[Hungerford] Section 4.3, #1. Find a monic associate of:

(a) $3x^3 + 2x^2 + x + 5$ in $\mathbb{Q}[x]$

(b) $3x^5 - 4x^2 + 1$ in $\mathbb{Z}_5[x]$

(c) $ix^3 + x - 1$ in $\mathbb{C}[x]$

Solution. In all these examples, the coefficient ring is a field, so you can just divide all coefficients by the leading coefficient. Doing so yields:

(a) $x^3 + \frac{2}{3}x^2 + \frac{1}{3}x + \frac{5}{3}$

(b) $x^5 - 3x^2 + 2$ or $x^5 + 2x^2 + 2$

(c) $x^3 - ix + i$ or $x^3 + \frac{1}{i}x - \frac{1}{i}$

[Hungerford] Section 4.3, #11 Show that $x^3 - 3$ is irreducible in $\mathbb{Z}_7[x]$.

Solution. We will do a proof by contradiction. So suppose that $x^3 - 3$ is reducible in $\mathbb{Z}_7[x]$. Then it can be factored as a product of two non-constant polynomials of degree less than 3. This leaves two possibilities:

- $x^3 - 3$ splits into 3 linear factors
- $x^3 - 3$ factors into a linear factor and a quadratic irreducible

In both cases, there must be a linear factor. But this would mean that there is a root $x^3 - 3$ in $\mathbb{Z}_7$, or equivalently, that $x^3 = 3$ for some $x \in \mathbb{Z}_7$. But we can easily check that this is not the case:

- $0^3 = 0$
- $1^3 = 1$
- $2^3 = 8 = 1$
- $3^3 = 27 = -1 = 6$
- $4^3 = (-3)^3 = -27 = 1$
- $5^3 = (-2)^3 = -8 = -1 = 6$
- $6^3 = (-1)^3 = -1 = 6$

[Hungerford] Section 4.3, #12 Express $x^4 - 4$ as a product of irreducibles in $\mathbb{Q}[x]$, $\mathbb{R}[x]$, and $\mathbb{C}[x]$.

Solution. In $\mathbb{Q}[x]$, we can factor $x^4 - 4$ using the "difference of squares" formula:

$$x^4 - 4 = (x^2 - 2)(x^2 + 2)$$

Both of these factors are irreducible in $\mathbb{Q}[x]$, since neither have rational roots. In $\mathbb{R}[x]$, however, the first factor has two roots $\pm\sqrt{2}$, so we can further factor it as

$$x^4 - 4 = (x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)$$

Finally, if we consider this as a polynomial in $\mathbb{C}[x]$, it will completely split into linear factors, since $\mathbb{C}$ is algebraically closed. Namely, $x^2 + 2$ has two roots $\pm i\sqrt{2}$, so we get

$$x^4 - 4 = (x - \sqrt{2})(x + \sqrt{2})(x - i\sqrt{2})(x + i\sqrt{2})$$