Chapter One

Vector Analysis

1. Definitions:

2. **A scalar quantity:** is a quantity which is completely characterized by its magnitude. Examples of physical quantities that are scalar are mass, time, temperature, volume, and work.
3. **A vector quantity:** is a quantity which is completely characterized by its magnitude and direction. Force, velocity, displacement, and acceleration are examples of vector quantities. A vector can be represented geometrically by an arrow whose direction is approximately chosen and whose length is proportional to the magnitude of the vector.
4. **Field:** If at each point of a region there is a corresponding value of some physical function, the region is called a field. Fields may be classified as either scalar or vector, depending upon the type of function involved.

If the value of the physical function at each point is a scalar quantity, then the field is a scalar function. The temperature of the atmosphere and density of a nonhomogeneous body are examples of scalar fields.

When the value of the function at each point is a vector quantity, the field is a vector field. The wind velocity of the atmosphere, the force of gravity on a mass in space, and the force on a charged body placed in an electric field, are examples of vector fields.

2. Vector Algebra

For this purpose a three-dimensional Cartesian coordinate system introduced to represent the vector. The variables of this system are x, y, z.

The sum of two vectors is defined as the vector whose components are the sums of the corresponding components of the original vectors.

$$\vec{C} = \vec{A} + \vec{B} \quad (1)$$

$$C_x = A_x + B_x \quad , \quad C_y = A_y + B_y \quad , C_z = A_z + B_z \quad (2)$$

When the order of the operation may be reversed with no effect on the result, the operation is said to obey the commutative law:

$$\vec{A} + \vec{B} = \vec{B} + \vec{A} \quad (3)$$

The operation of subtraction is defined as the addition of the negative. This is written as

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \qquad (4)$$

The vector addition and subtraction are associative. In vector notation this appears as

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C} = (\vec{A} + \vec{C}) + \vec{B} = \vec{A} + \vec{B} + +\vec{C} \qquad (5)$$

In other words, the parentheses are not needed.

Multiplication of a scalar and vector: When a vector is multiplied by a scalar,a new vector is produced whose direction is the same as the original vector and whose magnitude is the product of the magnitudes of the vectors and scalars. Thus,

$$\vec{C} = a\vec{B} \qquad (6)$$

$a\vec{B}$ is a vector $\vec{C}$, their components are

$$C_x = aB_x, C_y = aB_y, \quad C_z = aB_z$$

If $\vec{B}$ is a vector field and a a scalar field then $\vec{C}$ is a new vector field.

A three-dimensional vector is completely described by its projections on the x, y, z axes.

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

Where A_x, A_y, and A_z are the magnitudes of the projections of the vector on the x, y, and z axes respectively, and $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are unit vectors in the direction of the axes, as in the fig.(1).

If any two vectors $\vec{A}$ and $\vec{B}$ are added, the result is

$$\vec{A} + \vec{B} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} + B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k}$$

This shows that each of the three components of the resultant vector is found by adding the two corresponding components of the individual vectors.

The vector equation can be written as three separate and distinct equations. For example

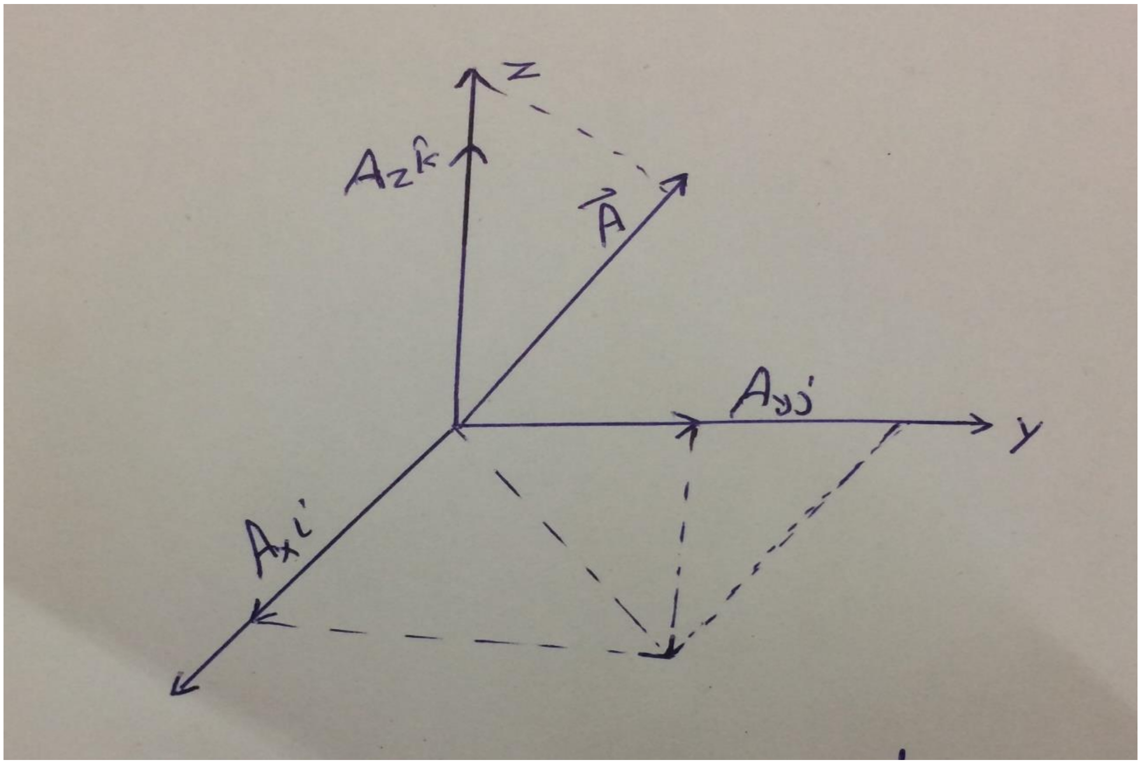


Fig.(1): A three Dimensional vector in rectangular coordinates.

$\vec{A} + \vec{B} = \vec{C} + \vec{D} + \vec{E}$

Could be written as three equations

$A_x + B_x = C_x + D_x + E_x$

$A_y + B_y = C_y + D_y + E_y$

$A_z + B_z = C_z + D_z + E_z$

Scalar Multiplication: Two types of vector multiplication have been defined, namely: "*scalar product*" and "*vector product*".

The scalar product of two vectors is a scalar quantity whose magnitude is equal to the product of the magnitudes of the two vectors and the cosine of the angle between them.

$\vec{A}.\vec{B} = AB\cos\theta$

The dot product obeys the commutative law, that is,

$\vec{A}.\vec{B} = \vec{B}.\vec{A}$

A physical example of the dot product can be found in the relationship between the force and the distance in terms of the work

$work = \vec{F}.\vec{D}$

Using ordinary algebraic rules

$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$

$$\vec{A}.\vec{B} = A_xB_x(\hat{\imath}.\hat{\imath}) + A_xB_y(\hat{\imath}.\hat{\jmath}) + A_xB_z(\hat{\imath}.\hat{k}) + A_yB_x(\hat{\jmath}.\hat{\imath}) + A_yB_y(\hat{\jmath}.\hat{\jmath}) + A_yB_z(\hat{\jmath}.\hat{k}) + A_zB_x(\hat{k}.\hat{\imath}) + A_zB_y(\hat{k}.\hat{\jmath}) + A_zB_z(\hat{k}.\hat{k})$$

From the dot product law, the unit vectors are satisfied the following

$\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$ and $\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = \hat{\jmath}.\hat{\imath} = \hat{k}.\hat{\jmath} = \hat{\imath}.\hat{k} = 0$

Thus;

$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$

<u>Vector multiplication:</u> The vector (cross) product of two vectors is defined as a vector whose magnitude is the product of the magnitudes of the two vectors and the sine of the angle between them, and whose direction is perpendicular to the plane containing the two vectors.

$\vec{A} \times \vec{B} = \vec{C}$

$|\vec{A} \times \vec{B}| = AB\sin\theta$

The vector $\vec{B} \times \vec{A}$ would have the same magnitude but the opposite direction, that is

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

The commutative law does not apply

$$\vec{A} \times \vec{B} = A_xB_x(\hat{\imath} \times \hat{\imath}) + A_xB_y(\hat{\imath} \times \hat{\jmath}) + A_xB_z(\hat{\imath} \times \hat{k}) + A_yB_x(\hat{\jmath} \times \hat{\imath}) + A_yB_y(\hat{\jmath} \times \hat{\jmath}) + A_yB_z(\hat{\jmath} \times \hat{k}) + A_zB_x(\hat{k} \times \hat{\imath}) + A_zB_y(\hat{k} \times \hat{\jmath}) + A_zB_z(\hat{k} \times \hat{k})$$

Thus, by using the definition of the vector product, may be obtain

$\hat{\imath} \times \hat{\jmath} = \hat{k} = -\hat{\jmath} \times \hat{\imath}$

$\hat{j} \times \hat{k} = \hat{\imath} = -\hat{k} \times \hat{j}$

$\hat{k} \times \hat{\imath} = \hat{j} = -\hat{\imath} \times \hat{k}$

$\hat{\imath} \times \hat{\imath} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

Therefore;

$$\vec{A} \times \vec{B} = \left(A_y B_z - A_z B_y\right)\hat{\imath} + (A_z B_x - A_x B_z)\hat{j} + \left(A_x B_y - A_y B_x\right)\hat{k}$$

Or using the determinant;

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Some useful expressions:

$\vec{C} \times \left(\vec{A} + \vec{B}\right) = \vec{C} \times \vec{A} + \vec{C} \times \vec{B}$ and $\vec{A}.\left(\vec{B} + \vec{C}\right) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$

$\vec{A}.\vec{A} = |\vec{A}||\vec{A}| = A^2$

<u>Vector and Scalar Triple Product</u>

1. Triple Scalar product

This product can be given by $\left(\vec{A} \times \vec{B}\right).\vec{C}$ which can be a volume, i.e.,

$$V = \left(\vec{A} \times \vec{B}\right).\vec{C}$$

Which can be written as

$$V = \left(\vec{A} \times \vec{B}\right).\vec{C} = \left(\vec{B} \times \vec{C}\right).\vec{A} = \left(\vec{C} \times \vec{A}\right).\vec{B}$$

In terms of the components, this product can be written as

$$\vec{A}.\left(\vec{B} \times \vec{C}\right) = A_x\left(B_y C_z - B_z C_y\right) + A_y(B_z C_x - B_x C_z) + A_z\left(B_x C_y - B_y C_x\right)$$

Also, this product can be represented in terms of determinant as follows

$$\vec{A}.\left(\vec{B} \times \vec{C}\right) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

2. Triple Vector Product

Triple vector product defined by

$\vec{Q} = \vec{A} \times (\vec{B} \times \vec{C})$, which satisfies the identity

$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$ (Prove that)

Differentiation:

The del operator $\vec{\nabla}$ has many important applications in physical problems. In Cartesian coordinates is defined as

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

There are three possible operations with $\vec{\nabla}$ which are:

1- If V is a scalar function, then

$$\vec{\nabla}V = \frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}$$

This operation is called the gradient of a scalar function, and it is abbreviated as

$$\vec{\nabla}V = \boldsymbol{grad}\, V$$

2- If $\vec{A}$ is a vector function, then

$$\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

This operation is called the divergence and is abbreviated

$$\vec{\nabla}.\vec{A} = div\vec{A}$$

3- If $\vec{A}$ is a vector function, then

$$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

This operation is called the curl and can be written as

$$\vec{\nabla} \times \vec{A} = curl\, \vec{A}$$

Some identities:

1- $div\ curl\vec{A} = \vec{\nabla}.\left(\vec{\nabla} \times \vec{A}\right) = 0$
2- $curlgrad\ V = \vec{\nabla} \times \left(\vec{\nabla}V\right) = 0$
3- $div\ grad\ V = \vec{\nabla}.\left(\vec{\nabla}V\right) = \nabla^2 V$

Where the operator ∇^2 is called the Laplacian and defined in Cartesian coordinates as follows

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

4- $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2\vec{A}$
5- $\vec{\nabla}.\vec{\mathrm{A}} \times \vec{B} = \vec{B}.\vec{\nabla} \times \vec{\mathrm{A}} - \vec{\mathrm{A}}.\vec{\nabla} \times \vec{\mathrm{B}}$
6- $\vec{\nabla}(ab) = a\vec{\nabla}b + b\vec{\nabla}a$
7- $\vec{\nabla}.\left(a\vec{\mathrm{B}}\right) = \vec{B}.\vec{\nabla}a + a\vec{\nabla}.\vec{B}$
8- $\vec{\nabla}.\left(a\vec{\nabla}b\right) = \vec{\nabla}a.\vec{\nabla}b + a\nabla^2 b$
9- $\vec{\nabla} \times \left(a\vec{B}\right) = \vec{\nabla}a \times \vec{B} + a\vec{\nabla} \times \vec{B}$
10- $\vec{\nabla} \times \left(\vec{\mathrm{A}} \times \vec{B}\right) = \vec{A}\left(\vec{\nabla}.\vec{B}\right) - \vec{B}\left(\vec{\nabla}.\vec{A}\right) + \left(\vec{B}.\vec{\nabla}\right)\vec{A} - \left(\vec{A}.\vec{\nabla}\right)\vec{B}$
11- $\vec{\nabla}\left(\vec{A}.\vec{B}\right) = \left(\vec{A}.\vec{\nabla}\right)\vec{B} + \left(\vec{B}.\vec{\nabla}\right)\vec{A} + \vec{A} \times \left(\vec{\nabla} \times \vec{B}\right) + \vec{B} \times \left(\vec{\nabla} \times \vec{A}\right)$

3. **The Gradient**

The partial derivatives $f_x(x, y)$ and $f_y(x, y)$ represent the rates of change of $f(x, y)$ in directions parallel to the x-and y-axes. *In this section we will investigate rates of change of* $f(x, y)$ in other directions.

The partial derivatives of a function give the instantaneous rates of change of that function in directions parallel to the coordinate axes. Directional derivatives allow us to compute the rates of change of a function with respect to distance in any direction.

The gradient of a scalar function φ is a vector whose magnitude is the maximum directional derivative at the point being considered and whose direction is the direction of the maximum directional derivative at the point.

The gradient has the direction normal to the level surface of φ through the point being considered.

If the scalar function φ represents temperature, then $\vec{\nabla}\varphi = grad\varphi$ is a temperature gradient, or rate of change of temperature with distance.

Temperature φ is a scalar quantity, the temperature gradient $\vec{\nabla}\varphi$ is a vector quantity.

In terms of the gradient the directional derivative is given by

$$\frac{d\varphi}{ds} = |grad\varphi| \cos\theta \qquad (1)$$

Where θ is the angle between the direction of $d\vec{S}$ and the direction of the gradient.

$$\frac{d\varphi}{ds} = grad\varphi . \frac{d\vec{S}}{ds} \qquad (2)$$

This equation enables us to find the explicit form of the gradient in any coordinate system in which we know the form of $d\vec{S}$.

In rectangular coordinates

$$d\vec{S} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$$

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

From equation (2) and the latter equation, we obtain

$$\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = (grad\ \varphi)_x dx + (grad\ \varphi)_y dy + (grad\ \varphi)_z dz$$

Equating coefficients on both sides, we obtain

$grad\ \varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$ this equation represents the gradient in rectangular coordinates.

In the same way, the gradient in spherical coordinates (r, θ, ϕ) is given by

$$grad\ \varphi = \hat{a}_r \frac{\partial\varphi}{\partial r} + \hat{a}_\theta \frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \hat{a}_\emptyset \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\emptyset}$$

Also, the gradient in cylindrical coordinates (r, θ, z) is given by

$$grad\ \varphi = \hat{a}_r \frac{\partial\varphi}{\partial r} + \hat{a}_\emptyset \frac{1}{r}\frac{\partial\varphi}{\partial\emptyset} + \hat{a}_z \frac{\partial\varphi}{\partial z}$$

Vector Integration: Consider three kinds of integrals: *Line, Surface, and Volume integral*. The integrand may be either a *vector* or *scalar*.

1- The line integral: If $\vec{F}$ is a vector, the *line integral* of $\vec{F}$ is written as

$$\int_{a\,C}^{b} \vec{F}.d\vec{l}$$

Where C is the curve along which the integration is performed, a and b the initial and final points of the curve, and $d\vec{l}$ is an infinitesimal vector displacement along the curve C. *Since* $\vec{F}.d\vec{l}$ *is a scalar, it is clear that the line integral is a scalar*.

The line integral depends not only on the endpoints a and b but also on the curve C along which the integration is to be done.

The line integral around a closed curve is defined by

$$\oint_C \vec{F}.d\vec{l}$$

The integral around a closed curve may or may not be zero.

2- <u>The Surface integral:</u> If $\vec{F}$ is a vector, the *line integral* of $\vec{F}$ is written as

$$\int_S \vec{F}.\vec{n}da$$

Where S is the surface over which the integration is be performed, *da,* is an infinitesimal area on S and $\vec{n}$ is a unit normal to *da*.

The surface integral of $\vec{F}$ over a closed surface S is sometimes denoted by

$$\oint_S \vec{F}.\vec{n}da$$

Note that the surface integral is a scalar.

3- <u>The Volume integral:</u>

If $\vec{F}$ is a vector and φ a scalar, then the two volume integrals in which we are interested are:

$$J = \int_v \varphi dv \qquad \text{and} \qquad \vec{K} = \int_v \vec{F}dv$$

J is a scalar and $\vec{K}$ is a vector.

4. <u>The Divergence of a Vector and Gauss Theorem</u>

The divergence of a vector $\vec{F}$ written as $div\vec{F}$, which is defined as follows:

$$div\vec{F} = \lim_{v\to 0}\frac{1}{v}\oint_S \vec{F}.\vec{n}da$$

The divergence is a scalar point function (scalar field)

The divergence in rectangular coordinates is found to be

$div\,\vec{F} = \vec{\nabla}.\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$ (Prove that)

The divergence in spherical coordinates is found to be

$div\,\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$ (Prove that)

While the divergence in cylindrical coordinates is

$div\,\vec{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial F_z}{\partial z}$ (Prove that)

Divergence Theorem:

This theorem states that the integral of the divergence of a vector over a volume *V* is equal to the surface integral of the normal component of the vector over the surface bounding *V*. That is,

$$\int_V div\,\vec{F}dv = \oint_S \vec{F}.\vec{n}da$$

In other words, the divergence theorem states that the flux of a vector field across a closed surface with outward orientation is equal to the triple integral of the divergence over the region enclosed by the surface, i.e.,

$$\iint_\sigma \vec{F}.\vec{n}dS = \iiint_G div\,\vec{F}dv$$

5. The Curl and Stokes Theorem

The third interesting vector differential operator is the *curl*. The *curl* of a vector written as $curl\,\vec{F}$ which is defined as:

$$curl\,\vec{F} = \lim_{V\to 0}\frac{1}{V}\oint_S \vec{n}\times\vec{F}\,da$$

In rectangular coordinates the *curl* is given by:

$$curl\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

In cylindrical coordinates the *curl* is given by:

$$curl\,\vec{F} = \vec{\nabla} \times \vec{F} = \frac{1}{r}\begin{vmatrix} \hat{a}_r & r\hat{a}_\varphi & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ F_r & rF_\varphi & F_z \end{vmatrix}$$

In spherical coordinates the *curl* is given by:

$$curl\,\vec{F} = \vec{\nabla} \times \vec{F} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & \hat{a}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_r & rF_\theta & r\sin\theta\, F_\varphi \end{vmatrix}$$

Stokes' theorem:

The line integral of a vector around a closed curve is equal to the integral of the normal component of its *curl* over any surface bounded by the curve. That is

$$\oint_C \vec{F}.d\vec{l} = \int_S curl\,\vec{F}.\vec{n}da$$

Where C is a closed curve which bounds the surface S.

H.W: Write the *curl of a vector in cylindrical and spherical coordinates.*

6. **Laplacian Operator ∇^2:**

The operator that results by taking the dot product of the del operator with itself is denoted by $\nabla^2$and is called the Laplacian operator. This operator has the form.

$$\nabla^2 = \vec{\nabla}.\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

When applied to $\emptyset(x, y, z)$ the Laplacian operator produces the function.

$$\nabla^2\emptyset = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2}$$

$\nabla^2\emptyset$ can also be expressed as $div(\vec{\nabla}\emptyset)$, the equation $\nabla^2\emptyset = 0$, or

$\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2} = 0$ is known as Laplace's equation.

7. Some Operations Involving del Operator

I. $\vec{\nabla}.\left(\vec{\nabla}\emptyset\right) = \nabla^2\emptyset = \vec{\nabla}.(grad\emptyset) = divgrad\emptyset$ or

$$\nabla^2\emptyset = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2}$$

II. $\nabla^2\emptyset$ is a scalar operator can operate on a vector, as follows:

$$\nabla^2\vec{A} = \frac{\partial^2\vec{A}}{\partial x^2} + \frac{\partial^2\vec{A}}{\partial y^2} + \frac{\partial^2\vec{A}}{\partial z^2}$$

III. $\vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) = grad\ div\ \vec{A}$

IV. $$\vec{\nabla} \times \left(\vec{\nabla}\emptyset\right) = curl\ grad\ \emptyset = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} & \frac{\partial\emptyset}{\partial z} \end{vmatrix} = 0$$

Thus, the curl of the gradient of any scalar field is zero.

Where $\frac{\partial^2}{\partial x\partial y} = \frac{\partial^2}{\partial y\partial x}$

The cross product of the operator $\vec{\nabla}$ with any vector $\vec{A}$ equal zero, $\vec{\nabla} \times \vec{A} = 0$, this means that the vector $\vec{A}$ must be a gradient of a scalar function, i.e., $\vec{A} = \vec{\nabla}\emptyset$, in this case the vector $\vec{A}$ is called non rotation vector.

V. $\vec{\nabla}.\left(\vec{\nabla} \times \vec{A}\right) = div\ curl\ \vec{A} = 0$

If there is a vector such $\vec{B}$, where $\vec{\nabla}.\vec{B} = 0$, this means that $\vec{B}$ must be a curl of a vector, such that $\vec{A}$, i.e., $\vec{B} = \vec{\nabla} \times \vec{A}$, and the vector $\vec{B}$ is called a rotational vector.

VI. $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = curl\ curl\ \vec{A} = grad\ div\ \vec{A} - \nabla^2\vec{A} = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \vec{\nabla}.\vec{\nabla}\vec{A}$

8. Some Formulas Involving Differential Operators

1-	$\vec{\nabla}(\varphi + \psi) = \vec{\nabla}\varphi + \vec{\nabla}\psi$
2-	$\vec{\nabla}\varphi\psi = \varphi\vec{\nabla}\psi + \psi\vec{\nabla}\varphi$
3-	$div\left(\vec{F} + \vec{G}\right) = div\vec{F} + div\vec{G}$
4-	$curl\left(\vec{F} + \vec{G}\right) = curl\vec{F} + curl\vec{G}$
5-	$\vec{\nabla}\left(\vec{F}.\vec{G}\right) = \left(\vec{F}.\vec{\nabla}\right)\vec{G} + \left(\vec{G}.\vec{\nabla}\right)\vec{F} + \vec{F} \times curl\ \vec{G} + \vec{G} \times curl\ \vec{F}$

6-	$div\ \varphi\vec{F} = \varphi\ div\ \vec{F} + \vec{F}.\vec{\nabla}\varphi$
7-	$div\ (\vec{F} \times \vec{G}) = \vec{G}.curl\ \vec{F} - \vec{F}.curl\ \vec{G}$
8-	$div\ curl\ \vec{F} = 0$
9-	$curl\ \varphi\vec{F} = \varphi\ curl\ \vec{F} + \vec{\nabla}\varphi \times \vec{F}$
10-	$curl\ (\vec{F} \times \vec{G}) = \vec{F}\ div\ \vec{G} - \vec{G}\ div\ \vec{F} + (\vec{G}.\vec{\nabla})\vec{F} - (\vec{F}.\vec{\nabla})\vec{G}$
11-	$curl\ curl\ \vec{F} = grad\ div\ \vec{F} - \nabla^2\vec{F}$
12-	$curl\ \vec{\nabla}\varphi = 0$
13-	$\oint_S\ \vec{F}.\vec{n}da = \int_V\ div\ \vec{F}\ dv$
14-	$\oint_C\ \vec{F}.d\vec{l} = \int_S\ curl\ \vec{F}.\vec{n}da$
15-	$\oint_S\ \varphi\vec{n}da = \int_V\ \vec{\nabla}\varphi dv$
16-	$\oint_S\ \vec{F}(\vec{G}.\vec{n})da = \int_V\ \vec{F}\ div\ \vec{G}\ dv + \int_V\ (\vec{G}.\vec{\nabla})\vec{F}dv$
17-	$\oint_S\ \vec{n} \times \vec{F}da = \int_V\ curl\ \vec{F}\ dv$
18-	$\oint_C\ \varphi d\vec{l} = \int_S\ \vec{n} \times \vec{\nabla}\varphi da$

9. Differential Length, Area, and Volume

Differential elements in length, area, and volume are useful in vector calculus. They are defined in the Cartesian, cylindrical, and spherical coordinate systems.

A. Cartesian Coordinates

From fig.(2), we notice that

(1) Differential displacement is given by

$$d\vec{l} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$$

(2) Differential normal area is given by

$$d\vec{S} = dxdy\hat{k} \quad , \quad d\vec{S} = dydz\hat{\imath} \quad , \quad d\vec{S} = dxdz\hat{\jmath}$$

(3) Differential volume is given by

$$dv = dxdydz$$

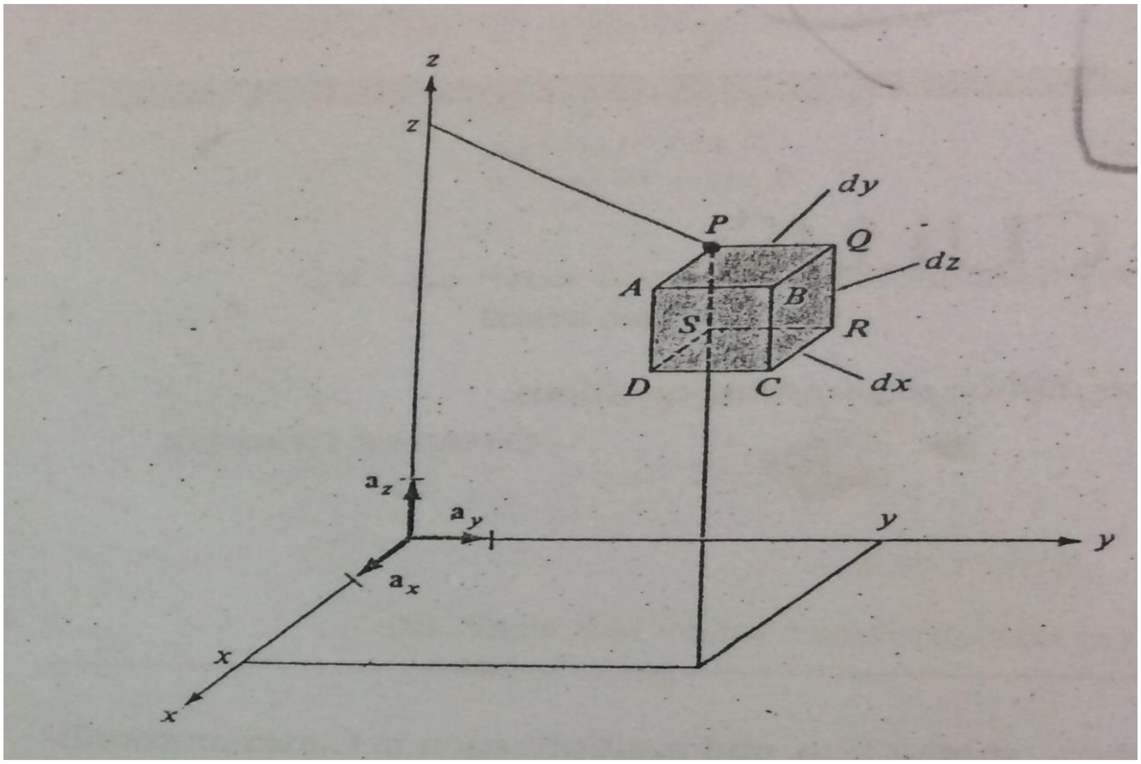


Fig.2: Differential elements in the right-handed Cartesian coordinates system.

B. Cylindrical Coordinates

Notice from fig.(3), that in cylindrical coordinates, differential elements can be found as follows:

(1) Differential displacement is given by

$$d\vec{l} = dr\hat{a}_r + rd\varphi\hat{a}_\varphi + dz\hat{a}_z$$

(2) Differential normal area is given by

$$d\vec{S} = rd\varphi dz\hat{a}_r \ , \quad d\vec{S} = drdz\hat{a}_\varphi \quad , \quad d\vec{S} = rd\varphi dr\hat{a}_z$$

(3) Differential volume is given by

$$dv = rdrd\varphi dz$$

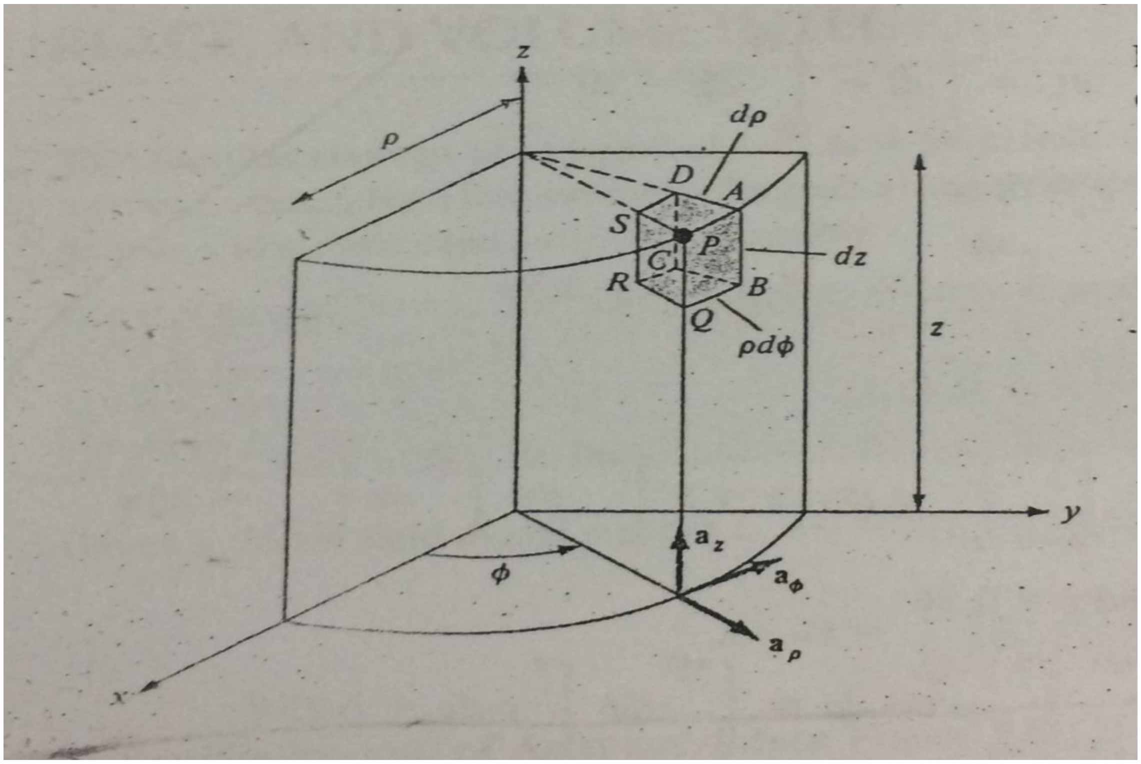


Fig.(3): Differential elements in cylindrical coordinate system.

C. Spherical Coordinates

From fig.(4), we notice that in spherical coordinates,

(1) The differential displacement is

$$d\vec{l} = dr\hat{a}_r + rd\theta\hat{a}_\theta + r\sin\theta\, d\varphi\hat{a}_\varphi$$

(2) The differential normal area is

$$d\vec{S} = r^2\sin\theta\, d\theta d\varphi\hat{a}_r \quad , \quad d\vec{S} = r\sin\theta\, drd\varphi\hat{a}_\theta \ , \quad d\vec{S} = rdrd\theta\hat{a}_\varphi$$

(3) The differential volume is

$$dv = r^2\sin\theta drd\theta d\varphi$$

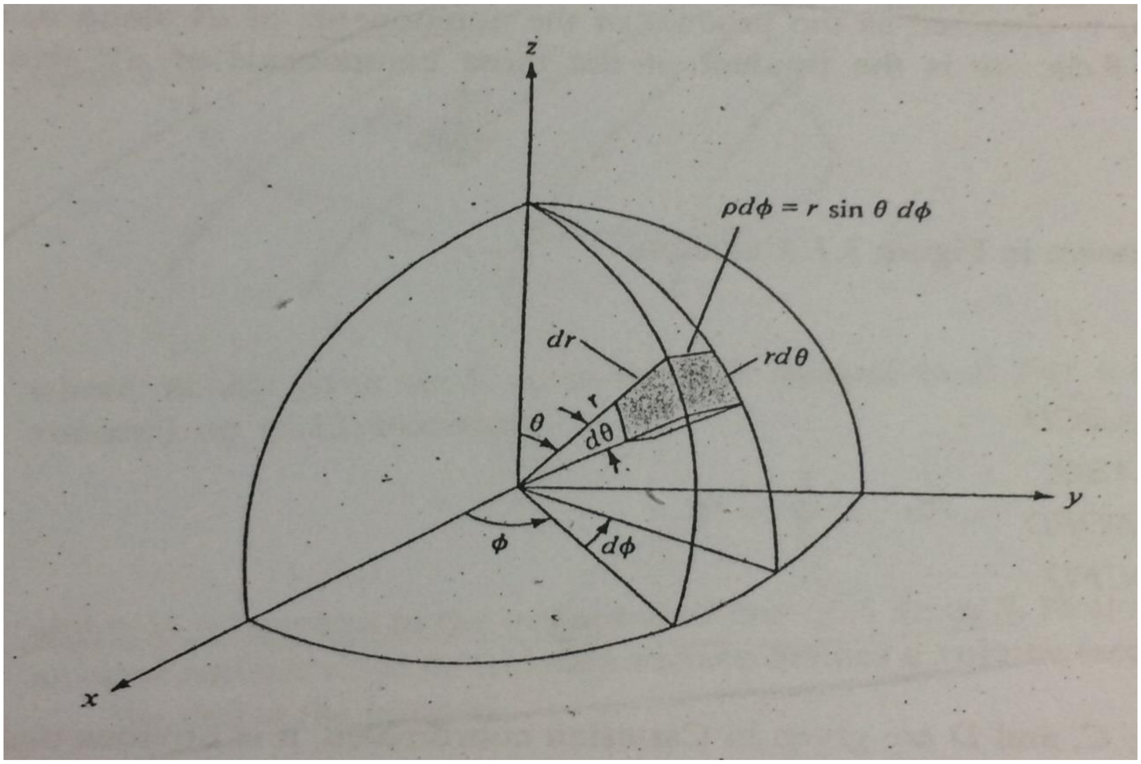


Fig.(4): Differential elements in spherical coordinate system.

Differential volume, surface, and line elements.

Coordinates System	Rectangular	Cylindrical	Spherical
Line element	$d\vec{l} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$	$d\vec{l} = dr\hat{a}_r + rd\varphi\hat{a}_\varphi + dz\hat{a}_z$	$d\vec{l} = dr\hat{a}_r + rd\theta\hat{a}_\theta +$ $r\sin\theta\, d\varphi\hat{a}_\varphi$
Volume element	$dv = dxdydz$	$dv = rdrd\varphi dz$	$dv = r^2 \sin\theta drd\theta d\varphi$
Surface element	$d\vec{S} = dxdy\hat{k}$ $d\vec{S} = dydz\hat{\imath}$ $d\vec{S} = dxdz\hat{\jmath}$	$d\vec{S} = rd\varphi dz\hat{a}_r$ $d\vec{S} = drdz\hat{a}_\varphi$ $d\vec{S} = rd\varphi dr\hat{a}_z$	$d\vec{S} = r^2\sin\theta\, d\theta d\varphi\hat{a}_r$ $d\vec{S} = r\sin\theta\, drd\varphi\hat{a}_\theta$ $d\vec{S} = rdrd\theta\hat{a}_\varphi$

H.W: Obtain each of the element in the table.

10. Green's Theorem

This theorem given by the following

$$\int_V (\psi\nabla^2\varphi - \varphi\nabla^2\psi)dv = \oint_S (\psi\, grad\, \varphi - \varphi grad\, \psi)\,.\vec{n}da$$

This theorem follows from the application of the divergence theorem to the vector

$$\vec{F} = \psi\, grad\, \varphi - \varphi grad\, \psi$$

Examples:

Example.1: If $\vec{r}$ a vector starting from the origin. Show that

1- $\vec{\nabla}.\vec{r} = 3$,
2- $\vec{\nabla} \times \vec{r} = 0$
3- $\vec{\nabla}(\vec{A}.\vec{r}) = \vec{A}$
4- $(\vec{A}.\vec{\nabla})\vec{r} = \vec{A}$

H.W

Solution:

1- $\vec{\nabla}.\vec{r} = 3$

Since, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\vec{\nabla}.\vec{r} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

2- $\vec{\nabla} \times \vec{r} = 0$

$$\vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{\imath}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

Example.2: If $\vec{r}$ a vector starting from the origin. Show that

1- $\nabla^2\left(\frac{1}{r}\right) = 0$
2- $\vec{\nabla}r = \frac{\vec{r}}{r}$
3- $\vec{\nabla} \times \left(\frac{\vec{r}}{r^3}\right) = 0$
4- $\vec{\nabla}.\left(\frac{\vec{r}}{r^3}\right) = 0$
5- $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

Solution:

1- $\vec{\nabla}r = \frac{\vec{r}}{r}$

Since $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $r = (x^2 + y^2 + z^2)^{1/2} = \sqrt{x^2 + y^2 + z^2}$

$$\vec{\nabla} r = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)(x^2 + y^2 + z^2)^{1/2}$$

$$= \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{1/2}\hat{\imath} + \frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{1/2}\hat{\jmath} + \frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{1/2}\hat{k}$$

$$= \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x\hat{\imath}) + \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2y\hat{\jmath}) + \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2z\hat{k})$$

$$= \frac{x\hat{\imath}}{(x^2+y^2+z^2)^{1/2}} + \frac{y\hat{\jmath}}{(x^2+y^2+z^2)^{1/2}} + \frac{z\hat{k}}{(x^2+y^2+z^2)^{1/2}}$$

$$= \frac{1}{(x^2+y^2+z^2)^{1/2}}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) = \frac{\vec{r}}{r}$$

2- $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

$$\vec{\nabla}\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)(x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$= -\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}}(2x\hat{\imath}) - \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}}(2y\hat{\jmath}) - \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}}(2z\hat{k})$$

$$= -\left[\frac{x\hat{\imath}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{y\hat{\jmath}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{z\hat{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}\right]$$

$$= -\frac{1}{\left[(x^2+y^2+z^2)^{\frac{1}{2}}\right]^3}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) = --\frac{\vec{r}}{r^3}$$

3- $\vec{\nabla}.\left(\frac{\vec{r}}{r^3}\right) = 0$

We have $|\vec{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}} = \sqrt{x^2 + y^2 + z^2}$ and $r^3 = (x^2 + y^2 + z^2)^{\frac{3}{2}}$

$$\vec{\nabla}.\left(\frac{\vec{r}}{r^3}\right) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(\frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) . \left\{\frac{x\hat{\imath}}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{y\hat{\jmath}}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{z\hat{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)$$

Each term can be evaluated alone as follows:

Taking the first term

$$\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) = \frac{(x^2+y^2+z^2)^{\frac{3}{2}}.1 - x\left(\frac{3}{2}\right)(x^2+y^2+z^2)^{\frac{1}{2}}.2x}{\left[(x^2+y^2+z^2)^{\frac{3}{2}}\right]^2}$$

$$= \frac{(x^2+y^2+z^2)^{\frac{3}{2}} - 3x^2(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}$$

The same procedure can be carryout to y and z terms, therefore

$$\vec{\nabla}.\left(\frac{\vec{r}}{r^3}\right) = \frac{\left[(x^2+y^2+z^2)^{\frac{3}{2}} - 3x^2(x^2+y^2+z^2)^{\frac{1}{2}}\right] + \left[(x^2+y^2+z^2)^{\frac{3}{2}} - 3y^2(x^2+y^2+z^2)^{\frac{1}{2}}\right] + \left[(x^2+y^2+z^2)^{\frac{3}{2}} - 3z^2(x^2+y^2+z^2)^{\frac{1}{2}}\right]}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{\frac{1}{2}}[x^2+y^2+z^2-3x^2+x^2+y^2+z^2-3y^2+x^2+y^2+z^2-3z^2]}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{\frac{1}{2}}[3x^2+3y^2+3z^2-3x^2-3y^2-3z^2]}{(x^2+y^2+z^2)^3} = 0$$

Thus, $\vec{\nabla}.\left(\frac{\vec{r}}{r^3}\right) = 0$

Example.3: Show that the following two vectors are perpendicular

$\vec{A} = \hat{\imath} + 4\hat{\jmath} + 3\hat{k}$

$\vec{B} = 4\hat{\imath} + 2\hat{\jmath} - 4\,\hat{k}$

Solution:

Using the dot product law $\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

To be the vectors perpendicular, the angle between them must be 90^0 , therefore, the result of the dot product of the two vectors must be zero, i.e., $\vec{A}.\vec{B} = 0$

$\vec{A}.\vec{B} = (\hat{\imath} + 4\hat{\jmath} + 3\hat{k}).(\vec{B} = 4\hat{\imath} + 2\hat{\jmath} - 4\,\hat{k}) = 4 + 8 - 12 = 0$

$\therefore \vec{A}$ and $\vec{B}$ are perpendicular.

<u>Example.4 :</u> If $\vec{r}$ is a vector from the origin to the point (x, y, z), prove that $(\vec{U}.\vec{\nabla})\vec{r} = \vec{U}$, where $\vec{U}$ is any vector.

<u>Solution:</u>

Let $\vec{U} = U_x\hat{\imath} + U_y\hat{\jmath} + U_z\hat{k}$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\vec{U}.\vec{\nabla} = (U_x\hat{\imath} + U_y\hat{\jmath} + U_z\hat{k}).\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)$

$= U_x\frac{\partial}{\partial x} + U_y\frac{\partial}{\partial y} + U_z\frac{\partial}{\partial z}$

$(\vec{U}.\vec{\nabla})\vec{r} = \left(U_x\frac{\partial}{\partial x} + U_y\frac{\partial}{\partial y} + U_z\frac{\partial}{\partial z}\right)(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$

$= U_x\frac{\partial}{\partial x}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) + U_y\frac{\partial}{\partial y}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) + U_z\frac{\partial}{\partial z}(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$

$= U_x\hat{\imath} + U_y\hat{\jmath} + U_z\hat{k} = \vec{U}$

<u>Example.5</u>: Use the divergence theorem to find the outward flux of the vector field $\vec{F}(x,y,z) = 2x\hat{\imath} + 3y\hat{\jmath} + z^2\hat{k}$ across the unit cube in fig.(5).

<u>Solution:</u>

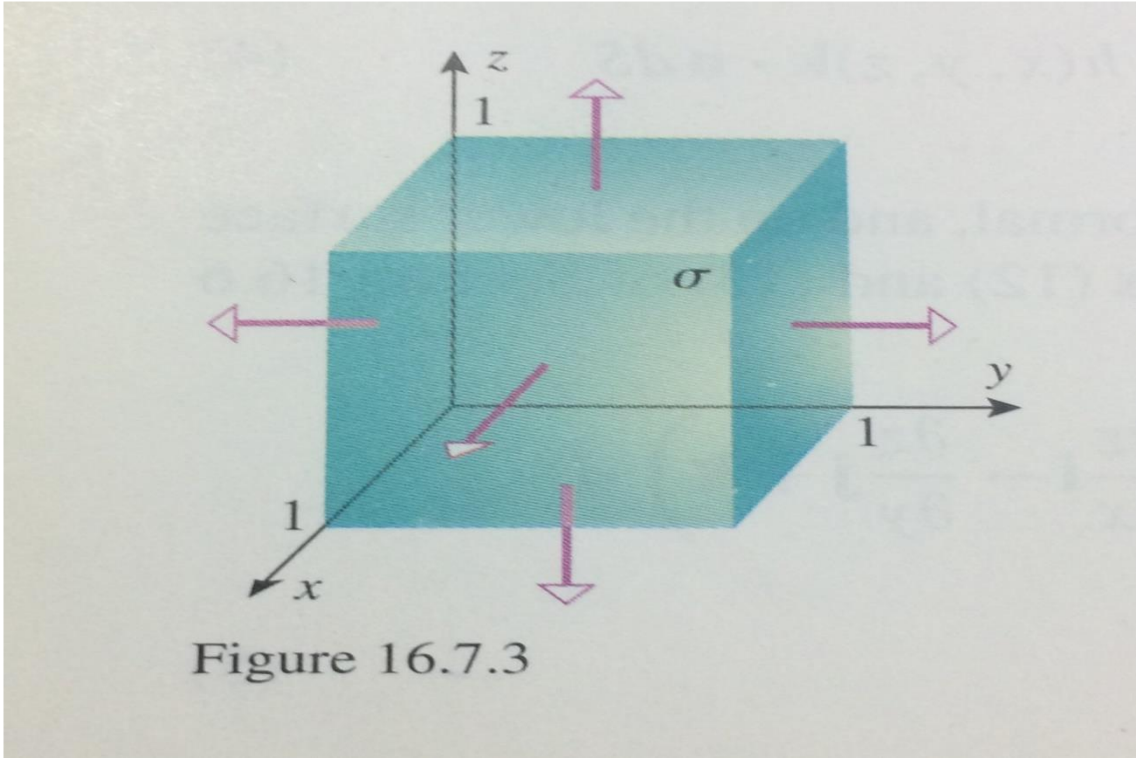
Let σ denoted the out-ward-oriented surface of the cube and G the region that it encloses.

$div\,\vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(z^2) = 5 + 2z$

The flux across σ is

$\Phi = \iint_\sigma \vec{F}.\vec{n}dS = \iiint_G (5 + 2z)dv = \int_0^1\int_0^1\int_0^1(5 + 2z)dzdydx$

$= \int_0^1\int_0^1[5z + z^2]_{z=0}^1 dydx = \int_0^1\int_0^1 6dydx = 6$



Fig(5): A unit cube in rectangular coordinates.

Example.6: Use the divergence theorem to find the outward flux of the vector field $\vec{F}(x, y, z) = x^3\hat{\imath} + y^3\hat{\jmath} + z^2\hat{k}$ across the surface of the region that is elncloed by the circular cylinder $x^2 + y^2 = 9$ and the planes z=0 and z=2 as in fig(6).

Solution:

$$div\,\vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^2) = 3x^2 + 3y^2 + 2z$$

The flux across σ is

$$\Phi = \iint_\sigma \vec{F}.\vec{n}dS = \iiint_G (3x^2 + 3y^2 + 2z)dv$$

$$= \int_0^{2\pi}\int_0^3\int_0^2 (3r^2 + 2z)rdzdrd\varphi \quad \text{using cylindrical coordinates}$$

$$= \int_0^{2\pi}\int_0^3 [3r^3z + z^2r]_{z=0}^2 drd\varphi$$

$$= \int_0^{2\pi}\int_0^3 (6r^3 + 4r)drd\varphi$$

$$= \int_0^{2\pi}\left[\frac{3r^4}{2} + 2r^2\right]_0^3 d\varphi$$

$$= \int_0^{2\pi} \frac{279}{2} d\varphi = 279\pi$$

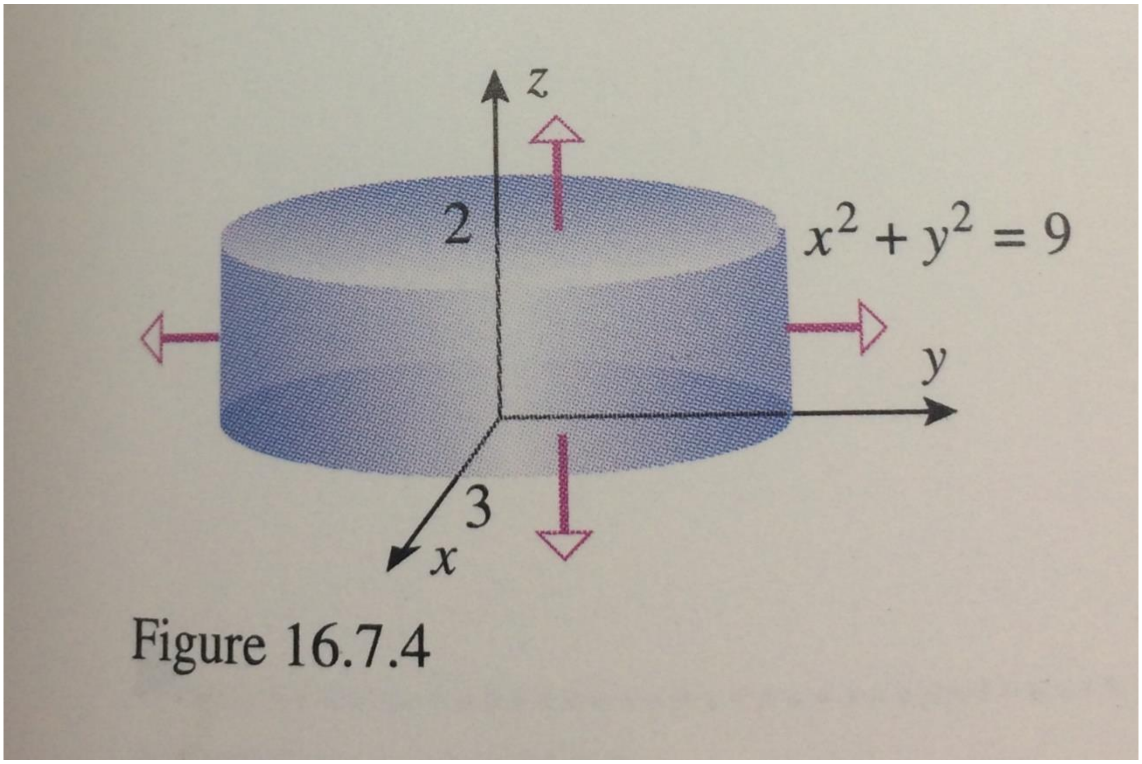


Fig.(6): Circular cylinder.

Example.7: Find the divergence and curl of the vector field

$\vec{F}(x, y, z) = x^2y\hat{\imath} + 2y^3z\hat{\jmath} + 3z\hat{k}$.

Solution:

$$div\,\vec{F} = \vec{\nabla}.\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(x^2y\hat{\imath} + 2y^3z\hat{\jmath} + 3z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y^3z) + \frac{\partial}{\partial z}(3z)$$

$$div\,\vec{F} = 2xy + 6y^2z + 3$$

$$curl\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2y^3z & 3z \end{vmatrix} = -2y^3\hat{\imath} - x^2\hat{k}$$

Example.8: Show that the divergence of the inverse-square field

$\vec{F}(x,y,z) = \frac{c}{(x^2+y^2+z^2)^{\frac{3}{2}}}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$ is zero.

Solution:

For simplicity let $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, then

$$\vec{F}(x,y,z) = \frac{cx\hat{\imath}+cy\hat{\jmath}+cz\hat{k}}{r^3} = \frac{cx}{r^3}\hat{\imath} + \frac{cy}{r^3}\hat{\jmath} + \frac{cz}{r^3}\hat{k} \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$div\,\vec{F} = c\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right]$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{r^3 - x(3r^2)(\frac{x}{r})}{(r^3)^2} = \frac{1}{r^3} - \frac{3x^2}{r^5}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{r^3 - y(3r^2)(\frac{y}{r})}{(r^3)^2} = \frac{1}{r^3} - \frac{3y^2}{r^5}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{r^3 - z(3r^2)(\frac{z}{r})}{(r^3)^2} = \frac{1}{r^3} - \frac{3z^2}{r^5}$$

$$div\,\vec{F} = c\left[\frac{3}{r^3} - \frac{3x^2+3y^2+3z^2}{r^5}\right] = c\left[\frac{3}{r^3} - \frac{3r^2}{r^5}\right] = 0$$