Problem 17

Consider an electric circuit containing a capacitor, resistor, and battery; see Figure 1.2.3. The charge $Q(t)$ on the capacitor satisfies the equation[5]

$$R\frac{dQ}{dt} + \frac{Q}{C} = V,$$

where R is the resistance, C is the capacitance, and V is the constant voltage supplied by the battery.

(a) If $Q(0) = 0$, find $Q(t)$ at any time t, and sketch the graph of Q versus t.

(b) Find the limiting value Q_L that $Q(t)$ approaches after a long time.

(c) Suppose that $Q(t_1) = Q_L$ and that at time $t = t_1$ the battery is removed and the circuit is closed again. Find $Q(t)$ for $t > t_1$ and sketch its graph.

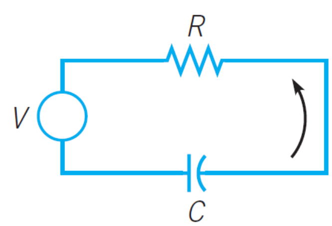

FIGURE 1.2.3 The electric circuit of Problem 17.

Solution

Part (a)

$$RQ' + \frac{Q}{C} = V$$

Bring Q/C to the right side.

$$RQ' = -\frac{Q}{C} + V$$

Divide both sides by R.

$$\begin{aligned} Q' &= -\frac{Q}{RC} + \frac{V}{R} \\ &= -\frac{1}{RC}(Q - CV) \end{aligned}$$

Divide both sides by $Q - CV$.

$$\frac{Q'}{Q - CV} = -\frac{1}{RC}$$

The left side can be written as $d/dt(\ln|Q - CV|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}\ln|Q - CV| = -\frac{1}{RC}$$

[5]This equation results from Kirchhoff's laws, which are discussed in Section 3.7.

Integrate both sides with respect to t.

$$\ln|Q - CV| = -\frac{1}{RC}t + C_1$$

Exponentiate both sides.

$$\begin{aligned} |Q - CV| &= \exp\left(-\frac{1}{RC}t + C_1\right) \\ &= e^{C_1}\exp\left(-\frac{1}{RC}t\right) \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$Q(t) - CV = \pm e^{C_1}\exp\left(-\frac{1}{RC}t\right)$$

Let $A = \pm e^{C_1}$ and add CV to both sides to solve for $Q(t)$.

$$Q(t) = CV + A\exp\left(-\frac{1}{RC}t\right)$$

Apply the initial condition here to determine A.

$$Q(0) = CV + A = 0 \quad \rightarrow \quad A = -CV$$

Therefore,

$$\begin{aligned} Q(t) &= CV - CV\exp\left(-\frac{1}{RC}t\right). \\ &= CV\left[1 - \exp\left(-\frac{1}{RC}t\right)\right]. \end{aligned}$$

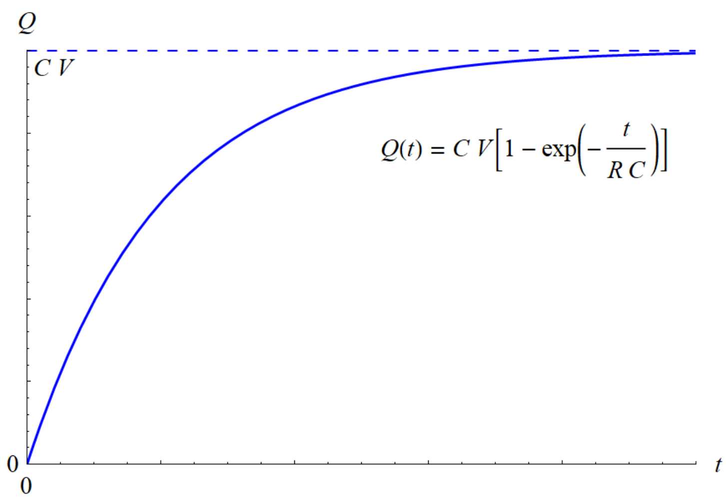

Figure 1: This figure shows the charge on the (charging) capacitor $Q(t)$ versus time t.

Part (b)

$$Q_L = \lim_{t\to\infty} Q(t) = \lim_{t\to\infty} CV\left[1 - \underbrace{\exp\left(-\frac{1}{RC}t\right)}_{=0}\right] = CV$$

Alternatively, we could argue that once the limiting value of charge on the capacitor is reached, equilibrium is reached and $dQ/dt = 0$. The original ODE becomes

$$0 + \frac{Q_L}{C} = V,$$

where we conclude that $Q_L = CV$ as $t \to \infty$.

Part (c)

The initial value problem here is the same as before but now $V = 0$ because the battery has been removed.

$$R\frac{dQ}{dt} + \frac{Q}{C} = 0, \quad Q(0) = Q_L = CV$$

Bring Q/C to the right side.

$$RQ' = -\frac{Q}{C}$$

Divide both sides by RQ.

$$\frac{Q'}{Q} = -\frac{1}{RC}$$

The left side can be written as $d/dt(\ln Q)$ by the chain rule.

$$\frac{d}{dt}\ln Q = -\frac{1}{RC}$$

Integrate both sides with respect to t.

$$\ln Q = -\frac{1}{RC}t + C_2$$

Exponentiate both sides.

$$\begin{aligned} Q(t) &= \exp\left(-\frac{1}{RC}t + C_2\right) \\ &= e^{C_2}\exp\left(-\frac{1}{RC}t\right) \end{aligned}$$

Let $B = e^{C_2}$.

$$Q(t) = B\exp\left(-\frac{1}{RC}t\right)$$

Apply the initial condition to determine B.

$$Q(0) = B = CV$$

Therefore,

$$Q(t) = CV\exp\left(-\frac{1}{RC}t\right), \quad t > t_1.$$

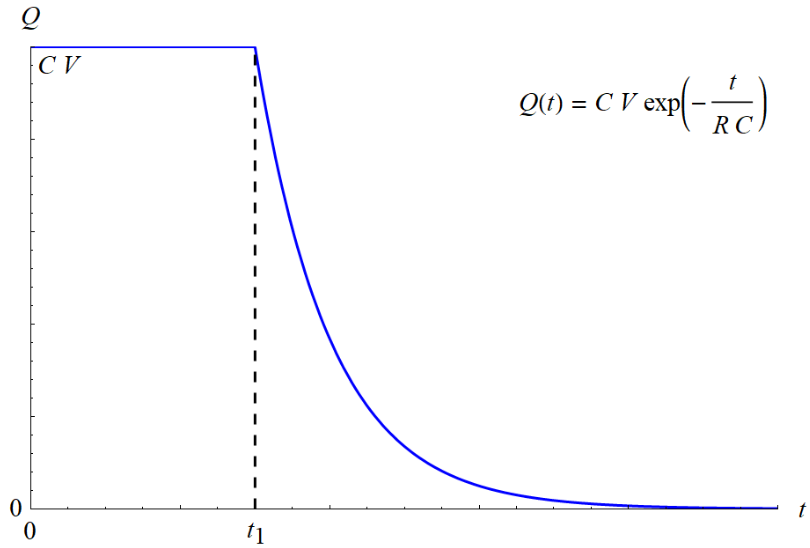

Figure 2: This figure shows the charge on the (discharging) capacitor $Q(t)$ versus time t.