CHAPTER THREE

3.1 INTRODUCTION

Heat transfer problems are also classified as being *one-dimensional, two dimensional,* or *three-dimensional,* depending on the relative magnitudes of heat transfer rates in different directions and the level of accuracy desired. For example, the steady temperature distribution in a long bar of rectangular cross section can be expressed as T(x, y) if the temperature variation in the z-direction (along the bar) is negligible and there is no change with time (Fig 3.1)

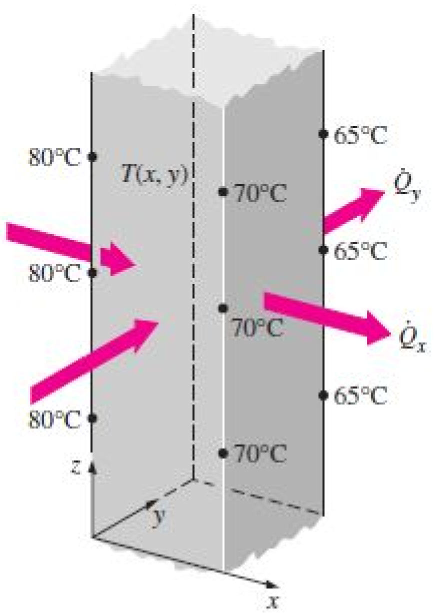


FIGURE 3.1 Two-dimensional heat transfer in a long rectangular bar.

3.2 ONE-DIMENSIONAL HEAT CONDUCTION EQUATION

Consider heat conduction through a large plane wall such as the wall of a house, the glass of a single pane window, the metal plate at the bottom of a pressing iron, a cast iron steam pipe, a cylindrical nuclear fuel element, an electrical resistance wire, the wall of a spherical container, or a spherical metal ball that is being quenched or tempered. Heat conduction in these and many other geometries can be approximated as being one-dimensional since heat conduction through these geometries will be dominant in one direction and negligible in other directions. Below we will develop the onedimensional heat conduction equation in rectangular, cylindrical, and spherical coordinates.

3.2.1Heat Conduction Equation in a Large Plane Wall

Consider a thin element of thickness Δx in a large plane wall, as shown in Figure 3.2 . Assume the density of the wall is , the specific heat is C, and the area of the wall normal to the direction of heat transfer is A. An *energy balance* on this thin element during a small time interval Δt can be expressed as

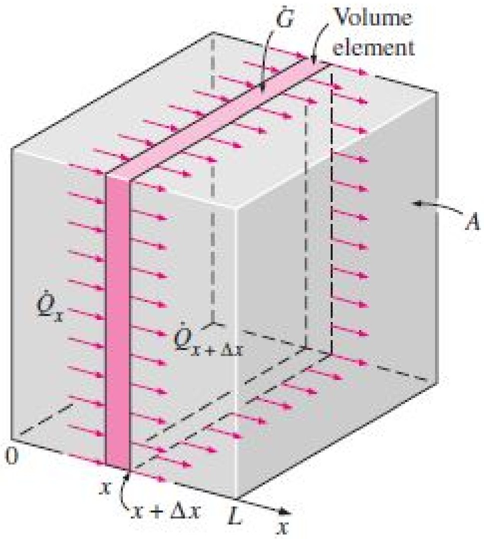


FIGURE 3.3 One-dimensional heat conduction through a volume element in a large plane wall.

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$

Rate of heat conduction at x

$$\dot{Q}\text{=E =}q_x = \text{KA}\,\frac{\text{dT}}{\text{dX}}$$

Rate of heat conduction at x+ Δx

$$\dot{Q}\text{=E=}\ q_{x+\Delta x} = q_x + \frac{\text{d}q_x}{\text{d}x}\text{d}x$$

Rate of heat generation (**energy source**) inside the element

$$\dot{Q}=E==\dot{G}=\dot{q}\,dxdydz$$

Rate of change of the energy content (**energy storage**) of the element

$$\dot{Q}=E=\rho\, C\,\frac{dT}{dt}\,dxdydz$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+g=\rho C\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

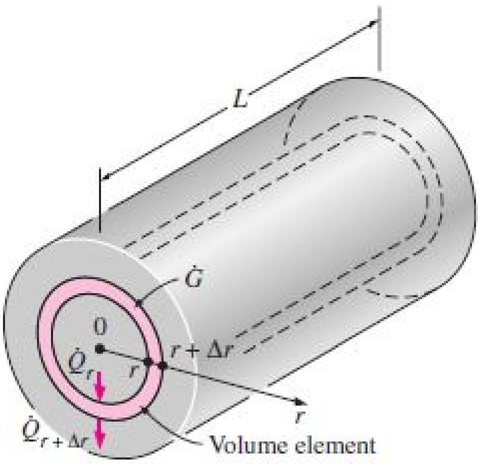
where the property α= $k/\rho C$ is the **thermal diffusivity** of the material and represents how fast heat propagates through a material. It reduces to the following forms under specified conditions.

(1) *Steady-state:* $(\partial/\partial t=0)$ $$\frac{d^2T}{dx^2}+\frac{\dot{g}}{k}=0$$

(2) *Transient, no heat generation:* $(g=0)$ $$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t=0$ and $g=0)$ $$\frac{d^2T}{dx^2}=0$$

3.2.2.Heat Conduction Equation in a Long Cylinder



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(1) *Steady-state:* $(\partial/\partial t = 0)$ $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{g}}{k}=0$$

(2) *Transient, no heat generation:* $(\dot{g} = 0)$ $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0 \text{ and } \dot{g} = 0)$ $$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$

(*a*) The form that is ready to integrate

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$

(*b*) The equivalent alternative form

$$r\frac{d^2T}{dr^2}+\frac{dT}{dr}=0$$

3.2.3 Heat Conduction Equation in a Sphere

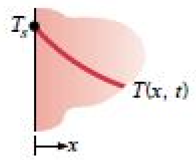
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

3.3 SOLUTION OF STEADY ONE-DIMENSIONAL HEAT CONDUCTION PROBLEMS

Boundary conditions

1. Constant surface temperature

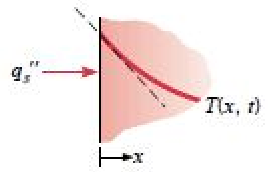
$$T(0, t) = T_s$$



2. Constant surface heat flux

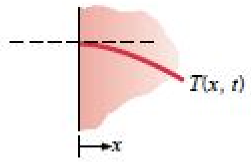
(a) Finite heat flux

$$-k\frac{\partial T}{\partial x}\Big|_{x=0} = q_s''$$



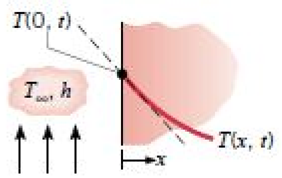
(b) Adiabatic or insulated surface

$$\frac{\partial T}{\partial x}\Big|_{x=0} = 0$$

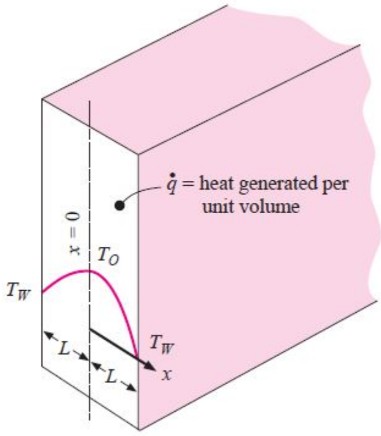


3. Convection surface condition

$$-k\frac{\partial T}{\partial x}\Big|_{x=0} = h[T_\infty - T(0, t)]$$



Case 1 PlaneWall with Heat Sources



$$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{q}}{K}=0$$

$$\frac{dT}{dx}=-\frac{\dot{q}}{K}x+C_1$$

$$T=-\frac{\dot{q}}{K}\frac{x^2}{2}+C_1x+C_2$$

B.C1: $x=\pm L \longrightarrow T=Tw$

B.C2: $x=0 \longrightarrow T=To$

B.C3: $x=0 \longrightarrow \frac{dT}{dx}=0$

at $x=0 \longrightarrow \frac{dT}{dx}=0 \longrightarrow \therefore C_1=0$

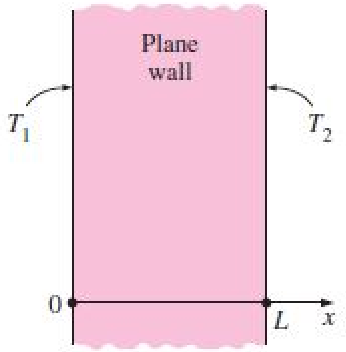
at $x=0 \longrightarrow T=To \longrightarrow \therefore C2=T_o$

$$\therefore T=-\frac{\dot{q}}{K}\frac{x^2}{2}+T_o \qquad \text{a}$$

at $x=\pm L \longrightarrow T=Tw \longrightarrow \therefore T_w=-\frac{\dot{q}}{K}\frac{L^2}{2}+T_o \qquad \text{b}$

By divided a to b : $\frac{T-T_o}{T_w-T_o}=\left(\frac{x}{L}\right)^2$

Case 2 PlaneWall with Heat Sources and one side insulation



$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{K} = 0$$

$$\frac{dT}{dx} = -\frac{\dot{q}}{K}x + C_1$$

$$T = -\frac{\dot{q}}{K}\frac{x^2}{2} + C_1 x + C_2$$

B.C1: x = 0 ⟹ $\frac{dT}{dx} = 0$

B.C2: x = L ⟹ T = Tw

at x= 0 ⟹ $\frac{dT}{dx} = 0$ ⟹ $\therefore C_1 = 0$

at x= L ⟹ T=Tw ⟹ $\therefore C2 = T_w + \frac{\dot{q}}{2K}L^2$

$$\therefore T = -\frac{\dot{q}}{K}\frac{x^2}{2} + T_w + \frac{\dot{q}}{2K}L^2$$

EXAMPLE 3.1

A plane wall 6.0 mm thick generates heat internally at the rate of 4 MW/m3. One side of the wall is insulated, and the other side is at 95∘C.
The thermal conductivity of the wall is 21 W/m • C. Calculate the temperature distribution in the wall.

Solution:-

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{K} = 0$$

$$\frac{dT}{dx} = -\frac{\dot{q}}{K}x + C_1$$

$$T = -\frac{\dot{q}}{K}\frac{x^2}{2} + C_1 x + C_2$$

B.C1: $x = 0 \rightarrow \frac{dT}{dx} = 0$

B.C2: $x = 0.006\text{m} \rightarrow T = 95\ C$

at $x = 0 \rightarrow \frac{dT}{dx} = 0 \rightarrow \therefore C_1 = 0$

at $0.006\text{m} \rightarrow T = 95\ C \rightarrow \therefore C2 = 95 + \frac{4\times10^6 \times (0.006)^2}{2\times21}$

$$\therefore C_2 = 98.42$$

$$\therefore T = \frac{\dot{q}}{K}\frac{x^2}{2} + 98.42$$

$\therefore T(0) = 98.42C$

$\therefore T(1) = 98.04C$

$\therefore T(2) = 96.89C$

$\therefore T(3) = 95C$

EXAMPLE3.2

A plane wall 6.0 cm thick generates heat internally at the rate of 0.3 MW/m3. One side of the wall is insulated, and the other side is exposed to an environment at 93C.
The convection heat-transfer coefficient between the wall and the environment is 570 W/m2 · C. The thermal conductivity of the wall is 21 W/m · C.
Calculate the maximum temperature in the wall.

Solution:

$\dot{q} = 0.30 \ \frac{\text{MW}}{\text{m}^3}$ Same as half of wall 15 cm thick with convection on each side.

$$T_0 - T_w = \frac{\dot{q}L^2}{2k} = \frac{(0.30\times10^6)(0.060)^2}{(2)(21)} = 25.7°\text{C}$$

$$\dot{q}LA = hA(T_w - T_\infty) \qquad T_w - T_\infty = \frac{(0.30\times10^6)(0.060)}{570} = 31.6°\text{C}$$

$$T_0 = T_{max} = 93 + 25.7 + 31.6 = 150.3°\text{C}$$

EXAMPLE 3.3

Heat is generated in a 2.5-cm-square copper rod at the rate of 35.3MW/m3. The rod is exposed to a convection environment at 20◦C, and the heat-transfer coefficient is 4000 W/m2 · C. Calculate the surface temperature of the rod.

Solution:

$$\dot{q}AL = hPL(T_w - T_\infty)$$

$$(35.3\times10^6)(0.025)^2 = (4000)(4)(0.025)(T_w - 20)$$

$$T_w = 75.16°\text{C}$$

EXAMPLE 3.4

A certain semiconductor material has a conductivity of 0.0124 W/cm · C. A rectangular bar of the material has a cross-sectional area of 1 cm² and a length of 3 cm.
One end is maintained at 300C and the other end at 100C, and the bar carries a current of 50 A. Assuming the longitudinal surface is insulated, calculate the midpoint temperature in the bar. Take the resistivity as (1.5×10−³Ω.cm).

Solution:

$$k = 0.0124 \ \frac{\text{W}}{\text{cm}\cdot°\text{C}} = 1.24 \ \frac{\text{W}}{\text{m}\cdot°\text{C}}$$

$$\rho = 1.5\times10^{-3} \ \Omega\cdot\text{cm}$$

$$R = (1.5\times10^{-3})\left(\frac{3}{1}\right) = 4.5\times10^{-3}$$

$$q = I^2R = (50)^2(4.5\times10^{-3}) = 11.25 \text{ W}$$

$$\dot{q} = \frac{q}{V} = \frac{11.25}{3\times10^{-6}} = 3.75 \ \frac{\text{MW}}{\text{m}^3} \qquad T = -\frac{\dot{q}}{2k}x^2 + c_1x + c_2$$

$$L = 1.5 \text{ cm} = 0.015 \text{ m} \qquad T = 300 \text{ at } x = -0.015$$

$$T = 100 \text{ at } x = +0.015$$

$$300 - 100 = c_1(-0.015 - 0.015) \qquad c_1 = -6667$$

$$300 = \frac{(-3.75\times10^{6})(0.015)^2}{(2)(1.24)} - (6667)(-0.015) + c_2 \qquad c_2 = 540.2$$

at $x = 0$ $\quad T = c_2 = 540.2°C$

General Heat Conduction Equation in Cylinder with Heat Sources

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{\dot{q}}{k} = 0$$

B.C1: r = 0 $\Longrightarrow \frac{dT}{dr} = 0$

B.C2: r = R $\Longrightarrow$ T = Tw

$$r\frac{d^2T}{dr^2} + \frac{dT}{dr} = \frac{-\dot{q}r}{k}$$

$$r\frac{d^2T}{dr^2} + \frac{dT}{dr} = \frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

By integration:

$$r\frac{dT}{dr} = \frac{-\dot{q}r^2}{2k} + C_1$$

$$T = \frac{-\dot{q}r^2}{4k} + C_1 \ln r + C_2$$

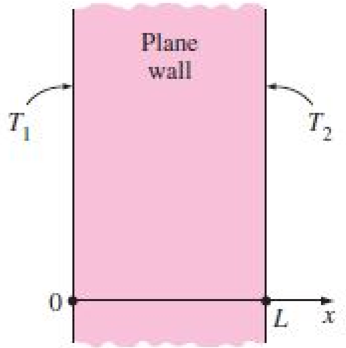
$$C_1 = 0 \qquad C_2 = T_w + \frac{\dot{q}R^2}{4k}$$

The final solution:

$$T - T_w = \frac{\dot{q}}{4k}(R^2 - r^2) \quad \Longrightarrow \quad (1)$$

EXAMPLE 3.5

Consider a large plane wall of thickness *L* _ 0.2 m, thermal conductivity *k* 1.2 W/m · °C, and surface area *A* 15 m2. The two sides of the wall are maintained at constant temperatures of *T*1 120°C and *T*2 50°C, respectively, as shown in Figure . Determine (*a*) the variation of temperature within the wall and the value of temperature at *x* 0.1 m and (*b*) the rate of heat conduction through the wall under steady conditions.



Sol

$$\frac{d^2T}{dx^2} = 0$$

with boundary conditions

$$T(0) = T_1 = 120°C$$
$$T(L) = T_2 = 50°C$$

$$\frac{dT}{dx} = C_1$$

$$T(x) = C_1x + C_2$$

$$T(0) = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_1$$

$$T(L) = C_1L + C_2 \quad \rightarrow \quad T_2 = C_1L + T_1 \quad \rightarrow \quad C_1 = \frac{T_2 - T_1}{L}$$

$$T(x) = \frac{T_2 - T_1}{L}x + T_1$$

$$T(0.1\text{ m}) = \frac{(50 - 120)°\text{C}}{0.2\text{ m}}(0.1\text{ m}) + 120°\text{C} = 85°\text{C}$$

(*b*) The rate of heat conduction anywhere in the wall is determined from Fourier's law to be

$$q = -kA\frac{dT}{dx} = -kAC_1 = -kA\frac{T_2 - T_1}{L} = kA\frac{T_1 - T_2}{L} \quad \textbf{(2-57)}$$

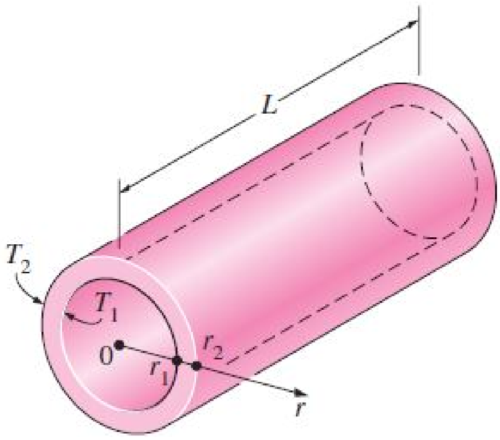
The numerical value of the rate of heat conduction through the wall is determined by substituting the given values to be

$$q = kA\frac{T_1 - T_2}{L} = (1.2\text{ W/m}\cdot°\text{C})(15\text{ m}^2)\frac{(120 - 50)°\text{C}}{0.2\text{ m}} = 6300\text{ W}$$

EXAMPLE 3.6

Consider a steam pipe of length L 20 m, inner radius $r1$ 6 cm, outer radius $r2$ 8 cm, and thermal conductivity k 20 W/m · °C. The inner and outer surfaces of the pipe are maintained at average temperatures of $T1$ 150°C and $T2$ 60°C, respectively. Obtain a general relation for the temperature distribution inside the pipe under steady conditions, and determine the rate of heat loss from the steam through the pipe.

Sol



$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

with boundary conditions

$$T(r_1) = T_1 = 150°\text{C}$$
$$T(r_2) = T_2 = 60°\text{C}$$

Integrating the differential equation once with respect to r gives

$$r\frac{dT}{dr} = C_1$$

where C_1 is an arbitrary constant. We now divide both sides of this equation by r to bring it to a readily integrable form,

$$\frac{dT}{dr} = \frac{C_1}{r}$$

$$T(r) = C_1 \ln r + C_2$$

We now apply both boundary conditions by replacing all occurrences of r and T (r)

$$T(r) = \left(\frac{\ln(r/r_1)}{\ln(r_2/r_1)}\right)(T_2 - T_1) + T_1$$

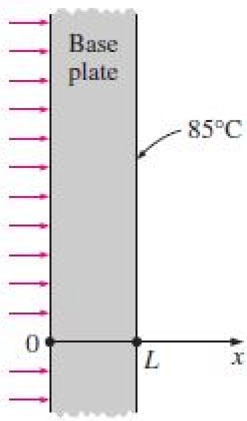
$$q_{\text{cylinder}} = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{C_1}{r} = -2\pi kLC_1 = 2\pi kL\frac{T_1 - T_2}{\ln(r_2/r_1)}$$

$$q = 2\pi(20 \text{ W/m}\cdot°\text{C})(20 \text{ m})\frac{(150 - 60)°\text{C}}{\ln(0.08/0.06)} = \mathbf{786\ kW}$$

HW sheet 3

1. Consider a large plane wall of thickness *L* _ 0.4 m, thermal conductivity *k* _ 2.3 W/m · °C, and surface area *A* _ 20 m2. The left side of the wall is maintained at a constant temperature of *T*1 _ 80°C while the right side loses heat by convection to the surrounding air at *T* _ 15°C with a heat transfer coefficient of *h* _ 24 W/m2 · °C. Assuming constant thermal conductivity and no heat generation in the wall, (*a*) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the wall, (*b*) obtain a relation for the variation of temperature in the wall by solving the differential equation, and (*c*) evaluate the rate of heat transfer through the wall.
Answer: (*c*) 6030 W

2. Consider the base plate of a 800-W household iron with a thickness of *L* _ 0.6 cm, base area of *A* _ 160 cm2, and thermal conductivity of *k* _ 20 W/m · °C. The inner surface of the base plate is subjected to uniform heat flux generated by the resistance heaters inside. When steady operating conditions are reached, the outer surface temperature of the plate is measured to be 85°C. Disregarding any heat loss through the upper part of the iron, (*a*) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the plate, (*b*) obtain a relation for the variation of temperature in the base plate by solving the differential equation, and (*c*) evaluate the inner surface temperature.
Answer: (*c*) 100°C



3. Consider a large plane wall of thickness *L* _ 0.3 m, thermal conductivity *k* _ 2.5 W/m · °C, and surface area *A* _ 12 m2. The left side of the wall at *x* _ 0 is subjected to a net heat flux of *q*· 0 _ 700 W/m2 while the temperature at that surface is measured to be *T*1 _ 80°C. Assuming constant thermal conductivity and no heat generation in the wall, (*a*) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the wall, (*b*) obtain a relation for the variation of temperature in the wall by solving the differential equation, and (*c*) evaluate the temperature of the right surface of the wall at *x* _ *L*. *Answer:* (*c*) _4°C

4. Aspherical container of inner radius *r*1 _ 2 m, outer radius *r*2 _ 2.1 m, and thermal conductivity *k* _ 30 W/m · °C is filled with iced water at 0°C. The container is gaining heat by convection from the surrounding air at *T* _ 25°C with a heat transfer coefficient of *h* _ 18 W/m2 · °C. Assuming the inner surface temperature of the container to be 0°C, (*a*) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the container, (*b*) obtain a relation for the variation of temperature in the container by solving the differential equation, and (*c*) evaluate the rate of heat gain to the iced water.

5. Consider a homogeneous spherical piece of radioactive material of radius *r*0 _ 0.04 m that is generating heat at a constant rate of *g*· _ 4 _ 107 W/m3. The heat generated is dissipated to the environment steadily. The outer surface of the sphere is maintained at a uniform temperature of 80°C and the thermal conductivity of the sphere is *k* _ 15 W/m · °C. Assuming steady one-dimensional heat transfer, (*a*) express the differential equation and the boundary conditions for heat conduction through the sphere, (*b*) obtain a relation for the variation of temperature in the sphere by solving the differential equation, and (*c*) determine the temperature at the center of the sphere.

6. A long homogeneous resistance wire of radius *r*0 _ 5 mm is being used to heat the air in a room by the passage of electric current. Heat is generated in the wire uniformly at a rate of *g*· _ 5 _ 107 W/m3 as a result of resistance heating. If the temperature of the outer surface of the wire remains at 180°C, determine the temperature at *r* _ 2 mm after steady operation conditions are reached. Take the thermal conductivity of the wire to be *k* _ 8 W/m · °C. *Answer:* 212.8°C