Question 2

Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2 - 2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.

(a) Find the sum of the areas of regions R and S.

(b) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

(c) Let h be the vertical distance between the graphs of f and g in region S. Find the rate at which h changes with respect to x when $x = 1.8$.
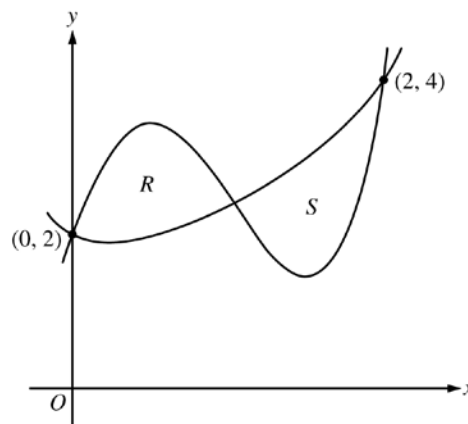
(a) The graphs of $y = f(x)$ and $y = g(x)$ intersect in the first quadrant at the points $(0, 2)$, $(2, 4)$, and $(A, B) = (1.032832, 2.401108)$.

$$\text{Area} = \int_0^A [g(x) - f(x)]\, dx + \int_A^2 [f(x) - g(x)]\, dx$$
$$= 0.997427 + 1.006919 = 2.004$$

$4 : \begin{cases} 1 : \text{limits} \\ 2 : \text{integrands} \\ 1 : \text{answer} \end{cases}$

(b) $\text{Volume} = \int_A^2 [f(x) - g(x)]^2\, dx = 1.283$
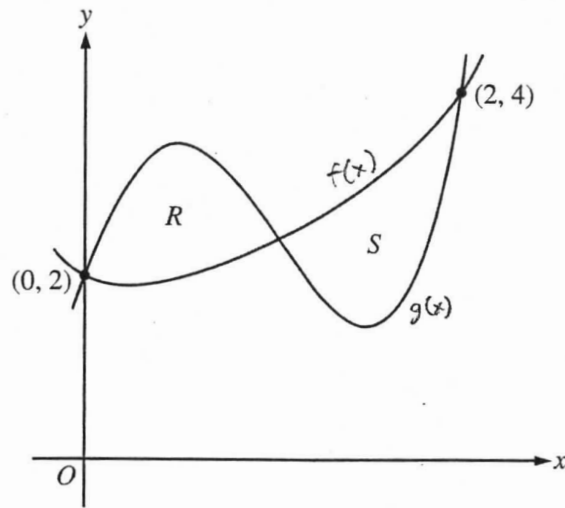
$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $h(x) = f(x) - g(x)$
$h'(x) = f'(x) - g'(x)$
$h'(1.8) = f'(1.8) - g'(1.8) = -3.812$ (or -3.811)

$2 : \begin{cases} 1 : \text{considers } h' \\ 1 : \text{answer} \end{cases}$

2. Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2 - 2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.

(a) Find the sum of the areas of regions R and S.

$$f(a) = g(a)$$

$$a = 1.032831888364 1$$

$$\int_0^a \left[g(x) - f(x) \right] dx \; + \; \int_a^2 \left[f(x) - g(x) \right] dx = \boxed{2.004}$$

-6-

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(b) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Find the ~~vertick~~ volume of the solid.

Cross section area $A(x) = \left(f(x) - g(x)\right)^2$ $a = 1.0328318883641$

$$\int_a^2 A(x)\, dx = \boxed{1.283}$$

(c) Let h be the vertical distance between the graphs of f and g in region S. Find the rate at which h changes with respect to x when $x = 1.8$.
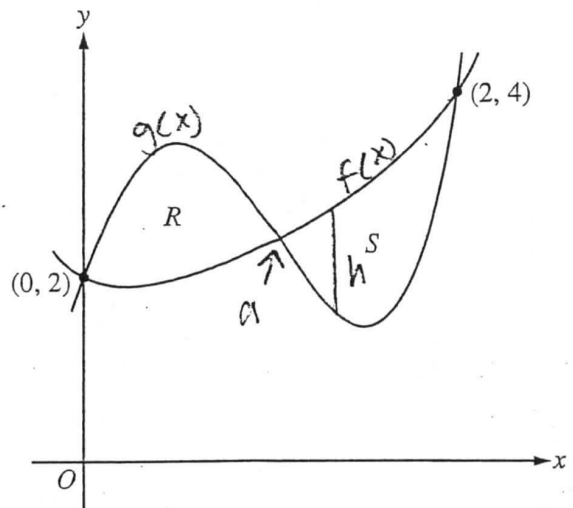
$$f'(1.8) = 2.1162821217136$$

$$g'(1.8) = 5.928$$

$$\frac{dh}{dx} = f'(1.8) - g'(1.8) = \boxed{-3.812}$$

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2. Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2 - 2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.

(a) Find the sum of the areas of regions R and S.

$$a = 1.0328319$$

$$R = \int_0^a (g(x) - f(x))\,dx = 0.997 \text{ un}^2$$

$$S = \int_a^2 (f(x) - g(x))\,dx = 1.007 \text{ un}^2$$

-6-

(b) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

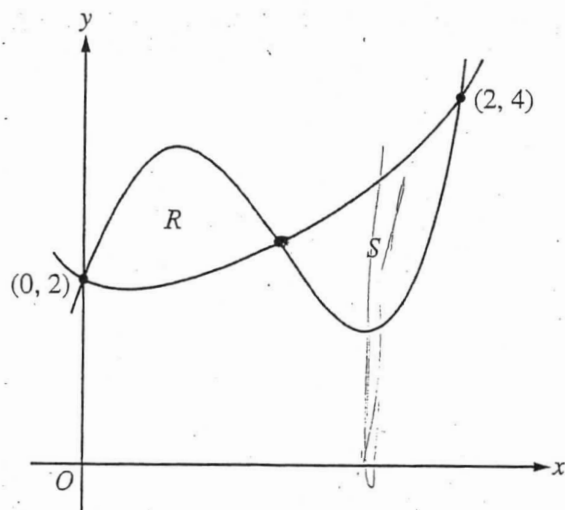$$\text{Volume} = \int_a^2 \left((f(x) - g(x))^2 \right) dx = 1.283 \, un^3$$

(c) Let h be the vertical distance between the graphs of f and g in region S. Find the rate at which h changes with respect to x when $x = 1.8$.

$$h = \int_a^2 (f(x) - g(x)) \, dx$$

$$\frac{dh}{dt} = \int_a^2 (f'(x) - g'(x)) \, dx =$$

$$\frac{dh}{dt} = f(1.8) - g(1.8) = 20.449$$

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2. Let f and g be the functions defined by $f(x) = 1 + x + e^{x^2 - 2x}$ and $g(x) = x^4 - 6.5x^2 + 6x + 2$. Let R and S be the two regions enclosed by the graphs of f and g shown in the figure above.

(a) Find the sum of the areas of regions R and S.

$$\int_0^1 \left((x^4 - 6.5x^2 + 6x + 2) - (1 + x + e^{x^2 - 2x}) \right) dx$$

$$\approx 1$$

$$\int_1^2 \left((1 + x + e^{x^2 - 2x}) - (x^4 - 6.5x^2 + 6x + 2) \right) dx$$

$$\approx 1$$

$$= \boxed{2}$$

-6-

(b) Region S is the base of a solid whose cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

$$\pi \int_1^2 \left(\left(1 + x + e^{x^2 - 2x}\right)^2 - \left(x^4 - 6.5x^2 + 6x + \tfrac{3}{2}\right) \right)$$

(c) Let h be the vertical distance between the graphs of f and g in region S. Find the rate at which h changes with respect to x when $x = 1.8$.

$$h = f - g$$
$$h' = f' - g'$$
$$h'(1.8) = x + e^{x^2 - 2x} - 4x^3 - 13x + 6$$
$$h'(1.8) = -38.23$$

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Question 2

Overview

In this problem students were given a graph of the boundary curves of two planar regions R and S in the first quadrant. One boundary curve is defined by $f(x) = 1 + x + e^{x^2 - 2x}$, and the other boundary is defined by $g(x) = x^4 - 6.5x^2 + 6x + 2$. In part (a) students were asked to find the sum of the areas of regions R and S. Two intersection points of the boundary curves, $(0, 2)$ and $(2, 4)$, are given, and students were expected to find the other point of intersection by using the calculator. The intersection point is $(A, B) = (1.032832, 2.401108)$. The sum of the areas of R and S is $\int_0^A (g(x) - f(x))\, dx + \int_A^2 (f(x) - g(x))\, dx$. Students were expected to use the calculator to evaluate the integrals. In part (b) students were asked to find the volume of a solid with S as its base. Students had to interpret the area of the cross sections as $[f(x) - g(x)]^2$, and use the calculator to evaluate the volume as $\int_A^2 [f(x) - g(x)]^2\, dx$. In part (c) students had to find the rate of change of the vertical distance, h, between the graphs of f and g at $x = 1.8$. Students were expected to recognize and communicate $h'(x) = f'(x) - g'(x)$, then evaluate $h'(1.8)$ using the numerical derivative at a point capability of the calculator.

Sample: 2A
Score: 9

The response earned all 9 points.

Sample: 2B
Score: 6

The response earned 6 points: 3 points in part (a), 3 points in part (b), and no points in part (c). In part (a) the student presents correct integrals for the areas of the two regions and earned the first 3 points. The student evaluates the areas of the two regions correctly. The student does not find the sum and did not earn the answer point. In part (b) the student's work is correct. In part (c) the student presents an incorrect expression for h'.

Sample: 2C
Score: 3

The response earned 3 points: 2 points in part (a), no points in part (b), and 1 point in part (c). In part (a) the student uses $x = 1$ as the x-coordinate of the point of intersection. The student did not earn the first point. For each of the regions, the student presents the correct integrand, so the second and third points were earned. The student is not eligible for the answer point. In part (b) the student presents an incorrect integrand. In part (c) the student considers $h' = f' - g'$ and earned the first point. The evaluation of $h'(1.8)$ is incorrect.