THE BOUNDEDNESS OF THE SOLUTIONS OF A DIFFERENTIAL EQUATION IN THE COMPLEX DOMAIN

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1. **Introduction.** Let $Q(z)$ be an analytic function of the complex variable z in a domain. In the following we shall be concerned with the differential equation

$$\frac{d^2W}{dz^2} + Q(z)\,W = 0. \tag{1}$$

Only those solutions $W(z)$ of (1) which are distinct from the trivial solution $(\equiv 0)$ shall be considered.

For a real-valued continuous solution $y(x) \not\equiv 0$ of the differential equation

$$\frac{d^2y}{dx^2} + f(x)\,y = 0, \tag{2}$$

where $f(x)$ is a real-valued piecewise continuous function of the real variable x for $0 \le x < \infty$, N. Levinson [1] has shown that the rapidity with which $y(x)$ can grow, and the rapidity with which it can tend to zero, both depend on the growth of $\alpha(x)$, where

$$\alpha(x) = \int_0^x |f(x) - a|\,dx, \tag{3}$$

and a is a real positive constant. More precisely, he showed that

$$y(x) = O\left(\exp\left[\frac{1}{2}\,a^{-1/2}\,\alpha(x)\right]\right), \tag{4}$$

and that if $\alpha(x) = O(x)$ as $x \longrightarrow \infty$, then

$$\limsup_{x \to \infty} |y(x)| \exp\left[\frac{1}{2}\,a^{-1/2}\,\alpha(x)\right] > 0. \tag{5}$$

If there exists a positive constant a such that $\alpha(x)$ converges as $x \longrightarrow \infty$, then

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from (4) it follows that every solution $y(x)$ of (2) is bounded as $x \longrightarrow \infty$. Levinson also showed that (4) and (5) are the best possible results of their types.

Along any line of the z-plane, for instance the real axis, the differential equation (1) has the form

$$\frac{d^2W}{dx^2} + Q(x)\,W = 0, \tag{6}$$

where x is real. Along a line, the growth of the solutions $W(x)$ of (6) also depends on the growth of a function similar to that in (3), and they also satisfy two relations like (4) and (5). These relations will be established in §2. From these results, we can obtain sufficient conditions for the boundedness of the solutions of (1) on a line, or on certain regions of the z-plane.

In §3 we shall investigate the asymptotic behavior of the solutions of (6) when they are bounded. In §6 we shall give a relation of the boundedness of the solutions of a self-adjoint differential equation of the third order and a differential equation of the second order.

2. Growth of the solutions along the real axis. We now consider equation (6) where x is real. Let $q_1(x)$ and $q_2(x)$ be, respectively, the real and imaginary parts of $Q(x)$. If

$$\phi(x) = \int_0^x [\,|a - q_1(x)| + |q_2(x)|\,]\,dx, \tag{7}$$

where a is a positive constant, then $\phi(x)$ determines not only how large a solution $W(x)$ of (6) can become, but also determines how small it can become. These results are contained in the following two theorems.

THEOREM 1. *If*

(a) $W(x)$ *is a solution of* (6),
(b) $\phi(x)$ *is defined as in* (7),

then

$$W(x) = O\left(\exp\left[\frac{1}{2}\,a^{-1/2}\,\phi(x)\right]\right),\quad W'(x) = O\left(\exp\left[\frac{1}{2}\,a^{-1/2}\,\phi(x)\right]\right). \tag{8}$$

An immediate consequence of this theorem is the following corollary.

COROLLARY 1.1. *Every solution* $W(x)$ *of the equation* (6) *and its derivative* $W'(x)$ *are bounded as* $x \longrightarrow \infty$ *provided there exists a positive constant* a *such that* $\phi(x)$ *converges as* $x \longrightarrow \infty$.

In Theorem 1 we cannot expect to replace $\phi(x)$ by a more symmetric form

$$\int_0^x [\,|a - q_1(x)| + |b - q_2(x)|\,]\, dx,$$

where $b \neq 0$ and is real, and $a > 0$. A counter-example is the differential equation

$$\frac{d^2W}{dx^2} + (1 + i)\, W = 0,$$

which has solutions unbounded as $x \longrightarrow \infty$.

THEOREM 2. *If*

(a) $W(x)$ *is a solution of* (6),

(b) $\phi(x) = O(x)$ *as* $x \longrightarrow \infty$, *where* $\phi(x)$ *is defined as in* (7),

then

$$\limsup_{x \to \infty} |W(x)| \exp\left[\frac{1}{2}\, a^{-1/2}\, \phi(x)\right] > 0. \tag{9}$$

Clearly $\limsup |W(x)| > 0$ as $x \longrightarrow \infty$ if $\phi(x)$ is convergent.

That (8) and (9) are the best possible results follows from the fact that (4) and (5) are the best possible results.

We shall now prove Theorem 1 and 2.

Proof of Theorem 1. Let the real and imaginary parts of a solution $W(x)$ of (6) be $u(x)$ and $v(x)$, respectively. Separating the real and imaginary part of (6), we obtain

$$u'' + q_1(x)\, u - q_2(x)\, v = 0, \tag{10}$$

$$v'' + q_2(x)\, u + q_1(x)\, v = 0. \tag{11}$$

Suppose $a > 0$, and let

$$H(x) = |W'(x)|^2 + a|W(x)|^2 = u'^2(x) + v'^2(x) + a[u^2(x) + v^2(x)]. \tag{12}$$

Then using (10) and (11), we have

$$\begin{aligned}\frac{dH}{dx} &= 2(u'u'' + v'v'') + 2a(uu' + vv') \\ &= 2[a - q_1(x)]\,(uu' + vv') + 2q_2(x)\,(u'v - uv').\end{aligned} \tag{13}$$

Using the following inequalities,

$$2uu' \leq a^{-1/2}(au^2 + u'^2), \qquad 2vv' \leq a^{-1/2}(av^2 + v'^2),$$

$$2u'v \leq a^{-1/2}(u'^2 + av^2), \qquad 2uv' \leq a^{-1/2}(v'^2 + au^2),$$

and (13), we see that

$$\text{(14)} \qquad \frac{dH}{dx} \leq a^{-1/2}[|a - q_1(x)| + |q_2(x)|]\,(u'^2 + v'^2 + au^2 + av^2)$$

$$= a^{-1/2}(|a - q_1(x)| + |q_2(x)|)\, H.$$

Since $H > 0$, we have

$$\text{(15)} \qquad \frac{1}{H}\frac{dH}{dx} \leq a^{-1/2}[|a - q_1(x)| + |q_2(x)|]$$

Integrating (15) from 0 to x, we obtain

$$\text{(16)} \qquad H(x) \leq H(0)\exp[a^{-1/2}\,\phi(x)].$$

In view of the definition of $H(x)$, the expression in (6) is equivalent to the two in (8). This completes the proof of Theorem 1.

Proof of Theorem 2. In much the same way as in the proof of Theorem 1, it is easy to show that

$$\frac{1}{H}\frac{dH}{dx} \geq -a^{-1/2}[|a - q_1(x)| + |q_2(x)|]$$

Consequently, we have

$$\text{(17)} \qquad H(x) = |W'(x)|^2 + a|W(x)|^2 \geq C\exp[-a^{-1/2}\,\phi(x)].$$

For each positive integer n, let x_n, x_n', x_n'' be points in the interval $n \leq x \leq n+1$ such that

$$|W(x_n)| = \max|W(x)|, \quad |u'(x_n')| = \min|u'(x)|, \quad |v'(x_n'')| = \min|v'(x)|$$

in the interval $n \leq x \leq n+1$. Integrating (10) from x_n' to x_n and (11) from x_n'' to x_n, we obtain

$$\text{(18)} \qquad u'(x_n) = u'(x_n') + \int_{x_n'}^{x_n}[-q_1(x)\,u(x) + q_2(x)\,v(x)]\,dx$$

$$\leq |u'(x_n')| + |W(x_n)|\int_n^{n+1}(|q_1(x)| + |q_2(x)|)\,dx,$$

$$\text{(19)} \qquad v'(x_n) = v'(x_n'') + \int_{x_n''}^{x_n}[-q_2(x)\,u(x) - q_1(x)\,v(x)]\,dx$$

$$\leq |v'(x_n'')| + |W(x_n)|\int_n^{n+1}[|q_1(x)| + |q_2(x)|]\,dx.$$

Since

$$|W'(x_n)| \le |u'(x_n)| + |v'(x_n)|,$$

(18) and (19) yield

$$(20)\qquad |W'(x_n)| \le |u'(x_n')| + |v'(x_n'')| + 2|W(x_n)| \int_n^{n+1} [|q_1(x)| + |q_2(x)|]\, dx.$$

Clearly either $|u'(x_n')| = 0$ or $u'(x)$ does not change sign in $n \le x \le n+1$. If $u'(x)$ does not change sign in $n \le x \le n+1$, we have

$$(21)\quad 2 \max_{n \le x \le n+1} |u(x)| \ge |u(n+1) - u(n)| = \left|\int_n^{n+1} u'(x)\, dx\right| \ge |u'(x_n')|.$$

Obviously (21) holds if $|u'(x_n')| = 0$. So (21) is always true. Hence

$$(22)\qquad 2|W(x_n)| \ge 2 \max_{n \le x \le n+1} |u(x)| \ge |u'(x_n')|.$$

Similarly,

$$(23)\qquad 2|W(x_n)| \ge |v'(x_n'')|.$$

Substitution of (22) and (23) into (20) yields

$$(24)\qquad |W'(x_n)| \le |W(x_n)| \left\{4 + 2 \int_n^{n+1} [|q_1(x)| + |q_2(x)|]\, dx\right\}.$$

From (17) and (24), we obtain

$$(25)\qquad |W(x_n)|^2 \left\{\left(4 + 2 \int_n^{n+1} [|q_1(x)| + |q_2(x)|)\, dx\right]^2 + a\right\} \ge C \exp[-a^{-1/2} \phi(x_n)].$$

Since $\phi(x) = O(x)$ as $x \longrightarrow \infty$, it is easy to show that, for an infinite number of n,

$$\int_n^{n+1} [|q_1(x)| + |q_2(x)|]\, dx$$

is bounded. Thus for an infinite number of n, we have the inequality

$$(26)\qquad |W(x_n)|^2 \exp[a^{-1/2} \phi(x_n)] \ge C_1$$

for some positive constant C_1. Consequently (26) yields the result

$$\limsup_{x\to\infty} |W(x)| \exp\left[\frac{1}{2} a^{-1/2} \phi(x)\right] > 0.$$

This completes the proof of Theorem 2.

3. **Asymptotic behavior of the solutions.** If $\phi(x)$ converges as $x \longrightarrow \infty$, the solutions $W(x)$ of (6) are not only bounded, but also resemble the solutions of the differential equation

$$\frac{d^2 W}{dx^2} + aW = 0. \tag{27}$$

This result is proved in the following theorem.

THEOREM 3. *If*

(a) $W(x)$ *is a solution of* (6),

(b) $\phi(x)$, *defined as in* (7), *converges as* $x \longrightarrow \infty$,

then for some complex constants A *and* B,

$$\lim_{x\to\infty} [W(x) - (A \sin\sqrt{a}\, x + B \cos\sqrt{a}\, x)] = 0. \tag{28}$$

Proof of Theorem 3. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the equation (27) such that

$$y_1(0) = 0, \quad y_1'(0) = 1; \quad y_2(0) = 1, \quad y_2'(0) = 0. \tag{29}$$

Rewrite (6) in the form

$$\frac{d^2 W}{dx^2} + aW = [a - Q(x)]\, W. \tag{30}$$

Then a solution $W(x)$ of (30) can be expressed as

$$\begin{aligned} W(x) &= A\, y_1(x) + B\, y_2(x) \\ &\quad + \int_x^\infty [a - Q(t)]\, W(t)\, [y_2(x)\, y_1(t) - y_1(x)\, y_2(t)]\, dt \end{aligned} \tag{31}$$

for some complex constants A and B, where the integral is convergent since $\phi(x)$ is convergent, $W(x)$ is bounded, and

$$y_1(x) = a^{-1/2} \sin\sqrt{a}\, x, \quad y_2(x) = \cos\sqrt{a}\, x;$$

(31) can be obtained by the method of variation of constants. Hence the absolute value of the integral in (31) can be arbitrarily small if x is large enough. In other

words,

$$\lim_{x\to\infty} \{W(x) - [A\ y_1(x) + B\ y_2(x)]\} = 0.$$

This completes the proof.

Differentiating (31) clearly yields

$$\lim_{x\to\infty} \{W'(x) - [A\ y_1'(x) + B\ y_2'(x)]\} = 0.$$

4. **Boundedness of the solutions in certain regions.** In this section we shall apply the results of Theorem 1 to obtain sufficient conditions for the boundedness of the solutions of the equation (1) in certain regions of the z-plane.

Let R be the region

$$z = x + iy, \quad 0 \le x < \infty, \quad \alpha \le y \le \beta. \tag{32}$$

On a half line $L(y_0)$, $z = x + iy_0$, in R, the differential equation (1) becomes

$$\frac{d^2 W}{dx^2} + Q(x + iy_0)\ W = 0. \tag{33}$$

Denoting the real and imaginary part of $Q(x + iy_0)$ by $q_1(x, y_0)$ and $q_2(x, y_0)$, respectively, we see that according to Theorem 1, the growth of a solution $W(x)$ of (1) on $L(y_0)$ depends on the growth of

$$\phi(x, y_0) = \int_0^x [|a - q_1(x, y_0)| + |q_2(x, y_0)|]\ dx, \tag{34}$$

where a is a positive constant. If $\phi(x, y_0)$ is convergent for some positive constant a, then $W(z)$ and $W'(z)$ are bounded on $L(y_0)$, and

$$\lim_{x\to\infty} [W(x + iy_0) - (A\ \sin\sqrt{a}\ x + B\ \cos\sqrt{a}\ x)] = 0$$

for some complex constants A and B. Let

$$\Phi(x, y_0) = \int_0^x |a - Q(x + iy_0)|\ dx. \tag{35}$$

Clearly the convergence of $\Phi(x, y_0)$ implies the convergence of $\phi(x, y_0)$. Let $\Phi(x, y_0)$ be uniformly bounded in R in the sense that for each y_0 ($\alpha \le y_0 \le \beta$), there exists a positive constant a such that sup a is finite and inf a is positive, and $\Phi(x, y_0) \le M$, M being some constant, for all x in $0 \le x < \infty$ and all y_0 in $\alpha \le y \le \beta$; then by applying (16) on each $L(y_0)$, it is easy to see that $W(z)$ and $W'(z)$ are bounded in R. If the condition that sup a is finite is removed, clearly we still have $W(z)$ bounded in R. This proves the following theorem.

THEOREM 4. *If*

(a) *R is a region defined as in* (32),
(b) *$\Phi(x, y_0)$, defined as in* (35), *is uniformly bounded in R in the sense defined above,*

then each solution $W(z)$ of (1) *and its derivative $W'(z)$ are bounded in R.*

Consider another region R,

$$z = x + re^{i\theta_0}, \qquad 0 \leq r < \infty, \ \alpha \leq x \leq \beta, \tag{36}$$

where θ_0 is a real constant. On a half line $L(x_0)$, $z = x_0 + r \exp(i\theta_0)$, in R, the equation (1) reduces to

$$\frac{d^2 W}{dr^2} + P(r, x_0) W = 0, \tag{37}$$

where $P(r, x_0) = Q[x_0 + r \exp(i\theta_0)] \exp(2i\theta_0)$.

THEOREM 5. *If*

(a) *R is a region defined as in* (36),
(b) *for each x_0, $\alpha \leq x_0 \leq \beta$, there exists a positive constant a such that* sup *a is finite and* inf *a is positive and*

$$\int_0^r |a - P(r, x_0)| \, dr \leq M,$$

M being some constant, for all r in $0 \leq r < \infty$ and all x_0 in $\alpha \leq x \leq \beta$,

then each solution $W(z)$ of (1) *and its derivative $W'(z)$ are bounded in R.*

The proofs of this theorem and of the following Theorem 6 are similar to that of Theorem 4.

Denote by S the sector

$$z = re^{i\theta}, \qquad 0 \leq r < \infty, \ \alpha \leq \theta \leq \beta. \tag{38}$$

On a fixed ray $\theta = \theta_0$ in S, equation (1) reduces to

$$\frac{d^2 W}{dr^2} + T(r, \theta_0) W = 0, \tag{39}$$

where $T(r, \theta_0) = Q(r \exp(i\theta_0)) \exp(2i\theta_0)$. We have the following result.

THEOREM 6. *If*

(a) S *is a region defined as in* (38),

(b) *for each* θ_0, $\alpha \leq \theta_0 \leq \beta$, *there exists a positive constant* a *such that* $\sup a$ *is finite and* $\inf a$ *is positive and*

$$\int_0^r |a - T(r, \theta_0)| \, dr \leq M,$$

M *being some constant, for all* r *in* $0 \leq r < \infty$ *and all* θ_0 *in* $\alpha \leq \theta \leq \beta$,

then each solution $W(z)$ *of* (1) *and its derivative* $W'(z)$ *are bounded in* S.

5. **Extension.** Let C be an analytic curve [2, p. 702]

$$x = f(t), \; y = g(t), \tag{40}$$

where t is real. Along C the equation (1) has the form

$$\frac{d^2 W}{dt^2} + A(t) \frac{dW}{dt} + B(t) W = 0. \tag{41}$$

It is well known that equation (41) can be reduced to the form of (6). It follows that our results apply to the solutions along a line or in regions bounded by lines as well as to the solutions along an analytic curve or in regions bounded by analytic curves.

6. **A self-adjoint differential equation of the third order.** Let $Y(z)$ be a solution of the self-adjoint differential equation

$$\frac{d^2 Y}{dz^3} + Q(z) \frac{dY}{dz} + \frac{1}{2} \frac{dQ(z)}{dz} Y = 0, \tag{42}$$

where $Q(z)$ is analytic in a region R. Let $W(z)$ be a solution of

$$\frac{d^2 W}{dz^2} + \frac{1}{4} Q(z) W = 0. \tag{43}$$

In Theorem 7 we shall prove that every solution $Y(z)$ of (42) is bounded in R if and only if every solution $W(z)$ of (43) is bounded in R. In fact the growth of the solutions of (43) determines and is determined by the growth of the solutions of (42).

THEOREM 7. *Every solution* $Y(z)$ *of* (42) *is bounded in* R *if and only if every solution* $W(z)$ *of* (43) *is bounded in* R.

Proof. Let $W_1(z)$ and $W_2(z)$ be any two linearly independent solutions of

(43). The theorem follows from the fact that $W_1^2(z)$, $W_1(z)\,W_2(z)$ and $W_2^2(z)$ are three linearly independent solutions of (42). That $W_1^2(z)$, $W_1(z)\,W_2(z)$ and $W_2^2(z)$ are solutions of (42) can be verified by substitution. We now show that they are linearly independent. If A, B, C are constants, and if

$$AW_1^2(z) + BW_1(z)\,W_2(z) + CW_2^2(z) \equiv 0, \tag{44}$$

then by factoring (44) we get

$$[AW_1(z) + bW_2(z)]\,[cW_1(z) + dW_2(z)] \equiv 0, \tag{45}$$

where a, b, c and d depend on A, B and C. Hence at least on the factors in (45) is identically zero. It follows that either $a = b = 0$ or $c = d = 0$. Consequently $A = B = C = 0$. This completes the proof.

7. **Added in proof.** With the aid of the Phragmén-Lindelöf theorems [see 3], the results of §4 can be greatly improved.

For example, let R be the region defined as in (32), with $\beta - \alpha = \pi h^{-1}$. Let there be a positive constant a such that as $x \longrightarrow \infty$,

$$\phi(x, y) = O(e^{kx}), \tag{46}$$

where $k < h$, uniformly for y in $\alpha \leq \beta$, and that

$$\phi(x, \alpha) = O(1), \quad \phi(x, \beta) = O(1). \tag{47}$$

Then, by Corollary 1.1, any solution $W(z)$ of (1) is bounded on $L(\alpha)$ and on $L(\beta)$, and so is bounded on these lines and on the segment $x = 0$ in R. From (46) and Theorem 1, we have

$$W(z) = O(e^{Me^{kx}})$$

uniformly in y, where M is some positive constant. By a theorem of Phragmén-Lindelöf, $W(z)$ is then bounded in R. Similarly $W'(z)$ is bounded in R.

Using Theorem 3, from (47), we see that

$$W(z) - (A_1 \sin a^{1/2} z + B_1 \cos a^{1/2} z) \tag{48}$$

tends to zero as $z \longrightarrow \infty$ on $L(\alpha)$ for some constants A_1 and B_1. Similarly (48) tends to zero on $L(\beta)$ if A_1 and B_1 are replaced, respectively, by some constants A_2 and B_2. Write

$$F_i(z) = A_i \sin a^{1/2} z + B_i \cos a^{1/2} z, \qquad (i = 1, 2).$$

Then

$$[W(z) - F_1(z)][W(z) - F_2(z)]$$

tends to zero as $z \longrightarrow \infty$ on $L(\alpha)$ and on $L(\beta)$; and since it is bounded in R, by another theorem of Phragmén-Lindelöf, it tends uniformly to zero as $z \longrightarrow \infty$. Thus to any ϵ there corresponds a segment $z = x_0 + iy$ in R on which

$$(49) \qquad |W(z) - F_1(z)| \; |W(z) - F_2(z)| \leq \epsilon.$$

At every point of this segment either

$$|W(z) - F_1(z)| \leq \epsilon^{1/2} \quad \text{or} \quad |W(z) - F_2(z)| \leq \epsilon^{1/2} \quad (\text{or both}),$$

and we may suppose that the former inequality holds at $y = \alpha$, the latter at $y = \beta$; let y_0 be the upper bound of values of y for which the former holds; then y_0 is either a point where the latter holds, or a limit of such points; hence, since both factors on the left side of (49) are continuous, both inequalities hold at y_0. At $z = x_0 + iy_0$, we then have

$$(50) \qquad |F_1(z) - F_2(z)| \leq |W(z) - F_1(z)| + |W(z) - F_2(z)| \leq 2\epsilon^{1/2}.$$

On the other hand, (49) holds on every segment $z = x_1 + iy$ if x_1 is large enough, and there is a point $z = x_1 + iy_1$ at which (50) holds. Consider an arbitary segment $z = x_2 + iy$. Since $F_1(z) - F_2(z)$ is a periodic function in x, there is a point on this segment at which (50) holds. But $F_1(z) - F_2(z)$ is continuous and ϵ is arbitary, so that $F_1(z) - F_2(z) = 0$ at some point on this segment, and therefore on every segment. If these points have a limit-point inside R, then $F_1(z) = F_2(z)$ in R; otherwise there is a segment on $y = \alpha$ or $y = \beta$ in which $F_1(z) - F_2(z) = 0$, then $A_1 = A_2$, $B_1 = B_2$, and hence $F_1(z) = F_2(z)$ in R. Thus as $z \longrightarrow \infty$ the function (48) tends to zero on $L(\alpha)$ and on $L(\beta)$, and since it is bounded in R, by a theorem of Phragmén-Lindelöf, it tends to zero uniformly in $\alpha \leq y \leq \beta$.

Similarly, as $z \longrightarrow \infty$, we see that

$$W'(z) - a^{1/2}(A_1 \cos a^{1/2} z - B_1 \sin a^{1/2} z)$$

tends to zero uniformly in $\alpha \leq y \leq \beta$.

References

1. N. Levinson, *The growth of the solutions of a differential equation*, Duke Math. J. 8 (1941), 1-10.

2. W. F. Osgood, *Lehrbuch der Funktionentheorie* I, Berlin, Teubner, 1928.

3. E. C. Titchmarsh, *The theory of functions*, Oxford University Press, London, 1939.

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ERRATA

J. L. Brenner, *Matrices of quaternions*,
Vol. I, p. 329 line 18, read "*sfield*" for "*fields*".

P. M. Pu, *Some inequalities in certain nonorientable Riemannian manifolds*,
Vol. II, p. 71 line 11, read $A \geq 3^{1/2}\ \alpha^2/2$ for $A \geq 3^1/_2\ \alpha^2/_2$.

Everett Larguier, *Homology bases with applications to local connectedness*,
Vol. II, p. 193 line 15, read $1/k$ for i/k.
Vol. II, p. 198 line 7, read $x \in S$ for $x \quad S$.

Lars V. Ahlfors, *Remarks on the Neumann-Poincaré integral equation*,
Vol. II, p. 271 last line should read *Pacific J. Math.* 2 (1952), 271-280.

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