J. Math. Soc. Japan
Vol. 46, No. 2, 1994

The versal deformation of the E_6-singularity and a family of cubic surfaces

By Jiro SEKIGUCHI*)

(Received Feb. 12, 1993)

1. Introduction.

The purpose of this paper is to clarify the relation between the versal deformation of the E_6-singularity and a family of cubic surfaces originally due to A. Cayley.

We consider the cubic surface $S(p_0, p_1, p_2, q_0, q_1, q_2)$ defined by

$$x^3-2yz^2-y^2+x(p_0w^2+p_1zw+p_2z^2)+q_0w^3+q_1zw^2+q_2z^2w=0 \tag{1.1}$$

in $\boldsymbol{P}^3$ with homogeneous coordinate $(x:y:z:w)$, where $p_0, p_1, p_2, q_0, q_1, q_2$ are parameters. We frequently write $pq=(p_0, p_1, p_2, q_0, q_1, q_2)$ for simplicity. If we put $w=1$, the family of surfaces $S(pq)$ is regarded as the versal deformation of the rational double point of type E_6:

$$x^3-2yz^2-y^2=0$$

(cf. [Sl]). On the other hand, there is a long history on the study of cubic surfaces. Among others, we recall the 4-dimensional family of cubic surfaces due to A. Cayley (cf. [C]). Modifying his family, we introduce a family of cubic surfaces of $\boldsymbol{P}^3$ with homogeneous coordinate $(X:Y:Z:W)$ depending on parameters $(\lambda, \mu, \nu, \rho)$ as follows (cf. [NS]):

$$\begin{aligned}\rho W[\lambda X^2+\mu Y^2+\nu Z^2+(\rho-1)^2(\lambda\mu\nu\rho-1)^2W^2+(\mu\nu+1)YZ+(\lambda\nu+1)ZX \\ +(\lambda\mu+1)XY-(\rho-1)(\lambda\mu\nu\rho-1)W\{(\lambda+1)X+(\mu+1)Y+(\nu+1)Z\}]+XYZ=0.\end{aligned} \tag{1.2}$$

Since the moduli space of the cubic surfaces is 4-dimensional, the family above has enough parameters. For this reason, writing down the defining equation (1.1) in the form (1.2), we obtain a map Ψ: $pq\to(\lambda, \mu, \nu, \rho)$ at least in principle. Since the map Ψ is multi-valued, we have to change the parameter space of $S(pq)$ to its covering space admitting a linear $W(E_6)$-action, where $W(E_6)$ is the Weyl group of type E_6, in order to define a single-valued map to the $(\lambda, \mu, \nu, \rho)$-space.

*) Partially supported by Grant-in-Aid for Scientific Research (No. 04640135), the Ministry of Education, Science and Culture.

One of the motivations of the present study is a suggestion by M. Yoshida concerning the configuration space $\boldsymbol{P}_2^6$ of 6 points of $\boldsymbol{P}^2$ which is, roughly speaking, identified with $\boldsymbol{C}^4$. In a private communication, he pointed out the possibility of the birational action of $W(E_6)$ on the space $\boldsymbol{P}_2^6$. On the other hand, there is another realization of $W(E_6)$ as a group of birational transformations of $\boldsymbol{C}^4$ related with the family of cubic surfaces (1.2). A conceptual explanation of an isomorphism between two realizations of $W(E_6)$ as groups of birational transformations of $\boldsymbol{C}^4$ is given in Hunt [**H**] as a conjecture. The author started the present study with determining a required $W(E_6)$-equivariant birational map.

We are now going to explain the main result of this paper briefly. From the definition, $W(E_6)$ is a finite reflection group on a 6-dimensional vector space. Let $\boldsymbol{P}^5$ be the projective space associated to the 6-dimensional linear space. Then $W(E_6)$ acts on $\boldsymbol{P}^5$ as a projective linear transformation group. Now we recall the configuration space $\boldsymbol{P}_2^6$ of 6 points of $\boldsymbol{P}^2$. Roughly speaking, a Zariski open subset of $\boldsymbol{P}_2^6$ consisting of 6 points in general position is identified with a quasi-affine subset of $\boldsymbol{C}^4$ (cf. section 4). To distinguish the coordinate system of $\boldsymbol{C}^4$ from $(\lambda, \mu, \nu, \rho)$, we write (x_1, x_2, y_1, y_2) for the coordinate of $\boldsymbol{P}_2^6$. There is a $W(E_6)$-action on $\boldsymbol{P}_2^6$ (cf. Theorem 4.2). On the other hand, $W(E_6)$ is realized as a group of birational transformations on the $(\lambda, \mu, \nu, \rho)$-space which is naturally obtained from the study of the family (5.2) (cf. [**NS**]). Now we freely use the notation in section 1 to state the main theorem. Let t be the projective coordinate of $\boldsymbol{P}^5$. We define two maps Φ_1 and Φ_2 as follows. The map Φ_1: $\boldsymbol{P}^5 \to \boldsymbol{C}^4$ is given by

$$\Phi_1(t) = (x_1(t),\ x_2(t),\ y_1(t),\ y_2(t)),$$

where

$$x_1(t) = \frac{h_{24}\cdot h_{234}\cdot h_{15}\cdot h_{135}}{h_{14}\cdot h_{134}\cdot h_{25}\cdot h_{235}}, \qquad x_2(t) = \frac{h_{24}\cdot h_{234}\cdot h_{16}\cdot h_{136}}{h_{14}\cdot h_{134}\cdot h_{26}\cdot h_{236}},$$

$$y_1(t) = \frac{h_{34}\cdot h_{234}\cdot h_{15}\cdot h_{125}}{h_{14}\cdot h_{124}\cdot h_{35}\cdot h_{235}}, \qquad y_2(t) = \frac{h_{34}\cdot h_{234}\cdot h_{16}\cdot h_{126}}{h_{14}\cdot h_{124}\cdot h_{36}\cdot h_{236}}.$$

We note that h, h_{ij}, h_{ijk} are roots of type E_6 whose precise definition is given in section 2. On the other hand, Φ_2 is a map from the (x_1, x_2, y_1, y_2)-space to the $(\lambda, \mu, \nu, \rho)$-space defined by

$$\Phi_2(x_1, x_2, y_1, y_2) = (\lambda, \mu, \nu, \rho),$$

where

$$\lambda = \frac{x_2(x_1-1)(y_1-y_2)(y_2-1)}{y_2(x_1-x_2)(x_2-1)(y_1-1)},$$

$$\mu = \frac{\{(y_1-1)(x_2-y_2)-(y_2-1)(x_1-y_1)\}\,x_2y_2}{x_1x_2y_1-x_1x_2y_2-x_1y_1y_2+x_1y_2+x_2y_1y_2-x_2y_1},$$

$$\nu = -\frac{(x_1y_2-x_2y_1)(x_2-1)(y_2-1)}{(x_1-x_2)(x_2-y_2)(y_1-y_2)},$$

$$\rho = \frac{(x_1-x_2)(x_2-y_2)(y_1-1)}{\{(x_1-1)(x_2-y_2)-(x_1-y_1)(x_2-1)\}(y_2-1)x_2}.$$

Moreover, we put $\Phi_3=\Phi_2\circ\Phi_1$. Then we can state the main theorem of this paper (see Theorems 4.4, 5.5).

MAIN THEOREM. *The three maps Φ_j $(j=1, 2, 3)$ are $W(E_6)$-equivariant.*

This in particular implies that Φ_3 is a required modification of the multi-valued map Ψ.

We start the proof of Main Theorem with determining the 45 triple tangent planes for the cubic surface $S(pq)$ with a generic parameter pq. To accomplish the computation, we are indebted to Shioda [**Sh**] in which a concrete description of 27 lines on $S(pq)$ is obtained. The triple tangent planes are given their namings in a natural manner by using three weights of a 27 dimensional irreducible representation of the Lie algebra of type E_6. For this reason, we give their namings: $\pi(ij)$, $\pi(i_1i_2.i_3i_4.i_5i_6)$. On the other hand, it is known by A. Cayley [**C**] (see also [**N**], [**H**]) that to each triple tangent plane there associates a cross ratio which is an invariant of a given general cubic surface. Noting this, we first define a linear transformation

$$T: (x:y:z:w)\longrightarrow(X:Y:Z:W)$$

of $\boldsymbol{P}^3$ in such a way that the namings of the 45 triple tangent planes for $S(pq)$ and those for the surface (5.2) with Shlaefli's namings (cf. section 5) are compatible. We next compute the cross ratios attached to some of triple tangent planes for $S(pq)$ and those for the surface (5.2) and last compare the cross ratios obtained in two ways. Along this idea, we can show Main Theorem.

In section 6, we will discuss a topic related with the unpublished note of B. Hunt [**H**] on the mapping degree of the map Ψ_1.

The author is indebted to Professors B. Hunt and M. Yoshida. In particular, parts of the contents are based on the communications with B. Hunt and his unpublished note [**H**].

2. The Weyl group of type E_6.

We define the notation on the root system of type E_6 in this section basically following B. Hunt [**H**].

Let $E_{\boldsymbol{R}}$ be a Cartan subalgebra of a compact Lie algebra of type E_6, i.e. $E_{\boldsymbol{R}}\cong\boldsymbol{R}^6$. Let $t=(t_1, t_2, t_3, t_4, t_5, t_6)$ be a coordinate system of $E_{\boldsymbol{R}}$ such that the roots of type E_6 are:

$$\pm(t_i \pm t_j), \qquad 1 \leqq i < j \leqq 5$$
$$\pm\frac{1}{2}(\delta_1 t_1 + \delta_2 t_2 + \delta_3 t_3 + \delta_4 t_4 + \delta_5 t_5 + \delta_6 t_6)$$

(where $\delta_j = \pm 1$ and $\prod_j \delta_j = 1$). Note that compared with the notation in [**B**], our variables $t_i = \varepsilon_i$, $i = 1, \cdots, 5$, while our coordinate t_6 is denoted $\varepsilon_6 - \varepsilon_7 - \varepsilon_8$ in [**B**].

We now introduce the following 36 linear forms on E_R:

$$h = -\frac{1}{2}(t_1 + \cdots + t_6), \quad h_{1j} = -t_{j-1} + h_0, \quad j = 2, \cdots, 6$$
$$h_{jk} = t_{j-1} - t_{k-1}, \quad 1 < j < k < 7, \quad h_{1jk} = -t_{j-1} - t_{k-1}, \quad 1 < j < k < 7$$
$$h_{jkl} = -t_{j-1} - t_{k-1} - t_{l-1} + h_0, \quad 1 < j < k < l < 7$$

where

$$h_0 = \frac{1}{2}(t_1 + \cdots + t_5 - t_6).$$

Then the totality of h, h_{ij}, h_{ijk} forms a set of positive roots of type E_6. (In the sequel, we frequently write

$$h_{ij} = h_{ji} \ (i \neq j), \qquad h_{ijk} = h_{ikj} = h_{jki}, \quad \text{etc.} \quad (i < j < k)$$

for simplicity.)

We introduce a positive definite quadratic form on E_R defined by

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_5^2 + \frac{1}{3} t_6^2.$$

This quadratic form defines an inner product on E_R. Then it is possible to define reflections with respect to hyperplanes. In particular, let s (resp. s_{ij}, s_{ijk}) be the reflection on E_R with respect to the hyperplane $h=0$ (resp. $h_{ij}=0$, $h_{ijk}=0$). Then the Weyl group of type E_6 which is denoted by $W(E_6)$ in this note is the group generated by the 36 reflections defined above.

As a system of simple roots, we take

$$\alpha_1 = h_{12}, \quad \alpha_2 = h_{123}, \quad \alpha_3 = h_{23}, \quad \alpha_4 = h_{34}, \quad \alpha_5 = h_{45}, \quad \alpha_6 = h_{56}.$$

Then the Dynkin diagram is:

$$\begin{tikzcd}[arrows=dash]
\alpha_1 \arrow[r] & \alpha_3 \arrow[r] & \alpha_4 \arrow[r] \arrow[d] & \alpha_5 \arrow[r] & \alpha_6 \\
& & \alpha_2 & &
\end{tikzcd}$$

Let g_j be the reflection on E_R with respect to the root α_j ($j=1, \cdots, 6$). Then, from the definition,

$$g_1 = s_{12}, \quad g_2 = s_{123}, \quad g_3 = s_{23}, \quad g_4 = s_{34}, \quad g_5 = s_{45}, \quad g_6 = s_{56}.$$

It is easy to describe the action of g_j on t. In fact, the action g_2 is the permutation between t_1 and $-t_2$ and so that between t_2 and $-t_1$. The action g_j ($j=3, 4, 5, 6$) is that between t_{j-1} and t_j. It is a little complicated to explain g_1 on t. We give the action of g_1 on the roots. In the below, we assume $i, j, k \in \{3, 4, 5, 6\}$. Then

$$g_1(h) = h, \qquad g_1(h_{1j}) = h_{2j}, \qquad g_1(h_{2j}) = h_{1j}, \qquad g_1(h_{ij}) = h_{ij},$$

$$g_1(h_{1jk}) = h_{2jk}, \qquad g_1(h_{2jk}) = h_{1jk}, \qquad g_1(h_{ijk}) = h_{ijk}.$$

Let E be the complexification of $E_{\boldsymbol{R}}$ and we extend the action of $W(E_6)$ on $E_{\boldsymbol{R}}$ to that on E in a natural manner. Moreover let $\boldsymbol{P}^5$ be the projective space associated to E. Then the $W(E_6)$-action on E induces a projective linear action of $W(E_6)$ on $\boldsymbol{P}^5$.

We next define the following 27 linear forms of $t=(t_1, t_2, t_3, t_4, t_5, t_6)$ by

$$a_1 = -\frac{2}{3}t_6, \qquad b_1 = \frac{1}{2}\Big(t_1+t_2+t_3+t_4+t_5-\frac{1}{3}t_6\Big),$$

$$b_j = t_{j-1}+\frac{1}{3}t_6, \qquad j=2, \cdots, 6, \qquad c_{1j} = -t_{j-1}+\frac{1}{3}t_6, \qquad j=2, \cdots, 6,$$

$$a_j = t_{j-1}-\frac{1}{2}(t_1+t_2+t_3+t_4+t_5+\frac{1}{3}t_6\Big), \qquad j=2, \cdots, 6,$$

$$c_{ij} = -t_{i-1}-t_{j-1}+\frac{1}{2}\Big(t_1+t_2+t_3+t_4+t_5-\frac{1}{3}t_6\Big), \qquad 1<i<j\leqq 6.$$

These are just the $W(E_6)$-orbit of the fundamental weight $a_1=(4\alpha_1+3\alpha_2+5\alpha_3+6\alpha_4+4\alpha_5+2\alpha_6)/3$. For simplicity, we denote by Ω_{27} the totality of the 27 weights above.

We say that a set $\{\omega, \omega', \omega''\}$ $(\omega, \omega', \omega'' \in \Omega_{27})$ is a tritangent triple (of weights) if they are satisfied with the condition (TP):

$$\text{(TP)} \qquad \omega+\omega'+\omega'' = 0.$$

It is easy to show that there are 30 tritangent triples

$$\{a_i, b_j, c_{ij}\}, \qquad i \neq j$$

and 15 tritangent triples

$$\{c_{i_1i_2}, c_{i_3i_4}, c_{i_5i_6}\} \qquad (\{i_1, i_2, i_3, i_4, i_5, i_6\} = \{1, 2, 3, 4, 5, 6\}).$$

As a result, there are totally 45 tritangent triples and they are transitive by $W(E_6)$-action.

We are going to define basic $W(E_6)$-invariant polynomials of t. Let ε_k be the k-th elementary symmetric polynomial of a_j, b_j, c_{ij} and let δ_k be the k-th power sum of a_j, b_j, c_{ij}, that is,

$$\delta_k = \sum_{j=1}^{6}(a_j^k+b_j^k)+\sum_{i<j}c_{ij}^k .$$

Then by direct computation, we obtain the following.

LEMMA 2.1.

$$\varepsilon_2 = -\frac{1}{2}\delta_2, \quad \varepsilon_5 = \frac{1}{5}\delta_5, \quad \varepsilon_6 = -\frac{1}{96}(16\delta_6+\delta_2^3),$$

$$\varepsilon_8 = -\frac{1}{8}\left(\delta_8-\frac{2}{3}\delta_2\delta_6-\frac{1}{576}\delta_2^4\right), \quad \varepsilon_9 = \frac{1}{9}\delta_9,$$

$$\varepsilon_{12} = \frac{1}{16588800}(25\delta_2^6-11520\delta_2^3\delta_6+95040\delta_2^2\delta_8+117504\delta_2\delta_5^2+230400\delta_6^2-1382400\delta_{12}).$$

LEMMA 2.2. *Let $\sigma_i=\sigma_i(t_1^2, \cdots, t_5^2)$ be the i-th elementary symmetric polynomial of $t_1^2, \cdots, t_5^2$ and $\sqrt{\sigma_5}=t_1\cdots t_5$. Then*

$$\delta_2 = 2t_6^2+6\sigma_1, \quad \delta_5 = -\frac{5}{54}t_6^5+\frac{5}{9}\sigma_1 t_6^3+\frac{5}{2}(\sigma_1^2-4\sigma_2)t_6+60\sqrt{\sigma_5},$$

$$\delta_6 = \frac{11}{108}t_6^6+\frac{5}{12}\sigma_1 t_6^4+\frac{5}{4}(\sigma_1^2-4\sigma_2)t_6^2-60\sqrt{\sigma_5}t_6+\frac{3}{4}(3\sigma_1^3-4\sigma_1\sigma_2+24\sigma_3),$$

The explicit form of δ_5 in Lemma 2.2 is already given in Hunt [**H**].

It is clear the subgroup generated by g_2, g_3, g_4, g_5 is identified with the Weyl group $W(D_4)$ of type D_4. We put

$$k_1 = g_1g_3g_4g_5g_2g_4g_3g_1, \qquad k_2 = g_6g_5g_4g_3g_2g_4g_5g_6 .$$

Then it is easy to show the following.

LEMMA 2.3.

(i) *$k_1^2=k_2^2=1$ and $k_1k_2k_1=k_2k_1k_2$.*

(ii) *Both k_1, k_2 normalize $W(D_4)$.*

(iii) *The group generated by $W(D_4)$ and k_1, k_2 is isomorphic to $W(F_4)$.*

In the sequel, we always identify $W(F_4)$ with the group generated by $W(D_4)$ and k_1, k_2 without any comment. It is easy to show that the isotropy of the tritangent triple $\{a_1, b_6, c_{16}\}$ is $W(F_4)$.

3. The construction theorem of elliptic curves due to T. Shioda.

It is known that there are 27 lines on a general cubic surface and 45 tritangent planes.

We are going to construct 45 tritangent planes for a cubic surface $S(pq)$

(pq: generic) to each tritangent triple using the construction of 27 lines on $S(pq)$ due to T. Shioda [**Sh**]. Before entering into the construction, we note that the notation here is slightly different from [**Sh**].

We first suppose that the line L defined by

$$x = uz + rw, \qquad y = dz + ew, \tag{3.1}$$

lies on $S(pq)$, where u, r, d, e are constants such that

$$d = (u^3 + up_2)/2, \qquad e = (3u^2 r - d^2 + up_1 + rp_2 + q_2)/2. \tag{3.2}$$

Let $F(x, y, z, w)$ be the polynomial in the defining equation (1.1) of $S(pq)$. Then solving the equation

$$F(uz + rw, dz + ew, z, w) = 0,$$

we obtain the relations on r and u as shown in [**Sh**, Theorem (E_6)]:

$$\sum_{k=0}^{27} C_{27-k}(pq) u^k = 0, \tag{3.3}$$

$$r = \frac{R_1(pq, u)}{R_2(pq, u)}, \tag{3.4}$$

where

$$\begin{aligned} R_1(pq, u) = {} & -64p_0p_1u - 16p_0p_2^2u^2 - 160p_0p_2u^4 - 336p_0u^6 + 32p_1^2p_2u^2 + 176p_1^2u^4 \\ & -8p_1p_2^2u^5 + 32p_1p_2q_2u + 80p_1p_2u^7 - 64p_1q_1 + 320p_1q_2u^3 + 88p_1u^9 - 2p_2^5u^4 \\ & -17p_2^4u^6 + 8p_2^3q_2u^2 - 72p_2^3u^8 - 16p_2^2q_1u + 16p_2^2q_2u^4 - 134p_2^2u^{10} - 160p_2q_1u^3 \\ & +104p_2q_2u^6 - 110p_2u^{12} - 576q_0u^2 - 336q_1u^5 + 144q_2^2u^2 + 96q_2u^8 - 33u^{14}, \end{aligned} \tag{3.5.1}$$

$$\begin{aligned} R_2(pq, u) = {} & 8(48p_0u^2 + 8p_1^2 - 2p_1p_2^2u - 20p_1p_2u^3 - 66p_1u^5 - p_2^4u^2 - 8p_2^3u^4 \\ & -28p_2^2u^6 - 24p_2q_2u^2 - 60p_2u^8 - 24q_1u - 96q_2u^4 - 39u^{10}), \end{aligned} \tag{3.5.2}$$

and certain polynomials $C_j(pq)$ of pq. (In [**Sh**], the explicit forms of $R_1(pq, u)$, $R_2(pq, u)$ were not written. But the determination of them are straightforward.)

We may take $u = a_j$, b_j, c_{ij} as the 27 solutions of equation (3.3). Then, comparing the coefficients of (3.3) with the definition of ε_k, we have

$$C_0(pq) = 1, \quad C_1(pq) = C_3(pq) = 0, \quad C_k(pq) = (-1)^k \varepsilon_k \quad (k = 2,\ k > 3).$$

Moreover, we have the following relations among p_0, p_1, p_2, q_0, q_1, q_2 and ε_k (cf. [**Sh**, (10.18)]):

$$p_2 = \frac{1}{12}\varepsilon_2, \quad p_1 = \frac{1}{48}\varepsilon_5, \quad q_2 = \frac{1}{96}(\varepsilon_6 - 168p_2^3),$$

$$p_0 = \frac{1}{480}(\varepsilon_8 - 294 p_2^4 - 528 p_2 q_2), \quad q_1 = \frac{1}{1344}(\varepsilon_9 - 1008 p_1 p_2^2),$$

$$q_0 = \frac{1}{17280}(\varepsilon_{12} - 608 p_1^2 p_2 - 4768 p_0 p_2^2 - 252 p_2^6 - 1200 p_2^3 q_2 + 1248 q_2^2).$$

Stressing the dependence of r, d, e on u, we put

$$r = r(u), \qquad d = d(u), \qquad e = e(u)$$

in the sequel and let $L(a_j)$ (resp. $L(b_j)$, $L(c_{ij})$) be the line of $\boldsymbol{P}^3$ defined by the equations

$$x = uz + r(u)w, \qquad y = d(u)z + e(u)w$$

with the value $u = a_j$ (resp. b_j, c_{ij}).

At the present stage, we study basic properties of the function $r(u)$ of u. It follows from [**Sh**] that $r(a_j)$, $r(b_j)$, $r(c_{ij})$ are polynomials of t. In particular, we have

LEMMA 3.1.

$$\begin{aligned} r(a_1) = \frac{1}{5184}(&81t_1^4 - 54t_1^2t_2^2 - 54t_1^2t_3^2 - 54t_1^2t_4^2 - 54t_1^2t_5^2 - 90t_1^2t_6^2 + 81t_2^4 - 54t_2^2t_3^2 \\ &- 54t_2^2t_4^2 - 54t_2^2t_5^2 - 90t_2^2t_6^2 + 81t_3^4 - 54t_3^2t_4^2 - 54t_3^2t_5^2 - 90t_3^2t_6^2 + 81t_4^4 \\ &- 54t_4^2t_5^2 - 90t_4^2t_6^2 + 81t_5^4 - 90t_5^2t_6^2 + 73t_6^4), \\ r(b_2) = \frac{1}{1296}(&81t_1^4 + 135t_1^3t_6 - 54t_1^2t_2^2 - 54t_1^2t_3^2 - 54t_1^2t_4^2 - 54t_1^2t_5^2 + 72t_1^2t_6^2 - 27t_1t_2^2t_6 \\ &- 27t_1t_3^2t_6 - 27t_1t_4^2t_6 - 27t_1t_5^2t_6 + 3t_1t_6^3 + 27t_2^2t_3^2 + 27t_2^2t_4^2 + 27t_2^2t_5^2 - 9t_2^2t_6^2 \\ &- 162t_2t_3t_4t_5 + 27t_3^2t_4^2 + 27t_3^2t_5^2 - 9t_3^2t_6^2 + 27t_4^2t_5^2 - 9t_4^2t_6^2 - 9t_5^2t_6^2 + t_6^4). \end{aligned}$$

Moreover, $r(c_{12})$ is obtained from $r(b_1)$ by changing t_1, t_2 with $-t_1$, $-t_2$.

One way to prove this lemma is to substitute $u = a_1, b_2, c_{12}$ in $R_1(pq, u)$, $R_2(pq, u)$ (cf. (3.4), (3.5.1), (3.5.2)) and compute the results. To accomplish this aim, the author needed a help of computer.

We now recall the definition of a tritangent plane for a general cubic surface S of $\boldsymbol{P}^3$. Let L, L', L'' be three lines on S such that L, L', L'' mutually intersect each other. Then there is a plane π containing L, L', L'' called a tritangent plane. It is known that there are totally 45 tritangent planes for a given general cubic surface. We are going to determine tritangent planes for $S(pq)$.

THEOREM 3.2. *If $\{\omega, \omega', \omega''\}$ is a tritangent triple, then $L(\omega)$, $L(\omega')$, $L(\omega'')$ are contained in a same tritangent plane for $S(pq)$.*

To prove this theorem, we need preparations.

We first define

$$\varphi_{ij} = a_i b_j + a_i c_{ij} + b_j c_{ij}, \qquad \phi_{ij} = a_i b_j c_{ij}.$$

The next lemma is a direct consequence of Lemma 3.1.

LEMMA 3.3.

$$r(a_1)+r(b_2)+r(c_{12}) = \frac{1}{4}(p_2-\varphi_{12})^2,$$

$$a_1 r(a_1)+b_2 r(b_2)+c_{12} r(c_{12}) = -p_1+\frac{1}{2}\phi_{12}(p_2-\varphi_{12}),$$

$$e(a_1)-\frac{1}{2}r(a_1)(p_2-\varphi_{12})$$

$$=\frac{1}{2}\left\{a_1^2 r(a_1)+b_2^2 r(b_2)+c_{12}^2 r(c_{12})+\frac{1}{4}\varphi_{12}(p_2-\varphi_{12})^2-\frac{1}{4}\phi_{12}^2+q_2\right\}.$$

LEMMA 3.4. *Let $\pi(12)$ be the plane defined by*

$$y = \frac{1}{2}(p_2-\varphi_{12})x+\frac{1}{2}\phi_{12}z+\tau_{12}w, \tag{3.6}$$

where

$$\tau_{12} = e(a_1)-\frac{1}{2}r(a_1)(p_2-\varphi_{12}). \tag{3.7}$$

Then the three lines $L(a_1)$, $L(b_2)$, $L(c_{12})$ are contained in $\pi(12)$.

PROOF. Let

$$y = \tau_x x+\tau_z z+\tau_w w \tag{3.8}$$

be a plane, where τ_x, τ_z, τ_w are constants. If the lines $L(a_1)$, $L(b_2)$ are on the plane (3.8), we obtain

$$a_1\tau_x+\tau_z = d(a_1), \qquad r(a_1)\tau_x+\tau_w = e(a_1),$$

$$b_2\tau_x+\tau_z = d(b_2), \qquad r(b_2)\tau_x+\tau_w = e(b_2).$$

Then, noting the definition of a_1, b_2, $d(a_1)$, $d(b_2)$, we have

$$\tau_x = \frac{1}{2}(p_2-\varphi_{12}), \qquad \tau_z = \frac{1}{2}\phi_{12}.$$

These imply

$$\tau_w = e(a_1)-r(a_1)\tau_x = e(a_1)-\frac{1}{2}r(a_1)(p_2-\varphi_{12}).$$

The computation above combined with Lemma 3.3 shows that the two lines

$L(a_1)$ and $L(b_2)$ actually intersect and lie on the plane (3.6).

By an argument parallel to above implies that $L(a_1)$, $L(b_2)$, $L(c_{12})$ lie on the plane (3.6).

PROOF OF THEOREM 3.2. As we remarked in section 2, any tritangent triple of weights is transformed to $\{a_1, b_2, c_{12}\}$ by a certain element of $W(E_6)$. This combined with Lemma 3.4 implies the required statement.

Let $\pi(ij)$ be the tritangent plane containing $L(a_i)$, $L(b_j)$, $L(c_{ij})$ and let $\pi(i_1i_2.i_3i_4.i_5i_6)$ be the tritangent plane containing $L(c_{i_1i_2})$, $L(c_{i_3i_4})$, $L(c_{i_5i_6})$. Noting Lemma 3.4, we can write down the explicit forms of the defining equations for them. For this purpose, we first put

$$\tau_{ij}=e(a_i)-\frac{1}{2}r(a_i)(p_2-\varphi_{ij})\,.$$

Then it is clear from Lemma 3.3 that

$$\tau_{ij}=e(b_j)-\frac{1}{2}r(b_j)(p_2-\varphi_{ij})=e(c_{ij})-\frac{1}{2}r(c_{ij})(p_2-\varphi_{ij})$$

and it follows from Lemma 3.4 that $\pi(ij)$ is defined by

$$y=\frac{1}{2}(p_2-\varphi_{ij})x+\frac{1}{2}\psi_{ij}z+\tau_{ij}w\,.$$

On the other hand, we put

$$\varphi_{i_1i_2i_3i_4i_5i_6}=c_{i_3i_4}c_{i_5i_6}+c_{i_5i_6}c_{i_1}c_{i_2}+c_{i_1i_2}c_{i_3i_4},\qquad \psi_{i_1i_2i_3i_4i_5i_6}=c_{i_1i_2}c_{i_3i_4}c_{i_5i_6}\,,$$

$$\tau_{i_1i_2i_3i_4i_5i_6}=e(c_{i_1i_2})-\frac{1}{2}r(c_{i_1i_2})(p_2-\varphi_{i_1i_2i_3i_4i_5i_6})\,.$$

Then $\pi(i_1i_2.i_3i_4.i_5i_6)$ is defined by

$$y=\frac{1}{2}(p_2-\varphi_{i_1i_2i_3i_4i_5i_6})x+\frac{1}{2}\psi_{i_1i_2i_3i_4i_5i_6}z+\tau_{i_1i_2i_3i_4i_5i_6}w\,.$$

REMARK 3.6. We consider the tritangent plane $\pi(16)$. Its defining equation is

$$y=\frac{1}{2}(p_2-\varphi_{16})x+\frac{1}{2}\psi_{16}z+\tau_{16}w\,.$$

It follows from the definition of $W(F_4)$ in section 2 that $\pi(16)$ is left fixed by $W(F_4)$. This in particular implies that $p_2-\varphi_{16}$, ψ_{16}, τ_{16} are $W(F_4)$-invariant polynomials.

We are going to define cross ratios for tritangent planes (cf. [**C**], [**N**], [**H**]). We take a line on the surface $S(pq)$, say, $L(a_1)$. Then there are five tritangent planes containing $L(a_1)$, in fact, $\pi(1j)$ $(j=2, 3, 4, 5, 6)$ are such tritangent planes. From four of the five planes, say, $\pi(1j)$ $(j=2, 3, 4, 5)$, it is possible to define a cross ratio in the following manner. Let L be a line of $\boldsymbol{P}^5$ and let z_j be the point on the line L which is the intersection of $\pi(1j)$ with L ($j=2, 3, 4, 5$). We take L so that z_2, z_3, z_4, z_5 are mutually different. Then we can define a cross ratio from z_2, z_3, z_4, z_5:

$$\widetilde{CR}(1, 6; 2, 3, 4, 5)=\frac{(z_2-z_5)(z_3-z_4)}{(z_2-z_4)(z_3-z_5)}. \tag{3.9}$$

We put $\alpha=\widetilde{CR}(1, 6; 2, 3, 4, 5)$ for a moment. Then, by permutations among 2, 3, 4, 5, we obtain α, $1-\alpha$, $1/\alpha$, $1/(1-\alpha)$, $\alpha/(\alpha-1)$, $(\alpha-1)/\alpha$.

Let $\{i_1, i_2, i_3, i_4, i_5, i_6\}$ be so taken that $\{i_1, i_2, i_3, i_4, i_5, i_6\}=\{1, 2, 3, 4, 5, 6\}$. Then, taking $L(a_{i_1})$, $\pi(i_1 i_j)$ ($j=2, 3, 4, 5, 6$) instead of $L(a_1)$ and $\pi(1j)$ ($j=2, 3, 4, 5, 6$), we can define $\widetilde{CR}(i_1, i_6; i_2, i_3, i_4, i_5)$ similarly.

We are going to compute cross ratios for some of four tritangent planes for $S(pq)$.

DEFINITION 3.7.

$$x_1(t)=\widetilde{CR}(3, 6; 1, 2, 4, 5), \qquad x_2(t)=\widetilde{CR}(3, 5; 1, 2, 4, 6),$$
$$y_1(t)=\widetilde{CR}(2, 6; 1, 3, 4, 5), \qquad y_2(t)=\widetilde{CR}(2, 5; 1, 3, 4, 6).$$

LEMMA 3.8.

$$x_1(t)=\frac{h_{24}\cdot h_{234}\cdot h_{15}\cdot h_{135}}{h_{14}\cdot h_{134}\cdot h_{25}\cdot h_{235}}, \qquad x_2(t)=\frac{h_{24}\cdot h_{234}\cdot h_{16}\cdot h_{136}}{h_{14}\cdot h_{134}\cdot h_{26}\cdot h_{236}},$$
$$y_1(t)=\frac{h_{34}\cdot h_{234}\cdot h_{15}\cdot h_{125}}{h_{14}\cdot h_{124}\cdot h_{35}\cdot h_{235}}, \qquad y_2(t)=\frac{h_{34}\cdot h_{234}\cdot h_{16}\cdot h_{126}}{h_{14}\cdot h_{124}\cdot h_{36}\cdot h_{236}}.$$

PROOF. It is possible to take $L: z=w=0$ as a generic line. Then it follows from the definition that $\pi(ij)\cap L=\{(1: \varphi_{ij}: 0: 0)\}$. Noting this, we find that

$$x_1(t)=\frac{(\varphi_{31}-\varphi_{35})(\varphi_{32}-\varphi_{34})}{(\varphi_{31}-\varphi_{34})(\varphi_{32}-\varphi_{35})}, \qquad x_2(t)=\frac{(\varphi_{31}-\varphi_{36})(\varphi_{32}-\varphi_{34})}{(\varphi_{31}-\varphi_{34})(\varphi_{32}-\varphi_{36})},$$
$$y_1(t)=\frac{(\varphi_{21}-\varphi_{25})(\varphi_{23}-\varphi_{24})}{(\varphi_{21}-\varphi_{24})(\varphi_{23}-\varphi_{25})}, \qquad y_2(t)=\frac{(\varphi_{21}-\varphi_{26})(\varphi_{23}-\varphi_{24})}{(\varphi_{21}-\varphi_{24})(\varphi_{23}-\varphi_{26})}.$$

On the other hand, it is easy to show that

$$\varphi_{ij}-\varphi_{ik}=\pm h_{jk}\cdot h_{ijk}.$$

These imply the lemma.

By an argument similar to the proof of Lemma 3.8, we can show the following.

THEOREM 3.9. If $\{i_1, i_2, i_3, i_4, i_5, i_6\}=\{1, 2, 3, 4, 5, 6\}$, then

$$\widetilde{CR}(i_3, i_6; i_1, i_2, i_4, i_5) = \pm\frac{h_{i_2i_4}\cdot h_{i_2i_3i_4}\cdot h_{i_1i_5}\cdot h_{i_1i_3i_5}}{h_{i_1i_4}\cdot h_{i_1i_3i_4}\cdot h_{i_2i_5}\cdot h_{i_2i_3i_5}}.$$

The following lemma will be used in the subsequent sections. Since its proof is straightforward, we omit it.

LEMMA 3.10.

$$x_1(t)-1 = -\frac{h_{123}\cdot h_{345}\cdot h_{12}\cdot h_{45}}{h_{14}\cdot h_{235}\cdot h_{25}\cdot h_{134}}, \qquad x_2(t)-1 = -\frac{h_{123}\cdot h_{346}\cdot h_{12}\cdot h_{46}}{h_{14}\cdot h_{236}\cdot h_{26}\cdot h_{134}},$$

$$y_1(t)-1 = -\frac{h_{123}\cdot h_{13}\cdot h_{245}\cdot h_{45}}{h_{14}\cdot h_{235}\cdot h_{124}\cdot h_{35}}, \qquad y_2(t)-1 = -\frac{h_{123}\cdot h_{13}\cdot h_{246}\cdot h_{46}}{h_{14}\cdot h_{236}\cdot h_{124}\cdot h_{36}},$$

$$x_1(t)-y_1(t) = \frac{h_{15}\cdot h_{234}\cdot h_{123}\cdot h_{23}\cdot h_{145}\cdot h_{45}}{h_{14}\cdot h_{235}\cdot h_{124}\cdot h_{25}\cdot h_{134}\cdot h_{35}},$$

$$x_2(t)-y_2(t) = \frac{h_{16}\cdot h_{234}\cdot h_{123}\cdot h_{23}\cdot h_{146}\cdot h_{46}}{h_{14}\cdot h_{236}\cdot h_{124}\cdot h_{26}\cdot h_{134}\cdot h_{36}},$$

$$y_1(t)-y_2(t) = \frac{h_{234}\cdot h_{123}\cdot h_{13}\cdot h_{256}\cdot h_{34}\cdot h_{56}}{h_{14}\cdot h_{235}\cdot h_{236}\cdot h_{124}\cdot h_{35}\cdot h_{36}},$$

$$x_1(t)-x_2(t) = \frac{h_{234}\cdot h_{123}\cdot h_{356}\cdot h_{12}\cdot h_{24}\cdot h_{56}}{h_{14}\cdot h_{235}\cdot h_{236}\cdot h_{25}\cdot h_{26}\cdot h_{134}}.$$

4. The configuration space of 6 points in P^2.

The purpose of this section is to define a $W(E_6)$-equivariant map from P^5 to the configuration space P_2^6 of 6 points in P^2 by using $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ introduced in the previous section.

For this purpose, we first introduce the linear space W of 3×6 matrices:

$$W = \left\{X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{pmatrix};\ x_{ij} \in \boldsymbol{C}\ (1 \leqq i \leqq 3,\ 1 \leqq j \leqq 6)\right\}.$$

Then W admits a left $GL(3, \boldsymbol{C})$-action and a right $GL(6, \boldsymbol{C})$-action in a natural way. For a moment, we identify $(\boldsymbol{C}^*)^6$ with the maximal torus of $GL(6, \boldsymbol{C})$ consisting of diagonal matrices and consider the action of $GL(3, \boldsymbol{C})\times(\boldsymbol{C}^*)^6$ on W instead of $GL(3, \boldsymbol{C})\times GL(6, \boldsymbol{C})$.

For simplicity, we write $X=(X_1, X_2)$ for the matrix $X\in W$, where both X_1, X_2 are 3×3 matrices. For any 3×3 matrix $Y=(y_{ij})_{1\leqq i,j\leqq 3}$ with the condition $y_{ij}\neq 0$ ($1\leqq i, j\leqq 3$), we define a 3×3 matrix

$$\sigma(Y) = \left(\frac{1}{y_{ij}}\right)_{1\leqq i,j\leqq 3}$$

following a suggestion of M. Yoshida. Moreover, we put

$$D(i_1, i_2, i_3) = \det\begin{pmatrix} x_{1i_1} & x_{1i_2} & x_{1i_3} \\ x_{2i_1} & x_{2i_2} & x_{2i_3} \\ x_{3i_1} & x_{3i_2} & x_{3i_3} \end{pmatrix}$$

for a given matrix $X \in W$.

Using these notation, we define subsets W', W_0 of W by

$$W' = \{X \in W\,;\ D(i_1, i_2, i_3) \neq 0\ (1 \leqq i_1 < i_2 < i_3 \leqq 6)\}$$

$$W_0 = \{(X_1, X_2) \in W'\,;\ (I_3, \mathrm{Cof}(X_1^{-1}X_2)), (I_3, \sigma(X_1^{-1}X_2)) \in W'\},$$

where $\mathrm{Cof}(Y) = (\det Y)^t Y^{-1}$ is the cofactor matrix of a given square matrix Y.

It is clear that the action of $GL(3, \boldsymbol{C}) \times (\boldsymbol{C}^*)^6$ on W naturally induces that on each of W', W_0. In the sequel, we mainly consider the quotient space of W_0 under the action of $GL(3, \boldsymbol{C}) \times (\boldsymbol{C}^*)^6$, that is,

$$W_Q = GL(3, \boldsymbol{C}) \backslash W_0 / (\boldsymbol{C}^*)^6.$$

It is clear from the definition that for any element $X \in W_0$, there are $(g, h) \in GL(3, \boldsymbol{C}) \times (\boldsymbol{C}^*)^6$ and $(x_1, x_2, y_1, y_2) \in \boldsymbol{C}^4$ such that

$$gXh = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & y_1 & y_2 \end{pmatrix}.$$

In particular (x_1, x_2, y_1, y_2) is uniquely determined for $X \in W_0$. In this sense, $W_Q = GL(3, \boldsymbol{C}) \backslash W_0 / (\boldsymbol{C}^*)^6$ is identified with an open subset of $\boldsymbol{C}^4$.

Changes of column vectors of $X \in W_0$ induce birational transformations on $\boldsymbol{C}^4$ with coordinate system (x_1, x_2, y_1, y_2). Let $s_j\ (1 \leqq j \leqq 5)$ be the birational transformation on $\boldsymbol{C}^4$ corresponding to the change of the j-th column vector and $(j+1)$-column vector of $X \in W_0$. Moreover W_Q admits an involution s_R induced from the action on W_0 defined by

$$\tilde{s}_R : (X_1, X_2) \longrightarrow (I_3, \sigma(X_1^{-1}X_2))$$

for any $(X_1, X_2) \in W_0$.

LEMMA 4.1. *The birational transformations s_j $(1 \leqq j \leqq 5)$ and s_R on $\boldsymbol{C}^4$ are given by*

$$s_1 : (x_1, x_2, y_1, y_2) \longrightarrow \left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{y_1}{x_1}, \frac{y_2}{x_2}\right),$$

$$s_2 : (x_1, x_2, y_1, y_2) \longrightarrow (y_1, y_2, x_1, x_2),$$

$$s_3 : (x_1, x_2, y_1, y_2) \longrightarrow \left(\frac{x_1 - y_1}{1 - y_1}, \frac{x_2 - y_2}{1 - y_2}, \frac{y_1}{y_1 - 1}, \frac{y_2}{y_2 - 1}\right),$$

$$s_4: (x_1, x_2, y_1, y_2) \longrightarrow \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \frac{1}{y_1}, \frac{y_2}{y_1}\right),$$

$$s_5: (x_1, x_2, y_1, y_2) \longrightarrow (x_2, x_1, y_2, y_1),$$

$$s_R: (x_1, x_2, y_1, y_2) \longrightarrow (1/x_1, 1/x_2, 1/y_1, 1/y_2).$$

The proof of this lemma is straightforward.

We define 15 hypersurfaces $T_j: f_j=0$ $(1 \leqq j \leqq 15)$, of $\boldsymbol{C}^4$, where

$$f_1 = x_1y_2 - x_2y_1 - x_1 + x_2 + y_1 - y_2, \quad f_2 = y_1 - 1, \quad f_3 = x_1 - 1,$$

$$f_4 = y_2 - 1, \quad f_5 = x_2 - 1, \quad f_6 = y_1 - y_2, \quad f_7 = x_1 - x_2, \quad f_8 = x_1 - y_1,$$

$$f_9 = x_2 - y_2, \quad f_{10} = x_1y_2 - x_2y_1, \quad f_{11} = x_2, \quad f_{12} = x_1, \quad f_{13} = y_2, \quad f_{14} = y_1,$$

$$f_{15} = x_1y_2(1-y_1)(1-x_2) - x_2y_1(1-x_1)(1-y_2).$$

It follows from the definition that $s_1, \cdots, s_5, s_R$ are biregular outside the union T of the hypersurfaces T_j $(1 \leqq j \leqq 15)$. For a moment, let $\tilde{G}$ be the group generated by $s_1, \cdots, s_5, s_R$.

The following theorem which seems known shows a concrete correspondence between $W(E_6)$ and the group $\tilde{G}$ defined above.

THEOREM 4.2. *The correspondence*

$$g_1 \longrightarrow s_1, \quad g_2 \longrightarrow s_R, \quad g_3 \longrightarrow s_2, \quad g_4 \longrightarrow s_3, \quad g_5 \longrightarrow s_4, \quad g_6 \longrightarrow s_5$$

induces a group isomorphism of $W(E_6)$ to the group $\tilde{G}$.

PROOF. From the construction of s_j, $j=1, 2, 3, 4, 5$, it is easy to show the relations:

$$s_js_k = s_ks_j \ (|j-k| > 1), \qquad s_js_ks_j = s_ks_js_k \ (|j-k| = 1).$$

Therefore it suffices to show

$$s_js_R = s_Rs_j \ (j = 1, 2, 4, 5), \qquad s_3s_Rs_3 = s_Rs_3s_R,$$

which are easy to check.

REMARK. In [**DO**], it is stated that there is a $W(E_6)$-action on W_Q. See also [**N**, Appendix], [**H**].

We are going to define cross ratios for 5 points in $\boldsymbol{P}^2$ following [**H**]. Let $\xi_i = [\xi_{1i} : \xi_{2i} : \xi_{3i}]$ $(1 \leqq i \leqq 5)$ be five points of $\boldsymbol{P}^2$ in a general position and let l be a line of $\boldsymbol{P}^2$. We denote by $P_i = [1 : z_i : w_i]$ the intersection of l and the line passing through the points ξ_1 and ξ_i. We take l so that the four points P_i $(i=2, 3, 4, 5)$ are mutually different. Then we define

$$CR(\xi_2, \xi_3, \xi_4, \xi_5\,;\,\xi_1) = \frac{(z_2-z_4)(z_3-z_5)}{(z_2-z_5)(z_3-z_4)}, \tag{4.1}$$

which is in fact a cross ratio of z_2, z_3, z_4, z_5.

Now we consider a matrix of the form

$$X = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & y_1 & y_2 \end{pmatrix}$$

which is a representative of a point of W_Q as explained before. From the matrix X, we define six points ξ_i $(i=1, \cdots, 6)$ in $\boldsymbol{P}^2$ in a usual manner, that is,

$$\xi_1 = [1:0:0], \qquad \xi_2 = [0:1:0], \qquad \xi_3 = [0:0:1],$$
$$\xi_4 = [1:1:1], \qquad \xi_5 = [1:x_1:y_1], \qquad \xi_6 = [1:x_2:y_2].$$

Then we can compute $CR(\xi_{i_2}, \xi_{i_3}, \xi_{i_4}, \xi_{i_5}\,;\,\xi_{i_1})$ explicitly for various i_1, i_2, i_3, i_4, i_5. In particular, the next lemma is a direct consequence of its definition.

LEMMA 4.3.

$$x_1 = CR(\xi_2, \xi_1, \xi_4, \xi_5\,;\,\xi_3), \qquad x_2 = CR(\xi_2, \xi_1, \xi_4, \xi_6\,;\,\xi_3),$$
$$y_1 = CR(\xi_1, \xi_3, \xi_4, \xi_5\,;\,\xi_2), \qquad y_2 = CR(\xi_1, \xi_3, \xi_4, \xi_6\,;\,\xi_2).$$

From the equations.

$$CR(\xi_{i_2}, \xi_{i_3}, \xi_{i_4}, \xi_{i_5}\,;\,\xi_{i_1}) = \widetilde{CR}(i_1, i_6\,;\,i_2, i_3, i_4, i_5), \tag{4.2}$$

we obtain various equalities. In particular, by computing the cases

$$(i_1, i_2, i_3, i_4, i_5, i_6) = (3,2,1,4,5,6),\ (3,2,1,4,6,5),\ (2,1,3,4,5,6),\ (2,1,3,4,6,5),$$

we have

$$x_1 = x_1(t), \qquad x_2 = x_2(t), \qquad y_1 = y_1(t), \qquad y_2 = y_2(t), \tag{4.3}$$

where $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ are the functions on $\boldsymbol{P}^5$ (cf. Definition 3.7).

The linear action of $W(E_6)$ on E defined in section 2 induces a projective linear action of $W(E_6)$ on $\boldsymbol{P}^5$ under the identification $\boldsymbol{P}^5 = \boldsymbol{P}(E)$. On the other hand, in virtue of Theorem 4.2, we obtain a birational action of $W(E_6)$ on $\boldsymbol{C}^4$ with coordinate (x_1, x_2, y_1, y_2).

THEOREM 4.4. *Let $\Phi_1(t)$ be a map from $\boldsymbol{P}^5$ to $\boldsymbol{C}^4$ with coordinate (x_1, x_2, y_1, y_2) defined by*

$$\Phi_1(t) = (x_1(t), x_2(t), y_1(t), y_2(t)).$$

Then $\Phi_1(t)$ is $W(E_6)$-equivariant.

PROOF. Noting the definition of g_j and that of birational transformations s_j, s_R, we can check the claim. The most complicated case is the implication

$$\Phi_1(g_4(t))=\left(\frac{x_1(t)-y_1(t)}{1-y_1(t)},\ \frac{x_2(t)-y_2(t)}{1-y_2(t)},\ \frac{y_1(t)}{y_1(t)-1},\ \frac{y_2(t)}{y_2(t)-1}\right),$$

which follows from Lemma 3.10.

5. Relations with a family of cubic surfaces due to A. Cayley.

The purpose of this section is to show a relation between the versal family of the E_6-singularity and the family of cubic surfaces originally due to A. Cayley.

We first recall the definition of the family of cubic surfaces due to Cayley [C]:

$$\begin{aligned}(5.1)\quad & w_1\Big[x_1^2+y_1^2+z_1^2+w_1^2+\left(mn+\frac{1}{mn}\right)y_1z_1+\left(nl+\frac{1}{nl}\right)z_1x_1+\left(lm+\frac{1}{lm}\right)x_1y_1\\ &+w_1\left\{\left(l+\frac{1}{l}\right)x_1+\left(m+\frac{1}{m}\right)y_1+\left(n+\frac{1}{n}\right)z_1\right\}\Big]+kx_1y_1z_1=0.\end{aligned}$$

(We use the homogeneous coordinate $(x_1:y_1:z_1:w_1)$ instead of $(X:Y:Z:W)$ in [C].)

Modifying his family, we introduce a family of cubic surfaces of $\boldsymbol{P}^3$ with homogeneous coordinate $(X:Y:Z:W)$ depending on parameters $(\lambda, \mu, \nu, \rho)$ as follows (cf. [NS]):

$$\begin{aligned}(5.2)\qquad & \rho W[\lambda X^2+\mu Y^2+\nu Z^2+(\rho-1)^2(\lambda\mu\nu\rho-1)^2W^2\\ &+(\mu\nu+1)YZ+(\lambda\nu+1)ZX+(\lambda\mu+1)XY\\ &-(\rho-1)(\lambda\mu\nu\rho-1)W\{(\lambda+1)X+(\mu+1)Y+(\nu+1)Z\}]+XYZ=0.\end{aligned}$$

The relation between (5.1) and (5.2) is given as follows (cf. [NS]):

$$(X, Y, Z, W)=\left(mnx_1,\ nly_1,\ lmz_1,\ -\frac{lmn}{\rho(\rho-1)(\lambda\mu\nu\rho-1)}w_1\right),$$

$$\lambda=l^2,\qquad \mu=m^2,\qquad \nu=n^2,\qquad k=-\frac{(\rho-1)(\lambda\mu\nu\rho-1)}{lmn\rho}.$$

In [C, pp. 376-378], there is a list of the defining equations of 45 tritangent planes and their namings for the surface (5.1). (See also [N, p. 10], where those of 45 tritangents are given for (5.2).) For our purpose, we change their namings into those due to Schlaefli following Hunt [H]. For the sake of convenience, we write the list in [H]. (Below, the left-hand side is Schlaefli's notation and the right-hand side is Cayley's.)

$$(12) \longleftrightarrow \zeta, \quad (13) \longleftrightarrow z, \quad (14) \longleftrightarrow \mathrm{z}, \quad (15) \longleftrightarrow \bar{\mathrm{z}}, \quad (16) \longleftrightarrow w$$

$$(21) \longleftrightarrow \bar{\mathrm{r}}, \quad (23) \longleftrightarrow \bar{\mathrm{n}}, \quad (24) \longleftrightarrow \bar{\mathrm{m}}_1, \quad (25) \longleftrightarrow \bar{\mathrm{q}}_1, \quad (26) \longleftrightarrow \bar{\mathrm{x}}$$

$$(31) \longleftrightarrow \mathrm{n}, \quad (32) \longleftrightarrow \mathrm{r}, \quad (34) \longleftrightarrow \mathrm{q}_1, \quad (35) \longleftrightarrow \mathrm{m}_1, \quad (36) \longleftrightarrow \mathrm{x}$$

$$(41) \longleftrightarrow \mathrm{l}_1, \quad (42) \longleftrightarrow \mathrm{h}, \quad (43) \longleftrightarrow \bar{\mathrm{g}}, \quad (45) \longleftrightarrow \mathrm{l}, \quad (46) \longleftrightarrow x$$

$$(51) \longleftrightarrow \bar{\mathrm{p}}_1, \quad (52) \longleftrightarrow \bar{\theta}, \quad (53) \longleftrightarrow \mathrm{f}, \quad (54) \longleftrightarrow \mathrm{p}, \quad (56) \longleftrightarrow \xi$$

$$(61) \longleftrightarrow \bar{\mathrm{y}}, \quad (62) \longleftrightarrow \bar{\mathrm{r}}_1, \quad (63) \longleftrightarrow \bar{\mathrm{n}}_1, \quad (64) \longleftrightarrow \bar{\mathrm{l}}, \quad (65) \longleftrightarrow \bar{\mathrm{p}},$$

$$(123456) \longleftrightarrow \theta, \quad (123546) \longleftrightarrow \bar{\mathrm{h}}, \quad (123645) \longleftrightarrow \mathrm{r}_1,$$

$$(132456) \longleftrightarrow \bar{\mathrm{f}}, \quad (132546) \longleftrightarrow \mathrm{g}, \quad (132645) \longleftrightarrow \mathrm{n}_1,$$

$$(142356) \longleftrightarrow \mathrm{p}_1, \quad (142536) \longleftrightarrow \mathrm{q}, \quad (142635) \longleftrightarrow \mathrm{m},$$

$$(152346) \longleftrightarrow \bar{\mathrm{l}}_1, \quad (152436) \longleftrightarrow \bar{\mathrm{m}}, \quad (152634) \longleftrightarrow \bar{\mathrm{q}},$$

$$(162345) \longleftrightarrow \mathrm{y}, \quad (162435) \longleftrightarrow y, \quad (162534) \longleftrightarrow \eta.$$

In particular,

$$(46): X=0, \quad (162435): Y=0, \quad (13): Z=0, \quad (16): W=0. \tag{5.3}$$

We recall the surface $S(pq)$ and its 45 tritangent planes which are written by

$$\pi(ij)\ (i \neq j), \qquad \pi(i_1 i_2 . i_3 i_4 . i_5 i_6).$$

Then, it follows from the definition that there is a projective linear map $T(x: y: z: w)=(X: Y: Z: W)$ such that T induces a transformation of the 45 tritangent planes for $S(pq)$ to those for (5.2) defined by

$$\pi(ij) \longrightarrow (ij),$$

$$\pi(i_1 i_2 . i_3 i_4 . i_5 i_6) \longrightarrow (i_1 i_2 i_3 i_4 i_5 i_6)$$

for all (ij) and $\{i_1, i_2, i_3, i_4, i_5, i_6\}$. Then (5.3) implies that T is defined by

$$\begin{cases} X = c_x\{(p_2-\varphi_{46})x/2-y+\phi_{46}z/2+\tau_{46}w\}, \\ Y = c_y\{(p_2-\varphi_{162435})x/2-y+\phi_{162435}z/2+\tau_{162435}w\}, \\ Z = c_z\{(p_2-\varphi_{13})x/2-y+\phi_{13}z/2+\tau_{13}w\}, \\ W = c_w\{(p_2-\varphi_{16})x/2-y+\phi_{16}z/2+\tau_{16}w\} \end{cases} \tag{5.4}$$

for some constants c_x, c_y, c_z, c_w depending on t. (Concrete expression of c_x, c_y, c_z, c_w will be given in section 7.)

By taking cross ratios for tritangents planes, we can obtain a map of $\boldsymbol{P}^5$

to the $(\lambda, \mu, \nu, \rho)$-space. We are going to determine the map in question.

For this purpose, we first recall the definitions of the tritangent planes (31), (32), (34), (35), (36), (21), (23), (24), (25), (26) (cf. [**N**, p. 10]):

$$\lambda X+\mu Y+\lambda\mu\nu\rho Z+(\lambda\mu\nu\rho-1)(\lambda\mu\rho-\lambda\rho-\mu\rho+1)W=0, \tag{21}$$

$$\lambda X+Y+\lambda\nu\rho Z-(\rho-1)(\lambda\mu\nu\rho-1)W=0, \tag{23}$$

$$\lambda\rho X+Y+\lambda\nu\rho Z-\lambda\rho(\rho-1)(\lambda\mu\nu\rho-1)W=0, \tag{24}$$

$$\lambda\nu\rho X+Y+\lambda\nu\rho Z-\rho(\lambda\mu\nu\rho-1)(\lambda\mu\rho-\lambda-\mu+1)W=0, \tag{25}$$

$$\lambda X-(\lambda\rho-1)(\lambda\mu\nu\rho-1)W=0, \tag{26}$$

$$X+\mu Y+\mu\nu\rho Z-(\rho-1)(\lambda\mu\nu\rho-1)W=0, \tag{31}$$

$$X+Y+\nu\rho Z-(\rho-1)(\mu\nu\rho+\lambda\nu\rho-\nu\rho-1)W=0, \tag{32}$$

$$\lambda\rho X+Y+\nu\rho Z-\rho(\rho-1)(\lambda\mu\nu\rho+\lambda\nu-\lambda-\nu)W=0, \tag{34}$$

$$\lambda\nu\rho X+Y+\nu\rho Z-\nu\rho(\rho-1)(\lambda\mu\nu\rho-1)W=0, \tag{35}$$

$$X-(\rho-1)(\mu\nu\rho-1)W=0. \tag{36}$$

Let L be a line of $\boldsymbol{P}^3$ and we put

$$w_{2,j}=L\cap(2j)\ (j=1, 3, 4, 5, 6), \qquad w_{3,j}=L\cap(3j)\ (j=1, 2, 4, 5, 6).$$

We take L so that $w_{2,j}$ $(j=1, 3, 4, 5, 6)$ are mutually different and that $w_{3,j}$ $(j=1, 2, 4, 5, 6)$ are mutually different. Identifying L with $\boldsymbol{P}^1=\boldsymbol{C}\cup\{\infty\}$, we regard $w_{2,j}$, $w_{3,j}$ as points of $\boldsymbol{C}\cup\{\infty\}$. Then we have the following lemma.

LEMMA 5.1.

$$\frac{(w_{2,1}-w_{2,5})(w_{2,3}-w_{2,4})}{(w_{2,1}-w_{2,4})(w_{2,3}-w_{2,5})}=\frac{(\mu\nu\rho-1)(\rho-1)}{(\mu\rho-1)(\nu\rho-1)}, \tag{5.5}$$

$$\frac{(w_{2,1}-w_{2,6})(w_{2,3}-w_{2,4})}{(w_{2,1}-w_{2,4})(w_{2,3}-w_{2,6})}=\frac{\mu(\rho-1)}{(\mu\rho-1)}, \tag{5.6}$$

$$\frac{(w_{3,1}-w_{3,5})(w_{3,2}-w_{3,4})}{(w_{3,1}-w_{3,4})(w_{3,2}-w_{3,5})}=\frac{(\lambda\rho-1)(\lambda\mu\nu\rho-1)}{(\lambda\mu\rho-1)(\lambda\nu\rho-1)}, \tag{5.7}$$

$$\frac{(w_{3,1}-w_{3,6})(w_{3,2}-w_{3,4})}{(w_{3,1}-w_{3,4})(w_{3,2}-w_{3,6})}=\frac{\mu(\lambda\rho-1)}{\lambda\mu\rho-1}. \tag{5.8}$$

PROOF. We may take the line $Y=Z=0$ as L and put $w_{2,j}=(v_{2,j}:0:0:1)$, $w_{3,j}=(v_{3,j}:0:0:1)$. Then, from the definition, we have

$$v_{2,1}=-(\lambda\mu\nu\rho-1)(\lambda\mu\rho-\lambda\rho-\mu\rho+1)/\lambda, \qquad v_{2,3}=(\rho-1)(\lambda\mu\nu\rho-1)/\lambda,$$

$$v_{2,4}=(\rho-1)(\lambda\mu\nu\rho-1), \qquad v_{2,5}=(\lambda\mu\nu\rho-1)(\lambda\mu\rho-\lambda-\mu+1)/(\lambda\nu),$$

$$v_{2,6}=(\lambda\rho-1)(\lambda\mu\nu\rho-1)/\lambda, \qquad v_{3,1}=(\rho-1)(\lambda\mu\nu\rho-1),$$

$$v_{3,2}=(\rho-1)(\mu\nu\rho+\lambda\nu\rho-\nu\rho-1), \qquad v_{3,4}=(\rho-1)(\lambda\mu\nu\rho+\lambda\nu-\lambda-\nu)/\lambda,$$

$$v_{3,5}=(\rho-1)(\lambda\mu\nu\rho-1)/\lambda, \qquad v_{3,6}=(\rho-1)(\mu\nu\rho-1).$$

Noting these, we obtain the lemma by direct computation.

It is clear from the definition that the left-sides of (5.5), (5.6), (5.7), (5.8) are cross ratios of tritangent planes (26), (25), (36), (35), respectively. Therefore, if the map T has the required properties, we obtain the following relations:

$$\widetilde{CR}(3, 6; 1, 2, 4, 5)=\frac{(\lambda\rho-1)(\lambda\mu\nu\rho-1)}{(\lambda\mu\rho-1)(\lambda\nu\rho-1)}, \quad \widetilde{CR}(3, 5; 1, 2, 4, 6)=\frac{\mu(\lambda\rho-1)}{\lambda\mu\rho-1},$$

$$\widetilde{CR}(2, 6; 1, 3, 4, 5)=\frac{(\mu\nu\rho-1)(\rho-1)}{(\mu\rho-1)(\nu\rho-1)}, \quad \widetilde{CR}(2, 5; 1, 3, 4, 6)=\frac{\mu(\rho-1)}{(\mu\rho-1)}.$$

At the present stage, we need a simple lemma to continue the discussion.

LEMMA 5.2. *The relations in* (i) *and* (ii) *on* (x_1, x_2, y_1, y_2) *and* $(\lambda, \mu, \nu, \rho)$ *are equivalent.*

(i)

$$\lambda=\frac{x_2(x_1-1)(y_1-y_2)(y_2-1)}{y_2(x_1-x_2)(x_2-1)(y_1-1)},$$

$$\mu=\frac{\{(y_1-1)(x_2-y_2)-(y_2-1)(x_1-y_1)\}x_2y_2}{x_1x_2y_1-x_1x_2y_2-x_1y_1y_2+x_1y_2+x_2y_1y_2-x_2y_1},$$

$$\nu=-\frac{(x_1y_2-x_2y_1)(x_2-1)(y_2-1)}{(x_1-x_2)(x_2-y_2)(y_1-y_2)},$$

$$\rho=\frac{(x_1-x_2)(x_2-y_2)(y_1-1)}{\{(x_1-1)(x_2-y_2)-(x_1-y_1)(x_2-1)\}(y_2-1)x_2}.$$

(ii)

$$x_1=\frac{(\lambda\rho-1)(\lambda\mu\nu\rho-1)}{(\lambda\mu\rho-1)(\lambda\nu\rho-1)}, \qquad x_2=\frac{(\lambda\rho-1)\mu}{\lambda\mu\rho-1},$$

$$y_1=\frac{(\mu\nu\rho-1)(\rho-1)}{(\mu\rho-1)(\nu\rho-1)}, \qquad y_2=\frac{(\rho-1)\mu}{\mu\rho-1}.$$

Let Φ_2 be a birational transformation of C^4 defined by

$$\Phi_2(x_1, x_2, y_1, y_2)=(\lambda, \mu, \nu, \rho),$$

where λ, μ, ν, ρ are rational functions of (x_1, x_2, y_1, y_2) defined in Lemma 5.2(i). Then Lemma 5.2 shows that Φ_2 is birational.

To continue the argument, we recall the $W(E_6)$-action on the family (5.2) of cubic surfaces given in [**NS**]. In particular, the $W(E_6)$-action in [**NS**] preserves the parameter space. In fact, we define the following six birational transformations on the $(\lambda, \mu, \nu, \rho)$-space:

$$\tilde{g}_1: \begin{cases} \lambda \longrightarrow \lambda\mu\nu\rho^2(1-\lambda)/(\lambda\mu\nu\rho^2-1) \\ \mu \longrightarrow (\lambda\mu\rho-1)(\lambda\mu\nu\rho-1)/(\mu(\lambda\rho-1)(\lambda\nu\rho-1)) \\ \nu \longrightarrow (\lambda\nu\rho-1)(\lambda\mu\nu\rho-1)/(\nu(\lambda\rho-1)(\lambda\mu\rho-1)) \\ \rho \longrightarrow (\lambda\rho-1)(\lambda\mu\nu\rho^2-1)/(\rho(\lambda-1)(\lambda\mu\nu\rho-1)) \end{cases}$$

$$\tilde{g}_2: (\lambda, \mu, \nu, \rho) \longrightarrow (\lambda, 1/\mu, \nu, \mu\rho)$$

$$\tilde{g}_3: (\lambda, \mu, \nu, \rho) \longrightarrow (1/\lambda, \mu, \nu, \lambda\rho)$$

$$\tilde{g}_4: (\lambda, \mu, \nu, \rho) \longrightarrow (\lambda\rho, \mu\rho, \nu\rho, 1/\rho)$$

$$\tilde{g}_5: (\lambda, \mu, \nu, \rho) \longrightarrow (\lambda, \mu, 1/\nu, \nu\rho)$$

$$\tilde{g}_6: \begin{cases} \lambda \longrightarrow (\lambda\nu\rho-1)(\lambda\mu\nu\rho-1)/(\lambda(\nu\rho-1)(\mu\nu\rho-1)) \\ \mu \longrightarrow (\mu\nu\rho-1)(\lambda\mu\nu\rho-1)/(\mu(\nu\rho-1)(\lambda\nu\rho-1)) \\ \nu \longrightarrow \lambda\mu\nu\rho^2(1-\nu)/(\lambda\mu\nu\rho^2-1) \\ \rho \longrightarrow (\nu\rho-1)(\lambda\mu\nu\rho^2-1)/(\rho(\nu-1)(\lambda\mu\nu\rho-1)). \end{cases}$$

Then the correspondence

$$g_j \longrightarrow \tilde{g}_j, \qquad j=1, \cdots, 6$$

induces an isomorphism between $W(E_6)$ and the group of birational transformations on the $(\lambda, \mu, \nu, \rho)$-space generated by $\tilde{g}_j$, $j=1, \cdots, 6$. In this manner, the $(\lambda, \mu, \nu, \rho)$-space admits a $W(E_6)$-action.

LEMMA 5.3. *The map Φ_2 is $W(D_4)$-equivariant.*

PROOF. It suffices to show the $W(D_4)$-equivariance Φ_2^{-1} whose explicit form is obtained by Lemma 5.2 (ii).

The lemma follows from Lemma 4.1 and the definition of $\tilde{g}_2$, $\tilde{g}_3$, $\tilde{g}_4$, $\tilde{g}_5$ given before the lemma.

We define another map Φ_3 from $\boldsymbol{P}^5$ to the $(\lambda, \mu, \nu, \rho)$-space as a composition of Φ_1 and Φ_2: $\Phi_3(t)=\Phi_2(\Phi_1(t))$.

LEMMA 5.4. *We define $\lambda(t)$, $\mu(t)$, $\nu(t)$, $\rho(t)$ by*

$$\Phi_3(t)=(\lambda(t), \mu(t), \nu(t), \rho(t)).$$

Then

$$\lambda(t)=\frac{h_{34}\cdot h_{345}\cdot h_{26}\cdot h_{256}}{h_{24}\cdot h_{245}\cdot h_{36}\cdot h_{356}}\cdot\frac{h_{13}\cdot h_{136}\cdot h_{24}\cdot h_{246}}{h_{12}\cdot h_{126}\cdot h_{34}\cdot h_{346}},$$

$$\mu(t)=\frac{h_{456}\cdot h_{235}\cdot h_{134}\cdot h_{126}}{h\cdot h_{15}\cdot h_{24}\cdot h_{36}}\cdot\frac{h_{16}\cdot h_{136}\cdot h_{24}\cdot h_{234}}{h_{14}\cdot h_{134}\cdot h_{26}\cdot h_{236}},$$

$$\nu(t)=\frac{h_{25}\cdot h_{235}\cdot h_{46}\cdot h_{346}}{h_{24}\cdot h_{234}\cdot h_{56}\cdot h_{356}}\cdot\frac{h_{15}\cdot h_{156}\cdot h_{24}\cdot h_{246}}{h_{14}\cdot h_{146}\cdot h_{25}\cdot h_{256}},$$

$$\rho(t)=\frac{h_{24}\cdot h_{245}\cdot h_{36}\cdot h_{356}}{h_{23}\cdot h_{235}\cdot h_{46}\cdot h_{456}}\cdot\frac{h_{14}\cdot h_{146}\cdot h_{23}\cdot h_{236}}{h_{13}\cdot h_{136}\cdot h_{24}\cdot h_{246}}.$$

PROOF. Since λ, μ, ν, ρ are contained in a $W(D_4)$-orbit, it suffices to show the formula for $\lambda(t)$. But it is easy to prove

$$\lambda(t)=\frac{h_{34}\cdot h_{345}\cdot h_{26}\cdot h_{256}}{h_{24}\cdot h_{245}\cdot h_{36}\cdot h_{356}}\cdot\frac{h_{13}\cdot h_{136}\cdot h_{24}\cdot h_{246}}{h_{12}\cdot h_{126}\cdot h_{34}\cdot h_{346}}$$

by using Lemma 3.10. Hence the lemma follows.

THEOREM 5.5. *The maps Φ_j $(j=2, 3)$ are $W(E_6)$-equivariant.*

PROOF. The $W(E_6)$-equivariance of Φ_2 is straightforward by using Lemma 5.2. Noting that $\Phi_3=\Phi_1\circ\Phi_2$, we imply the theorem.

6. A Conjecture of B. Hunt.

It is known (cf. [**B**]) that there is a unique $W(E_6)$-invariant homogeneous polynomial of $t=(t_1, \cdots, t_6)$ of degree 5 up to a constant factor. For example, we take $\delta_5(t)$ as such a polynomial.

Let I_5 be the hypersurface in $\boldsymbol{P}^5$ defined by $\delta_5(t)=0$. Since $\delta_5(t)$ is $W(E_6)$-invariant, so is I_5. Moreover, since $\dim I_5=4$, the restrictions $\Phi_1|I_5$, $\Phi_3|I_5$ are generically finite maps from I_5 to $\boldsymbol{C}^4$. In [**H**], B. Hunt stated conjectures on these maps which turn out to be one conjecture below.

CONJECTURE 6.1 ([**H**]). *Both $\Phi_1|I_5$, $\Phi_3|I_5$ are generically bijective.*

Since Φ_2 is birational, it suffices to show Conjecture 6.1 for one of $\Phi_1|I_5$, $\Phi_3|I_5$. Noting the definition of $\Phi_1(t)$, we find that Conjecture 6.1 is rewritten as follows:

PROBLEM 6.2. *Let x_1, x_2, y_1, y_2 be constants. At least assume that (x_1, x_2, y_1, y_2) is outside the set T (for the definition of T, see section 3). Using x_1, x_2, y_1, y_2, we define four polynomials of t by*

$$f_1=h_{24}\cdot h_{234}\cdot h_{15}\cdot h_{135}-x_1\cdot h_{14}\cdot h_{134}\cdot h_{25}\cdot h_{235},$$

$$f_2=h_{24}\cdot h_{234}\cdot h_{16}\cdot h_{136}-x_2\cdot h_{14}\cdot h_{134}\cdot h_{26}\cdot h_{236},$$

$$g_1=h_{34}\cdot h_{234}\cdot h_{15}\cdot h_{125}-y_1\cdot h_{14}\cdot h_{124}\cdot h_{35}\cdot h_{235},$$

$$g_2=h_{34}\cdot h_{234}\cdot h_{16}\cdot h_{126}-y_2\cdot h_{14}\cdot h_{124}\cdot h_{36}\cdot h_{236}.$$

Then how many solutions are there for the simultaneous equations of t defined by

$$f_1 = f_2 = g_1 = g_2 = \delta_5 = 0 \tag{6.1}$$

with the condition $\Phi_1(t) \notin T$?

Needless to say, there is a gap between Conjecture 6.1 and Problem 6.2, that is, conjecture 6.1 claims that for generic x_1, x_2, y_1, y_2, equation (6.1) has a unique projective solution. Since it is not clear whether Conjecture 6.1 is true or not, we reformulate it as a problem.

From now on, we are going to explain results related with Problem 6.2 and the moduli of cubic surfaces. We consider the hypersurface H in $\boldsymbol{P}^5$ defined by $\lambda(t)-1=0$, that is,

$$P(t) = h_{345}\cdot h_{26}\cdot h_{256}\cdot h_{13}\cdot h_{136}\cdot h_{246} - h_{245}\cdot h_{36}\cdot h_{356}\cdot h_{12}\cdot h_{126}\cdot h_{346} = 0\,. \tag{6.2}$$

For the polynomial $P(t)$, we have the following elementary but interesting lemma.

LEMMA 6.3. *The polynomial $P(t)$ of equation* (6.2) *is decomposed into two factors:*

$$P(t) = h_{23}\cdot P_5(t)\,,$$

where $P_5(t)$ is homogeneous of degree 5 *and*

$$P_5(t_1, t_2, t_3, t_4, t_5, t_6) = -\frac{1}{60}\delta_5(t_1, t_2, t_3, t_4, t_6, -3t_5)\,.$$

PROOF. By direct computation, we have

$$\begin{aligned}8P_5(t) = {}& t_1^4t_5-2t_1^2t_2^2t_5-2t_1^2t_3^2t_5-2t_1^2t_4^2t_5+2t_1^2t_5^3-2t_1^2t_5t_6^2-8t_1t_2t_3t_4t_6+t_2^4t_5-2t_2^2t_3^2t_5\\ &-2t_2^2t_4^2t_5+2t_2^2t_5^3-2t_2^2t_5t_6^2+t_3^4t_5-2t_3^2t_4^2t_5+2t_3^2t_5^3-2t_3^2t_5t_6^2+t_4^4t_5+2t_4^2t_5^3-2t_4^2t_5t_6^2\\ &-3t_5^5+2t_5^3t_6^2+t_5t_6^4\,.\end{aligned}$$

Since this implies in particular that $P_5(t)$ is symmetric with respect to t_1, t_2, t_3, t_4, t_6, we can prove the lemma by comparing P_5 with the definition of δ_5.

From this remarkable relation, we easily imply the following (cf. [**H**], [**N**]).

PROPOSITION 6.4. (i) *There are* 45 *hypersurfaces in $\boldsymbol{P}^5$ as the $W(E_6)$-orbit of H. Moreover, the isotropy subgroup of H in $W(E_6)$ is isomorphic to the Weyl group of type F_4.*

(ii) *The intersection $H \cap I_5$ is decomposed into two irreducible components. One is defined by $t_5=t_6=0$ and therefore is isomorphic to $\boldsymbol{P}^3$. The other is defined by an equation of degree* 24.

(iii) *If $t \in H$, then $\Phi_2(t)=(1, 1, 1, 1)$, that is, $\lambda(t)=\mu(t)=\nu(t)=\rho(t)=1$.*

PROOF. (i) It follows from Lemma 6.3 that $P_5(t)$ is $W(D_4)$-invariant. We now recall the definition of k_1, $k_2 \in W(E_6)$ (cf. section 2). By direct computation, we can show

$$P_5(k_1(t)) = -P_5(t), \qquad P_5(k_2(t)) = P_5(t).$$

Since $W(F_4)$ is generated by $W(D_4)$ and k_1, k_2, we conclude that the hypersurface I_5 is $W(F_4)$-invariant. Then (i) follows.

(ii) It is clear from Lemma 2.2 and Lemma 6.3 that $t_5=t_6=0$ implies $\delta_5(t)=P_5(t)=0$.

To find the second irreducible component of $I_5 \cap H$, we assume $t_5 \neq 0$ and erase t_6 from the equations $\delta_5(t)=P_5(t)=0$.

Since the computation is very complicated, we only reproduce here the outline of its proof. We first introduce symmetric polynomials of t_1, t_2, t_3, t_4 by

$$s_2 = t_1^2+t_2^2+t_3^2+t_4^2, \quad s_4 = t_1^2(t_2^2+t_3^2+t_4^2)+t_2^2(t_3^2+t_4^2)+t_3^2t_4^2, \quad s_4' = t_1t_2t_3t_4 .$$

Using s_2, s_4, s_4', we define the polynomial $R(t)$ of degree 24 by

$$R(t) = c_{10}t_5^{20}+c_9t_5^{18}+c_8t_5^{16}+c_7t_5^{14}+c_6t_5^{12}+c_5t_5^{10}+c_4t_5^8+c_3t_5^6+c_2t_5^4+c_1t_5^2+c_0 ,$$

where

$$c_{10} = 1728s_2^2, \qquad c_9 = 432s_2(-21s_2^2+20s_4),$$

$$c_8 = 27(4800s_4'^2+761s_2^4-1736s_2^2s_4+400s_4^2),$$

$$c_7 = 8s_2(-46656s_4'^2-3217s_2^4+12852s_2^2s_4-10368s_4^2),$$

$$c_6 = 2(-190080s_4'^2s_2^2-336960s_4'^2s_4+9251s_2^6-55955s_2^4s_4+91368s_2^2s_4^2-28080s_4^3),$$

$$c_5 = 2s_2(825360s_4'^2s_2^2-1582848s_4'^2s_4-3256s_2^6+27143s_2^4s_4-72496s_2^2s_4^2+61776s_4^3),$$

$$\begin{aligned} c_4 = {} & -59833728s_4'^4-1370994s_4'^2s_2^4+5809680s_4'^2s_2^2s_4-4732128s_4'^2s_4^2-193s_2^8 \\ & +3054s_2^6s_4-12981s_2^4s_4^2+10120s_2^2s_4^3+21168s_4^4 , \end{aligned}$$

$$\begin{aligned} c_3 = {} & 2s_2(-2191104s_4'^4+199476s_4'^2s_2^4-1263024s_4'^2s_2^2s_4+1990080s_4'^2s_4^2 \\ & +496s_2^8-7327s_2^6s_4+40443s_2^4s_4^2-98824s_2^2s_4^3+90160s_4^4), \end{aligned}$$

$$\begin{aligned} c_2 = {} & -907200s_4'^4s_2^2+2491776s_4'^4s_4-54714s_4'^2s_2^6+554274s_4'^2s_2^4s_4 \\ & -1854576s_4'^2s_2^2s_4^2+2051616s_4'^2s_4^3-256s_2^{10}+4640s_2^8s_4-33505s_2^6s_4^2 \\ & +120460s_2^4s_4^3-215600s_2^2s_4^4+153664s_4^5 , \end{aligned}$$

$$c_1 = 6s_4'^2s_2(-4968s_4'^2s_2^2+14688s_4'^2s_4-26s_2^6+285s_2^4s_4-1032s_2^2s_4^2+1232s_4^3),$$

$$c_0 = 27s_4'^4(192s_4'^2+s_2^4-8s_2^2s_4+16s_4^2).$$

Moreover,

$$N(t) = -2\{(5s_2^5-1602s_2^4t_5^2-34s_2^3s_4+4134s_2^3t_5^4+10037s_2^2s_4t_5^2-3005s_2^2t_5^6$$
$$+56s_2s_4^2-12820s_2s_4t_5^4+828s_2t_5^8-15764s_4^2t_5^2+1980s_4t_5^6-360t_5^{10})t_5^2$$
$$-(s_2^2+164s_2t_5^2-4s_4+7368t_5^4)s_4'^2\}\, s_4't_5,$$

$$D(t) = -\{3(31s_2^3+650s_2^2t_5^2-92s_2s_4+2320s_2t_5^4-1752s_4t_5^2+5648t_5^6)s_4'^2t_5^2$$
$$+2(2464s_4^2-2055s_4t_5^4+187t_5^8)s_2^2t_5^4-4(1687s_4^2-415s_4t_5^4+12t_5^8)s_2t_5^6$$
$$-(1465s_4-1044t_5^4)s_2^4t_5^4+15(269s_4-61t_5^4)s_2^3t_5^6-16s_4'^4+144s_2^6t_5^4$$
$$-599s_2^5t_5^6-5488s_4^3t_5^4+2072s_4^2t_5^8-120s_4t_5^{12}\}.$$

Then assuming $t_5\neq 0$, from the equations

$$P_5(t) = \delta_5(t) = 0,$$

we obtain

$$t_6 = N(t)/D(t), \qquad R(t) = 0.$$

The equation $R(t)=0$ defines the hypersurface of I_5 stated in Proposition 6.4(ii).

(iii) follows from direct computation.

REMARK 6.5. It follows from Proposition 6.4 (i) that there is a natural 1-1 correspondence between the $W(E_6)$-orbit of H and the 45 exceptional divisors of Naruki's cross ratio variety [**N**].

If we consider the equation $\lambda-1=0$ in the (x_1, x_2, y_1, y_2)-space, we obtain a hypersurface H_0 defined by

$$x_2(x_1-1)(y_1-y_2)(y_2-1)-y_2(x_1-x_2)(x_2-1)(y_1-1) = 0. \tag{6.3}$$

Now we formulate a problem simplified from Problem 6.2, noting Proposition 6.4 (ii). Namely, we consider Problem 6.2 in the case $t_5=t_6=0$ and $t_1=1$. (The condition $t_1=1$ is not essential. From the homogeneity, we may assume $t_j=1$ for some j.)

PROBLEM 6.2$'$. *Define four polynomials of t_2, t_3, t_4 by*

$$f_{10} = (t_2+t_3-t_4+1)^2(t_2+t_4)(t_3-1)-x_1(t_2+t_3)(t_2-t_3+t_4+1)^2(t_4-1),$$

$$f_{20} = (t_2+t_3+t_4+1)(t_2+t_3-t_4+1)(t_3-1)t_2$$
$$+x_2(t_2+t_3)(t_2-t_3+t_4+1)(t_2-t_3-t_4+1),$$

$$g_{10} = (t_2+t_3-t_4+1)^2(t_2-t_3)(t_4+1)-y_1(t_2-t_3+t_4+1)^2(t_2-t_4)(t_3+1),$$

$$g_{20}=(t_2+t_3+t_4+1)(t_2+t_3-t_4+1)(t_2-t_3)$$
$$-y_2(t_2-t_3+t_4+1)(t_2-t_3-t_4+1)(t_3+1)t_2,$$

where x_1, x_2, y_1, y_2 are constants with the condition (6.3) *and* $(x_1, x_2, y_1, y_2)\notin T$. (*In particular, we assume that x_1 is a rational function of x_2, y_1, y_2.*) *Then how many solutions are there for equations* (6.4) *of t_2, t_3, t_4 below*

$$f_{10}=f_{20}=g_{10}=g_{20}=0 \tag{6.4}$$

under the condition $t\notin T$?

It is possible to give an answer to Problem 6.2′. In fact, erasing t_3, t_4 from (6.4), we obtain an equation for t_2 defined by

$$\sum_{j=0}^{9} b_j t_2^j=0, \tag{6.5}$$

where

$$b_9=(x_2y_1y_2-x_2y_2-2y_1y_2+y_1+y_2^2)^2(x_2y_2-2y_2+1)y_2^4,$$

$$b_8=3(x_2y_1y_2-x_2y_2-2y_1y_2+y_1+y_2^2)^2(x_2y_2-2x_2+1)y_2^4,$$

$$b_6=-4(x_2^2y_1y_2-x_2^2y_2+x_2y_1^2+x_2y_1y_2^2-4x_2y_1y_2+x_2y_1+x_2y_2^2-y_1^2y_2+y_1y_2)$$
$$\times(x_2y_1y_2-x_2y_2-2y_1y_2+y_1+y_2^2)(x_2y_2-2x_2+1)y_2^3,$$

$$b_5=-6(x_2y_1y_2-x_2y_2-2y_1y_2+y_1+y_2^2)(x_2y_1+x_2y_2^2-2x_2y_2-y_1y_2+y_2)$$
$$\times(x_2y_2-2y_2+1)x_2y_1y_2^2,$$

$$b_4=6(x_2y_1y_2-x_2y_2-2y_1y_2+y_1+y_2^2)(x_2y_1+x_2y_2^2-2x_2y_2-y_1y_2+y_2)$$
$$\times(x_2y_2-2x_2+1)x_2y_1y_2^2,$$

$$b_3=4(x_2^2y_1y_2-x_2^2y_2+x_2y_1^2+x_2y_1y_2^2-4x_2y_1y_2+x_2y_1+x_2y_2^2-y_1^2y_2+y_1y_2)$$
$$\times(x_2y_1+x_2y_2^2-2x_2y_2-y_1y_2+y_2)(x_2y_2-2y_2+1)x_2y_1y_2,$$

$$b_1=-3(x_2y_1+x_2y_2^2-2x_2y_2-y_1y_2+y_2)^2(x_2y_2-2y_2+1)x_2^2y_1^2,$$

$$b_0=-(x_2y_1+x_2y_2^2-2x_2y_2-y_1y_2+y_2)^2(x_2y_2-2x_2+1)x_2^2y_1^2,$$

$$b_7=b_2=0.$$

Moreover, if t_2 is a solution of (6.5), t_3, t_4 are uniquely determined by (6.4).

It is provable that equation (6.5) for t_2 is irreducible of degree 9 and that for generic x_2, y_1, y_2, (6.5) has no multiple factor. As a consequence, we obtain the following.

THEOREM 6.6. *The restriction of Φ_1 to the subspace $t_5=t_6=0$ is generically*

9 *to* 1.

The author is not sure whether Theorem 6.6 induces the *invalidity* of Conjecture 6.1 or not.

7. Miscellaneous results and concluding remarks.

7.1. After the manuscript was written up, T. Shioda pointed out that Theorem 3.2 can be proved by using his theory of Mordell-Weil lattices.

On the other hand, B. Hunt pointed out that the $W(E_6)$-equivariance of the map Ψ_2 is a special case of a general result of E. Looijenga on a relation between double ratios of root systems and geometric double ratios on del Pezzo surfaces (cf. [**H**, p. 15]).

7.2. In section 5, we defined a linear map T (cf. (5.4)). We are going to determine the constants c_x, c_y, c_z, c_w.

We first note that the tritangent plane (36) for the surface (5.2) is defined by (cf. [**N**])

$$X-(\rho-1)(\mu\nu\rho-1)W=0.$$

This combined with (5.4) implies

$$\begin{aligned} &c_x\left\{\frac{1}{2}(p_2-\varphi_{46})x-y+\frac{1}{2}\phi_{46}z+\tau_{46}w\right\} \\ &\qquad -(\rho-1)(\mu\nu\rho-1)c_w\left\{\frac{1}{2}(p_2-\varphi_{16})x-y+\frac{1}{2}\phi_{16}z+\tau_{16}w\right\} \\ &=c\left\{\frac{1}{2}(p_2-\varphi_{36})x-y+\frac{1}{2}\phi_{36}z+\tau_{36}w\right\} \end{aligned}$$

for a constant c. Comparing the coefficients of x, y, we obtain

$$\begin{aligned} &c_x(p_2-\varphi_{46})-(\rho-1)(\mu\nu\rho-1)c_w(p_2-\varphi_{16})=c(p_2-\varphi_{36}), \\ &c_x-(\rho-1)(\mu\nu\rho-1)c_w=c. \end{aligned}$$

Solving the equations above, we obtain

$$c_x=(\rho-1)(\mu\nu\rho-1)\frac{\varphi_{36}-\varphi_{16}}{\varphi_{36}-\varphi_{46}}c_w. \tag{7.1}$$

In the same way, we obtain

$$c_y=(\rho-1)(\lambda\nu\rho-1)\frac{\varphi_{162345}-\varphi_{16}}{\varphi_{162345}-\varphi_{162435}}c_w, \tag{7.2}$$

$$c_z=(\rho-1)(\lambda\mu\rho-1)\frac{\varphi_{16}-\varphi_{14}}{\varphi_{13}-\varphi_{14}}\varphi_w. \tag{7.3}$$

By direct computation, we have

$$\frac{\varphi_{36}-\varphi_{16}}{\varphi_{36}-\varphi_{46}}=-\frac{h_{13}\cdot h_{245}}{h_{34}\cdot h_{125}},\ \frac{\varphi_{162345}-\varphi_{16}}{\varphi_{162345}-\varphi_{162435}}=-\frac{h_{236}\cdot h_{456}}{h_{25}\cdot h_{34}},\ \frac{\varphi_{16}-\varphi_{14}}{\varphi_{13}-\varphi_{14}}=-\frac{h_{46}\cdot h_{146}}{h_{34}\cdot h_{134}}$$

$$\rho-1=\frac{h_{23}\cdot P_5}{h_{245}\cdot h_{36}\cdot h_{356}\cdot h_{12}\cdot h_{126}\cdot h_{346}},\quad \mu\nu\rho-1=\frac{h_{125}\cdot P_5}{h\cdot h_{13}\cdot h_{14}\cdot h_{256}\cdot h_{26}\cdot h_{56}},$$

$$\lambda\nu\rho-1=\frac{h_{25}\cdot P_5}{h_{12}\cdot h_{126}\cdot h_{234}\cdot h_{356}\cdot h_{56}\cdot h_{456}},\quad \lambda\mu\rho-1=\frac{h_{134}\cdot P_5}{h\cdot h_{12}\cdot h_{15}\cdot h_{36}\cdot h_{46}\cdot h_{346}}.$$

(The polynomial P_5 is the one defined in section 6.)

From these equations, it is possible to determine c_x, c_y, c_z, c_w. (Since T is projective linear, we may assume that $c_w=1$.)

7.3. The polynomial (3.3) of u is related with a 27-dimensional irreducible representation of the Lie algebra $\underline{e}_6$ of type E_6. By an argument parallel to [Sl], *the following statement seems provable.*

Let (π, V) be an irreducible representation of $\underline{e}_6$ such that $\dim V=27$. Let x be a subregular nilpotent element of $\underline{e}_6$, that is, x is nilpotent such that its centralizer $Z_{\underline{e}_6}(x)$ has dimension $\operatorname{rank}\underline{e}_6+2=8$. Moreover, let h, y be elements of $\underline{e}_6$ such that $\{x, h, y\}$ is a TDS. Let $e_1, \cdots, e_8$ be a basis of $Z_{\underline{e}_6}(y)$. Taking $v=\sum_{j=1}^{8} w_j e_j \in Z_{\underline{e}_6}(y)$, we consider the characteristic polynomial

$$\chi(\Lambda;\ w_1, \cdots, w_8)=\det(\Lambda-\pi(x+v)).$$

Since $\deg_\Lambda \chi(\Lambda;\ w_1, \cdots, w_8)=27$, we put

$$\chi(\Lambda;\ w_1, \cdots, w_8)=\Lambda^{27}+C_1\Lambda^{26}+C_2\Lambda^{25}\cdots+C_{26}\Lambda+C_{27} \tag{7.4}$$

for some $C_j\ (j=1, 2, \cdots, 27)$. Then from (7.4), we obtain a lot of equations with respect to w_i and C_j. These equations are reduced to a unique equation which turns out to be equation (1.1) (with $w=1$) of Introduction by a certain change of variables.

In the argument above, the role of 27 weights a_j, b_j, c_{ij} is clear. But what are the roles of 27 lines and 45 tritangent planes?

7.4. It is possible to give an interpretation of the 76 divisors of Naruki's cross ratio variety (cf. [N]) in terms of root system Δ of type E_6. We are going to explain this briefly.

We first define a linear subspace $CR(\boldsymbol{P})$ of $\boldsymbol{P}^2$ with coordinate $(\xi_1 : \xi_2 : \xi_3)$ defined by the equation $\xi_1+\xi_2+\xi_3=0$. Clearly $CR(\boldsymbol{P})$ is equal to $\boldsymbol{P}^1$, but it is convenient to use $CR(\boldsymbol{P})$ for our purpose.

Let Z be the Zariski open subset of $\boldsymbol{P}^5$ defined by

$$h\cdot\prod_{j<k} h_{jk}\cdot\prod_{i<j<k} h_{ijk}\neq 0.$$

We first define a cross ratio map of Z to $CR(\boldsymbol{P})$ by

$$t \longrightarrow (h_{j_3j_5}h_{j_3j_4j_5}h_{j_2j_6}h_{j_2j_4j_6} : -h_{j_2j_5}h_{j_2j_4j_5}h_{j_3j_6}h_{j_3j_4j_6} : h_{j_2j_3}h_{j_2j_3j_4}h_{j_5j_6}h_{j_4j_5j_6}).$$

By permutations of indices among 1, 2, 3, 4, 5, 6, we obtain 30 maps of the form above. We need another cross ratio map defined by

$$t \longrightarrow (h_{j_1j_3j_5}h_{j_2j_4j_5}h_{j_2j_3j_6}h_{j_1j_4j_6} : -h_{j_2j_3j_5}h_{j_1j_4j_5}h_{j_1j_3j_6}h_{j_2j_4j_6} : h_{j_1j_2}h_{j_3j_4}h_{j_5j_6}h).$$

In this case, by permutations of indices among 1, 2, 3, 4, 5, 6, we obtain 15 maps of the form above. As a result, we obtain 45 ($=30+15$) cross ratio maps of Z to $CR(\boldsymbol{P})$.

Taking the product of these maps, we define a map cr_{E_6} of Z to $CR(\boldsymbol{P})^{45}$. Let $C'_{E_6}=cr_{E_6}(Z)$ and let C_{E_6} be its Zariski closure in $CR(\boldsymbol{P})^{45}$.

THEOREM 7.4.1 ([**N**]). (i) *C_{E_6} is 4-dimensional and non-singular.*

(ii) *The $W(E_6)$-action on C_{E_6} is biregular.*

(iii) *$C_{E_6}-C'_{E_6}$ is a divisor with normal crossings. There are* 76 *irreducible components of $C_{E_6}-C'_{E_6}$ each of which is smooth.*

In [**N**], C_{E_6} is denoted C and C'_{E_6} is equal to M. The variety C_{E_6} is called Naruki's cross ratio variety in [**H**].

We are now going to give a root system theoretic interpretation of the 76 divisors of $C_{E_6}-C'_{E_6}$. Let φ be one of root forms h, h_{jk}, h_{ijk}. Then taking the limit $\varphi\to 0$ in C_{E_6}, we obtain a hypersurface Y_φ in C_{E_6}. In this way, we obtain 36 divisors of C_{E_6}. Clearly these correspond to positive roots of the root system Δ. In the sequel, such a hypersurface is called a hypersurface of the 1^{st} kind. Any hypersurface of the 1^{st} kind admits a biregular Σ_6-action induced by the $W(E_6)$-action and is isomorphic to the 3-dimensional Terada model. Here the n-dimensional Terada model means the n-dimensional nonsingular variety T_n constructed in [**T**] which plays an important role in the study of Appell-Lauricella hypergeometric function $F_D(z_1, \cdots, z_n)$.

On the other hand, we take an A_2-subroot system of Δ, for example, $\Delta_1=\{\pm h_{12}, \pm h_{23}, \pm h_{13}\}$. Then, we put

$$u=\frac{h_{23}}{h_{12}}$$

and substitute

$$h_{23}=uh_{12}, \qquad h_{13}=(1+u)h_{12}$$

in C'_{E_6}. Next taking the limit $h_{12}\to 0$, we obtain a hypersurface X_1 in C_{E_6} which depends on the choice of the A_2-subroot system Δ_1.

There are two other A_2-subroot systems $\Delta_2=\{\pm h_{45}, \pm h_{56}, \pm h_{46}\}$, $\Delta_3=\{\pm h_{123}, \pm h, \pm h_{456}\}$. The triple $\{\Delta_1, \Delta_2, \Delta_3\}$ is characterized by the properties that they are mutually orthogonal and span the linear space E. Then we can construct a hypersurface X_2 (resp. X_3) of C_{E_6} from Δ_2 (resp. Δ_3) by an argument similar to do X_1.

Using the notation above, we can show the following.

(i) X_1 equals X_2, X_3 and is isomorphic to $T_1 \times T_1 \times T_1$.

(ii) There are totally 40 divisors of the form X_1 corresponding to subroot systems of Δ whose types are $A_2+A_2+A_2$.

A hypersurface constructed in this manner is called a hypersurface of the 2^{nd} kind.

In [**N**], hypersurfaces of the 1^{st} kind and hypersurfaces of the 2^{nd} kind are called A_1-divisors and non-normality divisors, respectively.

It is easy to describe the intersection relation among the 76 divisors in terms of root systems.

Since the Terada model is constructed in connection with Appell-Lauricella hypergeometric function, it is interesting to study the generalized hypergeometric function of type (3,6) as a function on Naruki's cross ratio variety (cf. [**MSY**]).

References

[B] N. Bourbaki, Groupes et Algèbres de Lie, Chaps. 4, 5, 6, Herman, Paris, 1968.

[C] A. Cayley, A memoir on cubic surfaces, Collected Papers VI.

[DO] I. Dolgachev and D. Ortland, Point sets in projective spaces and theta functions, Astérisque, **165**, 1988.

[H] B. Hunt, A remarkable quintic fourfold in $\boldsymbol{P}^5$ and its dual variety (Update: 7.1. 1992), in preparation.

[MSY] K. Matsumoto, T. Sasaki and M. Yoshida, The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type (3, 6), International J. of Math., **3** (1992), 1-164.

[N] I. Naruki, Cross ratio variety as a moduli space of cubic surfaces, Proc. London Math. Soc., **45** (1982), 1-30.

[NS] I. Naruki and J. Sekiguchi, A modification of Cayley's family of cubic surfaces and birational action of $W(E_6)$ over it, Proc. Japan Acad. Ser. A, **56** (1980), 122-125.

[Se] J. Sekiguchi, The configuration space of 6 points in $\boldsymbol{P}^2$, the moduli space of cubic surfaces and the Weyl group of type $W(E_6)$, RIMS Kokyuroku, **848** (1993), 74-85.

[Sh] T. Shioda, Construction of elliptic curves with high rank via the invariants of the Weyl groups, J. Math. Soc. Japan, **43** (1991), 673-719.

[Sl] P. Slodowy, Simple singularities and simple algebraic groups, Lecture Notes in Math., **815**.

[T] T. Terada, Fonction hypergéométriques F_1 et fonctions automorphes I, J. Math., Soc. Japan, **35** (1983), 451-475.

Jiro SEKIGUCHI
Department of Mathematics
University of Electro-Communications
Chofu, Tokyo 182
Japan