximal torus $T_n(\overline{\mathbf{F}}_s)$ of $\mathrm{Sp}_{2n}(\overline{\mathbf{F}}_s)$, described for instance in [16, Exercises 10.19, 10.29], is maximally split with respect to $F_q$ [16, Definition 21.13,Example 21.14] and the Weyl group $W(C_n)$ of $\mathrm{Sp}_{2n}(\overline{\mathbf{F}}_s)$ acts as the standard representation of the signed permutation group $C_2 \wr \Sigma_n$ in the $n$-dimensional real vector space $X(T_n(\overline{\mathbf{F}}_s)) \otimes \mathbf{R}$ spanned by the character group $X(T_n(\overline{\mathbf{F}}_s))$. As usual, $T_n(\overline{\mathbf{F}}_s)_w$ denotes the $F_q$-stable maximal torus of $\mathrm{GL}_n(\overline{\mathbf{F}}_s)$ corresponding to the Weyl group element $w \in W(C_n)$ [16, Proposition 25.1]. The number of elements in $T_n(\overline{\mathbf{F}}_s)_w$ that are fixed by the Frobenius endomorphism $F_q$ is $|T_n(\overline{\mathbf{F}}_s)_w^{F_q}| = \det(q - w^{-1})$ where the determinant is computed in $X(T_n(\overline{\mathbf{F}}_s)) \otimes \mathbf{R}$ [16, Proposition 25.3.(c)].

***Proof of Theorem 1.6*** Conjugacy classes in $W(C_n) = C_2 \wr \Sigma_n$ are in bijective correspondence with partitions, $(\lambda_-, \lambda_+)$, of $n$ into parts of two kinds [15, Chapter I, Appendix B] [26, Theorem 3.5]. If $w \in W(C_n)$ is in the conjugacy class of $(\lambda^-, \lambda^+)$ then (cf. [16, Example 25.4.(2)])

$$\begin{aligned}&\det(w^{-1})\det(q-w^{-1})^r \\ &\quad = (-1)^{n+|\lambda^-|} \prod_{d^- \in B(\lambda^-)} (q^{d^-} - 1)^{rE(\lambda^-,d^-)} \prod_{d^+ \in B(\lambda^+)} (q^{d^+} + 1)^{rE(\lambda^+,d^+)}\end{aligned}$$

The claim of the theorem is thus that

$$\begin{aligned}-\chi_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q)) = &\sum_{(\lambda^-,\lambda^+)\vdash n} (-1)^{|\lambda^-|} 2^{-|\lambda^- + \lambda^+|} \prod_{d^- \in B(\lambda^-)} \frac{(q^{d^-} - 1)^{rE(\lambda^-,d^-)}}{|C_{d^-} \wr \Sigma_{E(\lambda^-,d^-)}|} \\ &\times \prod_{d^+ \in B(\lambda^+)} \frac{(q^{d^+} + 1)^{rE(\lambda^+,d^+)}}{|C_{d^+} \wr \Sigma_{E(\lambda^+,d^+)}|}\end{aligned} \tag{6.3}$$

as the group $W(C_n)$ contains

$$\frac{|C_2 \wr \Sigma_n|}{\prod_{d^- \in B(\lambda^-)} |(C_2 \times C_{d^-}) \wr \Sigma_{E(\lambda^-,d^-)}| \prod_{d^+ \in B(\lambda^+)} |C_{2d^+} \wr \Sigma_{E(\lambda^+,d^+)}|}$$

elements in the conjugacy class $(\lambda^-, \lambda^+)$.

By Corollary 6.2, it suffices to show

$$\sum_{\substack{(\lambda^-,\lambda^+)\vdash n\\ \lambda^-+\lambda^+=\lambda}} (-1)^{|\lambda^+|} \prod_{d^-\in B(\lambda^-)} \frac{((q^{d^-}-1)^r)^{E(\lambda^-,d^-)}}{|C_{d^-}\wr \Sigma_{E(\lambda^-,d^-)}|} \prod_{d^+\in B(\lambda^+)} \frac{((q^{d^+}+1)^r)^{E(\lambda^+,d^+)}}{|C_{d^+}\wr \Sigma_{E(\lambda^+,d^+)}|}$$
$$= \prod_{d\in B(\lambda)} \frac{((q^d-1)^r-(q^d+1)^r)^{E(\lambda,d)}}{|C_d\wr\Sigma_{E(\lambda,d)}|}$$

for all partitions $\lambda\vdash n$ and all integers $r\geq 1$. Introducing the the coefficients

$$c(\lambda^-,\lambda^+) = (-1)^{|\lambda^+|}\frac{\prod_{d\in B(\lambda)}|C_d\wr\Sigma_{E(\lambda,d)}|}{\prod_{d^-\in B(\lambda^-)}|C_{d^-}\wr\Sigma_{E(\lambda^-,d^-)}|\prod_{d^-\in B(\lambda^+)}|C_{d^+}\wr\Sigma_{E(\lambda^+,d^+)}|}$$

we need to show

$$\sum_{\lambda^-+\lambda^+=\lambda} c(\lambda^-,\lambda^+) \prod_{d^-\in B(\lambda^-)} ((q^{d^-}-1)^r)^{E(\lambda^-,d^-)} \prod_{d^+\in B(\lambda^+)} ((q^{d^+}+1)^r)^{E(\lambda^+,d^+)}$$
$$= \prod_{d\in B(\lambda)} ((q^d-1)^r-(q^d+1)^r)^{E(\lambda,d)}$$

That reason that this is true is that the binomial formula

$$(a_1-b_1)^n = \sum_{1\leq i\leq n}(-1)^{n-i}\frac{|C_1\wr\Sigma_n|}{|C_1\wr\Sigma_i||C_1\wr\Sigma_{n-i}|}a_1^i b_1^{n-i}$$

generalizes to the identity

$$\prod_{d\in B(\lambda)}(a_d-b_d)^{E(\lambda,d)} = \sum_{\lambda^-+\lambda^+=\lambda} c(\lambda^-,\lambda^+) \prod_{d^-\in B(\lambda^-)} a_{d^-}^{E(\lambda^-,d^-)} \prod_{d^+\in B(\lambda^+)} b_{d^+}^{E(\lambda^+,d^+)}$$

in the polynomial ring $\mathbf{Z}[a_d,b_d|d\in B(\lambda)]$ with the $2|B(\lambda)|$ indeterminates $a_d,b_d$, $d\in B(\lambda)$. □

The right hand side of the identity from Theorem 1.6 is

$$\frac{(-1)^n}{|W(C_n)|}\sum_{w\in W(C_n)}\det(w)\det(q-w)^r$$
$$=\frac{(-1)^n}{|W(C_n)|}\sum_{w\in W(C_n)}\det(w^{-1})\det(q-w^{-1})^r$$
$$=\frac{(-1)^n}{|W(C_n)|}\sum_{w\in W(C_n)}\det(w)\det(q-w^{-1})^r$$
$$=\frac{(-1)^n}{|W(C_n)|}\sum_{w\in W(C_n)}\det(w)|T_n(\overline{\mathbf{F}}_s)_w^{F_q}|^r$$

where we used [16, Proposition 25.3(c)] and $\det(w)=\det(w^{-1})$.

**Corollary 6.4** *The generating functions for the sequences* $(-\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q)))_{r\geq 0}$ *and* $(-\tilde{\chi}_{r+1}(p,\mathrm{Sp}_{2n}(\mathbf{F}_q)))_{r\geq 0}$ *(with fixed* $n\geq 1$*) are*

$$\sum_{r\geq 0} -\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q))x^r = \frac{(-1)^n}{|W(C_n)|}\sum_{w\in W(C_n)} \frac{\det(w)}{1-x\det(q-w)}$$

$$\sum_{r\geq 0} -\tilde{\chi}_{r+1}(p,\mathrm{Sp}_{2n}(\mathbf{F}_q))x^r = \frac{(-1)^n}{|W(C_n)|}\sum_{w\in W(C_n)} \frac{\det(w)}{1-x\det(q-w)_p}$$

By considering conjugacy classes rather than the individual elements in $W(C_n)$, the formulas of Corollary 6.4 can also be written as

$$\sum_{r\geq 0} -\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q))x^r = \sum_{(\lambda^-,\lambda^+)\vdash n} (-1)^{|\lambda^-|}\frac{1}{T(\lambda^-,\lambda^+)}\frac{1}{1-xU(\lambda^-,\lambda^+)} \tag{6.5}$$

$$\sum_{r\geq 0} -\tilde{\chi}_{r+1}(p,\mathrm{Sp}_{2n}(\mathbf{F}_q))x^r = \sum_{(\lambda^-,\lambda^+)\vdash n} (-1)^{|\lambda^-|}\frac{1}{T(\lambda^-,\lambda^+)}\frac{1}{1-xU(\lambda^-,\lambda^+)_p} \tag{6.6}$$

where

$$T(\lambda^-,\lambda^+) = \prod_{d^-\in B(\lambda^-)} |C_{d^-}\wr \Sigma_{E(\lambda^-,d^-)}| \prod_{d^-\in B(\lambda^-)} |C_{d^-}\wr \Sigma_{E(\lambda^-,d^-)}|$$

$$U(\lambda^-,\lambda^+) = \prod_{d^-\in B(\lambda^-)} (q^{d^-}-1)^{E(\lambda^-,d^-)} \prod_{d^+\in B(\lambda^+)} (q^{d^+}+1)^{E(\lambda^+,d^+)}$$

for every partition $(\lambda^-,\lambda^+)$ of $n$ into parts of two kinds.

**Corollary 6.7** *Let* $\rho : W(C_n) = C_2\wr\Sigma_n \to W(A_n) = \Sigma_n$ *denote the projection with kernel* $C_2^n$*. Then*

$$1+\sum_{n\geq 1}\frac{x^n}{|W(C_n)|}\sum_{w\in W(C_n)} \det(\rho(w))\det(w)\det(q-w)^r = \prod_{\substack{0\leq j\leq r \\ j\equiv r \bmod 2}} (1-q^j x)^{\binom{r}{j}}$$

$$1+\sum_{n\geq 1}\frac{x^n}{|W(C_n)|}\sum_{w\in W(C_n)} \det(\rho(w))\det(w)\det(q-w)_p^r = \mathrm{FSp}_{r+1}(p,q,x)^{-1}$$

***Proof*** We find expresssions for the reciprocal power series $\mathrm{FSp}_{r+1}(q,x)^{-1}$ and $\mathrm{FSp}_{r+1}(p,q,x)^{-1}$. As in Corollary 6.2, we have

$$\begin{aligned}\mathrm{FSp}_{r+1}(q,x)^{-1} &= \exp\left(\sum_{n\geq 1} H(r,q^n)\frac{x^n}{n}\right)\\ &= \sum_{\lambda\vdash n} x^n \prod_{d\in B(\lambda)} \frac{((q^d-1)^r/2-(q^d+1)^r/2)^{E(\lambda,d)}}{|C_d \wr \Sigma_{E(\lambda,d)}|}\end{aligned}$$

and we can identify the coefficients of this power series as sums indexed by $W(C_n)$ as in the proof of Theorem 1.6. □

***Example 6.8*** Corollary 6.2 for $n=1,2,3$ shows

$$-\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q)) = \begin{cases} -H(r,q) & n=1\\ \frac{1}{2}H(r,q)^2-\frac{1}{2}H(r,q^2) & n=2\\ -\frac{1}{6}H(r,q)^3+\frac{1}{2}H(r,q)H(r,q^2)-\frac{1}{3}H(r,q^3) & n=3\end{cases}$$

where $H(r,q)=\frac{1}{2}(q-1)^r-\frac{1}{2}(q+1)^r$. Similar formulas hold for the $p$-primary equivariant Euler characteristics $-\tilde{\chi}_{r+1}(p,\mathrm{Sp}_{2n}(\mathbf{F}_q))$, $n=1,2,3$, where now $H(r,q)=\frac{1}{2}(q-1)_p^r-\frac{1}{2}(q+1)_p^r$. With fixed $n=1,2$, Theorem 1.6 (in the formulation of (6.3)) shows that

$$\begin{aligned}&-\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q))\\ &= \begin{cases}\frac{1}{2}(q+1)^r-\frac{1}{2}(q-1)^r & n=1\\ \frac{1}{8}(q-1)^{2r}+\frac{1}{8}(q+1)^{2r}+\frac{1}{4}(q^2+1)^r-\frac{1}{4}(q^2-1)^r-\frac{1}{4}(q-1)^r(q+1)^r & n=2\end{cases}\end{aligned}$$

and with fixed $r=1,2,3$ it shows that

$$\frac{(-1)^n}{|W(C_n)|}\sum_{w\in W(C_n)}\det(w)\det(q-w)^r = \begin{cases}1 & r=1\\ nq^{n-1} & r=2\\ \sum_{0\leq j\leq n}\binom{2+j}{2}q^{2j} & r=3\end{cases}$$

for all $n\geq 1$. From Corollary 6.4 (in the formulation of (6.5)) for $n=1,2$ we get

$$\begin{aligned}&\sum_{r\geq 0}-\tilde{\chi}_{r+1}(\mathrm{Sp}_{2n}(\mathbf{F}_q))x^r\\ &= \begin{cases}\frac{\frac{1}{2}}{1-(q+1)x}-\frac{\frac{1}{2}}{1-(q-1)x} & n=1\\ \frac{\frac{1}{8}}{1-x(q-1)^2}+\frac{\frac{1}{8}}{1-x(q+1)^2}+\frac{\frac{1}{4}}{1-x(q^2+1)}-\frac{\frac{1}{4}}{1-x(q^2-1)}-\frac{\frac{1}{4}}{1-x(q-1)(q+1)} & n=2\end{cases}\end{aligned}$$

In the $p$-primary case, when $p=2$ and $q=3^{2^e}$ with $e>0$, $(q-1)_2=2^{2+e}$, $(q+1)_2=2$, $(q^2-1)_2=2^{3+e}$, $(q-1)_2(q+1)_2=2^{3+e}$, $(q^2+1)_2=2$, and we get

$$\sum_{r\geq 0} -\tilde{\chi}_{r+1}(2,\mathrm{Sp}_2(\mathbf{F}_{3^{2^e}}))x^r = \frac{\frac{1}{2}}{1-2x} - \frac{\frac{1}{2}}{1-2^{2+e}x}$$

$$\sum_{r\geq 0} -\tilde{\chi}_{r+1}(2,\mathrm{Sp}_4(\mathbf{F}_{3^{2^e}}))x^r = \frac{\frac{1}{8}}{1-4^{2+e}x} + \frac{\frac{1}{8}}{1-4x} + \frac{\frac{1}{4}}{1-2x} - \frac{\frac{1}{4}}{1-2^{3+e}x} - \frac{\frac{1}{4}}{1-2^{3+e}x}$$

from Corollary 6.4 (in the formulation of (6.6)). Corollary 6.7 with $r=1,2$ and Example 5.19 show that

$$1+\sum_{n\geq 1}\frac{x^n}{|W(C_n)|}\sum_{w\in W(C_n)}\det(\rho(w))\det(w)\det(q-w)^r = \begin{cases} 1-x & r=1 \\ 1-2qx+x^2 & r=2 \end{cases}$$

$$1+\sum_{n\geq 1}\frac{x^n}{|W(C_n)|}\sum_{w\in W(C_n)}\det(\rho(w))\det(w)\det(3-w)_2^r = \begin{cases} \dfrac{1-x}{\prod_{n\geq 0} Q(x^{2^{n+1}})} & r=1 \\ \dfrac{(1-x)^2Q(x)^2}{\prod_{n\geq 0} Q(x^{2^{n+1}})^{2^{n+3}}} & r=2 \end{cases}$$

where $Q(x)=\frac{1-x}{1+x}$. Consequently, $\sum_{w\in W(C_n)}\det(\rho(w))\det(w)\det(q-w)^r=0$ for all $n>r$ if $r=1,2$.

## 7 Polynomial Identities for Partitions into Parts of Two Kinds

For any polynomial sequence $S$ and any rational number $m$ the $T_{mS}$-transform of $1\pm x$ is [20, Lemma 7.1]

$$T_{mS}(1\pm x)=\prod_{d\geq 1}(1\pm x^d)^{mS(d)(q)}=\prod_{d\geq 1}\sum_E\binom{mS(d)(q)}{\pm E}x^{dE}=\sum_{n\geq 0}x^n\sum_{\lambda\vdash n}\prod_{d\in B(\lambda)}\binom{mS(d)(q)}{\pm E(\lambda,d)}$$

The classical polynomial identity $T_{\mathrm{IM}(q)}(1-x)=\frac{1-qx}{1-x}$ gives the polynomial identity

$$\sum_{n\geq 0}x^n\sum_{\lambda\vdash n}\prod_{d\in B(\lambda)}\binom{m\mathrm{IM}_d(q)}{-E(\lambda,d)}=\left(\frac{1-qx}{1-x}\right)^m$$

for partitions. The cases $m=\pm 1$ are Thévenaz' polynomial identities [27, Theorems A–B] [20, Corollary 7.2].

The identities $T_{-\mathrm{SRIM}^-(q)}(1+x)T_{-\mathrm{SRIM}^+(q)}(1-x)=1-x$ and $T_{\mathrm{SRIM}^-(q)}(1-x)T_{-\mathrm{SRIM}^-(q)}(1+x)=\frac{1-qx}{1-x}$ (for odd $q$) of Lemma 4.9 translate into the following polynomial identities

$$\sum_{n\geq 0} x^n \sum_{(\lambda^-,\lambda^+)\vdash n} \prod_{d^-\in B(\lambda^-)} \binom{-m\mathrm{SRIM}^-_{d^-}(q)}{E(\lambda^-,d^-)} \prod_{d^+\in B(\lambda^+)} \binom{-m\mathrm{SRIM}^+_{d^+}(q)}{-E(\lambda^+,d^+)} = (1-x)^m$$

$$\sum_{n\geq 0} x^n \sum_{(\lambda^-,\lambda^+)\vdash n} \prod_{d^-\in B(\lambda^-)} \binom{m\mathrm{SRIM}^-_{d^-}(q)}{-E(\lambda^-,d^-)} \prod_{d^+\in B(\lambda^+)} \binom{-m\mathrm{SRIM}^-_{d^+}(q)}{E(\lambda^+,d^+)}$$
$$= \begin{cases} \left(\dfrac{1-qx}{1-x}\right)^m & 2\nmid q \\ (1-qx)^m & 2\mid q \end{cases}$$

for partitions into parts of two kinds.

***Example 7.1*** Based on the partitions, $\{(1^1,\emptyset),(\emptyset,1^1)\}$ and $\{(2^1,\emptyset),(1^2,\emptyset),(1^1,1^1),(\emptyset,1^2),(\emptyset,2^1)\}$, of 1 and 2 into parts of two kinds, we have the identities, valid for any rational number $m$,

$$\binom{m\mathrm{SRIM}^+_1(q)}{-1} + \binom{m\mathrm{SRIM}^-_1(q)}{1} = -\binom{-m}{1}$$

$$\binom{m\mathrm{SRIM}^+_2(q)}{-1} + \binom{m\mathrm{SRIM}^+_1(q)}{-2} + \binom{m\mathrm{SRIM}^-_1(q)}{1}\binom{m\mathrm{SRIM}^+_1(q)}{-1} + \binom{m\mathrm{SRIM}^-_2(q)}{1} + \binom{m\mathrm{SRIM}^-_1(q)}{2} = \binom{-m}{2}$$

$$\binom{m\mathrm{SRIM}^-_1(q)}{-1} + \binom{-m\mathrm{SRIM}^-_1(q)}{1} = \begin{cases} -mq+m & 2\nmid q \\ -mq & 2\mid q \end{cases}$$

$$\binom{m\mathrm{SRIM}^-_2(q)}{-1} + \binom{m\mathrm{SRIM}^-_1(q)}{-2} + \binom{m\mathrm{SRIM}^-_1(q)}{-1}\binom{-m\mathrm{SRIM}^-_1(q)}{1} + \binom{-m\mathrm{SRIM}^-_1(q)}{2} + \binom{-m\mathrm{SRIM}^-_2(q)}{1}$$
$$= \begin{cases} \binom{m}{2}q^2 - m^2q + \binom{m+1}{2} & 2\nmid q \\ \binom{m}{2}q^2 & 2\mid q \end{cases}$$

by comparing coefficients of $x^n$ for $n = 1, 2$.

## Appendix A. Equivariant Euler Characteristics of Posets

This appendix contains a few elementary observations about equivariant Euler characteristics for group actions on posets.

Let $S$ be a finite set and $\dim : S \to \mathbf{Z}$ a function associating an integer $\geq -1$ to every element of $S$. The Euler characteristic and the reduced Euler characteristic of the graded set $(S, \dim)$ are the alternating sums

$$\chi(S,\dim) = \sum_{d\geq 0}(-1)^d|\dim^{-1}(d)|$$
$$\tilde{\chi}(S,\dim) = \sum_{d\geq -1}(-1)^d|\dim^{-1}(d)| = \chi(S,\dim) - |\dim^{-1}(-1)|$$

of the numbers of $d$-dimensional elements of $S$ for $d\geq 0$ or $d\geq -1$.

Let $\Pi$ be a finite poset. A simplex in $\Pi$ is a totally ordered subset of $\Pi$. The set $|\Pi|$ of all simplices in $\Pi$ (including the empty simplex) is graded by the function $\dim : |\Pi| \to \mathbf{Z}$ taking a simplex $\sigma\subseteq\Pi$ to one less than its cardinality, $\dim\sigma = |\sigma| - 1$.

**Definition A.1** The Euler characteristic of the poset $\Pi$ is $\chi(\Pi) = \chi(|\Pi|,\dim)$ and the reduced Euler characteristic is $\tilde{\chi}(\Pi) = \tilde{\chi}(|\Pi|,\dim) = \chi(\Pi) - 1$.

Let $G$ be a finite group. Write $\mathrm{Hom}(\mathbf{Z}^r, G)$ for the set of homomorphisms of $\mathbf{Z}^r$ to $G$ and $\mathrm{Hom}(\mathbf{Z}^r, G)/G$ for the set of conjugacy classes of such homomorphisms. Equivalently, $\mathrm{Hom}(\mathbf{Z}^r, G)$ is the set of commuting $r$-tuples of elements in $G$ and $\mathrm{Hom}(\mathbf{Z}^r, G)/G$ is the set of conjugacy classes of commuting $r$-tuples.

Suppose now that $G$ acts on the poset $\Pi$ through order preserving bijections. For any subset $X$ of $G$, let $C_\Pi(X) = \{u\in\Pi | \forall g\in X : u^g = u\}$ denote the full subposet of elements of $\Pi$ fixed under the action from $X$. For any prime number $p$, let $Z_p^r = \mathbf{Z}\times\mathbf{Z}_p^{r-1}$ where $\mathbf{Z}_p$ is the abelian group of $p$-adic integers.

**Definition A.2** [2, 25] The $r$th, $r\geq 1$, ($p$-primary) equivariant Euler characteristic and ($p$-primary) reduced equivariant Euler characteristic of the $G$-poset $\Pi$ are

$$\chi_r(\Pi, G) = \frac{1}{|G|}\sum_{X\in\mathrm{Hom}(\mathbf{Z}^r,G)}\chi(C_\Pi(X(\mathbf{Z}^r)))$$
$$\chi_r(p,\Pi, G) = \frac{1}{|G|}\sum_{X\in\mathrm{Hom}(Z_p^r,G)}\chi(C_\Pi(X(Z_p^r)))$$
$$\tilde{\chi}_r(\Pi, G) = \frac{1}{|G|}\sum_{X\in\mathrm{Hom}(\mathbf{Z}^r,G)}\tilde{\chi}(C_\Pi(X(\mathbf{Z}^r)))$$
$$\tilde{\chi}_r(p,\Pi, G) = \frac{1}{|G|}\sum_{X\in\mathrm{Hom}(Z_p^r,G)}\tilde{\chi}(C_\Pi(X(Z_p^r)))$$

We note that $\tilde{\chi}_1(p,\Pi,G) = \tilde{\chi}_1(\Pi,G)$ for all primes $p$ since $Z_p^1 = \mathbf{Z}$. Also, $\chi_r(\Pi,G) = \tilde{\chi}_r(\Pi,G) + |\mathrm{Hom}(\mathbf{Z}^r,G)|/|G|$ and $\chi_r(p,\Pi,G) = \tilde{\chi}_r(p,\Pi,G) + |\mathrm{Hom}(Z_p^r,G)|/|G|$. The numbers of conjugacy classes of $r$-tuples of commuting elements of $G$ and commuting $p$-power order elements are

$$|\mathrm{Hom}(\mathbf{Z}^{r-1},G)/G| = |\mathrm{Hom}(\mathbf{Z}^{r},G)|/|G|,$$
$$|\mathrm{Hom}(\mathbf{Z}_p^{r-1},G)/G| = |\mathrm{Hom}(Z_p^{r},G)|/|G| \qquad (r \geq 1)$$

as $|\mathrm{Hom}(K,G)/G| = |\mathrm{Hom}(\mathbf{Z}\times K,G)|/|G|$ for any group $K$ [10, Lemma 4.13].

The equivariant Euler characteristics satisfy a recurrence relation.

**Lemma A.3** *For all* $r \geq 1$,

$$\begin{aligned}\tilde{\chi}_{r+1}(\Pi,G) &= \sum_{X\in\mathrm{Hom}(\mathbf{Z},G)/G} \tilde{\chi}_r(C_\Pi(X),C_G(X))\\ &= \sum_{X\in\mathrm{Hom}(\mathbf{Z}^r,G)/G} \tilde{\chi}_1(C_\Pi(X),C_G(X))\\ \tilde{\chi}_{r+1}(p,\Pi,G) &= \sum_{X\in\mathrm{Hom}(\mathbf{Z}_p,G)/G} \tilde{\chi}_r(p,C_\Pi(X),C_G(X))\\ &= \sum_{X\in\mathrm{Hom}(\mathbf{Z}_p^r,G)/G} \tilde{\chi}_1(C_\Pi(X),C_G(X))\end{aligned}$$

*and similar formulas are true for* $\chi_{r+1}(\Pi,G)$ *and* $\chi_{r+1}(p,\Pi,G)$.

***Proof*** A little more generally, we consider $\tilde{\chi}_{r_1+r_2}(p,\Pi,G)$ for $r_1 \geq 1$ and $r_2 \geq 2$. The $p$-primary equivariant Euler characteristic is

$$\begin{aligned}\tilde{\chi}_{r_1+r_2}(p,\Pi,G) &= \frac{1}{|G|}\sum_{X\in\mathrm{Hom}(\mathbf{Z}\times\mathbf{Z}_p^{r_1+r_2-1},G)} \tilde{\chi}(C_\Pi(X))\\ &= \frac{1}{|G|}\sum_{X_1\in\mathrm{Hom}(\mathbf{Z}_p^{r_1},G)}\ \sum_{X_2\in\mathrm{Hom}(\mathbf{Z}\times\mathbf{Z}^{r_2-1},C_G(X_1))} \tilde{\chi}(C_{C_\Pi(X_1)}(X_2))\\ &= \frac{1}{|G|}\sum_{X_1\in\mathrm{Hom}(\mathbf{Z}_p^{r_1},G)} |C_G(X_1)|\,\tilde{\chi}_{r_2}(p,C_\Pi(X_1),C_G(X_1))\\ &= \sum_{X_1\in\mathrm{Hom}(\mathbf{Z}_p^{r_1},G)/G} \tilde{\chi}_{r_2}(p,C_\Pi(X_1),C_G(X_1))\end{aligned}$$

where we use that the conjugacy class of $X_1(\mathbf{Z}_p^{r_1})$ contains $|G : C_G(X_1)|$ elements. □

The set $|C_\Pi(X)|/C_G(X)$ of $C_G(X)$-orbits of $C_\Pi(X)$-simplices, for any $X \subseteq G$, has Euler characteristic relative to the dimension function induced by $\dim : |\Pi| \to \{-1,0,1,\dots\}$, $\dim\sigma = |\sigma| - 1$.

**Lemma A.4** *For all* $r \geq 0$,

$$\begin{aligned}\tilde{\chi}_{r+1}(\Pi,G) &= \sum_{X\in\mathrm{Hom}(\mathbf{Z}^r,G)/G} \tilde{\chi}(|C_\Pi(X)|/C_G(X)),\\ \tilde{\chi}_{r+1}(p,\Pi,G) &= \sum_{X\in\mathrm{Hom}(\mathbf{Z}_p^r,G)/G} \tilde{\chi}(|C_\Pi(X)|/C_G(X))\end{aligned}$$

***Proof*** We first consider the case $r = 0$. The orbit counting formula shows that

$$\begin{aligned}\tilde{\chi}(|\Pi|/G) &= \sum_{d \geq -1} (-1)^d |\dim^{-1}(d)/G| \\ &= \frac{1}{|G|} \sum_{d \geq -1} (-1)^d \sum_{g \in G} ||C_\Pi(g)| \cap \dim^{-1}(d)| \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{d \geq -1} (-1)^d ||C_\Pi(g)| \cap \dim^{-1}(d)| \\ &= \frac{1}{|G|} \sum_{g \in G} \tilde{\chi}(C_\Pi(g)) = \tilde{\chi}_1(\Pi, G)\end{aligned}$$

Consequently, for all $r \geq 1$,

$$\begin{aligned}\tilde{\chi}_{r+1}(\Pi, G) &= \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)/G} \tilde{\chi}_1(C_\Pi(X), C_G(X)) \\ &= \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)/G} \tilde{\chi}(|C_\Pi(X)|/C_G(X)) \\ \tilde{\chi}_{r+1}(p, \Pi, G) &= \sum_{X \in \mathrm{Hom}(\mathbf{Z}_p^r, G)/G} \tilde{\chi}_1(C_\Pi(X), C_G(X)) \\ &= \sum_{X \in \mathrm{Hom}(\mathbf{Z}_p^r, G)/G} \tilde{\chi}(|C_\Pi(X)|/C_G(X))\end{aligned}$$

by Lemma A.3. □

It is clear from Lemma A.4, but maybe not from Definition A.2, that all ($p$-primary) equivariant Euler characteristics are integers.

## Appendix B. Eulerian Functions of Groups

Let $G$ be a finite group acting on a finite poset $\Pi$. For any natural number $r \geq 1$, the $r$th reduced equivariant Euler characteristic (Defintion A.1) and the $p$-primary $r$th equivariant reduced Euler characteristic are

$$\tilde{\chi}_r(\Pi, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(\mathbf{Z}^r, G)} \tilde{\chi}(C_\Pi(X)) = \frac{1}{|G|} \sum_{B \leq G} \varphi_{\mathbf{Z}^r}(B)\, \tilde{\chi}(C_\Pi(B)) \tag{B.1}$$

$$\tilde{\chi}_r(p, \Pi, G) = \frac{1}{|G|} \sum_{X \in \mathrm{Hom}(Z_p^r, G)} \tilde{\chi}(C_\Pi(X)) = \frac{1}{|G|} \sum_{B \leq G} \varphi_{Z_p^r}(B)\, \tilde{\chi}(C_\Pi(B)) \tag{B.2}$$

where $\varphi_{\mathbf{Z}^r}(B)$ ($\varphi_{Z_p^r}(B)$) is the number of epimorphisms of the abelian group $\mathbf{Z}^r$ ($Z_p^r = \mathbf{Z} \times \mathbf{Z}_p^{r-1}$) onto the subgroup $B$ of $G$. In this appendix, we recall some of the properties, helpful for concrete computer assisted calculations of equivariant Euler characteristics, of the eulerian function $\varphi_{\mathbf{Z}^r}(B)$ [9].

For any finite group $B$, let $\mathrm{Hom}(\mathbf{Z}^r, B)$ and $\mathrm{Epi}(\mathbf{Z}^r, B)$ be the set of homomorphisms or epimorphisms of $\mathbf{Z}^r$ to $B$. Then $\mathrm{Hom}(\mathbf{Z}^r, B) = \coprod_{A \leq B} \mathrm{Epi}(\mathbf{Z}^r, A)$ and $\varphi_{\mathbf{Z}^r}(B) = |\mathrm{Epi}(\mathbf{Z}^r, B)|$. (When $r = 1$ and $C_n$ is cyclic of order $n$, $\varphi_{\mathbf{Z}^1}(C_n)$ is Euler's totient function $\varphi(n)$.) We observe that $\varphi_{\mathbf{Z}^r}$ is multiplicative.

**Lemma B.3** *Let $B_1$ and $B_2$ be two finite groups of coprime order.*

(1) *For any subgroup $A$ of $B_1 \times B_2$, $A = A_1 \times A_2$ where $A_i$ is the image of $A$ under the projection $B_1 \times B_2 \to B_i$, $i = 1, 2$.*
(2) $\varphi_{\mathbf{Z}^r}(B_1 \times B_2) = \varphi_{\mathbf{Z}^r}(B_1) \times \varphi_{\mathbf{Z}^r}(B_2)$ *for any $r \geq 1$*

***Proof*** Let $g_i$ be the order of $B_i$, $i = 1, 2$. The order of $A$, which divides $g_1 g_2$, is of the form $k_1 k_2$ where $k_1$ divides $g_1$ and $k_2$ divides $g_2$. The order of $A_i$ divides $k_1 k_2$ and $g_i$. Thus $|A_i|$ divides $k_i$. It follows that the order of $A_1 \times A_2$ divides the order of $A$. But $A$ is a subgroup of $A_1 \times A_2$ so $|A| = |A_1 \times A_2|$ and $A = A_1 \times A_2$. □

Next, we compute $\varphi_{\mathbf{Z}^r}(C_p^d)$ where $C_p^d$ is elementary abelian of order $p^d$. First, $\mathrm{Epi}(\mathbf{Z}^r, C_p^d) = \mathrm{Epi}(C_p^r, C_p^d)$, the set of epimorphisms of $C_p^r$ onto $C_p^d$. Next, note that there is a bijection between the orbit set $\mathrm{Epi}(C_p^r, C_p^d)/\mathrm{Aut}(C_p^d)$ and the set of $(r-d)$-dimensional subspaces of $\mathbf{F}_p^r$ (kernels of epimorphisms). The number of such subspaces is the Gaussian binomial coefficient $\binom{r}{r-d}_p = \binom{r}{d}_p$ [24, Proposition 1.3.18]. Thus

$$\varphi_{\mathbf{Z}^r}(C_p^d) = |\mathrm{Epi}(\mathbf{Z}^r, C_p^d)| = \binom{r}{d}_p |\mathrm{GL}_d^+(\mathbf{F}_p)| = \prod_{j=0}^{d-1} (p^r - p^j) \tag{B.3}$$

In the general case, the number of homomorphism of $\mathbf{Z}^r$ to $B$ is

$$|\mathrm{Hom}(\mathbf{Z}^r, B)| = \sum_{A \leq B} |\mathrm{Epi}(\mathbf{Z}^r, A)| = \sum_{A \leq G} |\mathrm{Epi}(\mathbf{Z}^r, A)| \zeta(A, B)$$

where $\zeta(A, B) = 1$ if $A \leq B$ and $\zeta(A, B) = 0$ otherwise. The number of epimorphism of $\mathbf{Z}^r$ onto $B$ is

$$\varphi_{\mathbf{Z}^r}(B) = |\mathrm{Epi}(\mathbf{Z}^r, B)| = \sum_{A \leq G} |\mathrm{Hom}(\mathbf{Z}^r, A)| \mu(A, B)$$

by Möbius inversion. Of course, $\varphi_{\mathbf{Z}^r}(B) > 0$ if and only if $B$ is abelian and generated by $r$ of its elements. Assuming $B$ *is* abelian, $|\mathrm{Hom}(\mathbf{Z}^r, A)| = |A|^r$ for any $A \leq B$ so that [7, 9, 30]

$$\varphi_{\mathbf{Z}^r}(B) = |\mathrm{Epi}(\mathbf{Z}^r, B)| = \sum_{A \leq B} |A|^r \mu(A, B)$$

The Möbius function $\mu(A, B) = 0$ unless $\Phi(B) \leq A \leq B$ and then $\mu_B(A, B) = \mu_{B/\Phi(B)}(A/\Phi(B), B/\Phi(B))$ where $\Phi(B)$ is the Frattini subgroup [7]. Therefore

| $A$ | $C_1$ | $C_3$ | $C_3$ | $C_5$ | $C_3 \times C_3$ |
|---|---|---|---|---|---|
| $\lvert G : C_G(A)\rvert$ | 1 | 20 | 20 | 36 | 10 |
| $\widetilde{\chi}(C_L(A))$ | $-16$ | 2 | 2 | $-1$ | $-1$ |
| $\varphi_{\mathbf{Z}^{r+1}}(A)$ | 1 | $3^{r+1}-1$ | $3^{r+1}-1$ | $5^{r+1}-1$ | $(3^{r+1}-1)(3^{r+1}-3)$ |
| $\varphi_{Z_3^{r+1}}(A)$ | 1 | $3^{r+1}-1$ | $3^{r+1}-1$ | 4 | $(3^{r+1}-1)(3^{r+1}-3)$ |
| $\varphi_{Z_5^{r+1}}(A)$ | 1 | 2 | 2 | $5^{r+1}-1$ | 0 |

**Fig. 2** Abelian $2'$-subgroups of $\mathrm{Sp}_4(\mathbf{F}_2)$

$$\varphi_{\mathbf{Z}^r}(B) = \sum_{A \le B} |A|^r \mu_B(A,B) = |\Phi(B)|^r \sum_{A \le B/\Phi(B)} |A|^r \mu_{B/\Phi(B)}(A, B/\Phi(B))$$
$$= |\Phi(B)|^r \varphi_{\mathbf{Z}^r}(B/\Phi(B))$$

The abelian group $B$ is the product, $B = \prod B_p$, of its Sylow $p$-subgroups, $B_p$. By multiplicativity (Lemma B.3.(2)),

$$\varphi_{\mathbf{Z}^r}(B) = \prod_p \varphi_{\mathbf{Z}^r}(B_p)$$

The Frattini quotient $B_p/\Phi(B_p)$ is an elementary abelian $p$-group of order, say, $p^d$. We conclude that

$$\varphi_{\mathbf{Z}^r}(B_p) = |\Phi(B_p)|^r |\mathrm{Epi}(\mathbf{Z}^r, C_p^d)| \overset{(\mathrm{B}.4)}{=} |\Phi(B_p)|^r \prod_{j=0}^{d-1}(p^r - p^j)$$
$$= |B_p|^r \prod_{j=0}^{d-1}(1 - p^{j-r})$$

For the final equality, use that if $B_p$ has order $p^m$, then the order of the Frattini subgroup is $p^{m-d}$ so that $|\Phi(B_p)|^r = p^{r(m-d)}$.

For a prime $p$, recall that $Z_p^r = \mathbf{Z} \times \mathbf{Z}_p^{r-1}$. In particular, $Z_p^1 = \mathbf{Z}$ is independent of $p$. The number of epimorphisms of $Z_p^r$ onto $B$ is

$$\varphi_{Z_p^r}(B) = \varphi_{Z_p^r}\left(\prod_s B_s\right) = \prod_s \varphi_{Z_p^r}(B_s) = \varphi_{\mathbf{Z}^r}(B_p) \prod_{s \ne p} \varphi_{\mathbf{Z}}(B_s)$$

where $B_s$ is the Sylow $s$-subgroup of $B$. Here, $\varphi_{\mathbf{Z}}(B_s) = |B_s|(1 - p^{-1})$ if $B_s$ is cyclic and $\varphi_{\mathbf{Z}}(B_s) = 0$ otherwise. Thus $\varphi_{Z_q^r}(B) > 0$ if and only if $B_q$ can be generated by $r$ of its elements and $B_s$ is cyclic for all primes $s \ne q$.

***Example B.5*** The symplectic group $G = \mathrm{Sp}_4(\mathbf{F}_2)$, of order 720, acts on the discrete poset $L = \mathrm{L}_4^*(\mathbf{F}_2)$ of 30 totally isotropic subspaces. Equation (B.1) with the data of Figure 2, found with the help of the computer algebra system Magma [3], shows that

$$\begin{aligned}
-\tilde{\chi}_{r+1}(L,G) &= -\frac{1}{720}(-16+80(3^{r+1}-1)-36(5^{r+1}-1)-10(3^{r+1}-1)(3^{r+1}-3)) \\
&= \frac{1}{8}(3^r-1)^2-\frac{1}{4}(3^r-5^r) \\
-\tilde{\chi}_{r+1}(3,L,G) &= -\frac{1}{720}(-16+80(3^{r+1}-1)-36\cdot 4-10(3^{r+1}-1)(3^{r+1}-3)) \\
&= \frac{1}{8}(3^r-1)^2-\frac{1}{4}(3^r-1) \\
-\tilde{\chi}_{r+1}(5,L,G) &= -\frac{1}{720}(-16+160-36(5^{r+1}-1)) = \frac{1}{4}(5^r-1)
\end{aligned}$$

in accordance with Example 6.8. By Lemma 4.2, in (B.1) we only need abelian subgroups of $G$ of order prime to 2.

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