AP® CALCULUS AB
2016 SCORING GUIDELINES

Question 4

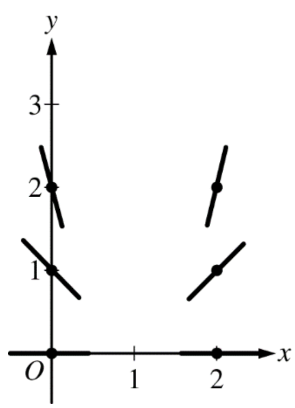
Consider the differential equation $\dfrac{dy}{dx} = \dfrac{y^2}{x - 1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \end{cases}$

(b) $\left.\dfrac{dy}{dx}\right|_{(x,\, y)=(2,\, 3)} = \dfrac{3^2}{2 - 1} = 9$

An equation for the tangent line is $y = 9(x - 2) + 3$.

$f(2.1) \approx 9(2.1 - 2) + 3 = 3.9$

$2 : \begin{cases} 1 : \text{tangent line equation} \\ 1 : \text{approximation} \end{cases}$

(c) $\dfrac{1}{y^2}\,dy = \dfrac{1}{x - 1}\,dx$

$$\int \frac{1}{y^2}\,dy = \int \frac{1}{x - 1}\,dx$$

$$-\frac{1}{y} = \ln|x - 1| + C$$

$$-\frac{1}{3} = \ln|2 - 1| + C \Rightarrow C = -\frac{1}{3}$$

$$-\frac{1}{y} = \ln|x - 1| - \frac{1}{3}$$

$$y = \frac{1}{\frac{1}{3} - \ln(x - 1)}$$

Note: This solution is valid for $1 < x < 1 + e^{1/3}$.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad\ \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

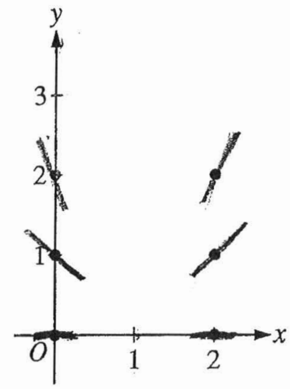
Note: max $3/5$ [1-2-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

NO CALCULATOR ALLOWED

4. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

$$\frac{dy}{dx} = \frac{y^2}{x-1}\Big|_{(x,y)=(2,3)} = \frac{9}{2-1} = \frac{9}{1} = 9$$

$$y - 3 = 9(x-2)$$

$$y = 9x - 18 + 3 = 9x - 15$$

$$y = 9(2.1) - 15 = 18.9 - 15 = 3.9$$

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Continue problem 4 on page 17.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

$$\frac{dy}{dx} = \frac{y^2}{x-1}$$

$$\int \frac{dy}{y^2} = \int \frac{dx}{x-1}$$

$$\frac{y^{-1}}{-1} = \ln|x-1| + C$$

$$\frac{-1}{y} = \ln|x-1| + C$$

$$\frac{-1}{3} = \ln|2-1| + C$$

$$\frac{-1}{3} = \cancel{\ln 1}^{0} + C$$

$$C = \frac{-1}{3}$$

$$\frac{-1}{y} = \ln(x-1) + \left(-\frac{1}{3}\right)$$

$$-1 = \left(\ln(x-1) - \frac{1}{3}\right) y$$

$$y = \frac{-1}{\ln(x-1) - \frac{1}{3}}$$

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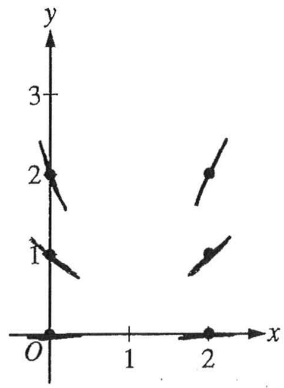
4. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



$\frac{0^2}{0-1} = 0$

$\frac{1}{0-1} = -1$

$\frac{4}{0-1} = -4$

$\frac{1}{2-1} = 1$ $\frac{0}{2-1} = 0$

$\frac{4}{2-1} = 4$

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

$\frac{dy}{dx} = \frac{y^2}{x-1}$

$y - 3 = 9(x-2)$

$\frac{dy}{dx} = \frac{9}{2-1} = 9$

$L(a) = f(a) + f'(a)(x-a)$

$3 + 9(2.1-2)$

$3 + 9(.1)$

$f(2.1) \approx 3.9$

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Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

$$\frac{dy}{dx} = \frac{y^2}{x-1}$$

$$dy = \frac{y^2 dx}{x-1}$$

$$(x-1)^{-1} dy = y^2 dx$$

$$\int y^{-2} dy = \int (x-1) dx$$

$$-y^{-1} = \frac{1}{2}x^2 - x + C$$

$$-\frac{1}{3} = \frac{1}{2}(2)^2 - 2 + C$$

$$-\frac{1}{3} = 2 - 2 + C$$

$$C = -\frac{1}{3}$$

$$-\frac{1}{y} = \frac{1}{2}x^2 - x - \frac{1}{3}$$

$$-1 = \left(\frac{1}{2}x^2 - x - \frac{1}{3}\right)y$$

$$y = \frac{-1}{\frac{1}{2}x^2 - x - \frac{1}{3}}$$

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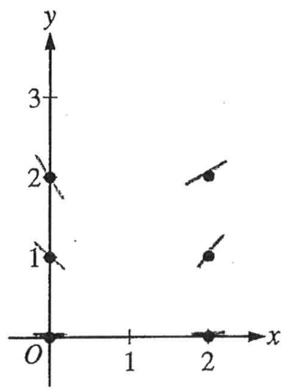
4. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

$\frac{0^2}{0-1} = \frac{0}{-1} = 0$

$\frac{1}{-1} = -1$

$\frac{4}{-1} = -4$



$\frac{1}{2-1} = 1$

$\frac{4}{2-1} = \frac{1}{2}$

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

$x = 2 \quad y = 3$

$\frac{y^2}{x-1} \rightarrow \frac{3^2}{2-1} = \frac{9}{1} = 9 = m$

$y - y_1 = m(x - x_1)$

$y - 3 = 9(x - 2)$

$y = 9x - 18 + 3$

$\boxed{y = 9x - 15}$

$f(2.1) \approx 9(2.1) - 15$

$= 3.9$

$2.1 \times 9 = 18.9$

$18.9 - 15.0 = 3.9$

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Continue problem 4 on page 17.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

$X = 2 \quad y = 3$

$\frac{dy}{dx} = \frac{y^2}{x-1} \qquad (x-1)(dy) = (y^2)(dx)$

$$\frac{(x-1)}{dx} = \frac{y^2}{dy}$$

$$\int x-1 \cdot \frac{1}{dx} = \int y^2 \cdot \frac{1}{dy}$$

$$\frac{1}{2}x^2 - x = \frac{1}{3}y^3 + c$$

$C: \quad \frac{1}{2}(2)^2 - 2 = \frac{1}{3}(3)^3 + c$

$$0 = 9 + c \qquad c = -9$$

$$\frac{1}{2}x^2 - x = \frac{1}{3}y^3 - 9$$

$\frac{1}{2}x^2 - x + 9 = \frac{1}{3}y^3$

$3\left(\frac{1}{2}x^2 - x + 9\right) = y^3$

$\frac{3}{2}x^2 - 3x + 27 = y^3$

$\sqrt[3]{3/2\, x^2 - 3x + 27} = y$

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Question 4

Overview

In this problem students were presented with a first-order separable differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$. In part (a) students needed to sketch a slope field at six points in the *xy*-plane provided: $(0, 0)$, $(0, 1)$, $(0, 2)$, $(2, 0)$, $(2, 1)$, and $(2, 2)$. In part (b) students were given that $y = f(x)$ is the particular solution to the differential equation with the initial condition $f(2) = 3$. Students needed to write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$, where the slope is computed using the given $\frac{dy}{dx}$. The value of $f(2.1)$ is approximated using the tangent line. In part (c) students were expected to use separation of variables to find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 3$.

Sample: 4A
Score: 9

The response earned all 9 points.

Sample: 4B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), and 2 points in part (c). In parts (a) and (b) the student's work is correct. In part (c) the student incorrectly separates the differential equation. The student was eligible for and earned 1 of the 2 antiderivatives points for the correct antidifferentiation of $y^{-2}\,dy$. This side of the equation is consistent with a correct separation of variables. Because the student earned at least 1 of the first 3 points, the student was eligible for and earned the fourth point. The student was not eligible for the last point.

Sample: 4C
Score: 3

The response earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the student earned the first point. The student has an inconsistent slope at $(2, 2)$, so the second point was not earned. In part (b) the student's work is correct. In part (c) the student does not have a correct approach for separation of variables. The student was not eligible for any points.