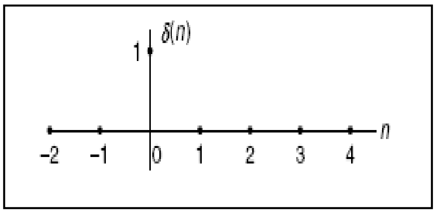
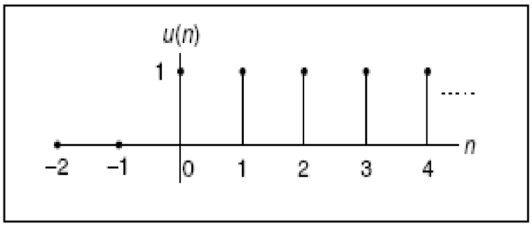
Lec. 3 Digital Signals and Systems

3.1 Digital Signals





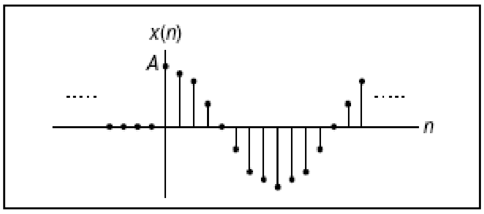
1- Digital unit-impulse function

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

2- Digital unit-step function

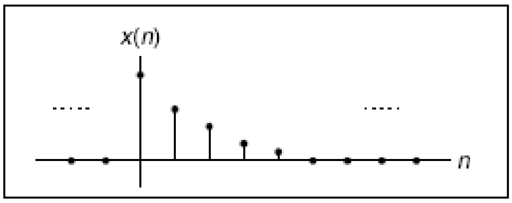
$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

3- Sinusoidal sequence



$x(n) = \cos wn \;, \quad 0 \leq n \leq \infty$

4- Exponential sequence



$x(n) = e^{-jwn}, \quad 0 \leq n \leq \infty$

Fig. (3.1) Some digital signals

3.2 Generation of Digital Signals

To develop the digital sequence from its analog signal function is by applying:

$$x(n) = x(t)|_{t=nT} = x(nT). \tag{3.1}$$

Example(1): assuming a DSP system with a sampling time interval of 125 microseconds, Convert each of the following analog signals x(t) to the digital signal x(n).

1. $x(t) = 10e^{-5000t}u(t)$
2. $x(t) = 10\sin(2000\pi t)u(t)$

Solution:

1. $x(n) = x(nT) = 10e^{-5000\times 0.000125n}u(nT) = 10e^{-0.625n}u(n).$
2. $x(n) = x(nT) = 10\sin(2000\pi \times 0.000125n)u(nT) = 10\sin(0.25\pi n)u(n).$

3.3 Power Signals:

Periodic signals are power signals because their energy per cycle is finite.

$$power = \frac{1}{T}\int_0^T | f(t) |^2 \, dt = \sum_{n=-\infty}^{\infty} | C_n |^2 = \varphi(\tau) \qquad (3.2)$$

Where:

$$C_n = \frac{1}{T}\int_0^T f(t)\, e^{-jnw_o t}\, dt \qquad , w_o = 2\,\pi\, f_o \qquad (3.3)$$

$$f(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{jnw_o t} \qquad (3.4)$$

$$\varphi(\tau) = \frac{1}{T}\int_0^T f(t)\; f(t \pm \tau)\, dt \qquad (3.5)$$

3.4 Energy Signals:

Non-periodic signals are called an energy signals because their power → 0

$$energy = \int_{-\infty}^{\infty} | f(t) |^2 \, dt = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} | F(W) |^2 \, dW = \lambda(\tau) \qquad (3.6)$$

Where:

$$F(W) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt \qquad (3.7)$$

$$\lambda(\tau) = \int_{-\infty}^{\infty} f(t)\; f(t \pm \tau)\, dt \qquad (3.8)$$

3.5 Classification of Systems

3.5.1 Linear System

Figure 3.2 illustrates that the system output due to the weighted sum inputs $\alpha\, x_1(n) \pm \beta\, x_2(n)$ is equal to the same weighted sum of the individual outputs obtained from their corresponding inputs, that is, $y(n) = \alpha\, y_1(n) \pm \beta\, y_2(n)$, where α and β are constants. Here, the principle of "superposition" is applied.

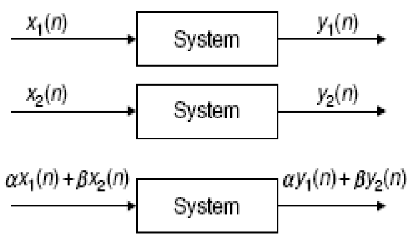


Fig. (3.2) Digital linear system

3.5.2 Time-Invariant System

A time-invariant system is illustrated in Figure 3.3. If the system is time invariant and $y_1(n)$ is the system output due to the input $x_1(n)$, then the shifted system input $x_1(n - n_0)$ will produce a shifted system output $y_1(n - n_0)$.

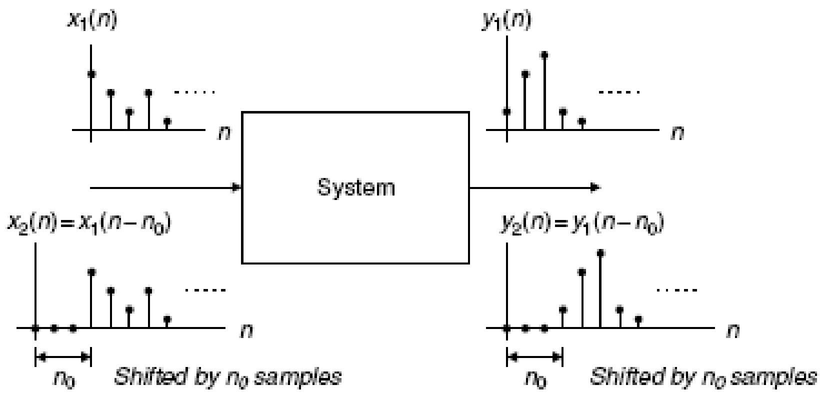


Fig. 3.3 Illustration of linear time-invariant system

Example 2: Given the linear systems:

a. $y(n) = 2x(n - 5)$

b. $y(n) = 2x(3n)$,

Determine whether each of the following systems is time invariant.

Solution:

a) Let the input and output be $x_1(n)$ and $y_1(n)$, respectively; then the system output is $y_1(n) = 2x_1(n - 5)$. Again, let $x_2(n) = x_1(n - n_0)$ be the shifted input and $y_2(n)$ be the output due to the shifted input. We determine the system output using the shifted input as

$$y_2(n) = 2x_2(n - 5) = 2x_1(n - n_0 - 5):$$

Meanwhile, shifting $y_1(n) = 2x_1(n - 5)$ by n_0 samples leads to

$$y_1(n - n_0) = 2x_1(n - 5 - n_0)$$

We can verify that $y_2(n) = y_1(n - n_0)$. Thus the shifted input of n_0 samples causes the system output to be shifted by the same n_0 samples, thus the system is *time invariant*.

b) Let the input and output be $x_1(n)$ and $y_1(n)$, respectively; then the system output is $y_1(n) = 2x_1(3n)$. Again, let the input and output be $x_2(n)$ and $y_2(n)$, where $x_2(n) = x_1(n - n_0)$, a shifted version, and the corresponding output is $y_2(n)$. We get the output due to the shifted input

$x_2(n) = x_1(n - n_0)$ and note that $x_2(3n) = x_1(3n - n_0)$:

$y_2(n) = 2x_2(3n) = 2x_1(3n - n_0)$:

On the other hand, if we shift $y_1(n)$ by n_0 samples, which replaces n in

$y_1(n) = 2x_1(3n)$ by $n - n_0$, it yield

$y_1(n - n_0) = 2x_1(3(n - n_0)) = 2x_1(3n - 3n_0)$:

Clearly, we know that $y_2(n) \neq y_1(n - n_0)$. Since the system output $y_2(n)$ using the input shifted by n_0 samples is not equal to the system output $y_1(n)$ shifted by the same n_0 samples, the system is *not time invariant*.

3.5.3 Causal System:

A causal system is one in which the output y(n) at time n depends only on the current input x(n) at time n, its past input sample values such as x(n − 1), x(n− 2), . . . : Otherwise, if a system output depends on the future input values, such as x(n + 1), x(n + 2), . . . , the system is noncausal. The noncausal system cannot be realized in real time.

Example 3: Given the following linear systems,

a. $y(n) = 0.5x(n) + 2.5x(n - 2)$, for $n \geq 0$

b. $y(n) = 0.25x(n - 1) + 0.5x(n + 1) - 0.4y(n - 1)$, for $n \geq 0$,

Determine whether each is causal.

Solution:

a) Since for $n \geq 0$, the output y(n) depends on the current input x(n) and its past value x(n − 2), the system is causal.

b) Since for $n \geq 0$, the output y(n) depends on the current input x(n) and its future value x(n + 2), the system is noncausal.

3.5.4. Stability:

A stable system is one for which every bounded input produces a bounded output (BIBO). The system is stable, if its transfer function vanishes after a sufficiently long time. For a stable system:

$$S = \sum_{k=-\infty}^{\infty} | h(k) | \langle \infty \qquad (3.9)$$

Where *h(k)* = unit impulse response

3.6 Difference Equations and Impulse Responses

A causal, linear, time-invariant system can be described by a difference equation having the following general form:

$$y(n) + a_1 y(n-1) + \ldots + a_N y(n-N)$$

$$= b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M), \qquad (3.10)$$

Where $a_1, \ldots, a_N$ and $b_0, b_1, \ldots, b_M$ are the coefficients of the difference equation. Equation (3.10) can further be written as:

$$y(n) = -a_1 y(n-1) - \ldots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M)$$

$$y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{j=0}^{M} b_j x(n-j). \qquad (3.11)$$

Notice that y(n) is the current output, which depends on the past output samples y(n − 1), . . . , y(n − N), the current input sample x(n), and the past input samples, x(n−1), . . . , x(n − M).

Example4: Given a linear system described by the difference equation

y(n) = x(n) + 0.5x(n − 1), Determine the nonzero system coefficients.

Solution: a. By comparing Equation (3.11), we have, $b_0 = 1$, and $b_1 = 0.5$

3.7 System Representation Using Its Impulse Response

A linear time-invariant system can be completely described by its unit-impulse response, which is defined as the system response due to the impulse input δ(n) with zero initial conditions, depicted in Figure 3.3. Here x(n) = δ(n) and y(n) = h(n).

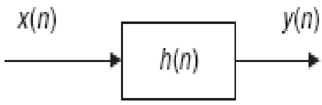


Fig. 3.4 Representation of a linear time-invariant system using the impulse response.

Example 5: Given the linear time-invariant system

y(n) = 0.5x(n) + 0.25x(n − 1) with an initial condition x(−1) = 0

a. Determine the unit-impulse response h(n).

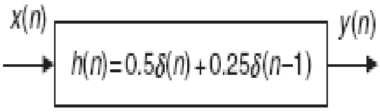
b. Draw the system block diagram.

c. Write the output using the obtained impulse response.

Solution:

a. h(n) = 0.5 δ(n) + 0.25 δ(n − 1) , where h(0)= 0.5, h(1) = 0.25 and h(n) = 0 elsewhere.

b.



c. $y(n) = h(0)\, x(n) + h(1)\, x(n - 1)$

From this result, it is noted that if the difference equation without the past output terms, $y(n - 1), \ldots, y(n - N)$, that is, the corresponding coefficients $a_1, \ldots, a_N$, are zeros, the impulse response h(n) has a finite number of terms. We call this a finite impulse response (FIR) system.

In general, we can express the output sequence of a linear time-invariant system from its impulse response and inputs as:

$$y(n) = \ldots + h(-1)\, x(n+1) + h(0)\, x(n) + h(1)\, x(n-1) + h(2)\, x(n-2) + \ldots \quad (3.12)$$

Equation (3.12) is called the ***digital convolution sum.***

Example 6: Given the difference equation

$y(n)= 0.25\, y(n - 1) + x(n)$ for $n \geq 0$ and $y(-1) = 0$,

a. Determine the unit-impulse response h(n).

b. Draw the system block diagram.

c. Write the output using the obtained impulse response.

d. For a step input x(n) = u(n), verify and compare the output responses for the first three output samples using the difference equation and digital convolution sum (Equation 3.12).

Solution:

a. Let $x(n) = \delta(n)$, then $h(n) = 0.25\, h(n - 1) + \delta(n)$

To solve for h(n), we evaluate

$h(0) = 0.25\, h(-1) + \delta(0) = 0.25\,(0) + 1 = 1$

$h(1) = 0.25\, h(0) + \delta(1) = 0.25\,(1) + 0 = 0.25$

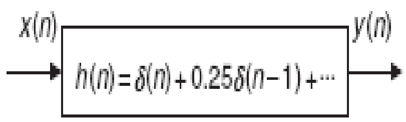
$h(2) = 0.25\, h(1) + \delta(2) = 0.25\,(0.5) + 0 = 0.0625$

. . .

With the calculated results, we can predict the impulse response as:

$h(n) =(0.25)^n u(n) = \delta(n) + 0.25\, \delta(n - 1) + 0.0625\, \delta(n - 2) + \ldots \ldots$

b. The system block diagram is given below



c. The output sequence is a sum of infinite terms expressed as

$y(n) = h(0)\, x(n) + h(1)\, x(n - 1) + h(2)x(n - 2) + \ldots$

$= x(n) + 0.25x(n - 1) + 0.0625x(n - 2) + \ldots$

d. From the difference equation and using the zero-initial condition, we have

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0$$
$$n = 0, y(0) = 0.25y(-1) + x(0) = u(0) = 1$$
$$n = 1, y(1) = 0.25y(0) + x(1) = 0.25 \times u(0) + u(1) = 1.25$$
$$n = 2, y(2) = 0.25y(1) + x(2) = 0.25 \times 1.25 + u(2) = 1.3125$$
$$\ldots$$

Applying the convolution sum in Equation (3.12) yields:

$$y(n) = x(n) + 0.25x(n-1) + 0.0625x(n-2) + \ldots$$

$$n = 0,\ y(0) = x(0) + 0.25x(-1) + 0.0625x(-2) + \ldots$$
$$= u(0) + 0.25 \times u(-1) + 0.125 \times u(-2) + \ldots = 1$$

$$n = 1,\ y(1) = x(1) + 0.25x(0) + 0.0625x(-1) + \ldots$$
$$= u(1) + 0.25 \times u(0) + 0.125 \times u(-1) + \ldots = 1.25$$

$$n = 2,\ y(2) = x(2) + 0.25x(1) + 0.0625x(0) + \ldots$$
$$= u(2) + 0.25 \times u(1) + 0.0625 \times u(0) + \ldots = 1.3125$$
$$\ldots$$

Notice that this impulse response h(n) contains an infinite number of terms in its duration due to the past output term y(n − 1). Such a system as described in the preceding example is called an infinite impulse response (IIR) system.

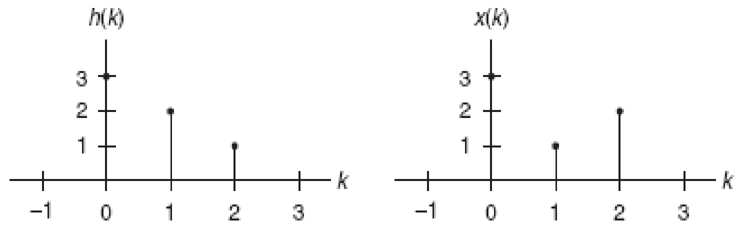
3.8 Digital Convolution

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \qquad (3.13)$$

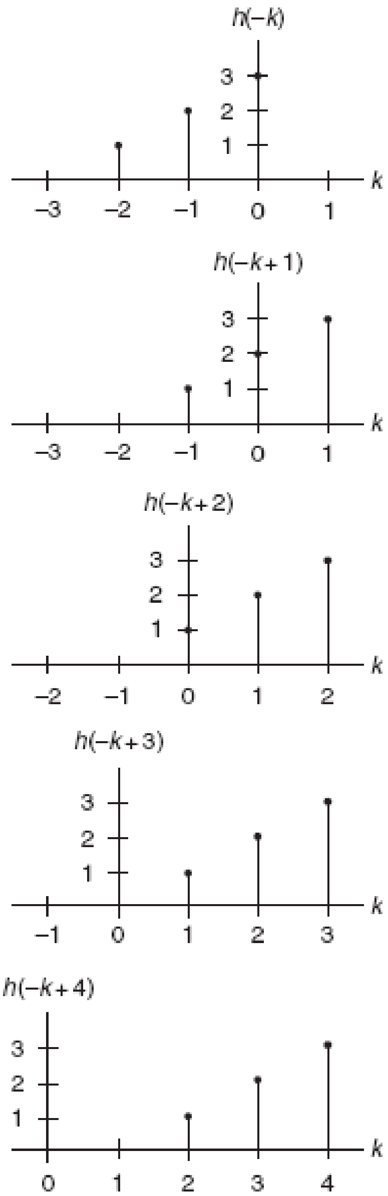
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

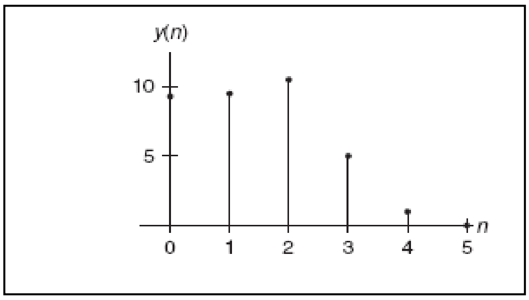
$N = N_1 + N_2 - 1$. Where N_1 = number of samples of x(n), N_2 = number of samples of h(n), and N = total number of samples.

3.8.1 Graphical method:



Example7: Find $y(n) = x(n) \otimes h(n)$ using graphical method





$$n = 0,\ y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3 \times 3 + 1 \times 0 + 2 \times 0 = 9,$$

$$n = 1,\ y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3 \times 2 + 1 \times 3 + 2 \times 0 = 9,$$

$$n = 2,\ y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11,$$

$$n = 3,\ y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3 \times 0 + 1 \times 1 + 2 \times 2 = 5.$$

$$n = 4,\ y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3 \times 0 + 1 \times 0 + 2 \times 1 = 2,$$

$$n \geq 5,\ y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3 \times 0 + 1 \times 0 + 2 \times 0 = 0.$$

3.8.2 Table lookup method

y(0) = 9

y(1) = 9

y(2) = 11

y(3) = 5

y(4) = 2

	3	2	1
3	9	6	3
1	3	2	1
2	6	4	2

3.8.3 Matrix by Vector method

Example7: If x(n) = [0.5 0.5 0.5], and h(n) = [3 2 1]

$$\begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2.5 \\ 3 \\ 1.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

3.8.4 Linear convolution and circular convolution

Linear convolution:

$$x_1(n) \otimes x_2(n) = \sum_{k=-\infty}^{\infty} x_1(n-k)\, x_2(k) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \qquad (3.14)$$

Circular convolution:

$$x_1(n) \otimes_N x_2(n) = \sum_{k=0}^{N-1} x_1(\,(n-k) \quad \mathrm{mod} N)\, x_2(k) = \sum_{k=0}^{N-1} x_1(k)\, x_2(\,(n-k)\, \mathrm{mod} N) \qquad (3.15)$$

If both $x_1(n)$ and $x_2(n)$ are of *finite length* N_1 and N_2 and defined on [0 N_1-1], and [0 N_2-1] respectively, the value of N needed so that circular and linear convolution are the same on [0 N-1] is : $N \geq N_1 + N_2 - 1$

Example 8: If x(n) = [1 2 3 2], and h(n) = [1 1 2]. Find y(n) such that linear and circular convolution are the same.

Solution:

N = 4 + 3 – 1 = 6

Then x(n) = [1 2 3 2 0 0] and h(n) = [1 1 2 0 0 0] x(n) is arranged in clockwise direction (italic numbers),while h(n) is arranged in the opposite clockwise direction (bold numbers). Each time, <u>only</u> h(n) will be shifted with the ***clockwise direction*** to find y(n). <u>*Note*</u>: the reference point is * and, the arrows represent multiplication process. Finally, addition process is performed.

2	**1**	**1***
0	*0*	*1**
2	*3*	*2*
0	**0**	**0**

y(0) = 1(1) = 1

0	**2**	**1***
0	*0*	*1**
2	*3*	*2*
0	**0**	**1**

y(1)= 1(1) + 2(1) = 3

0	**0**	**2***
0	*0*	*1**
2	*3*	*2*
0	**1**	**1**

y(2)= 2(1)+2(1)+3(1)=7

0	**0**	**0***
0	*0*	*1**
2	*3*	*2*
1	**1**	**2**

y(3)= 2(2)+3(1)+2(1)=9

1	**0**	**0***
0	*0*	*1**
2	*3*	*2*
1	**2**	**0**

y(4)= 3(2)+2(1)= 8

1	**1**	**0***
0	*0*	*1**
2	*3*	*2*
2	**0**	**0**

y(5)= 2(2)= 4

Using table lookup method:

y(0)= 1
y(1)= 3
y(2)= 7
y(3)= 9
y(4)= 8
y(5)= 4

	1	1	2
1	1	1	2
2	2	2	4
3	3	3	6
2	2	2	4

<u>**Example(9):**</u> Use graphical method to find circular convolution $x_1(n) \otimes_N x_2(n)$, if N = 4, x_1(n)= [1 2 2 0] and x_2(n)= [0 1 2 3]

Solution: Applying eq. (3.15), then

$$y(n) = \sum_{k=0}^{3} x_1(k)\ x_2(\ (n-k) \bmod 4\ )$$

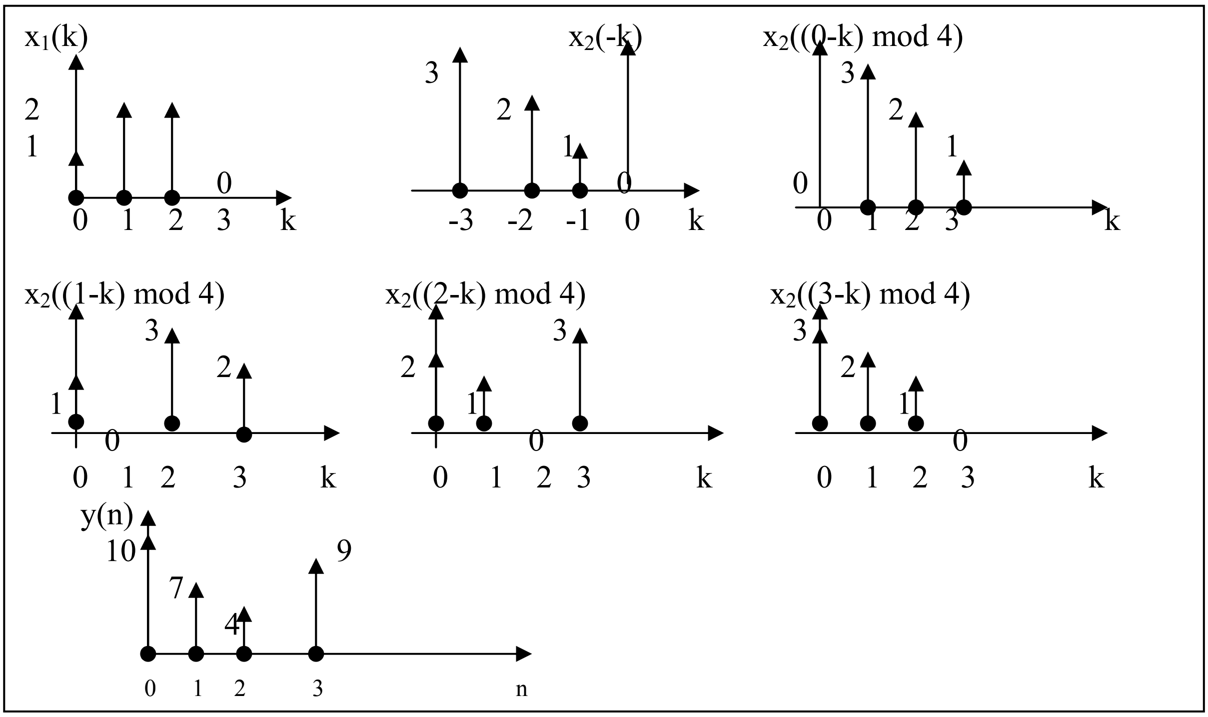
$$y(0) = \sum_{k=0}^{3} x_1(k)\ x_2(\ (-k) \bmod 4\ )$$

y(0) = x_1(0) x_2(-0 ● 4) + x_1(1) x_2(-1 ● 4) + x_1(2) x_2(-2 ● 4) + x_1(3) x_2(-3 ● 4)

● = mod addition

$y(0) = x_1(0)\,x_2(0) + x_1(1)\,x_2(3) + x_1(2)\,x_2(2) + x_1(3)\,x_2(1) = 1(0) + 2(3) + 2(2) + 0\,(1) = 10$

And so on



3.9 Deconvolution:

3.9.1 Iterative approach

Using equation (3.14) and assuming causal system (started at k =0), then:

$y(0) = x(0)\,h(0)$, then $x(0) = y(0) / h(0)$

$y(1) = h(1)\,x(0) + h(0)\,x(1)$, then $x(1) = (\,y(1) – h(1)\,x(0)\,) / h(0)$

3.9.2 Polynomial Approach:

A long division process is applied between two polynomials. For causal system, the remainder is always *zero*.

If y(n) = [12 10 14 6] and h(n) = [4 2]

Then $y = 12 + 10\,x + 14\,x^2 + 6\,x^3$, and $h = 4 + 2\,x$. Applying long division, we obtain i/p = $3 + x + 3\,x^2$. Then x(n) = [3 1 3]

3.9.3 Graphical method

$$\begin{bmatrix} 4 & \times & b_0 & 12 \\ 2 & & & 10 \\ & & & 14 \\ & & & 6 \end{bmatrix} \quad \begin{bmatrix} 4 & \times & b_1 & 12 \\ 2 & \times & 3 & 10 \\ & & & 14 \\ & & & 6 \end{bmatrix} \quad \begin{bmatrix} 4 & \times & b_2 & 12 \\ 2 & \times & 1 & 10 \\ 0 & \times & 3 & 14 \\ & & & 6 \end{bmatrix} \quad \begin{bmatrix} 4 & \times & b_3 & 12 \\ 2 & \times & 3 & 10 \\ 0 & \times & 1 & 14 \\ 0 & \times & 3 & 6 \end{bmatrix}$$

$4\,b_0 = 12$	$4\,b_1 + 2(3) = 10$	$4\,b_2 + 2(1) + 0\,(3)=14$	$4\,b_3 +6+0+0 = 6$
$b_0 = 3$	$b_1 = 1$	$b_2 = 3$	$b_3 = 0$

So, x(n) = [3 1 3]