ECE 301: Signals and Systems
Homework Assignment #2

Due on September 30, 2015

Professor: ***Aly El Gamal***
TA: ***Xianglun Mao***

Problem 1

Show that causality for a continuous-time linear system is equivalent to the following statement:

For any time t_0 and any input $x(t)$ such that $x(t) = 0$ for $t < t_0$, the corresponding output $y(t)$ must also be zero for $t < t_0$.

Solution

Assumption: If $x(t) = 0$ for $t < t_0$, then $y(t) = 0$ for $t < t_0$. To prove that: The system is causal.
Let us consider an arbitrary signal $x_1(t)$. Then, let us consider another signal $x_2(t)$ which is the same as $x_1(t)$ for $t < t_0$. But for $t > t_0$, $x_2(t) \neq x_1(t)$. Since the system is linear,

$$x_1(t) - x_2(t) \to y_1(t) - y_2(t).$$

Since $x_1(t) - x_2(t) = 0$ for $t < t_0$, by our assumption $y_1(t) - y_2(t) = 0$ for $t < t_0$. This implies that $y_1(t) = y_2(t)$ for $t < t_0$. In other words, the output is not affected by input values for $t \geq t_0$. Therefore, the system is causal.

Assumption: The system is causal. To prove that: If $x(t) = 0$ for $t < t_0$, then $y(t) = 0$ for $t < t_0$.
Let us assume that the signal $x(t) = 0$ for $t < t_0$. Then we may express $x(t)$ as $x(t) = x_1(t) - x_2(t)$, where $x_1(t) = x_2(t)$ for $t < t_0$. Since the system is linear, the output to $x(t)$ will be $y(t) = y_1(t) - y_2(t)$. Now, since the system is causal, $x_1(t) = x_2(t)$ for $t < t_0$ implies that $y_1(t) = y_2(t)$ for $t < t_0$. Therefore, $y(t) = 0$ for $t < t_0$.

Problem 2

The initial rest assupmtion corresponds to a zero-valued auxiliary condition being imposed at a time determined in accordance with the input signal. In this problem we show that if the auxiliary condition used is nonzero or if it is always applied at a fixed time (regardless of the input signal) the corresponding system cannot be LTI. Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation:

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \tag{1}$$

(a) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not linear.

(b) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not time invariant.

(c) Given the auxiliary condition $y(1) = 1$, show that the system is incrementally linear.

(d) Given the auxiliary condition $y(1) = 0$, show that the system is linear but not time invariant.

(e) Given the auxiliary condition $y(0) + y(4) = 0$, show that the system is linear but not time invariant.

Solution

(a) Consider $x_1(t) \xrightarrow{S} y_1(t)$ and $x_2(t) \xrightarrow{S} y_2(t)$. We know that $y_1(1) = y_2(1) = 1$. Now consider a third input to the system which is $x_3(t) = x_1(t) + x_2(t)$. Let the corresponding output be $y_3(t)$. Now, note that $y_3(1) = 1 \neq y_1(1) + y_2(1)$. Therefore, the system is not linear. A specific example follows: Consider an input signal $x_1(t) = e^{2t}u(t)$, the corresponding output for $t > 0$ is

$$y_1(t) = \frac{1}{4}e^{2t} + Ae^{-2t}.$$

Using the fact that $y_1(1) = 1$, we get for $t > 0$

$$y_1(t) = \frac{1}{4}e^{2t} + (1 - \frac{e}{4})e^{-2(t-1)}$$

Now, consider a second signal $x_2(t) = 0$. Then, the corresponding output is

$$y_2(t) = Be^{-2t}$$

for $t > 0$. Using the fact that $y_2(1) = 1$, we get for $t > 0$

$$y_2(t) = e^{-2(t-1)}.$$

Now consider a third signal $x_3(t) = x_1(t) + x_2(t) = x_1(t)$. Note that the output we get still be $y_3(t) = y_1(t)$ for $t > 0$. Clearly, $y_3(t) \neq y_1(t) + y_2(t)$ for $t > 0$. Therefore, the system is not linear.

(b) Again consider an input signal $x_1(t) = e^{2t}u(t)$. We know that the corresponding output for $t > 0$ with $y_1(1) = 1$ is

$$y_1(t) = \frac{1}{4}e^{2t} + (1 - \frac{e}{4})e^{-2(t-1)}.$$

Now, consider an input signal of the form $x_2(t) = x_1(t-T) = e^{2(t-T)}u(t-T)$. The output for $t > T$ is

$$y_2(t) = \frac{1}{4}e^{2(t-T)} + Ae^{-2t}.$$

Using the fact that $y_2(1) = 1$ and also assuming that $T < 1$, we get for $t > T$

$$y_2(t) = \frac{1}{4}e^{2(t-T)} + (1 - \frac{1}{4}e^{2(1-T)})e^{-2t}.$$

Now note that $y_2(t) \neq y_1(t-T)$ for $t > T$. Therefore, the system is not time invariant.

(c) In order to show that system is incrementally linear with the auxiliary condition specified as $y(1) = 1$, we need to first show that the system is linear with the axiliary condition specified as $y(1) = 0$.

For an input-output pair $x_1(t)$ and $y_1(t)$, we may use (1) and the initial rest condition to write

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t),\ y_1(1) = 0$$

For an input-output pair $x_2(t)$ and $y_2(t)$, we may use (1) and the initial rest condition to write

$$\frac{dy_2(t)}{dt} + 2y_2(t) = x_2(t),\ y_2(1) = 0$$

Scaling the first equation by α and second equation by β and summing, we get

$$\frac{d}{dt}\{\alpha y_1(t) + \beta y_2(t)\} + 2\{\alpha y_1(t) + \beta y_2(t))\} = \alpha x_1(t) + \beta x_2(t)$$

and

$$y_3(1) = y_1(1) + y_2(1) = 0$$

By inspection, it is clear that the output is $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ when the input is $x_3(t) = \alpha x_1(t) + \beta x_2(t)$. Furthermore, $y_3(1) = 0 = y_1(1) + y_2(1)$. Therefore, the system is linear.

Therefore, the overall system may be treated as the cascade of a linear system with an adder which adds the response of the system to the auxiliary conditions alone.

(d) In the previous part, we showed that the system is linear when $y(1) = 0$. In order to show that the system is not time invariant, consider an input of the form $x_1(t) = e^{2t}u(t)$. From part (a), we know that the corresponding output will be

$$y_1(t) = \frac{1}{4}e^{2t} + Ae^{-2t}.$$

Using the fact that $y_1(1) = 0$, we get for $t > 0$

$$y_1(t) = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2(t-2)}$$

Now consider an input of the form $x_2(t) = x_1(t - 1/2)$. Note that $y_2(1) = 0$. Clearly, $y_2(1) \neq y_1(1 - 1/2) = (1/4)(e - e^3)$. Therefore, $y_2(t) \neq y_1(t - 1/2)$ for all t. This implies that the system is not time invariant.

(e) A proof which is similar to the proof for linearity used in part (c) may be used here. We may show that the system is not time invariant by using the method outlined in part (d).

To prove the linearity, we may use the similar method outlined in part (c). For an input-output pair $x_1(t)$ and $y_1(t)$, we may use (1) and the initial rest condition to write

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t),\ y_1(0) + y_1(4) = 0$$

For an input-output pair $x_2(t)$ and $y_2(t)$, we may use (1) and the initial rest condition to write

$$\frac{dy_2(t)}{dt} + 2y_2(t) = x_2(t),\ y_2(0) + y_2(4) = 0$$

Problem 2 continued on next page...

Scaling the first equation by α and second equation by β and summing, we get

$$\frac{d}{dt}\{\alpha y_1(t) + \beta y_2(t)\} + 2\{\alpha y_1(t) + \beta y_2(t))\} = \alpha x_1(t) + \beta x_2(t)$$

and

$$y_3(0) + y_3(4) = y_1(0) + y_1(4) + y_2(0) + y_2(4) = 0$$

By inspection, it is clear that the output is $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ when the input is $x_3(t) = \alpha x_1(t) + \beta x_2(t)$. Furthermore, $y_3(0) + y_3(4) = 0 = y_1(0) + y_1(4) + y_2(0) + y_2(4)$. Therefore, the system is linear.

To show the system is not time invariant, we again consider an input of the form $x_1(t) = e^{2t}u(t)$. From part (a), we know that the corresponding output will be

$$y_1(t) = \frac{1}{4}e^{2t} + Ae^{-2t}.$$

Using the fact that $y_1(0) + y_1(4) = 0$, we get for $t > 0$

$$y_1(t) = \frac{1}{4}e^{2t} - \frac{1}{4} \cdot \frac{1+e^8}{1+e^{-8}}e^{-2t}$$

Now, consider an input signal of the form $x_2(t) = x_1(t-T) = e^{2(t-T)}u(t-T)$. The output for $t > T$,

$$y_2(t) = \frac{1}{4}e^{2(t-T)} + Ae^{-2t}.$$

Using the fact that $y_2(0) + y_2(4) = 0$ and also assuming that $T < 1$, we get for $t > T$

$$y_2(t) = \frac{1}{4}e^{2(t-T)} - \frac{1}{4} \cdot \frac{e^{2(4-T)} + e^{-2T}}{1+e^{-8}}e^{-2t}.$$

Now note that $y_2(t) \neq y_1(t-T)$ for $t > T$. therefore, The system is not time invariant.

Problem 3

Let

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 9 \\ 0, & \text{elsewhere} \end{cases} \quad \text{and } h[n] = \begin{cases} 1, & 0 \leq n \leq N \\ 0, & \text{elsewhere} \end{cases}$$

where $N \leq 9$ is an integer. Determine the value of N, given that $y[n] = x[n] * h[n]$ and

$$y[4] = 5, \; y[14] = 0$$

Solution

The signal $y[n]$ is

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

In this case, this summation reduces to

$$y[n] = \sum_{k=0}^{9} x[k]h[n-k] = \sum_{k=0}^{9} h[n-k]$$

From this it is clear that $y[n]$ is a summation of shifted replicas of $h[n]$. Since the last replica will begin at $n = 9$ and $h[n]$ is zero for $n > N$, $y[n]$ is zero for $n > N + 9$. Using this and the fact that $y[14] = 0$, we may conclude that N can *at most* be 4. Furthermore, since $y[4] = 5$, we can conclude that $h[n]$ has *at least* 5 non-zero points. The only value of N which satisfies both these conitions is 4.

Problem 4

One of the important properties of convolution, in both continuous and discrete time, is the associativity property. In this problem, we will check and illustrate this property.

(a) Prove the equality

$$[x(t) * h(t)] * g(t) = x(t) * [h(t) * g(t)] \tag{2}$$

by showing that both sides of (2) equal

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\delta)g(t-\delta-\tau)d\tau d\delta$$

(b) Consider two LTI systems with the unit sample responses $h_1[n]$ and $h_2[n]$ shown in Figure 1(a). These two systems are cascaded as shown in Figure 1(b). Let $x[n] = u[n]$.

(i) Compute $y[n]$ by first computing $w[n] = x[n] * h_1[n]$ and then computing $y[n] = w[n] * h_2[n]$; that is, $y[n] = [x[n] * h_1[n]] * h_2[n]$

(ii) Now find $y[n]$ by first convolving $h_1[n]$ and $h_2[n]$ to obtain $g[n] = h_1[n]*h_2[n]$ and then convolving $x[n]$ with $g[n]$ to obtain $y[n] = x[n] * (h_1[n] * h_2[n])$.

The answers to (i) and (ii) should be identical, illustrating the associativity property of discrete-time convolution.

(c) Consider the cascade of two LTI system as in Figure 1(b), where in this case

$$h_1[n] = sin(8n)$$

and

$$h_2[n] = a^n u[n], |a| < 1$$

and where the input is

$$x[n] = \delta[n] - a\delta[n-1]$$

Determine the output $y[n]$. (*Hint*: The use of the associative and communicative properties of convolution should greatly facilitate the solution.)

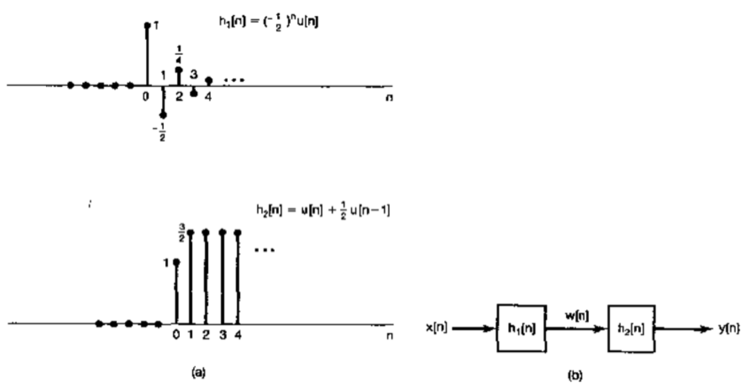

Figure 1: The discrete-time signal $h_1(t)$, $h_2(t)$ (a) and the cascaded system (b).

Solution

(a) We first have

$$[x(t) * h(t)] * g(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\delta' - \tau)g(t - \delta')d\tau d\delta'$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\delta)g(t - \delta - \tau)d\tau d\delta$$

Also,

$$x(t) * [h(t) * g(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t - \tau')h(\tau)g(\delta' - \tau)d\delta' d\tau$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\delta)h(\tau)g(t - \tau - \delta)d\tau d\delta$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\delta)g(t - \delta - \tau)d\tau d\delta$$

The equality is proved.

(b) (i) We first have

$$w[n] = u[n] * h_1[n] = \sum_{k=0}^{n}(-\frac{1}{2})^k = \frac{2}{3}[1 - (-\frac{1}{2})^{n+1}]u[n]$$

Now,

$$y[n] = w[n] * h_2[n] = (n + 1)u[n]$$

(ii) We first have

$$g[n] = h_1[n] * h_2[n] = \sum_{k=0}^{n}(-\frac{1}{2})^k + \frac{1}{2}\sum_{k=0}^{n-1}(-\frac{1}{2})^k = u[n]$$

Now,

$$y[n] = u[n] * g[n] = u[n] * u[n] = (n + 1)u[n]$$

The same result was obtained in both parts (i) and (ii).

(c) Note that

$$x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n].$$

Also note that

$$x[n] * h_2[n] = \alpha^n u[n] - \alpha^n u[n-1] = \delta[n].$$

Therefore,

$$x[n] * h_1[n] * h_2[n] = \delta[n] * sin(8n) = sin(8n).$$

Problem 5

In the text, we showed that if $h[n]$ is absolutely summable, i.e., if

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

then the LTI system with impulse response $h[n]$ is stable. This means that absolute summability is a *sufficient* condition for stability. In this problem, we shall show that it is also a *necessary* condition. Consider an LTI system with impulse response $h[n]$ that is not absolutely summable; that is,

$$\sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

(a) Suppose that the input to this system is

$$x[n] = \begin{cases} 0, & \text{if } h[-n] = 9 \\ \frac{h[-n]}{|h[-n]|}, & \text{if } h[-n] \neq 9 \end{cases}$$

Does this input signal represent a bounded input? If so, what is the smallest number B such that

$$|x[n]| \leq B \text{ for all } n?$$

(b) Calculate the output at $n = 0$ for this particular choice of input. Does the result prove the statement that absolute summability is a necessary condition for stability?

(c) In a similar fashion, show that a continuous-time LTI system is stable if any only if its impulse response is absolutely integrable.

Solution

(a) It is a bounded input. $|x[n]| \leq 1 = B_x$ for all n.

(b) Consider

$$\begin{aligned} y[0] &= \sum_{k=-\infty}^{\infty} x[-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} \frac{h^2[k]}{|h[k]|} \\ &= \sum_{k=-\infty}^{\infty} |h[k]| \to \infty \end{aligned}$$

Therefore, the output is not bounded. Thus, the system is not stable and absolute summability is necessary.

(c) Let

$$x[n] = \begin{cases} 0, & \text{if } h(-t) = 0 \\ \frac{h(-t)}{|h(-t)|}, & \text{if } h(-t) \neq 0 \end{cases}$$

Now, $|x(t)| \leq 1$ for all t. Therefore, $x(t)$ is a bounded input. Now,

$$\begin{aligned} y(0) &= \int_{-\infty}^{\infty} x(\tau)h(\tau)d\tau \\ &= \int_{-\infty}^{\infty} \frac{h^2(\tau)}{|h(\tau)|}d\tau \\ &= \int_{-\infty}^{\infty} |h(t)|dt = \infty \end{aligned}$$

Therefore, the system is unstable if the impulse response is not absolutely integrable.