Question 4

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1}\left(\frac{g(x) - 2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

(a) $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$	2 : $\frac{d^2y}{dx^2}$ in terms of x and y
(b) $\left.\frac{dy}{dx}\right\rvert_{(x,\,y)=(-2,\,8)} = (-2)^2 - \frac{1}{2}\cdot 8 = 0$ $\left.\frac{d^2y}{dx^2}\right\rvert_{(x,\,y)=(-2,\,8)} = 2(-2) - \frac{1}{2}\left((-2)^2 - \frac{1}{2}\cdot 8\right) = -4 < 0$ Thus, the graph of f has a relative maximum at the point $(-2, 8)$.	2 : conclusion with justification
(c) $\lim_{x \to -1}(g(x) - 2) = 0$ and $\lim_{x \to -1} 3(x+1)^2 = 0$ Using L'Hospital's Rule, $\lim_{x \to -1}\left(\frac{g(x) - 2}{3(x+1)^2}\right) = \lim_{x \to -1}\left(\frac{g'(x)}{6(x+1)}\right)$ $\lim_{x \to -1} g'(x) = 0$ and $\lim_{x \to -1} 6(x+1) = 0$ Using L'Hospital's Rule, $\lim_{x \to -1}\left(\frac{g'(x)}{6(x+1)}\right) = \lim_{x \to -1}\left(\frac{g''(x)}{6}\right) = \frac{-2}{6} = -\frac{1}{3}$	3 : $\begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$
(d) $h\left(\frac{1}{2}\right) \approx h(0) + h'(0)\cdot\frac{1}{2} = 2 + (-1)\cdot\frac{1}{2} = \frac{3}{2}$ $h(1) \approx h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right)\cdot\frac{1}{2} \approx \frac{3}{2} + \left(-\frac{1}{2}\right)\cdot\frac{1}{2} = \frac{5}{4}$	2 : $\begin{cases} 1 : \text{Euler's method} \\ 1 : \text{approximation} \end{cases}$

NO CALCULATOR ALLOWED

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

$$\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx}$$

$$= 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$$

$$= \boxed{2x - \frac{1}{2}x^2 + \frac{1}{4}y}$$

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

$$\left.\frac{d^2y}{dx^2}\right|_{(-2,8)} = 2(-2) - \frac{1}{2}(-2)^2 + \frac{1}{4}(8)$$

$$= -4 - 2 + 2 = -4$$

$$\left.\frac{dy}{dx}\right|_{(-2,8)} = (-2)^2 - \frac{1}{2}(8) =$$

$$= 4 - 4 = 0$$

f has a relative maximum at point $(-2,8)$ because $\frac{dy}{dx} = 0$ at $(-2,8)$ and $\frac{d^2y}{dx^2} < 0$ at $(-2,8)$.

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(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1}\left(\frac{g(x) - 2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

$$= \lim_{x\to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right) ; \frac{0}{0}$$

$$g'(x)\Big|_{(-1,2)} = \frac{dy}{dx}\Big|_{(-1,2)} = (-1)^2 - \frac{1}{2}(2) = 1 - 1 = 0$$

$$= \lim_{x\to -1}\left(\frac{g'(x)}{3\cdot 2(x+1)}\right) ; \frac{0}{0}$$

$$g''(x) = \frac{d^2y}{dx^2}\Big|_{(-1,2)} = 2(-1) - \frac{1}{2}\cdot 0 = -2$$

$$= \lim_{x\to -1}\left(\frac{g''(x)}{6}\right) = \frac{-2}{6} = \boxed{-\frac{1}{3}}$$

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

$$\frac{dy}{dx}\Big|_{(0,2)} = 0 - \frac{1}{2}(2) = -1$$

$$h(0.5) \approx 2 + (0.5)(-1) = 2 - 0.5 = 1.5$$

$$\frac{dy}{dx}\Big|_{(0.5,1.5)} = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{3}{2}\right) = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$$

$$h(1) \approx 1.5 + (0.5)\left(-\frac{1}{2}\right) = 1.5 - 0.25 = 1.25$$

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NO CALCULATOR ALLOWED

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

$\frac{dy}{dx} = x^2 - \frac{1}{2}y$

$\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx}$

$\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

$\frac{dy}{dx} = y' = f'(x$

$f'(x)$ 0 at -2

$\frac{dy}{dx} = x^2 - \frac{1}{2}y$

$\frac{dy}{dx} - x^2 = -\frac{1}{2}y$

$-2\frac{dy}{dx} + 2x^2 = y$

@ $x = -2$ $\frac{dy}{dx} = (-2)^2 - \frac{1}{2}(8)$

$= 4 - 4 = 0$

@ $x = -3$

@ $x = 0$

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Continue problem 4 on page 17.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x\to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

$g'' = y'' = \frac{d^2y}{dx^2}$

$\lim_{x\to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right) \xrightarrow{\text{L'Hopital}} \lim_{x\to -1}\frac{g'(x)}{6(x+1)} \overset{\text{L'Hopital}}{\Rightarrow} \lim_{x\to -1}\frac{g''(x)}{6} = \frac{-2}{6} = -\frac{1}{3}$

$\frac{d^2y}{dx^2}(-1) = 2(-1) - \frac{1}{2}\left((-1)^2 - \frac{1}{2}(2)\right)$

$= -2 - \cancel{\frac{1}{2}(1-1)}^{0}$

$= -2$

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

$h(0) = 2$

$h(0.5) = 2 + m\Delta x = 2 + (0.5)(-1) = 1.5$

$h(1) = 1.5 + m\Delta x = 2 + (0.5)(.5) = 2 - .25 = 1.75$

$y'(0) = 0 - \frac{1}{2}(2) = -1$

$y'(1.5) = (0.5)^2 - \frac{1}{2}(1.5)$

$= 0.25 - .75 = .5$

1.5
$.5$
$.25$

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

$$\boxed{\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx}}$$

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

$$\left.\frac{dy}{dx}\right|_{(-2,8)} = 4 - 4 = 0$$

the graph of f has a relative maximum at the point (-2, 8).

$$\left.\frac{dy}{dx}\right|_{(-3,8)} = 9 - 4 = 5 \leftarrow \text{pos} \therefore \text{inc}$$

$$\left.\frac{dy}{dx}\right|_{(\cancel{-3}\,-1,8)} = 1 - 4 = -3 \leftarrow \text{neg} \therefore \text{dec}$$

the graph goes from increasing to decreasing at (-2, 8), so the point is a relative maximum.

Continue problem 4 on page 17.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = 1 - 1 = 0$$

$$\lim_{x \to -1}\left(\frac{0-2}{3(x+1)^2}\right) = \lim_{x \to -1}\left(\frac{-2}{3(x+1)^2}\right) = \boxed{\frac{-2}{3}}$$

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

step size $= 1/2$

$$h\left(\tfrac{1}{2}\right) = h(0) + \left(0^2 - \tfrac{1}{2}(2)\right)\left(\tfrac{1}{2}\right)$$

$$h\left(\tfrac{1}{2}\right) = 2 + \left(-\tfrac{1}{2}\right) = \tfrac{3}{2}$$

$$h(1) \approx h\left(\tfrac{1}{2}\right) + \left(\left(\tfrac{1}{2}\right)^2 - \tfrac{1}{2}\left(\tfrac{3}{2}\right)\right)\left(\tfrac{1}{2}\right)$$

$$h(1) \approx \tfrac{3}{2} + \left(\tfrac{1}{4} - \tfrac{3}{4}\right)\left(\tfrac{1}{2}\right) \approx \tfrac{3}{2} + \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)$$

$$\tfrac{3}{2} - \tfrac{1}{4} = \tfrac{6}{4} - \tfrac{1}{4} = \tfrac{5}{4}$$

$$\boxed{h(1) \approx \tfrac{5}{4}}$$

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Question 4

Overview

In this problem students were presented with a differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$. In part (a) students needed to find the second derivative $\frac{d^2y}{dx^2}$ in terms of x and y using implicit differentiation. In part (b) students were given that $y = f(x)$ is the particular solution to the differential equation whose graph passes through $(-2, 8)$. Students needed to use both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ to determine if the graph of f has a relative minimum, a relative maximum, or neither at the point $(-2, 8)$, and justify their answer. In part (c) students were given that $y = g(x)$ is the particular solution to the differential equation with $g(-1) = 2$. Students needed to compute a limit. After indicating that L'Hospital's Rule is required, students were expected to apply L'Hospital's Rule twice. In part (d) students were given that $y = h(x)$ is the particular solution to the differential equation with $h(0) = 2$. Students needed to use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

Sample: 4A
Score: 9

The response earned all 9 points.

Sample: 4B
Score: 6

The response earned 6 points: 2 points in part (a), no points in part (b), 3 points in part (c), and 1 point in part (d). In part (a) the student's work is correct. In part (b) the student correctly evaluates the first derivative at $(-2, 8)$. The student does not consider values of the second derivative and was not eligible for any points. Note that for this problem, use of the First Derivative Test to justify a correct conclusion is very difficult. The Second Derivative Test is much more appropriate here. In part (c) the student's work is correct. In part (d) the student finds a correct approximation for $h(0.5)$ and sets up a second iteration of Euler's method. The student earned the first point. The student makes an error in computing the second iteration and has an incorrect approximation.

Sample: 4C
Score: 3

The response earned 3 points: 1 point in part (a), no points in part (b), no points in part (c), and 2 points in part (d). In part (a) the student differentiates correctly with respect to x and earned 1 point. The student does not present the second derivative in terms of x and y. In part (b) the student correctly evaluates the first derivative at $(-2, 8)$. The student does not consider values of the second derivative and was not eligible for any points. Note that for this problem, use of the First Derivative Test to justify a correct conclusion is very difficult. The Second Derivative Test is much more appropriate here. In part (c) the student makes an incorrect attempt to evaluate the limit and earned no points. In part (d) the student's work is correct.