Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

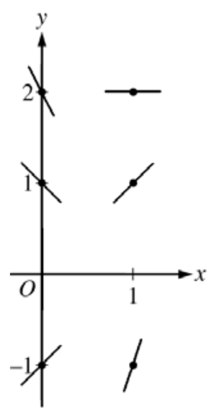
(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

(a)



$2 : \begin{cases} 1 : \text{slopes where } x = 0 \\ 1 : \text{slopes where } x = 1 \end{cases}$

(b) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y$

In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$.
Therefore, all solution curves are concave up in Quadrant II.

$2 : \begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{concave up with reason} \end{cases}$

(c) $\left.\frac{dy}{dx}\right|_{(x,\, y)=(2,\, 3)} = 2(2) - 3 = 1 \neq 0$

Therefore, f has neither a relative minimum nor a relative maximum at $x = 2$.

$2 : \begin{cases} 1 : \text{considers } \left.\frac{dy}{dx}\right|_{(x,\, y)=(2,\, 3)} \\ 1 : \text{conclusion with justification} \end{cases}$

(d) $y = mx + b \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(mx + b) = m$

$2x - y = m$

$2x - (mx + b) = m$

$(2 - m)x - (m + b) = 0$

$2 - m = 0 \Rightarrow m = 2$

$b = -m \Rightarrow b = -2$

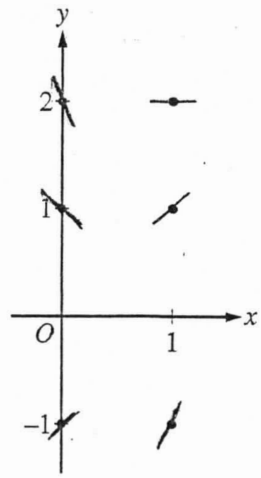
Therefore, $m = 2$ and $b = -2$.

$3 : \begin{cases} 1 : \frac{d}{dx}(mx + b) = m \\ 1 : 2x - y = m \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

$$\frac{dy}{dx} = 2x - y$$

$$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y)$$

$$\boxed{\frac{d^2y}{dx^2} = 2 - 2x + y}$$

In Quadrant II, $x < 0$ and $y > 0$,

so $\frac{d^2y}{dx^2} = 2 - 2x + y > 0$,

Thus $\boxed{\text{all solution curves in Quadrant II are concave up.}}$

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NO CALCULATOR ALLOWED

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

$$\frac{dy}{dx} = 2x - y = 2 \cdot 2 - 3 = 1$$

$\boxed{\text{Neither}}$, as $\frac{dy}{dx} \neq 0$ at $x = 2$.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

$\frac{dy}{dx} = 2x - y, \quad y = mx + b$

$\frac{dy}{dx} = m = 2x - y$

$m = 2x - (mx + b)$

$m = (2 - m)x - b$, equate coefficients

$2 - m = 0$

$m = 2$

$-b = m$

$b = -m = -2$

$\boxed{m = 2, b = -2}$

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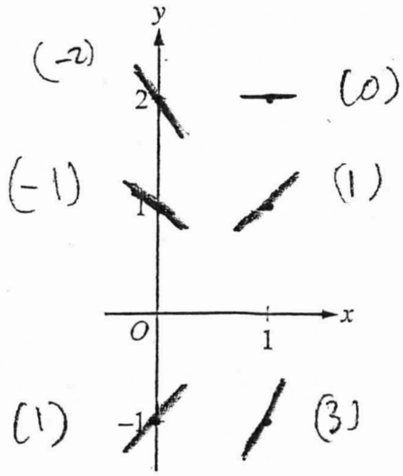
4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

$$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = 2 - 2x + y$$

always negative

always positive

−+ | ++

−− | +−

$$= 2 - 2(x) + y$$

$$= 2 + 2|x| + y = > 0$$

∴ concavity in Quadrant II is always concave up

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Continue problem 4 on page 17.

4B2
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(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

$\frac{d^2y}{dx^2} = 2 - 2(2) + 3 = 2 - 4 + 3 = -2 + 3 = 1 > 0$

concave up means minimum

According to the second derivative test, at $x = 2$ the second derivative is positive and $\therefore$ there is a minimum at that point

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

$m = 2x - y$

$y = (2x - y)x + b$

$y = 2x^2 - \cancel{yx} + b$

$y = 2x - \frac{dy}{d\cancel{x}}$

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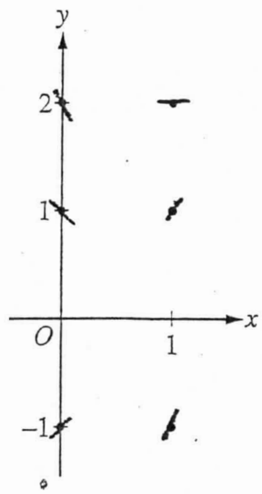
4C1

1 of 2

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

$y \backslash x$	0	1
-1	1	3
1	-1	1
2	-2	0



(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

$\frac{dy}{dx} = 2x - y$

$\frac{d^2y}{dx^2} = 2 - y'$

$0 = 2 - y'$

$0 = 2 - (2x - y)$

$0 = 2 - 2x + y$

$y = 2x - 2$

$x = \frac{y+2}{2}$

$2x - 2 = 0$

$2x = 2$

$x = 1$

$x = \frac{y+2}{2}$

$0 = y + 2$

$y = 2$

concave down at $(1, 2)$ because its a relative max.

Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

$\frac{dy}{dx} = 2x - y$

$\int dy + y = \int 2x\,dx$

$y + y' = x^2$

$y' = x^2 - y$

$y' = (2)^2 - (3)$

$y' = 1$

f has a relative min at $x = 2$ because at $f(2)$ it is 3.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

$\frac{dy}{dx} = 2x - y$

$\int dy + y = \int 2x\,dx$

$y + y' = x^2 + c$

$y' = x^2 - y + c$

$y' = (2)^2 - 3 + c$

$y' = c$

$y - 3 = m(x - 2)$

$y = x + 1$

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Question 4

Overview

In this problem students were to consider the first-order differential equation $\frac{dy}{dx} = 2x - y.$ In part (a) students were given an *xy*-plane with 6 labeled points and were expected to sketch a slope field by drawing a short line segment at each of the six points with slopes of $2x - y.$ In part (b) students needed to use implicit differentiation and the fact that $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ to obtain $\frac{d^2y}{dx^2} = 2 - 2x + y.$ Students were expected to explain that for points in Quadrant II, $x < 0$ and $y > 0$ so $\frac{d^2y}{dx^2} > 0.$ Thus, any solution curve for the differential equation that passes through a point (x, y) in Quadrant II must be concave up at $(x, y).$ In part (c) students were asked to consider the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 3.$ Students had to determine if $(2, 3)$ is the location of a relative minimum, a relative maximum, or neither for f and justify the answer. Students were expected to show that $\frac{dy}{dx} \neq 0$ at $(2, 3)$ and conclude that $(2, 3)$ is neither the location of a relative minimum nor a relative maximum. In part (d) students were asked to find the values of the constants m and b so that the linear function $y = mx + b$ satisfies the differential equation $\frac{dy}{dx} = 2x - y.$ Students were expected to show that if $y = mx + b,$ then $\frac{dy}{dx} = m.$ Using a substitution in $\frac{dy}{dx} = 2x - y$ leads to $2x - y = m$ and thus $2x - (mx + b) = m.$ This equation enabled the student to find the values of m and $b.$

Sample: 4A
Score: 9

The response earned all 9 points.

Sample: 4B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In part (a) the student's work is correct. In part (b) the student's work is correct, including correct reasoning about the sign of the second derivative using the signs of x and y in Quadrant II. In part (c) the student does not consider $\frac{dy}{dx},$ so the first point was not earned. The student considers $\frac{d^2y}{dx^2},$ which cannot be used as justification, and the student incorrectly identifies $x = 2$ as a minimum. The second point was not earned. In part (d) the student earned the first 2 points for declaring that $m = 2x - y.$ In doing so, the student communicates that the derivative of the linear function is its slope m (the first point) and connects the differential equation and its linear solution by equating the derivatives (the second point). The student does not arrive at an answer.

Sample: 4C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first point for the correct second derivative in x and y, shown in the work where the student writes $0 = 2 - (2x - y).$ In part (c) the student incorrectly solves

Question 4 (continued)

the differential equation and then, from that work, finds an incorrect expression for the first derivative. The student is not eligible for any points. In part (d) the student attempts to solve the differential equation by separation of variables and uses the point $(2, 3)$, which is not relevant to the question asked.