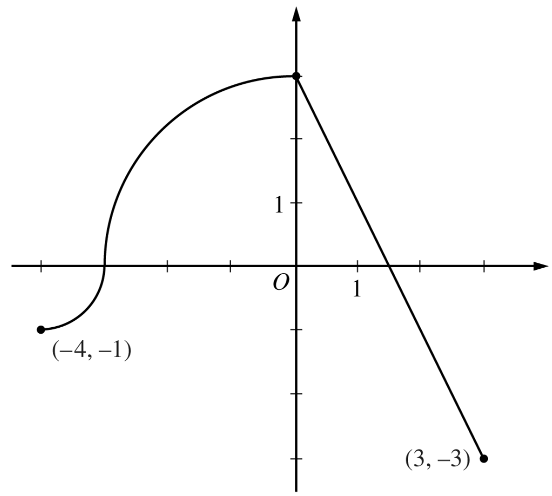
Question 4

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above.



Graph of f

Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

(a) $g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt = -6 - \dfrac{9\pi}{4}$

$g'(x) = 2 + f(x)$

$g'(-3) = 2 + f(-3) = 2$

$3 : \begin{cases} 1 : g(-3) \\ 1 : g'(x) \\ 1 : g'(-3) \end{cases}$

(b) $g'(x) = 0$ when $f(x) = -2$. This occurs at $x = \dfrac{5}{2}$.

$g'(x) > 0$ for $-4 < x < \dfrac{5}{2}$ and $g'(x) < 0$ for $\dfrac{5}{2} < x < 3$.

Therefore g has an absolute maximum at $x = \dfrac{5}{2}$.

$3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : \text{identifies interior candidate} \\ 1 : \text{answer with justification} \end{cases}$

(c) $g''(x) = f'(x)$ changes sign only at $x = 0$. Thus the graph of g has a point of inflection at $x = 0$.

1 : answer with reason

(d) The average rate of change of f on the interval $-4 \le x \le 3$ is $\dfrac{f(3) - f(-4)}{3 - (-4)} = -\dfrac{2}{7}$.

To apply the Mean Value Theorem, f must be differentiable at each point in the interval $-4 < x < 3$. However, f is not differentiable at $x = -3$ and $x = 0$.

$2 : \begin{cases} 1 : \text{average rate of change} \\ 1 : \text{explanation} \end{cases}$

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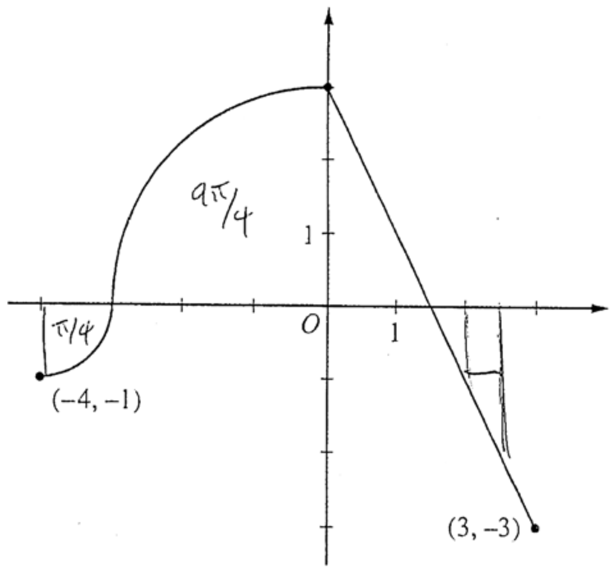

Graph of f

$\frac{\pi}{4}$ $\frac{\pi r^2}{4}$ $\frac{9\pi}{4}$

Work for problem 4(a)

$g(-3) = 2\cdot(-3) + \int_0^{-3} f(t)\,dt = \underline{-6 - \frac{9\pi}{4}}$

$g'(x) = \frac{d}{dx}\left(2x + \int_0^{x} f(t)\,dt\right) = \underline{2 + f(x)}$

$g'(-3) = 2 + f(-3) = 2 + 0 = \underline{2}$

Work for problem 4(b)

$g'(x) = 0$ $\quad 2 + f(x) = 0$

$f(x) = -2$

$x = 5/2$

$g'(x)$

$+ + + \mid - - - -$

$5/2$

check endpoints

$g(-4) = -8 + \int_0^{-4} f(t)\,dt$

$= -8 - 2\pi$

$g(5/2) = 5 + \int_0^{5/2} f(t)\,dt = 5 + \frac{5}{4}$

$g(3) = 6 + \int_0^{3} f(t)\,dt = 6$

$x = 5/2$, because g' going from + to − proves it as the only relative maximum and $g(5/2)$ is greater than g at either endpoint.

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Work for problem 4(c)

$g''(x) = \frac{d}{dx}(g'(x)) = f'(x)$

$f'(x)$

+ + + + + + | − − − − −

−4 −3 0 3/2 3

The only point of inflection for g is at $x = 0$, since $f'(x)$, which is equivalent to g'', only changes signs at $x = 0$ on the interval $-4 \le x \le 3$

Work for problem 4(d)

Avg. Rate of change $= \dfrac{f(3) - f(-4)}{3 - -4}$

$= \dfrac{-3 - -1}{3+4} = \underline{\dfrac{-2}{7}}$

Because Mean Value Theorem only applies when the function is continuous AND differentiable on the interval, which doesn't apply here since $f(x)$ isn't differentible at $x = 0$.

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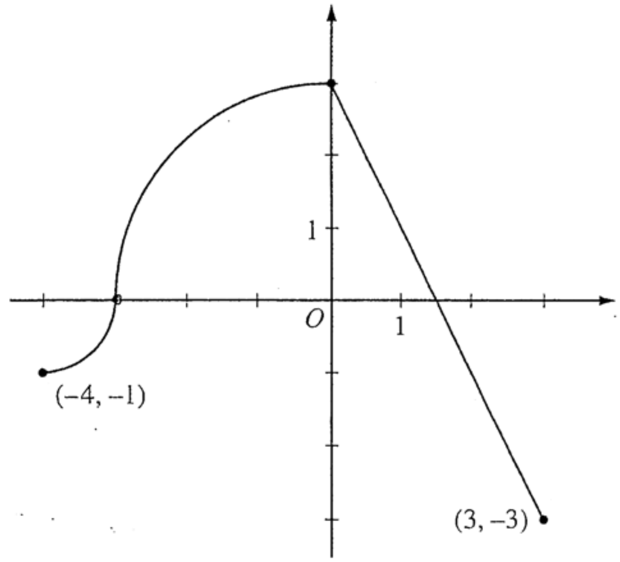

Graph of f

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Work for problem 4(a)

a. $g(-3) =$ πr^2

$g(-3) = 2(-3) + \int_{-3}^{0} f(t)\,dt$

-6

It's the area of a circle w/ the radius of 3, ~~and~~ divided by 4.

$\boxed{g'(x) = 2 + f(x)}$

$g'(-3) = 2 + 0$

$\boxed{g'(-3) = 2}$

$\frac{9\pi}{4}$

$-6 + \frac{9\pi}{4} =$

$\rightarrow \boxed{g(-3) = -6 + \frac{9\pi}{4}}$

Work for problem 4(b)

absolute maximum can be found by equaling the derivative of g to zero

$g'(x) = 2 + f(x) = 0$

$-2 = f(t)$

g has an absolute maximum when $x = 2.5$.

Continue problem 4 on page

Work for problem 4(c)

where g has a point of inflection =

when $g''(x)$ is zero.

$g(x) = 2 + f(x)$

$g''(x) = f'(x)$.

$0 = f'(x)$

so when the f is experiencing its max/min.

g has points of inflection when x is equal to -4, 0, and -3.

because the point of inflection is found when the second derivative is equal to zero (or changes signs).

$g''(x)$ ended up to be $f'(t)$. I equated $f'(t)$ to zero, ~~because~~ when the functions would be experiencing its points of inflection, the second deriv. = 0.

∴ I found x value to be -4, 0, and 3.

Work for problem 4(d)

the average rate of change

$$\frac{f(3) - f(-4)}{3 - -4} = \frac{-3 + +1}{7} = \frac{-2}{7}$$

the average rate of change is $\frac{-2}{7}$

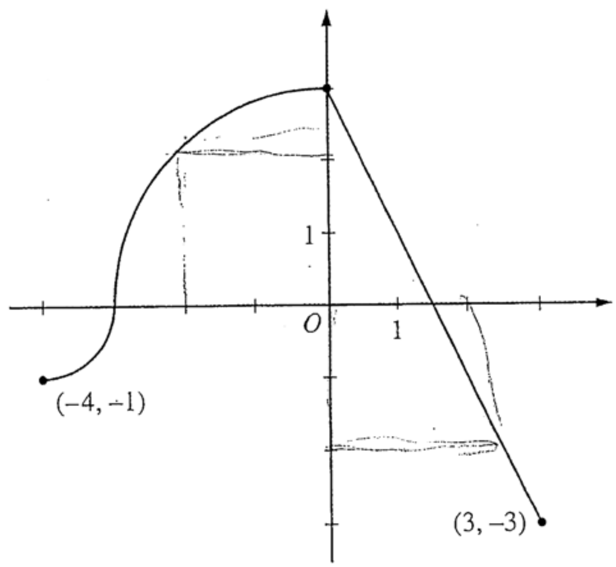
the mean value theorem states

1. if all value of x ~~are~~ is differentiable in the function, f in the closed interval [a,b]
2. if all value of x in the function, f is continuous in the open interval (a,b)

then there exists a value $f'(c) = \frac{f(b) - f(a)}{b-a}$.

However, f does not meet the requirements of the mean value theorem because it is not differentiable when x=0.

NO CALCULATOR ALLOWED



Graph of f

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Work for problem 4(a)

$g(x) = 2(x) + \int_0^x f(t)\,dt$

$g(-3) = -6 + \frac{1}{4}\pi(3)^2$

$\boxed{g(-3) = -6 + \frac{9}{4}\pi}$

$g'(x) = 2 + f(x)$

$g'(-3) = 2 + f(-3)$

$\boxed{g'(-3) = 2}$

Work for problem 4(b)

max occurs where $g'(x) = 0$ and where $g'(x)$ changes sign from + to −

$g'(x) = 2 + f(x)$

$0 = 2 + f(x)$

$-2 = f(x)$

$\boxed{\text{at } x = 2}$ there's a max

Continue problem 4 on page

Work for problem 4(c)

$g''(x) = 0$ means there's an inflection point

$g''(x) = f'(x)$

$0 = f'(x)$

So spots where slope $f(x) = 0$ or DNE,

the only such spot occurs at $\boxed{x=0}$

and at $\boxed{x=-3}$ where it's undefined

Work for problem 4(d)

$$\text{avg rate of change} = \frac{1}{3-(-4)} \int_{-4}^{3} f(x)\,dx$$

~~2quarter circles~~

2 quarter circle

1 a triangle

6

3

$$\frac{1}{7}\left[\frac{1}{4}\pi + \frac{9}{4}\pi + 9\right]$$

$$\frac{1}{7}\left[\frac{10}{4}\pi + 9\right] = \text{avg rate of change}$$

$A = \frac{1}{2}(3)(6)$

$A = 9$

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Question 4

Overview

This problem provided the graph of a continuous function *f*, defined for $-4 \le x \le 3.$ The graph consisted of two quarter circles and one line segment. The function g is defined by $g(x) = 2x + \int_0^x f(t)\, dt.$ Part (a) asked for $g(-3)$, an expression for $g'(x),$ and the value of $g'(-3).$ These items tested the interpretation of a definite integral in terms of the area of a region enclosed by the *x*-axis and the graph of the function given in the integrand, as well as the application of the Fundamental Theorem of Calculus to differentiate a function defined by an integral with a variable upper limit of integration. Part (b) asked for the *x*-coordinate of the point at which g attains an absolute maximum for $-4 \le x \le 3.$ Several approaches were possible, but they all begin with identification of candidates using the expression for $g'(x)$ found in part (a). Part (c) asked for locations of points of inflection for the graph of *g*, involving another analysis of $g'(x).$ Part (d) asked for the average rate of change of f on $-4 \le x \le 3,$ and tested knowledge of the hypotheses of the Mean Value Theorem to explain why that theorem is not contradicted given the fact that its conclusion does not hold for f on $-4 \le x \le 3.$

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), no point in part (c), and 2 points in part (d). In part (a) the student makes a sign error in evaluating $g(-3)$ but correctly determines $g'(x)$ and evaluates $g'(-3)$, thus earning 2 of the 3 points. In part (b) the student earned the first 2 points for considering where $g'(x) = 0$ and correctly identifying 2.5 as the interior candidate for the *x*-coordinate of the absolute maximum. The student does not justify this as giving the absolute maximum, and so the final point in part (b) was not earned. In part (c) the student gives incorrect *x*-coordinates for the point of inflection. In part (d) the student's work is correct.

Sample: 4C
Score: 3

The student earned 3 points: 2 points in part (a), 1 point in part (b), no point in part (c), and no points in part (d). In part (a) the student makes a sign error in evaluating $g(-3)$ but correctly determines $g'(x)$ and evaluates $g'(-3)$, thus earning earned 2 of the 3 points. In part (b) the student earned the first point for $g'(x) = 0.$ The student solves the equation incorrectly. In part (c) the student gives an incorrect *x*-coordinate for the point of inflection. In part (d) the student does not correctly compute the average rate of change and does not provide an explanation for why the Mean Value Theorem does not apply.