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ON SUPERCONVERGENCE RESULTS AND NEGATIVE NORM ESTIMATES FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. The purpose of this paper is to show how known negative norm estimates and superconvergence results applied to parabolic equations can be carried over to integro-differential equations of parabolic type. A quasi projection technique introduced earlier by Douglas, Dupont and Wheeler is modified to establish negative norm estimates in several space variables. Further, in a single space variable, knot superconvergence is also established. Finally, interior superconvergence estimates are also derived.

1. Introduction. In this paper, we discuss some superconvergence results and negative norm estimates for the following parabolic integro-differential equation

$$
\begin{aligned}
u_t + A(t)u &= \int_0^t B(t,\tau)u(\tau)\,d\tau + f \quad \text{in } \Omega \times J,\\
u &= 0 \quad \text{on } \partial\Omega \times J,\\
u(\cdot,0) &= u_0 \quad \text{in } \Omega\ .
\end{aligned}
\tag{1.1}
$$

Here, $u = u(x,t)$ and $f = f(x,t)$ are real valued functions in $\Omega \times J$, where Ω is a bounded domain in R^d with smooth boundary $\partial\Omega$, $J = (0,T]$, $T < \infty$ and $u_t = \partial u/\partial t$. Further, $A(t)$ is a selfadjoint, uniformly positive definite second order elliptic partial differential operator of the form

$$
A(t) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial}{\partial x_i}\right) + a_0(x,t)I,
$$

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and $B(t,\tau)$ is a general second-order partial differential operator

$$B(t,\tau) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(b_{ij}(x;t,\tau)\frac{\partial}{\partial x_i}\right) + \sum_{j=1}^{d} b_j(x;t,\tau)\frac{\partial}{\partial x_j} + b_0(x;t,\tau)I.$$

The nonhomogeneous term f and the coefficients of $A(t)$ and $B(t,\tau)$ are assumed to be smooth.

Let $H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\}$. Further, let $A(t;\cdot,\cdot)$ and $B(t,\tau;\cdot,\cdot)$ be bilinear forms on $H_0^1(\Omega)\times H_0^1(\Omega)$ corresponding to operators $A(t)$ and $B(t,\tau)$, i.e.,

$$A(t;u,v) = \int_\Omega \left(\sum_{i,j=1}^{d} a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0 uv\right)dx$$

and

$$B(t,\tau;u(\tau),v) = \int_\Omega \left(\sum_{i,j=1}^{d} b_{ij}(x;t,\tau)\frac{\partial u(\tau)}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{j=1}^{d} b_j(x;t,\tau)\frac{\partial u(\tau)}{\partial x_j} v + b_0(x;t,\tau)u(\tau)v\right)dx.$$

The weak formulation of the problem (1.1) may be stated as: Find $u : \bar{J} \to H_0^1(\Omega)$ such that

$$\begin{aligned} (u_t,v) + A(t;u,v) &= \int_0^t B(t,\tau;u(\tau),v)\,d\tau \\ &\quad + (f,v), \quad \forall\, v \in H_0^1(\Omega), \quad t \in J, \\ u(0) &= u_0. \end{aligned} \tag{1.2}$$

Here and below, we denote $(\cdot,\cdot)$ and $\|\cdot\|$ by L^2 inner product and the induced norm on $L^2 = L^2(\Omega)$. The error analysis will be carried out in usual Sobolev spaces $W^{s,p}(\Omega)$, s a nonnegative integer and $1 \le p \le \infty$, with $H^s = H^s(\Omega)$ denoting $W^{s,2}(\Omega)$. The normed dual of $H^s(\Omega)$, denoted by $H^{-s}(\Omega)$. We shall use the notation $\|\cdot\|_{s,\Omega}$ for norm in the Hilbert space $H^s(\Omega)$. If a norm is taken over the entire domain we suppress the subscript Ω in $H^s(\Omega)$. Sometimes we shall use the notation H_0^1 instead of $H_0^1(\Omega)$.

Let S_h, $0 < h < 1$, be a family of finite dimensional subspaces of H^1 with the following approximation property: For a given integer $r \geq 1$

(1.3)
$$\inf_{\chi \in S_h} \{\|v-\chi\| + h\|v-\chi\|_1\} \leq Ch^s\|v\|_s, \quad v \in H^s(\Omega), \quad 1 \leq s \leq r+1.$$

Now, setting $S_h^0 = S_h \cap H_0^1$, (1.3) holds for $v \in H^s \cap H_0^1$.

The standard semidiscrete finite element approximation is then defined as a function $u_h : \bar{J} \to S_h^0$ such that

(1.4)
$$\begin{aligned} (u_{h,t}, \chi) + A(t; u_h, \chi) &= \int_0^t B(t, \tau; u_h(\tau), \chi)\, d\tau \\ &\quad + (f, \chi), \quad \forall\, \chi \in S_h^0, \quad t \in J, \\ u_h(0) &= u_{0,h}, \end{aligned}$$

where $u_{0,h}$ is a suitable approximation of u_0 in S_h^0.

Earlier, Douglas, Dupont and Wheeler [**5**] have introduced quasi-projection technique for parabolic and hyperbolic equations in several space variables and have derived optimal negative norm estimates. In case of a single space variable, they have also established the knot superconvergence results with initial approximation for the Galerkin solution obtained through a sequence of elliptic projections. Recently, Thomée [**12**] has discussed superconvergence results for $t \geq \delta > 0$ with initial data being the L^2-projection of u. For further references related to knot superconvergence, see Arnold and Douglas [**1**] for quasilinear parabolic problems, and Jones and Pani [**6**] for nonlinear Stefan problems.

More recently, considerable attention has been devoted to the optimal error analysis for partial integro-differential equations, see, for example, Yanik and Fairweather [**16**], Cannon and Lin [**3, 4**], Lin et al. [**7**], Pani et al. [**11**], Pani and Peterson [**9**], Thomée and Zhang [**15**] and Zhang [**17**]. The main tool used for parabolic integro-differential equations is the Ritz-Volterra projection as against Ritz or elliptic projection for parabolic equations. In this paper, based on a sequence of Ritz-Volterra projections the quasi-projection technique is employed to derive knot superconvergence results for parabolic integro-differential equation in one space dimension. Moreover, in case of several space variables optimal negative norm estimates are also established. Finally, following Bramble and Schatz [**2**] an averaging operator is introduced and estimates on difference quotients are derived. These estimates then imply interior superconvergence estimates for the post-processed approximations.

The layout of this paper is as follows. In Section 2, the quasi-projection technique is introduced and its related estimates are explored. Optimal order negative norm estimates for the error in the Galerkin solution are derived in Section 3. In Section 4, the quasi-projection is applied to derive knot superconvergence of order $O(h^{2r})$ in the case of a single space variable. Finally, Section 5 is devoted to interior superconvergence estimates.

Throughout this paper C denotes a generic constant whose dependence can be traced easily from the proofs.

2. Quasi-projection technique and related estimates. The purpose of this paper is to discuss to what extent known negative norm estimates and superconvergence results for the case of parabolic differential equation, i.e., $B(t,s)=0$ (Thomée [**14**]) carry over to the present situation. A principal tool used in [**7**] was a generalization of elliptic or Ritz projection called Ritz-Volterra projection W_h : $C(\bar{J}, H_0^1(\Omega)) \to C(\bar{J}, S_h)$ defined by

$$A(t;(W_h u - u)(t), \chi) = \int_0^t B(t,\tau;(W_h u - u)(\tau), \chi)\, d\tau, \quad \forall\, \chi \in S_h^0,\ t \in \bar{J}. \tag{2.1}$$

We recall the following lemma from Arnold and Douglas [**1**], which will be used in our subsequent analysis.

Lemma 2.1. *Let there be given a linear functional* $F : H_0^1(\Omega) \to R$ *and numbers* $M_1 \geq M_2 \geq \cdots \geq M_{q+1},\ 0 \leq q \leq r$, *with*

$$|F(\eta)| \leq M_p \|\eta\|_p, \quad \forall\, \eta \in H^p(\Omega) \cap H_0^1(\Omega), \quad p = 1, 2, \dots, q+1.$$

Suppose $\Phi \in H_0^1(\Omega)$ *satisfies*

$$A(t;\Phi,\chi) = F(\chi), \quad \forall\, \chi \in S_h^0.$$

Then, for $s = -1, 0, \dots, q-1$,

$$\|\Phi\|_{-s} \le C\Big[(M_1 + \inf_{\chi \in S_h^0} \|\Phi - \chi\|_1) h^{s+1} + M_{s+2}\Big].$$

We have the following estimate for Ritz-Volterra projection.

Theorem 2.1. *Let* $k \ge 0$, $1 \le q \le r+1$, *and let* $\partial^k u/\partial t^k \in H^q(\Omega)$, *for* $t \in J$. *Then, for* $\rho = W_h u - u$ *and* $-1 \le s \le r-1$,

$$\left\| \frac{\partial^k \rho}{\partial t^k} \right\|_{-s} \le C h^{s+q} \left(\sum_{i=0}^{k} \left\| \frac{\partial^i u}{\partial t^i} \right\|_q + \int_0^t \|u\|_q \, d\tau \right).$$

Proof. We now apply Lemma 2.1 with $\Phi = \rho$ and $F(\eta) = \int_0^t B(t,\tau;\rho(\tau),\eta)\, d\tau$. Note that

$$|F(\eta)| \le C\left(\int_0^t \|\rho\|_1 \, d\tau \right) \|\eta\|_1,$$

and

$$\begin{aligned} |F(\eta)| &= \left| \int_0^t (\rho(\tau), B^*(t,\tau)\eta)\, d\tau \right| \\ &\le C\left(\int_0^t \|\rho(\tau)\|_{-s} \, d\tau \right) \|\eta\|_{s+2}, \end{aligned}$$

where $B^*(t,\tau)$ is the adjoint of $B(t,\tau)$. Setting $M_1 = C\left(\int_0^t \|\rho\|_1 \, d\tau\right)$ and $M_{s+2} = C\left(\int_0^t \|\rho(\tau)\|_{-s}\, d\tau\right)$, we therefore obtain, using approximation property,

$$\|\rho\|_{-s} \le C h^{s+1} \left(\int_0^t \|\rho\|_1 \, d\tau + h^{q-1} \|u(t)\|_q \right) + C \int_0^t \|\rho\|_{-s} \, d\tau.$$

Hence, by Gronwall's lemma the desired estimate for $k = 0$ follows. For $k = 1$, we differentiate (2.1) to have

$$\text{(2.2)} \qquad \begin{aligned} A\Big(t; \frac{\partial \rho}{\partial t}, \chi\Big) &= -A_t(t; \rho, \chi) + B(t, t; \rho(t), \chi) \\ &\quad + \int_0^t B_t(t, \tau; \rho(\tau), \chi)\, d\tau, \end{aligned}$$

where A_t and B_t correspond to the operator obtained by differentiating the coefficients of A and B, respectively, with respect to the first variable t. Let $F(\eta)$ be given by the righthand side (2.2). Further, an application of Lemma 2.1 with $\Phi = \partial\rho/\partial t$ yields

$$\begin{aligned} \Big\|\frac{\partial \rho}{\partial t}\Big\|_{-s} &\le C h^{s+1}\Big(\|\rho\|_1 + \int_0^t \|\rho\|_1\, d\tau + h^{q-1}\Big\|\frac{\partial u}{\partial t}\Big\|_q\Big) \\ &\quad + C\|\rho\|_{-s} + C\int_0^t \|\rho\|_{-s}\, d\tau. \end{aligned}$$

The proof is now completed by treating higher order temporal derivatives in a similar way. □

We shall modify the usual initial values in order to achieve superconvergence result. For this, let us define below the quasi projections.

Let $\rho_0 = W_h u - u = \rho$, $\theta_0 = u_h - W_h u$. Then it is easy to verify from (1.1), (1.4) and (2.1) that

$$\text{(2.3)} \qquad \begin{aligned} \Big(\frac{\partial \theta_0}{\partial t}, \chi\Big) + A(t; \theta_0, \chi) &= \int_0^t B(t, \tau; \theta_0, \chi)\, d\tau \\ &\quad - \Big(\frac{\partial \rho_0}{\partial t}, \chi\Big), \quad \chi \in S_h^0. \end{aligned}$$

Define maps $\rho_j : J \to S_h^0$ with $j \ge 1$, recursively by

$$\text{(2.4)} \qquad A(t; \rho_j, \chi) = \int_0^t B(t, \tau; \rho_j, \chi)\, d\tau - \Big(\frac{\partial \rho_{j-1}}{\partial t}, \chi\Big), \quad \chi \in S_h^0.$$

Differentiating (2.4) k times with respect to time, we obtain

$$
\begin{aligned}
A\Big(t;\frac{\partial^k\rho_j}{\partial t^k},\chi\Big) &= -\sum_{i=1}^{k}\binom{k}{i}A_{t^i}\Big(t;\frac{\partial^{k-i}\rho_j}{\partial t^{k-i}},\chi\Big)\\
&\quad+\sum_{i=0}^{k-1}\binom{k}{i}B_{t^i}\Big(t,t;\frac{\partial^{k-i-1}\rho_j}{\partial t^{k-i-1}},\chi\Big)\\
&\quad+\int_0^t B_{t^k}(t,\tau;\rho_j,\chi)\,d\tau-\Big(\frac{\partial^{k+1}\rho_{j-1}}{\partial t^{k+1}},\chi\Big),
\end{aligned}
\tag{2.5}
$$

where A_{t^i} and B_{t^i} are obtained by differentiating the coefficients of the operator $A_{t^{i-1}}$ and $B_{t^{i-1}}$, respectively, with respect to the first variable t.

We shall now have the following lemma to estimate ρ_j along with its temporal derivatives in negative norms.

Theorem 2.2. *Let $j,k\ge 0$, $1\le q\le r+1$, and let $\partial^{j+k}u/\partial t^{j+k}\in H^q(\Omega)$, for $t\in J$. Then, for $-1\le s+2j\le r-1$,*

$$
\Big\|\frac{\partial^k\rho_j}{\partial t^k}\Big\|_{-s}\le Ch^{q+s+2j}\Big(\sum_{i=0}^{j+k}\Big\|\frac{\partial^i u}{\partial t^i}\Big\|_q+\int_0^t\sum_{i=0}^{j}\Big\|\frac{\partial^i u}{\partial t^i}\Big\|_q\,d\tau\Big),\qquad t\in J.
$$

Proof. The case $j=0$ is covered by Theorem 2.1. The rest of the proof is completed by induction on j. For $j\ge 1$ and $k=0$, let $F(\eta)=\int_0^t B(t,\tau;\rho_j,\eta)\,d\tau-(\partial\rho_{j-1}/\partial t,\eta)$. Then

$$
|F(\eta)|\le C\Big(\int_0^t\|\rho_j\|_1\,d\tau+\Big\|\frac{\partial\rho_{j-1}}{\partial t}\Big\|_{-1}\Big)\|\eta\|_1
$$

and

$$
|F(\eta)|\le C\Big(\int_0^t\|\rho_j\|_{-s}\,d\tau+\Big\|\frac{\partial\rho_{j-1}}{\partial t}\Big\|_{-s-2}\Big)\|\eta\|_{s+2}.
$$

The inductive hypotheses imply that F fulfills the hypotheses of Lemma 2.1 with $\Phi = \rho_j$ and $q = r - 2j$. Therefore,

(2.6)
$$\|\rho_j\|_{-s} \leq C\left(\int_0^t \|\rho_j\|_1\,d\tau + \left\|\frac{\partial \rho_{j-1}}{\partial t}\right\|_{-1} + \inf_{\chi\in S_h^0}\|\rho_j-\chi\|_1\right)h^{s+1} + \left\|\frac{\partial \rho_{j-1}}{\partial t}\right\|_{-s-2} + \int_0^t \|\rho_j\|_{-s}\,d\tau.$$

Choose $\chi = \partial^k\rho_j/\partial t^k$ in (2.5); then we find easily

$$\left\|\frac{\partial^k \rho_j}{\partial t^k}\right\|_1 \leq C\left(\sum_{i=1}^{k+1}\left\|\frac{\partial^i \rho_{j-1}}{\partial t^i}\right\|_{-1} + \int_0^t \left\|\frac{\partial \rho_{j-1}}{\partial t}\right\|_{-1} d\tau\right).$$

On substituting the above estimate for $k = 0$ in (2.6), an application of Gronwall's lemma yields

$$\|\rho_j\|_{-s} \leq C\left(\left\|\frac{\partial \rho_{j-1}}{\partial t}\right\|_{-1} + \int_0^t \left\|\frac{\partial \rho_{j-1}}{\partial t}\right\|_{-1} d\tau\right)h^{s+1} + \left\|\frac{\partial \rho_{j-1}}{\partial t}\right\|_{-s-2} + \int_0^t \left\|\frac{\partial \rho_{j-1}}{\partial t}\right\|_{-s-2} d\tau.$$

Apply induction on j and Theorem 2.1 to obtain an estimate for ρ_j. For the time derivatives of ρ_j, we use (2.5) and follow the proof of Theorem 2.1 to obtain the desired estimates. This completes the proof. □

3. Optimal negative norm estimates. Set $u_j = W_h u + \rho_1 + \cdots + \rho_j$, $j \geq 1$, and define $\theta_j = u_h - u_j$, $j \geq 1$. A simple induction argument using (2.3) and (2.4) shows that

$$\left(\frac{\partial \theta_j}{\partial t}, \chi\right) + A(t;\theta_j,\chi) = \int_0^t B(t,\tau;\theta_j,\chi)\,d\tau - \left(\frac{\partial \rho_j}{\partial t}, \chi\right), \quad \chi \in S_h^0. \tag{3.1}$$

The estimates of Theorem 2.2 can be applied to (3.1) to bound θ_j after a choice of the initial value is made. Now we have the following theorem in order to estimate θ_j.

Theorem 3.1. *For $2k \le r-1$, let*

$$u_h(0) = u_k(0) = W_h u(0) + \rho_1(0) + \cdots + \rho_k(0). \tag{3.2}$$

Then there is a constant C independent of h such that

$$\begin{aligned}\|\theta_k\|_{L^\infty(L^2(\Omega))} \le & Ch^{q+\min(2k+1,r-1)}\Bigg(\sum_{i=0}^{k+1}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q \\ & + \int_0^t \sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q d\tau\Bigg), \\ & 1 \le q \le r+1.\end{aligned}$$

Proof. Choose $\chi = \theta_k$ in (3.1) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta_k\|^2 + \alpha\|\theta_k\|_1^2 \le C\left(\left(\int_0^t \|\theta_k(\tau)\|_1\, d\tau\right)\|\theta_k\|_1 + \left\|\frac{\partial \rho_k}{\partial t}\right\|_{-1}\|\theta_k\|_1\right).$$

Here α is the coercivity constant of A. Integrating both side from 0 to t, and using the inequality $ab \le a^2/\varepsilon + \varepsilon b^2$ with $\theta_k(0) = 0$, we have

$$\begin{aligned}\|\theta_k\|^2 + 2\alpha\int_0^t \|\theta_k\|_1^2\, d\tau \le & C_\varepsilon \int_0^t\int_0^\tau \|\theta_k(\tau')\|_1^2\, d\tau'\, d\tau \\ & + C_\varepsilon\int_0^t \left\|\frac{\partial \rho_k}{\partial t}\right\|_{-1}^2 d\tau + \varepsilon\int_0^t \|\theta_k\|_1^2\, d\tau.\end{aligned}$$

Choose ε appropriately, apply Gronwall's lemma and Theorem 2.2 to complete the rest of the proof. □

Theorem 3.2. *Let $1 \le q \le r+1$. Further, let $u_h(0)$ be defined by (3.2). Then θ_k satisfies*

$$\|\theta_k\|_{L^\infty(H^1(\Omega))} \le \begin{cases} Ch^{q+2k}\left(\sum_{i=0}^{k+1}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q + \int_0^t \sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q d\tau\right), & 2k \le r-1,\ r \text{ odd}, \\ Ch^{q+2k+1}\left(\sum_{i=0}^{k+2}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q + \int_0^t \sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q d\tau\right), & 2k \le r-2,\ r \text{ even}.\end{cases}$$

Proof. Setting $\chi = \partial\theta_k/\partial t$ in (3.1), and integrating both sides from 0 to t to obtain

(3.3)
$$\begin{aligned}\int_0^t \left\|\frac{\partial\theta_k}{\partial t}\right\|^2 d\tau + \frac{1}{2}A(t;\theta_k,\theta_k) = &\int_0^t B(t,\tau;\theta_k(\tau),\theta_k(t))\,d\tau\\ &-\int_0^t B(\tau,\tau;\theta_k(\tau),\theta_k(\tau))\,d\tau\\ &-\int_0^t\int_0^\tau B_t(\tau,\tau';\theta_k(\tau'),\theta_k(\tau))\,d\tau'\,d\tau\\ &-\frac{1}{2}\int_0^t A_t(t;\theta_k,\theta_k)\,d\tau\\ &-\int_0^t\left(\frac{\partial\rho_k}{\partial t},\frac{\partial\theta_k}{\partial t}\right)d\tau.\end{aligned}$$

Using coercivity condition for A and inequality $ab \le a^2/2 + b^2/2$, it follows that

$$\begin{aligned}\int_0^t \left\|\frac{\partial\theta_k}{\partial t}\right\|^2 d\tau + \alpha\|\theta_k\|_1^2 \le C&\int_0^t\int_0^\tau \|\theta_k(\tau')\|_1^2\,d\tau'\,d\tau\\ &+C\int_0^t\left(\|\theta_k\|_1^2 + \left\|\frac{\partial\rho_k}{\partial t}\right\|^2\right)d\tau.\end{aligned}$$

An application of Gronwall's lemma with Theorem 2.2 for $s = 0$ now yields

$$\|\theta_k\|_{L^\infty(H^1(\Omega))} \le Ch^{q+2k}\left(\sum_{i=0}^{k+1}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q + \int_0^t\sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q d\tau\right), \tag{3.4}$$

provided $2k \le r-1$. This completes the proof of the first estimate. For odd r, the choice $2k = r-1$ and $q = r+1$ produces an $O(h^{2r})$ estimate for θ_k in $H^1(\Omega)$. However, if r is even, only an $O(h^{2r-1})$ estimate can be obtained from (3.4). But the following arguments regain an $O(h^{2r})$ estimate in $H^1(\Omega)$. Now integrate the last term on the righthand side of (3.3) by parts with respect to time to have

$$\|\theta_k(t)\|_1^2 \le C\int_0^t \|\theta_k\|_1^2\,d\tau + C\left(\left\|\frac{\partial\rho_k}{\partial t}\right\|_{-1} + \int_0^t\left\|\frac{\partial^2\rho_k}{\partial t^2}\right\|_{-1}d\tau\right)\|\theta_k\|_1.$$

An application of the inequality $ab \le a^2/\varepsilon + \varepsilon b^2$ with Gronwall's lemma yields

$$\|\theta_k(t)\|_1^2 \le C\bigg(\bigg\|\frac{\partial \rho_k}{\partial t}\bigg\|_{-1}^2 + \int_0^t \bigg\|\frac{\partial \rho_k}{\partial t}\bigg\|_{-1}^2 \, d\tau + \int_0^t \bigg\|\frac{\partial^2 \rho_k}{\partial t^2}\bigg\|_{-1}^2 \, d\tau\bigg).$$

The second inequality now follows from Theorem 2.2 provided $2k \le r - 2$. □

Finally, we have the following theorem for the error estimates of $e = u_h - u$ in negative norms.

Theorem 3.3. *Let u and u_h be the solution of* (1.1) *and* (1.4), *respectively. Further, let $u_h(0)$ be given by* (3.2). *Then for $2k \le r - 1$ and $1 \le q \le r + 1$, the following negative norm estimate*

$$\|u_h - u\|_{L^\infty(H^{-s}(\Omega))} \le Ch^{s+q}\bigg(\sum_{i=0}^{k+1} \bigg\|\frac{\partial^i u}{\partial t^i}\bigg\|_q + \int_0^t \sum_{i=0}^{k} \bigg\|\frac{\partial^i u}{\partial t^i}\bigg\|_q \, d\tau\bigg)$$

holds for $0 \le s \le \min(2k+1, r-1)$.

Proof. We write error e with W_h defined by (2.1) as

$$\begin{aligned} e &= u_h - u = (u_h - W_h u) + (W_h u - u) \\ &= \theta_k + \sum_{i=0}^{k} \rho_i. \end{aligned} \tag{3.5}$$

The rest of the proof is an immediate consequence of Theorem 2.2, Theorem 3.1 and the following imbedding result

$$\|\theta_k\|_{L^\infty(H^{-s}(\Omega))} \le C\|\theta_k\|_{L^\infty(L^2(\Omega))}.$$

This completes the proof of the theorem. □

4. Knot superconvergence results. We now consider the case when $\Omega = I = (0,1)$. Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of I with $\max(x_i - x_{i-1}) = h$, and let

$$S_h^0 = \{v \in C(I) \mid \; v|_{(x_i, x_{i-1})} \in P_r; v(0) = v(1) = 0\}.$$

The elements of S_h^0 are assumed to be C^{p_i} functions, $0 \leq p_i \leq r$. However, at any knot at which superconvergence is sought, the smoothness constraint on S_h^0 reduces to simple continuity.

Let $\bar{x} \in (0,1)$ be a nodal point (knot) in each of the partitions, i.e., for each h there exists $i(h)$ so that $\bar{x} = x_{i(h)}$. For $s \geq 0$, we define the space $\tilde{H}^s$ as

$$\tilde{H}^s = \{u : u|_{(0,\bar{x})} \in H^s((0,\bar{x})), \quad u|_{(\bar{x},1)} \in H^s((\bar{x},1))\},$$

and norm on $\tilde{H}^s$ by

$$|||u|||_s^2 = \|u\|^2_{H^s((0,\bar{x}))} + \|u\|^2_{H^s((\bar{x},1))}.$$

The following lemma will be useful to produce a superconvergence result for ρ_j at any knot point $x_{i(h)}$.

Lemma 4.1. *Let $\Phi \in H_0^1$ satisfy*

$$A(t;\Phi,\chi) = F(\chi), \quad \chi \in S_h^0,$$

where F is a linear functional on H_0^1. Further, let there exist constants $M_1 \geq M_2 \geq \cdots \geq M_{q+1}$, with $0 \leq q \leq r$, such that

$$|F(\Phi)| \leq M_p |||\Phi|||_p, \quad \Phi \in \tilde{H}^p \cap H_0^1, \quad p = 1, 2 \ldots q+1.$$

Then for $0 \leq s \leq r-1$,

$$|\Phi(\bar{x})| \leq C\Big[(M_1 + \inf_{\chi \in S_h^0} \|\Phi - \chi\|_1) h^{s+1} + M_{s+2}\Big].$$

Proof. Let $\Psi \in \tilde{H}^s$, $s \geq 0$, be the solution of the following problem

$$\begin{aligned} A(t)\Psi &= 0 \quad \text{in } I/\{\bar{x}\}, \\ \Psi &= 0 \quad x = 0,1, \\ a_{11}\frac{\partial \Psi}{\partial x}\Big|_{\bar{x}-0}^{\bar{x}+0} &= -1. \end{aligned} \tag{4.1}$$

The solution Ψ of (4.1) satisfies the following regularity condition

$$|||\Psi|||_{s+2} \le C, \tag{4.2}$$

where the constant C is independent of the position of $\bar{x}$. Multiply equation (4.1) by $\Phi \in \tilde{H}^s \cap C(I)$, and integrate by parts to obtain

$$0 = (\Phi, A(t)\Psi) = A(t; \Phi, \Psi) + \left(a_{11} \frac{\partial \Psi}{\partial x} \Big|_{\bar{x}-0}^{\bar{x}+0} \right) \Phi(\bar{x}).$$

Use of jump condition at $x = \bar{x}$ yields

$$\Phi(\bar{x}) = A(t; \Phi, \Psi),$$

and therefore,

$$\begin{aligned} |\Phi(\bar{x})| &= |A(t; \Phi, \Psi - \chi) + F(\chi - \Psi) + F(\Psi)| \\ &\le C\|\Phi\|_1 \inf_{\chi \in S_h^0} \|\Psi - \chi\|_1 \\ &\quad + M_1 \inf_{\chi \in S_h^0} \|\Psi - \chi\|_1 + M_{s+2}|||\Psi|||_{s+2}. \end{aligned}$$

From Lemma 2.1, we have

$$\|\Phi\|_1 \le C(M_1 + \inf_{\chi \in S_h^0} \|\Phi - \chi\|_1),$$

and an application of (1.3) and (4.2) now completes the proof. □

Below, we shall discuss superconvergence results for ρ_j at nodal points $\bar{x}$.

Theorem 4.1. *Let $j \ge 0$, $1 \le q \le r+1$ and $0 \le s \le r - 2j - 1$. Then*

$$\left| \frac{\partial^k \rho_j(\bar{x}, t)}{\partial t^k} \right| \le Ch^{s+q+2j} \left(\sum_{i=0}^{k+j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_q + \int_0^t \sum_{i=0}^{j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_q d\tau \right).$$

In particular, for the choice $s = r - 2j - 1$,

$$\left| \frac{\partial^k \rho_j(\bar{x}, t)}{\partial t^k} \right| \le Ch^{q+r-1} \left(\sum_{i=0}^{k+j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_q + \int_0^t \sum_{i=0}^{j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_q d\tau \right).$$

Proof. Take $j = 0$ and $k = 0$. Apply Lemma 4.1 with $\Phi = \rho$ and $F(\eta) = \int_0^t B(t,\tau;\rho(\tau),\eta)\,d\tau$ to have

$$|F(\eta)| = C\biggl(\int_0^t \|\rho\|_1\,d\tau\biggr)\|\eta\|_1,$$

and

$$|F(\eta)| = C\biggl(\int_0^t \|\rho\|_{-s}\,d\tau\biggr)\|\eta\|_{s+2}.$$

Now setting $M_1 = C(\int_0^t \|\rho\|_1\,d\tau)$, and $M_{s+2} = C(\int_0^t \|\rho\|_{-s}\,d\tau)$ together with the approximation property (1.3), we obtain

$$|\rho(\bar{x},t)| \le C\biggl(\int_0^t \|\rho\|_1\,d\tau + h^{q-1}\|u(t)\|_q\biggr)h^{s+1} + C\int_0^t \|\rho\|_{-s}\,d\tau.$$

Further, using Theorem 2.1 for $j = 0$ and $k = 0$, we obtain the required estimate. For $j = 0$ and $k = 1$, we differentiate (2.1) to have (2.2). Let $F(\eta)$ now be given by the righthand side of this equation. Then, apply Lemma 4.1 with $\Phi = \partial\rho/\partial t$ and finally Theorem 2.1 to obtain the desired estimate for $\partial\rho/\partial t$. Similarly, the estimates for higher order time derivatives of ρ can be easily derived. For $j > 0$ and $k = 0$, we use Lemma 4.1 with $\Phi = \rho_j$ and

$$F(\eta) = \int_0^t B(t,\tau;\rho_j,\eta)\,d\tau - \biggl(\frac{\partial\rho_{j-1}}{\partial t},\eta\biggr).$$

Then induction argument on j can be repeated to obtain the estimate for ρ_j. Apply Lemma 4.1 with $\Phi = \partial\rho_j/\partial t$ and

$$\begin{aligned} F(\eta) = &-A_t(t;\rho_j,\eta) + B(t,t;\rho_j,\eta) \\ &+ \int_0^t B_t(t,\tau;\rho_j(\tau),\eta)\,d\tau - \biggl(\frac{\partial^2\rho_{j-1}}{\partial t^2},\eta\biggr), \end{aligned}$$

to complete the proof for $j > 0$ and $k = 1$. The proof of the theorem is completed by treating higher order time derivatives in a similar way. □

The following theorem shows the superconvergence results for the error $e = u_h - u$ at the nodal points $x = \bar{x}$.

Theorem 4.2. *Let $1 \le q \le r+1$, and $u_h(0)$ be defined by* (3.2). *Then*

$$|(u_h - u)(\bar{x}, t)| \le Ch^{q+r-1}\left(\sum_{i=0}^{k} \left\| \frac{\partial^i u}{\partial t^i} \right\|_q + \int_0^t \sum_{i=0}^{k} \left\| \frac{\partial^i u}{\partial t^i} \right\|_q d\tau \right) + Q,$$

where

$$Q = \begin{cases} Ch^{q+2k}(\sum_{i=0}^{k+1} \|\frac{\partial^i u}{\partial t^i}\|_q + \int_0^t \sum_{i=0}^{k} \|\frac{\partial^i u}{\partial t^i}\|_q \, d\tau), & 2k \le r-1, \\ & r \text{ odd}, \\ Ch^{q+2k+1}(\sum_{i=0}^{k+2} \|\frac{\partial^i u}{\partial t^i}\|_q + \int_0^t \sum_{i=0}^{k} \|\frac{\partial^i u}{\partial t^i}\|_q \, d\tau), & 2k \le r-2, \\ & r \text{ even}. \end{cases}$$

Proof. Now, write $u_h - u$ in the form

$$(u_h - u)(\bar{x}, t) = \theta_k(\bar{x}, t) + \sum_{i=0}^{k} \rho_j(\bar{x}, t).$$

Since $|\theta_k(\bar{x}, t)| \le \|\theta_k(t)\|_1$, the desired result now follows from Theorem 3.2 and Theorem 4.1. □

Remark. If $2k = r-1$ for r odd and $2k = r-2$ for r even, then for $q = r+1$

$$|(u_h - u)(\bar{x}, t)| = O(h^{2r}).$$

This yields a superconvergence result for $r \ge 2$.

In the rest of this section, the superconvergence results for the flux at the nodal point $x = \bar{x}$ are discussed. For this purpose, let us define a Galerkin approximation $w_h : J \to S_h$ as a solution of

(4.3)

$$(w_{h,t}, \chi) + A(t; w_h, \chi) = \int_0^t B(t, \tau; w_h(\tau), \chi) \, d\tau + (f, \chi) + a_{11} \frac{\partial u}{\partial x} \chi \Big|_0^1 - \int_0^t \left(b_{11} \frac{\partial u}{\partial x} \chi \right) \Big|_0^1 d\tau,$$

$$\chi \in S_h, \ 0 < t \le T,$$

where $w_h(0)$ is a suitably chosen initial condition. Denote the corresponding quasi projection

$$w_k = \tilde{W}_h u + y_1 + \cdots + y_k,$$

where $\tilde{W}_h$ satisfies (2.1) and $y_j, j = 1, \ldots, k$, satisfies (2.4) now with $\chi \in S_h$ instead of $\chi \in S_h^0$.

Lemma 4.2. *Let $\mu_j = y_j - \rho_j$, $j = 0, \ldots, k$. Then there is a positive constant C independent of h such that*

$$\left\| \frac{\partial^m \mu_j}{\partial t^m} \right\|_1 \le Ch^{r+q-1} \left(\sum_{i=0}^{m+j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_q + \int_0^t \sum_{i=0}^{j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_q d\tau \right), \quad t \in J.$$

Proof. Observe that

$$A(t; \mu_0, \chi) = \int_0^t B(t, \tau; \mu_0(\tau), \chi)\, d\tau, \quad \chi \in S_h^0, \tag{4.4}$$

and

$$\begin{aligned} A(t; \mu_j, \chi) = & \int_0^t B(t, \tau; \mu_j(\tau), \chi)\, d\tau \\ & - \left(\frac{\partial \mu_{j-1}}{\partial t}, \chi \right), \quad \chi \in S_h^0, \quad j = 1, \ldots, k. \end{aligned} \tag{4.5}$$

Choose $\chi = \mu_0 - y_0(0,t)(1-x) - y_0(1,t)x$ in (4.4) and obtain

$$\begin{aligned} \alpha \|\mu_0\|_1^2 & \le A(t; \mu_0, \mu_0) \\ & \le C(|y_0(0,t)| + |y_0(1,t)|) \left(\|\mu_0\|_1 + \int_0^t \|\mu_0\|_1\, d\tau \right) \\ & \quad + C \left(\int_0^t \|\mu_0\|_1\, d\tau \right) \|\mu_0\|_1. \end{aligned}$$

Now, using the inequality $ab \le a^2/\varepsilon + \varepsilon b^2$ and with an appropriate choice of ε, it follows that

$$\|\mu_0\|_1^2 \le C(|y_0(0,t)|^2 + |y_0(1,t)|^2) + C \int_0^t \|\mu_0\|_1^2\, d\tau.$$

Then, an application of Gronwall's lemma yields

$$\begin{aligned}\|\mu_0\|_1^2 &\le C(|y_0(0,t)|^2 + |y_0(1,t)|^2)\\ &\quad + C\int_0^t (|y_0(0,\tau)|^2 + |y_0(1,\tau)|^2)\,d\tau.\end{aligned}$$

Finally, use of Theorem 4.1 completes the proof for $j = 0$ and $m = 0$.

For time derivatives of μ_0, differentiate (4.4) m times and choose

$$\chi = \frac{\partial^m \mu_0}{\partial t^m} - \frac{\partial^m y_0}{\partial t^m}(0,t)(1-x) - \frac{\partial^m y_0}{\partial t^m}(1,t)x$$

in the resulting equation. Then use Theorem 4.1 to obtain

$$\begin{aligned}\left\|\frac{\partial^m \mu_0}{\partial t^m}\right\|_1 &\le C\Bigg[\sum_{i=0}^m \left(\left|\frac{\partial^i y_0}{\partial t^i}(0,t)\right| + \left|\frac{\partial^i y_0}{\partial t^i}(1,t)\right|\right)\\ &\qquad + \int_0^t (|y_0(0,\tau)| + |y_0(1,\tau)|)\,d\tau\Bigg]\\ &\le Ch^{r+q-1}\left(\sum_{i=0}^m \left\|\frac{\partial^i u}{\partial t^i}\right\|_q + \int_0^t \|u\|_q\,d\tau\right).\end{aligned}$$

Similarly, for $j \ge 1$ and $m = 0$ choose $\chi = \mu_j - y_j(0,t)(1-x) - y_j(1,t)x$ in (4.5) to have

$$\|\mu_j\|_1^2 \le C\left(|y_j(0,t)|^2 + |y_j(1,t)|^2 + \left\|\frac{\partial \mu_{j-1}}{\partial t}\right\|_1^2\right) + C\int_0^t \|\mu_j\|_1^2\,d\tau.$$

An application of Gronwall's lemma yields

$$\begin{aligned}\|\mu_j\|_1^2 &\le C\left(|y_j(0,t)|^2 + |y_j(1,t)|^2 + \left\|\frac{\partial \mu_{j-1}}{\partial t}\right\|_1^2\right)\\ &\quad + C\int_0^t \left(|y_j(0,\tau)|^2 + |y_j(1,\tau)|^2 + \left\|\frac{\partial \mu_{j-1}}{\partial t}\right\|_1^2\right)d\tau.\end{aligned}$$

Again differentiate (4.5) m times with respect to time, and choose

$$\chi = \frac{\partial^m \mu_j}{\partial t^m} - \frac{\partial^m y_j}{\partial t^m}(0,t)(1-x) - \frac{\partial^m y_j}{\partial t^m}(1,t)x$$

in the resulting equation. Finally, use Theorem 4.1 to obtain

$$\begin{aligned}\left\|\frac{\partial^m \mu_j}{\partial t^m}\right\|_1^2 \le C\Big[\sum_{i=1}^{m}\left(\left|\frac{\partial^i y_j}{\partial t^i}(0,t)\right|^2+\left|\frac{\partial^i y_j}{\partial t^i}(1,t)\right|^2\right)\\ +\sum_{i=1}^{m+1}\left\|\frac{\partial^i \mu_{j-1}}{\partial t^i}\right\|_1^2+\int_0^t(|y_j(0,\tau)|^2+|y_j(1,\tau)|^2)\,d\tau\\ +\int_0^t\left\|\frac{\partial \mu_{j-1}}{\partial t}\right\|_1^2 d\tau\Big]\\ \le Ch^{2(r+q-1)}\left(\sum_{i=0}^{m+j}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q^2+\int_0^t\sum_{i=0}^{j}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q^2 d\tau\right)\\ +C\sum_{i=1}^{m+1}\left\|\frac{\partial^i \mu_{j-1}}{\partial t^i}\right\|_1^2+C\int_0^t\left\|\frac{\partial \mu_{j-1}}{\partial t}\right\|_1^2 d\tau.\end{aligned}$$

Now, apply induction on j to complete the rest of the proof. □

For flux superconvergence result at nodal points $\bar{x}\in[0,1]$, we first discuss this for $\bar{x}=1$.

Theorem 4.3. *Let $u_h : J\to S_h^0$ be the solution of* (1.4) *and* (3.2), *and define*

$$\begin{aligned}\Gamma_1(t) &= (u_{h,t},x)+A(t;u_h,x)\\ &\quad -\int_0^t B(t,\tau;u_h(\tau),x)\,d\tau-(f,x),\quad t\in J.\end{aligned}\tag{4.6}$$

If $1\le q\le r+1$ *and* $2k\le r-1$, *then*

$$\begin{aligned}\left|\Gamma_1(t)-\left(a_{11}\frac{\partial u}{\partial x}\right)(1,t)+\int_0^t\left(b_{11}\frac{\partial u}{\partial x}\right)(1,\tau)\,d\tau\right|\\ \le C\Big[\left(\sum_{i=0}^{k+1}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q+\int_0^t\sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q d\tau\right)h^{q+r-1}\\ +\left(\sum_{i=0}^{k+1}\left\|\frac{\partial^i u}{\partial t^i}(0)\right\|_q+\sum_{i=0}^{k+2}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q\\ +\int_0^t\sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q d\tau\right)h^{q+\min(2k,r-1)}\Big].\end{aligned}$$

In particular, if r is odd and $2k = r-1$ or if r is even and $2k = r$,

$$\left|\Gamma_1(t) - \left(a_{11}\frac{\partial u}{\partial x}\right)(1,t) + \int_0^t \left(b_{11}\frac{\partial u}{\partial x}\right)(1,\tau)\,d\tau\right| \le Ch^{q+r-1}\left(\sum_{i=0}^{k+1}\left\|\frac{\partial^i u}{\partial t^i}(0)\right\|_q + \sum_{i=0}^{k+2}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q + \int_0^t \sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q d\tau\right).$$

Proof. Choose $\chi = x$ in (4.3) to have

$$\left(a_{11}\frac{\partial u}{\partial x}\right)(1,t) - \int_0^t \left(b_{11}\frac{\partial u}{\partial x}\right)(1,\tau)\,d\tau = (w_{h,t},x) + A(t;w_h,x) - \int_0^t B(t,\tau;w_h(\tau),x)\,d\tau - (f,x). \tag{4.7}$$

The function w_h in (4.7) is not computable. But w_h and the Galerkin approximation u_h with the Dirichlet data are very nearly the same. To analyze the error in the approximation of the flux, we write $\xi = u_h - w_h$. Then, from (4.6)–(4.7), we have

$$\Gamma_1(t) - \left[\left(a_{11}\frac{\partial u}{\partial x}\right)(1,t) - \int_0^t \left(b_{11}\frac{\partial u}{\partial x}\right)(1,\tau)\,d\tau\right] = (\xi_t,x) + A(t;\xi,x) - \int_0^t B(t,\tau;\xi(\tau),x)\,d\tau.$$

Here,

$$A(t;\xi,x) = (\xi,(-a_{11}' + a_0x)) + a_{11}\xi|_0^1$$

and

$$B(t,\tau;\xi,x) = (\xi, -(b_1x+b_{11})' + b_0x) + b_{11}\xi|_0^1 + xb_1\xi|_0^1.$$

Using Schwarz's inequality, it follows that

$$\left|\Gamma_1(t) - \left[\left(a_{11}\frac{\partial u}{\partial x}\right)(1,t) - \int_0^t \left(b_{11}\frac{\partial u}{\partial x}\right)(1,\tau)\,d\tau\right]\right| \le C\left[\|\xi_t\| + \|\xi\| + |\xi(0,t)| + |\xi(1,t)| + \int_0^t \|\xi\|\,d\tau + \int_0^t (|\xi(0,\tau)| + |\xi(1,\tau)|)\,d\tau\right]. \tag{4.8}$$

For the estimation of ξ in (4.8), set $\psi_k = w_k - w_h$ and $\theta_k = u_k - u_h$. Hence, write

$$\xi = \psi_k - \theta_k - \mu_0 - \sum_1^k \mu_j .$$

Using Theorem 3.1 and Lemma 4.2, we obtain

$$\begin{aligned}\|\xi\| &\le Ch^{q+r-1}\Big(\sum_{i=0}^{k} \Big\| \frac{\partial^i u}{\partial t^i}\Big\|_q + \int_0^t \sum_{i=0}^{k} \Big\| \frac{\partial^i u}{\partial t^i}\Big\|_q \, d\tau \Big) \\ &\quad + Ch^{q+\min(2k+1,r-1)}\Big(\sum_{i=0}^{k+1} \Big\| \frac{\partial^i u}{\partial t^i}\Big\|_q + \int_0^t \sum_{i=0}^{k} \Big\| \frac{\partial^i u}{\partial t^i}\Big\|_q \, d\tau \Big).\end{aligned} \tag{4.9}$$

The estimation of $\partial\xi/\partial t$ requires the estimation of $\partial\psi_k/\partial t$ and $\partial\theta_k/\partial t$. From (3.1) with $\psi_k(0) = \theta_k(0) = 0$, it follows that

$$\begin{aligned}\Big\| \frac{\partial \theta_k}{\partial t}(0)\Big\| + \Big\| \frac{\partial \psi_k}{\partial t}(0)\Big\| &\le C\Big(\Big\| \frac{\partial \rho_k}{\partial t}(0)\Big\| + \Big\| \frac{\partial y_k}{\partial t}(0)\Big\| \Big) \\ &\le Ch^{q+\min(2k,r-1)}\Big(\sum_{i=0}^{k+1} \Big\| \frac{\partial^i u}{\partial t^i}(0)\Big\|_q \Big).\end{aligned}$$

Now, differentiate (3.1) with respect to time and then choose $\chi = \partial\theta_k/\partial t$ to obtain

$$\begin{aligned}\Big\| \frac{\partial \theta_k(t)}{\partial t}\Big\|_{L^2(I)} &\le C\Big(\Big\| \frac{\partial \theta_k}{\partial t}(0)\Big\| + \Big\| \frac{\partial^2 \rho_k}{\partial t^2}\Big\| \Big) \\ &\le Ch^{q+\min(2k,r-1)}\Big(\sum_{i=0}^{k+1} \Big\| \frac{\partial^i u}{\partial t^i}(0)\Big\|_q \\ &\qquad + \sum_{i=0}^{k+2} \Big\| \frac{\partial^i u}{\partial t^i}\Big\|_q + \int_0^t \sum_{i=0}^{k} \Big\| \frac{\partial^i u}{\partial t^i}\Big\|_q \, d\tau \Big).\end{aligned} \tag{4.10}$$

A similar estimate is valid for $\partial\psi_k/\partial t$. Hence,

$$\begin{aligned}\left\|\frac{\partial\xi(t)}{\partial t}\right\| \le C\bigg[&\bigg(\sum_{i=0}^{k+1}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q + \int_0^t \sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q d\tau\bigg)h^{q+r-1}\\ &+\bigg(\sum_{i=0}^{k+1}\left\|\frac{\partial^i u}{\partial t^i}(0)\right\|_q + \sum_{i=0}^{k+2}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q\\ &+\int_0^t \sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q d\tau\bigg)h^{q+\min(2k,r-1)}\bigg]. \end{aligned} \tag{4.11}$$

Since $\xi(x,t) = -w_h(x,t)$ and $\mu_j(x) = y_j(x)$ for $x = 0$ and 1, an application of Theorem 4.1 with ρ_j replaced by y_j yields,

(4.12)

$$\begin{aligned}|\xi(0,t)| + |\xi(1,t)| \le Ch^{q+r-1}&\bigg(\sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q + \int_0^t \sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q d\tau\bigg)\\ &+ Ch^{q+\min(2k,r-1)}\bigg(\sum_{i=0}^{k+1}\left\|\frac{\partial^i u}{\partial t^i}(0)\right\|_q + \sum_{i=0}^{k+1}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q\\ &+\int_0^t \sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_q d\tau\bigg).\end{aligned}$$

Combine the inequalities (4.8)–(4.12) to complete the rest of the proof. □

Remark. For the superconvergence of flux at $x = 0$, choose $\chi = 1 - x$ in (4.3) instead of x and make necessary changes in the proof of Theorem 4.3.

The flux superconvergence of an order $O(h^{2r})$ can be obtained at an interior knot $\bar{x}$ of I. Let

$$\begin{aligned}\Gamma_{\bar{x}}(t) = \bar{x}^{-1}\{&(u_{h,t},x)_{\bar{x}} + A_{\bar{x}}(t;u_h,x)\\ &- \int_0^t B_{\bar{x}}(t,\tau;u_h(\tau),x)\,d\tau - (f,x)_{\bar{x}}\}, \quad t\in J,\end{aligned}$$

where the subsript $\bar{x}$ indicates that the integrals are to be taken over the interval $(0,\bar{x})$. Define

$$S_{h,\bar{x}} = \{v \mid v|_{[0,\bar{x}]} \in S_h\}.$$

Let $\hat{w}_h : J \to S_{h,\bar{x}}$ satisfy

$$(\hat{w}_{h,t}, \chi)_{\bar{x}} + A_{\bar{x}}(t; \hat{w}_h, \chi) = \int_0^t B_{\bar{x}}(t, \tau; \hat{w}_h(\tau), \chi)\, d\tau + (f, \chi)_{\bar{x}}$$
$$+ a_{11} \frac{\partial u}{\partial x} \chi \Big|_0^{\bar{x}} - \int_0^t \left(b_{11} \frac{\partial u}{\partial x} \chi \right) \Big|_0^{\bar{x}} d\tau,$$
$$\chi \in S_{h,\bar{x}}, \quad 0 < t \le T,$$

where $\hat{w}_h(0)$ is a suitably chosen initial condition on $(0, \bar{x})$. An approximation to $(a_{11}\partial u/\partial x)(\bar{x}, t) - \int_0^t (b_{11}\partial u/\partial x)(\bar{x}, \tau)\, d\tau$ can be evaluated by choosing $\chi = x$. The error in the approximation of the flux, i.e., $|\Gamma_{\bar{x}}(t) - (a_{11}\partial u/\partial x)(\bar{x}, t) + \int_0^t (b_{11}\partial u/\partial x)(\bar{x}, \tau)\, d\tau|$ can be easily proved in a similar way by replacing the function w_h by $\hat{w}_h$ in Theorem 4.3. So we will not pursue it further.

5. Interior superconvergence estimates. In this Section, the estimates for difference quotients of ρ_j, $j \ge 0$, in the interior $\Omega_0 \subset\subset \Omega \subset R^d$ will be carried out. These estimates will then be used to prove interior superconvergence results. For this purpose, we shall assume that the finite dimensional subspaces are based on uniform partitions in a specific sense in some interior domain $\Omega_0 \subset\subset \Omega \subset R^d$. We shall not describe the uniformity assumption here, but for a detailed description we refer to Nitsche and Schatz [**8**]. For any domain $\Omega_1 \subset \Omega$, let $\overset{0}{S}_h(\Omega_1) = \{v \in S_h(\Omega) : \operatorname{supp} v \subset \Omega_1\}$.

We shall use the following notations for the rest of this Section. For a linear functional F, define

$$|||F|||_{-1,\Omega_1} = \sup_{0 \ne \eta \in H_0^1(\Omega_1)} \frac{|F(\eta)|}{\|\eta\|_{1,\Omega_1}}.$$

Further, we shall denote for an integer s

$$|||\Phi|||_{-s,\Omega_1} = \sup_{0 \ne \eta \in H_0^s(\Omega_1)} \frac{(\Phi, \eta)}{\|\eta\|_{s,\Omega_1}}.$$

Below, we shall state two lemmas without proof. The proof can be easily found modifying the analysis of Nitsche and Schatz [**8**, Lemma 4.2 and Lemma 5.1]. For a detailed proof, see Pani and Sinha [**10**].

Lemma 5.1. *For* $\Omega_0 \subset\subset \Omega' \subset\subset \Omega_1 \subset \Omega$, *let* $\omega \in C_0^\infty(\Omega')$ *be such that* $\omega \equiv 1$ *on* Ω_0. *Let* $F : H_0^1(\Omega_1) \to R$ *be a bounded linear functional satisfying*

$$|F(\omega^2\eta)| \leq M_q\|\eta\|_{q,\Omega_1}, \quad \forall\, \eta \in H^q(\Omega_1) \cap H_0^1(\Omega_1), \quad 2 \leq q \leq r+1.$$

Further, let Φ *satisfy*

$$A(t;\Phi,\chi) = F(\chi), \quad \forall\, \chi \in \overset{0}{S}_h(\Omega_1).$$

Then for $0 \leq s \leq r-1$, *we have*

$$\begin{aligned}|||\Phi|||_{-s,\Omega_0} &\leq |||\omega\Phi|||_{-s,\Omega_1} \\ &\leq C[(\|\Phi\|_{1,\Omega_1} + |||F|||_{-1,\Omega_1})h^{s+1} \\ &\qquad + M_{s+2} + |||\Phi|||_{-s-1,\Omega_1}].\end{aligned}$$

Lemma 5.2. *For* $\Omega_0 \subset\subset \Omega' \subset\subset \Omega_1$, *let* $\omega \in C_0^\infty(\Omega')$ *be such that* $\omega \equiv 1$ *on* Ω_0. *Let* $F : H_0^1(\Omega_1) \to R$ *be a bounded linear functional satisfying*

$$|F(\omega\eta)| \leq M_i\|\eta\|_{i,\Omega_1}, \quad \forall\, \eta \in H^i(\Omega_1) \cap H_0^1(\Omega_1), \quad i \in \{1,2\}.$$

Further, let $\Phi_h \in S_h$ *satisfy*

$$A(t;\Phi_h,\chi) = F(\chi), \quad \forall\, \chi \in \overset{0}{S}_h(\Omega_1).$$

Then

$$\begin{aligned}\|\Phi_h\|_{1,\Omega_0} \leq \|\omega\Phi_h\|_{1,\Omega_1} \leq & C[h(\|\Phi_h\|_{1,\Omega_1} + |||F|||_{-1,\Omega_1}) \\ & + M_1 + M_2 + |||\Phi_h|||_{-1,\Omega_1}].\end{aligned}$$

Although the argument of Pani and Sinha [**10**] can be easily modified to prove the interior estimates of $\partial^k\rho_j/\partial t^k$, but for clarity we prefer to present below a short proof.

Lemma 5.3. *Let $k \geq 0$, $1 \leq q \leq r+1$ and $p \geq 0$ be a fixed but arbitrary integer. Set $\rho_{-1} = 0$. Then for $\Omega_0 \subset\subset \Omega' \subset\subset \Omega_1$, $0 \leq s \leq r-1$ and $j \geq 0$, we have*

(5.1)

$$\begin{aligned}
\left|\left|\left|\frac{\partial^k \rho_j}{\partial t^k}\right|\right|\right|_{-s,\Omega_0} \leq C\Bigg[\Bigg(\sum_{i=0}^{k}\left\|\frac{\partial^i \rho_j}{\partial t^i}\right\|_{1,\Omega_1} &+ \int_0^t \left(\|\rho_j\|_{1,\Omega_1} + \left|\left|\left|\frac{\partial \rho_{j-1}}{\partial t}\right|\right|\right|_{-1,\Omega_1}\right) d\tau \\
&+ \sum_{i=1}^{k+1}\left|\left|\left|\frac{\partial^i \rho_{j-1}}{\partial t^i}\right|\right|\right|_{-1,\Omega_1}\Bigg) h^{s+1} \\
&+ \sum_{i=1}^{k+1}\left|\left|\left|\frac{\partial^i \rho_{j-1}}{\partial t^i}\right|\right|\right|_{-s-2,\Omega_1} + \int_0^t \left|\left|\left|\frac{\partial \rho_{j-1}}{\partial t}\right|\right|\right|_{-s-2,\Omega_1} d\tau \\
&+ \sum_{i=0}^{k}\left|\left|\left|\frac{\partial^i \rho_j}{\partial t^i}\right|\right|\right|_{-s-p,\Omega_1} + \int_0^t |||\rho_j|||_{-s-p,\Omega_1}\, d\tau\Bigg].
\end{aligned}$$

Proof. To apply Lemma 5.1, let $F(\chi)$ be the righthand side of the equation (2.4) with $\chi \in \overset{0}{S}_h(\Omega_1)$. Note that

$$\begin{aligned}
|F(\omega^2\eta)| &= \left|\int_0^t B(t,\tau;\omega\rho_j,\omega\eta)\, d\tau + \int_0^t I_1\, d\tau + \left(\frac{\partial \rho_{j-1}}{\partial t}, \omega^2\eta\right)\right| \\
&\leq \int_0^t |(\omega\rho_j, B^*(t,\tau)\omega\eta)|\, d\tau \\
&\quad + \int_0^t |I_1|\, d\tau + \left|\left(\frac{\partial \rho_{j-1}}{\partial t}, \omega^2\eta\right)\right|,
\end{aligned}$$

where $I_1 = \int_{\Omega_1} \rho_j[\sum_{i,j=1}^d (-\partial/\partial x_i)(b_{ij}(\partial\omega/\partial x_j)\omega\eta) - b_{ij}(\partial\omega/\partial x_i)(\partial(\omega\eta)/\partial x_j)) - \sum_{j=1}^d b_j(\partial\omega/\partial x_j)(\omega\eta)]dx$. Therefore,

$$\begin{aligned}
|F(\omega^2\eta)| \leq C\Bigg(\int_0^t (|||\omega\rho_j|||_{-s,\Omega_1} + |||\rho_j|||_{-s-1,\Omega_1})\, d\tau \\
+ \left|\left|\left|\frac{\partial \rho_{j-1}}{\partial t}\right|\right|\right|_{-s-2,\Omega_1}\Bigg) |||\eta|||_{s+2,\Omega_1}.
\end{aligned}$$

Setting $|||F|||_{-1,\Omega_1} \le C(\int_0^t \|\rho_j\|_{1,\Omega_1}\, d\tau + |||\partial\rho_{j-1}/\partial t|||_{-1,\Omega_1})$, and $M_{s+2} = C(\int_0^t (|||\omega\rho_j|||_{-s,\Omega_1} + |||\rho_j|||_{-s-1,\Omega_1})\, d\tau + |||\partial\rho_{j-1}/\partial t|||_{-s-2,\Omega_1})$, apply Lemma 5.1 with $\Phi = \rho_j$ to obtain

$$|||\omega\rho_j|||_{-s,\Omega_1} \le C\Bigg[\left(\|\rho_j\|_{1,\Omega_1} + \int_0^t \|\rho_j\|_{1,\Omega_1}\, d\tau + \left|\left|\left|\frac{\partial\rho_{j-1}}{\partial t}\right|\right|\right|_{-1,\Omega_1}\right) h^{s+1} + \left|\left|\left|\frac{\partial\rho_{j-1}}{\partial t}\right|\right|\right|_{-s-2,\Omega_1} + |||\rho_j|||_{-s-1,\Omega_1} + \int_0^t |||\rho_j|||_{-s-1,\Omega_1}\, d\tau\Bigg] + C\int_0^t |||\omega\rho_j|||_{-s,\Omega_1}\, d\tau.$$

An application of Gronwall's lemma yields

(5.2)
$$|||\rho_j|||_{-s,\Omega_0} \le C\Bigg[\left(\|\rho_j\|_{1,\Omega_1} + \int_0^t \left(\|\rho_j\|_{1,\Omega_1} + \left|\left|\left|\frac{\partial\rho_{j-1}}{\partial t}\right|\right|\right|_{-1,\Omega_1}\right) d\tau + \left|\left|\left|\frac{\partial\rho_{j-1}}{\partial t}\right|\right|\right|_{-1,\Omega_1}\right) h^{s+1} + \left|\left|\left|\frac{\partial\rho_{j-1}}{\partial t}\right|\right|\right|_{-s-2,\Omega_1} + \int_0^t \left|\left|\left|\frac{\partial\rho_{j-1}}{\partial t}\right|\right|\right|_{-s-2,\Omega_1} d\tau + |||\rho_j|||_{-s-1,\Omega_1} + \int_0^t |||\rho_j|||_{-s-1,\Omega_1}\, d\tau\Bigg].$$

Let $\Omega_0 = \Omega^0 \subset\subset \Omega^1 \subset\subset \cdots \subset\subset \Omega^p = \Omega_1$ be a sequence of domains. Apply estimate (5.2) to the subdomains Ω^l and Ω^{l+1} to obtain

$$|||\rho_j|||_{-s,\Omega^l} \le C\Bigg[\left(\|\rho_j\|_{1,\Omega^{l+1}} + \int_0^t \|\rho_j\|_{1,\Omega^{l+1}}\, d\tau + \left|\left|\left|\frac{\partial\rho_{j-1}}{\partial t}\right|\right|\right|_{-1,\Omega^{l+1}} + \int_0^t \left|\left|\left|\frac{\partial\rho_{j-1}}{\partial t}\right|\right|\right|_{-1,\Omega^{l+1}} d\tau\right) h^{s+1} + \left|\left|\left|\frac{\partial\rho_{j-1}}{\partial t}\right|\right|\right|_{-s-2,\Omega^{l+1}} + \int_0^t \left|\left|\left|\frac{\partial\rho_{j-1}}{\partial t}\right|\right|\right|_{-s-2,\Omega^{l+1}} d\tau + |||\rho_j|||_{-s-1,\Omega^{l+1}} + \int_0^t |||\rho_j|||_{-s-1,\Omega^{l+1}}\, d\tau\Bigg].$$

Starting with $l = 0$ iterating p times and a repeated application of the estimate (5.2) to the last two terms of the above equation we obtain the required estimate for $k = 0$. For $k = 1$, let $F(\chi)$ be given by the righthand side of (2.5) with $\chi \in \overset{0}{S}_h(\Omega_1)$. Replacing χ by $\omega^2\eta$, apply Lemma 5.1 with $\Phi = \frac{\partial \rho_j}{\partial t}$ and (5.2) (by taking an intermediate subdomain) to obtain the desired result. The higher order time derivatives of ρ_j can be treated in a similar way, and this completes the rest of the proof. □

Below, we shall discuss an estimate of $\partial^k\rho_j/\partial t^k$. Using Lemma 5.2, and the analysis of Nitsche and Schatz [**8**, Lemma 5.1 and Theorem 5.2] with appropriate modifications, we have the following error estimate.

Lemma 5.4. *Let $\Omega_0 \subset\subset \Omega_1$, $k \geq 0$, $1 \leq q \leq r+1$ and $p \geq 0$ be a fixed but arbitrary integer. Then for $j = 0$*

$$\left\|\frac{\partial^k \rho}{\partial t^k}\right\|_{1,\Omega_0} \leq C\Big\{\Big[\sum_{i=0}^{k}\left\|\frac{\partial^i u}{\partial t^i}\right\|_{q,\Omega_1} + \int_0^t \|u\|_{q,\Omega_1}\,d\tau\Big]h^{q-1} + \sum_{i=0}^{k}\left|\left|\left|\frac{\partial^i \rho}{\partial t^i}\right|\right|\right|_{-p,\Omega_1} + \int_0^t |||\rho|||_{-p,\Omega_1}\,d\tau\Big\},$$

and for $j \geq 1$

$$\left\|\frac{\partial^k \rho_j}{\partial t^k}\right\|_{1,\Omega_0} \leq C\Big\{\Big[\sum_{i=0}^{k+j}\left\|\frac{\partial^i u}{\partial t^i}\right\|_{q,\Omega_1} + \int_0^t \sum_{i=0}^{j}\left\|\frac{\partial^i u}{\partial t^i}\right\|_{q,\Omega_1} d\tau\Big]h^{q-1} + \sum_{i=0}^{k}\left|\left|\left|\frac{\partial^i \rho_j}{\partial t^i}\right|\right|\right|_{-p,\Omega_1} + \int_0^t |||\rho_j|||_{-p,\Omega_1}\,d\tau\Big\}.$$

Proof. Let $\Omega_0 \subset\subset \Omega_0' \subset\subset \Omega_1' \subset\subset \Omega_1 \subset \Omega$, and let $\omega \in C_0^\infty(\Omega_1')$ be such that $\omega \equiv 1$ on Ω_0'. For $j \geq 1$, set $\tilde{\rho_j} = \omega\rho_j$. Let $T_h\tilde{\rho_j} \in \overset{0}{S}_h(\Omega_1)$ be the unique solution of

$$A(t;T_h\tilde{\rho_j},\chi) = \int_0^t B(t,\tau;T_h\tilde{\rho_j},\chi)\,d\tau - \Big(\frac{\partial\tilde{\rho}_{j-1}}{\partial t},\chi\Big), \quad \forall\,\chi \in \overset{0}{S}_h(\Omega_1). \tag{5.3}$$

$$\|\rho_j\|_{1,\Omega_0} \le \|\rho_j - T_h\tilde{\rho_j}\|_{1,\Omega_0} + \|T_h\tilde{\rho_j}\|_{1,\Omega_0}. \tag{5.4}$$

Since the equation (2.4) and (5.3) holds for all $\chi \in \overset{0}{S}_h(\Omega_0')$, on subtraction we have

$$A(t;\rho_j - T_h\tilde{\rho_j},\chi) = \int_0^t B(t,\tau;\rho_j - T_h\tilde{\rho_j},\chi)\,d\tau, \quad \forall\,\chi \in \overset{0}{S}_h(\Omega_0').$$

Let $F(\omega\eta)$ be as in the right hand side of the above equation with replacing χ by $\omega\eta$. Set $\tau_h = \rho_j - T_h\tilde{\rho_j}$. Apply Lemma 5.2 with $\Phi_h = \tau_h$ to have

$$\begin{aligned}\|\tau_h\|_{1,\Omega_0} \le C\bigg[h\bigg(\|\tau_h\|_{1,\Omega_0'} + \int_0^t \|\tau_h\|_{1,\Omega_0'}\,d\tau\bigg) \\ + |||\tau_h|||_{-1,\Omega_0'} + \int_0^t |||\tau_h|||_{-1,\Omega_0'}\,d\tau\bigg].\end{aligned}$$

For the last two terms, again apply Lemma 5.1 and then take a sequence of subdomains between Ω_0 and Ω, and use inverse estimate (see Nitsche and Schatz [**8**, Lemma 5.2]) to obtain

$$\|\tau_h\|_{1,\Omega_0} \le C\bigg(|||\tau_h|||_{-p,\Omega_0'} + \int_0^t |||\tau_h|||_{-p,\Omega_0'}\,d\tau\bigg),$$

where p is an arbitrary and fixed integer. With triangle inequality and imbedding result

$$\begin{aligned}|||\tau_h|||_{-p,\Omega_0'} &= |||\rho_j - T_h\tilde{\rho_j}|||_{-p,\Omega_0'} \\ &\le |||\rho_j|||_{-p,\Omega_0'} + |||T_h\tilde{\rho_j}|||_{-p,\Omega_0'} \\ &\le C(|||\rho_j|||_{-p,\Omega_1} + \|T_h\tilde{\rho_j}\|_{1,\Omega_1}).\end{aligned}$$

Therefore, (5.4) becomes

$$\begin{aligned}\|\rho_j\|_{1,\Omega_0} \le C\bigg(\|T_h\tilde{\rho_j}\|_{1,\Omega_1} + \int_0^t \|T_h\tilde{\rho_j}\|_{1,\Omega_1}\,d\tau \\ + |||\rho_j|||_{-p,\Omega_1} + \int_0^t |||\rho_j|||_{-p,\Omega_1}\,d\tau\bigg).\end{aligned} \tag{5.5}$$

To estimate $\|T_h\tilde{\rho}_j\|_1$, choose $\chi = T_h\tilde{\rho}_j$ in (5.3), and use Cauchy-Schwarz's inequality and Gronwall's lemma to have

$$(5.6)\quad \|T_h\tilde{\rho}_j\|_{1,\Omega_0} \le C\left(\left|\left|\left| \frac{\partial \rho_{j-1}}{\partial t} \right|\right|\right|_{-1,\Omega_1} + \int_0^t \left|\left|\left| \frac{\partial \rho_{j-1}}{\partial t} \right|\right|\right|_{-1,\Omega_1} d\tau \right).$$

Combining (5.5)–(5.6) and an induction on j, the desired estimate for $k = 0$ and $j \ge 1$ follows. Differentiating (2.4) and (5.3) with respect to t, the proof for $k \ge 1$ can be easily derived in a similar way.

Now it remains to estimate for $j = 0$. For $j = 0$, we note the following changes in the arguments. Set $\tilde{u} = \omega u$ and let $T_h\tilde{u} \in \overset{0}{S}_h(\Omega_1)$ be an auxiliary projection defined by

$$(5.7)\qquad A(t; T_h\tilde{u} - \tilde{u}, \chi) = \int_0^t B(t,\tau; T_h\tilde{u} - \tilde{u}, \chi)\, d\tau, \quad \chi \in \overset{0}{S}_h(\Omega_1).$$

Instead of (5.4), we have now

$$\|\rho\|_{1,\Omega_0} \le \|W_h u - T_h\tilde{u}\|_{1,\Omega_0} + \|T_h\tilde{u} - \tilde{u}\|_{1,\Omega_0},$$

where $W_h u$ is the Ritz-Volterra projection defined by (2.1) with $\chi \in \overset{0}{S}_h(\Omega_1)$. Subtracting (2.1) and (5.7), it follows that

$$A(t; W_h u - T_h\tilde{u}, \chi) = \int_0^t B(t,\tau; W_h u - T_h\tilde{u}, \chi)\, d\tau, \quad \chi \in \overset{0}{S}_h(\Omega_0').$$

Repeating the arguments as in case of $j \ge 1$, we obtain the desired estimates, and this completes the proof. □

We now shall prove the following theorem.

Theorem 5.1. *Let $k, j \ge 0$ and $1 \le q \le r+1$. Further, let $\partial^{k+j}u/\partial t^{k+j} \in H^q(\Omega_1)$. Then for $\Omega_0 \subset\subset \Omega_1 \subset \Omega$ and $0 \le s+2j \le r-1$, we have*

$$\left|\left|\left| \frac{\partial^k \rho_j}{\partial t^k} \right|\right|\right|_{-s,\Omega_0} \le C\left[\left(\sum_{i=0}^{k+j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{q,\Omega_1} + \int_0^t \sum_{i=0}^{j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{q,\Omega_1} d\tau \right) h^{s+q+2j} + \sum_{i=0}^{k} \left\| \frac{\partial^i \rho_j}{\partial t^i} \right\|_{-p} + \int_0^t \|\rho_j\|_{-p}\, d\tau \right].$$

Proof. Let $\Omega_0 \subset\subset \Omega_0' \subset\subset \Omega_1$. With $j = 0$, apply (5.1) to subdomains $\Omega_0 \subset\subset \Omega_0'$ to have

$$\left|\left|\left|\frac{\partial^k \rho}{\partial t^k}\right|\right|\right|_{-s,\Omega_0} \le C\Big[\Big(\sum_{i=0}^{k}\left\|\frac{\partial^i \rho}{\partial t^i}\right\|_{1,\Omega_0'} + \int_0^t \|\rho\|_{1,\Omega_0'}\, d\tau\Big)h^{s+1} + \sum_{i=0}^{k}\left|\left|\left|\frac{\partial^i \rho}{\partial t^i}\right|\right|\right|_{-s-p,\Omega_1} + \int_0^t |||\rho|||_{-s-p,\Omega_1}\, d\tau\Big].$$

Note that for any nonnegative integer l

$$|||\phi|||_{-l,\Omega_1} \le C\|\phi\|_{-l}. \tag{5.8}$$

The desired estimate for $j = 0$ follows from Lemma 5.4, and estimate (5.8). Similarly, for $j \ge 1$, use Lemmas 5.3–5.4 with estimate (5.8) to obtain

$$\begin{aligned}\left|\left|\left|\frac{\partial^k \rho_j}{\partial t^k}\right|\right|\right|_{-s,\Omega_0} \le C\Big[\Big(&\sum_{i=1}^{k+1}\left|\left|\left|\frac{\partial^i \rho_{j-1}}{\partial t^i}\right|\right|\right|_{-1,\Omega_0'} \\ &+ \int_0^t \left|\left|\left|\frac{\partial \rho_{j-1}}{\partial t}\right|\right|\right|_{-1,\Omega_0'} d\tau\Big)h^{s+1} \\ &+ \sum_{i=1}^{k+1}\left|\left|\left|\frac{\partial^i \rho_{j-1}}{\partial t^i}\right|\right|\right|_{-s-2,\Omega_0'} + \sum_{i=0}^{k}\left\|\frac{\partial^i \rho_j}{\partial t^i}\right\|_{-s-p} \\ &+ \int_0^t \Big(\left|\left|\left|\frac{\partial \rho_{j-1}}{\partial t}\right|\right|\right|_{-s-2,\Omega_0'} + \|\rho_j\|_{-s-p}\Big)\, d\tau\Big].\end{aligned}$$

An induction on j and Theorem 2.2 now complete the rest of the proof. □

Let $\mu = (\mu_1, \ldots, \mu_d)$ be a multi-integer, and let the translation operator T_h^μ be defined by

$$T_h^\mu v(x) = v(x + \mu h).$$

Set the forward difference quotient as

$$\partial_{h,j} v = h^{-1}(T_h^{e_j} - I)v,$$

where e_j is the unit vector in the direction of x_j, and I is the identity operator. For an arbitrary multi-index α, write

$$\partial_h^\alpha = \partial^\alpha = \partial_{h,1}^{\alpha_1} \cdots \partial_{h,d}^{\alpha_d}.$$

In the following theorem, the estimate for the difference quotients of $\partial^k \rho_j / \partial t^k$, $k, j \geq 0$, will be discussed.

Theorem 5.2. *Let* $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$, $\partial^{k+j} u / \partial t^{k+j} \in H^{q+|\alpha|}(\Omega_1)$, $k, j \geq 0$, *α a multi-index and* $1 \leq q \leq r+1$. *Suppose that the equations* (2.1) *and* (2.4) *are satisfied. Then for* $0 \leq s + 2j \leq r-1$ *and small* h, *we have*

$$\begin{aligned} \left|\left|\left| \partial^\alpha \left(\frac{\partial^k \rho_j}{\partial t^k} \right) \right|\right|\right|_{-s,\Omega_0} \leq & C \bigg(\sum_{i=0}^{k+j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{q+|\alpha|,\Omega_1} \\ & + \int_0^t \sum_{i=0}^{j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{q+|\alpha|,\Omega_1} d\tau \bigg) h^{s+q+2j} \\ & + \sum_{i=0}^{k} \left\| \frac{\partial^i \rho_j}{\partial t^i} \right\|_{-p} + \int_0^t \|\rho_j\|_{-p} \, d\tau. \end{aligned}$$

Proof. We shall prove the result using induction. For $j = 0$, and $k = 0, 1$, now from equation (2.1) with χ in $\overset{0}{S}_h(\Omega_1)$ we form an equation in $\partial^\alpha \rho$ and obtain using Lemmas 5.1–5.2, as in Theorem 5.1

$$\begin{aligned} |||\partial^\alpha \rho|||_{-s,\Omega_0} \leq C \bigg[\bigg(& \|u\|_{q+|\alpha|,\Omega_1} + \int_0^t \|u\|_{q+|\alpha|,\Omega_1} d\tau \bigg) h^{s+q} \\ & + \|\rho\|_{-p} + \int_0^t \|\rho\|_{-p} \, d\tau \bigg], \end{aligned}$$

and

$$\begin{aligned} |||\partial^\alpha \rho_t|||_{-s,\Omega_0} \leq C \bigg[\bigg(& \|u\|_{q+|\alpha|,\Omega_1} + \|u_t\|_{q+|\alpha|,\Omega_1} \\ & + \int_0^t \|u\|_{q+|\alpha|} \, d\tau \bigg) h^{s+q} \\ & + \|\rho\|_{-p} + \|\rho_t\|_{-p} + \int_0^t \|\rho\|_{-p} \, d\tau \bigg], \end{aligned}$$

where p is an arbitrary but fixed integer. The higher order time derivatives for $j = 0$ can be derived analogously. For $j \geq 1$, the proof will follow similarly as in the case of ρ and by induction on j. In this case using Lemma 5.2 with (5.8), it is easy to show that

$$\left\| \partial^\alpha \left(\frac{\partial^k \rho_j}{\partial t^k} \right) \right\|_{1,\Omega_0} \leq C \Bigg[\sum_{i=1}^{k} \left|\left|\left| \partial^\alpha \left(\frac{\partial^i \rho_{j-1}}{\partial t^i} \right) \right|\right|\right|_{-1,\Omega_1} + \int_0^t \left|\left|\left| \partial^\alpha \left(\frac{\partial \rho_{j-1}}{\partial t} \right) \right|\right|\right|_{-1,\Omega_1} d\tau + \sum_{i=0}^{k} \left\| \frac{\partial^i \rho_j}{\partial t^i} \right\|_{-p} + \int_0^t \|\rho_j\|_{-p}\, d\tau \Bigg]. \tag{5.9}$$

Let $\Omega_0 \subset\subset \Omega_0' \subset\subset \Omega_1' \subset\subset \Omega_1$. Apply Theorem 5.1 with appropriate changes and replacing $\partial^k \rho_j / \partial t^k$ by $\partial^\alpha(\partial^k \rho_j/\partial t^k)$ (replacing Ω_0 by Ω_0' and Ω_1 by Ω_1'), and use (5.9) to obtain

$$\begin{aligned} &\left|\left|\left| \partial^\alpha \left(\frac{\partial^k \rho_j}{\partial t^k} \right) \right|\right|\right|_{-s,\Omega_0'} \\ &\leq C \Bigg[\left(\sum_{i=0}^{k+j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{q+|\alpha|,\Omega_1'} + \int_0^t \sum_{i=0}^{j} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{q+|\alpha|,\Omega_1'} d\tau \right) h^{s+q+2j} \\ &\quad + \sum_{i=0}^{k} \left\| \frac{\partial^i \rho_j}{\partial t^i} \right\|_{-p} + \int_0^t \|\rho_j\|_{-p}\, d\tau \\ &\quad + \sum_{i=0}^{k} \sum_{\beta<\alpha} \left|\left|\left| \partial^\beta \left(\frac{\partial^i \rho_j}{\partial t^i} \right) \right|\right|\right|_{-s,\Omega_1'} + \int_0^t \sum_{\beta<\alpha} |||\partial^\beta \rho_j|||_{-s,\Omega_1'}\, d\tau \Bigg]. \end{aligned}$$

An induction argument on α now completes the proof of the theorem. □

We shall conclude this Section by showing the interior superconvergence error estimate by considering certain averages of u_h as an approximation to u. More precisely, the averages are formed by computing $K_h * u_h$, where the kernel K_h is defined as follows: For t real, let

$$\chi(t) = \begin{cases} 1, & |t| \leq 1/2, \\ 0, & |t| > 1/2, \end{cases}$$

and for an integer l, set $\psi_1^{(l)}(t) = \chi * \chi * \cdots * \chi$, i.e., convolution $l-1$ times of χ. The function $\psi_1^{(l)}$ is in fact a one-dimensional B-spline basis of order l. For $x \in R^d$ and $l = r-1$ define K_h by

$$K_h(x) = \prod_{m=1}^{d} \Bigg(\sum_{j=-(r-1)}^{r-1} h^{-1} c'_j \psi_1^{(r-1)}(h^{-1}x_m - j) \Bigg),$$

where the constants c'_j are given by $c'_0 = c_0$ and for $1 \le j \le r-1$, $c'_{-j} = c'_j = c_{j/2}$. Here c_j, $0 \le j \le r-1$, are determined as the unique solution of the linear system of algebraic equations

$$\sum_{j=0}^{r-1} c_j \int_R \psi_1^{(r-1)}(y)(y+j)^{2m}\, dy = \delta_{0,m}, \quad 0 \le m \le r-1.$$

We shall recall the following known estimates from Bramble and Schatz [**2**] for our future use. For $\Omega_0 \subset\subset \Omega_1 \subset \Omega$, it is known that for h sufficiently small, (see Lemmas 5.2–5.3 in [**2**])

$$\|K_h * v - v\|_{\Omega_0} \le Ch^q \|v\|_{q,\Omega_1}, \quad 0 \le q \le 2r, \quad v \in H^q(\Omega_1),$$

and for any fixed integer s and $v \in H^s(\Omega_1)$

$$\|D^\alpha (K_h * v)\|_{s,\Omega_0} \le C \|\partial^\alpha v\|_{s,\Omega_1}, \tag{5.10}$$

where $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d})$ and ∂^α is the corresponding difference operator with step size h. Further, let s be an arbitrary but fixed nonnegative integer. Then there is a constant C such that (see Lemma 4.2 in [**2**])

$$\|v\|_{\Omega_0} \le C \sum_{|\alpha| \le s} |||D^\alpha v|||_{-s,\Omega_1}. \tag{5.11}$$

Finally, we shall prove the main result of this Section.

Theorem 5.3. *Let u and u_h be the solution of* (1.1) *and* (1.4) *respectively. Further, let $\partial^k u/\partial t^k \in H^{2r}(\Omega_1) \cap H^{r+1}(\Omega)$, $2k \le r-1$. Then for $\Omega_0 \subset\subset \Omega_1 \subset \Omega$ and small h, we have*

$$\|u - K_h * u_h\|_{0,\Omega_0} \le Ch^{2r} \Bigg[\sum_{i=0}^{k} \left(\left\| \frac{\partial^i u}{\partial t^i} \right\|_{2r,\Omega_1} + \left\| \frac{\partial^i u}{\partial t^i} \right\|_{r+1} \right) + \int_0^t \sum_{i=0}^{k} \left(\left\| \frac{\partial^i u}{\partial t^i} \right\|_{2r,\Omega_1} + \left\| \frac{\partial^i u}{\partial t^i} \right\|_{r+1} \right) d\tau \Bigg].$$

Proof. By triangle inequality

$$\|u - K_h * u_h\|_{0,\Omega_0} \leq \|u - K_h * u\|_{0,\Omega_0} + \|K_h * e\|_{0,\Omega_0}.$$

Since $\|u - K_h * u\|_{0,\Omega_0} \leq Ch^{2r}\|u\|_{2r,\Omega_1}$, it is sufficient to estimate the last term. From (3.5), (5.10) and (5.11) with $s = r - 2j - 1$ we write

$$\begin{aligned}\|K_h * e\|_{0,\Omega_0} &\leq \|K_h * \theta_k\|_{0,\Omega_0} + \sum_{j=0}^{k} \|K_h * \rho_j\|_{0,\Omega_0} \\ &\leq C\|\theta_k\| + \sum_{|\alpha|\leq r-1} \sum_{j=0}^{k} |||D^\alpha (K_h * \rho_j)|||_{-(r-2j-1),\Omega_1} \\ &\leq C\|\theta_k\| + \sum_{|\alpha|\leq r-1} \sum_{j=0}^{k} |||\partial^\alpha (\rho_j)|||_{-(r-2j-1),\Omega_1}.\end{aligned}$$

Use Theorems 3.1 and 5.2 with $s = r-2j-1$, $q = r+1$ and $p = r-2j-1$ with estimate of ρ_j from Theorem 2.2 to complete the proof. □

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