On structures of certain L^2-well-posed mixed problems for hyperbolic systems of first order

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§ 1. Introduction and results.

Let P be an x_0-strictly hyperbolic $2m\times 2m$-system of differential operators of first order defined over a C^∞-cylinder $\boldsymbol{R}^1\times\Omega\subset\boldsymbol{R}^{n+1}$. Let B be an $m\times 2m$-system of functions defined on the boundary Γ of $\boldsymbol{R}^1\times\Omega$. We consider the following mixed problem:

$$(P, B)\ \begin{cases} P(x, D)\,u(x)=f(x) & x\in \boldsymbol{R}^1\times\Omega \quad (x_0>0)\,, \\ B(x)\,u(x)=g(x) & x\in\Gamma \quad (x_0>0)\,, \\ u(x)=h(x) & x\in\boldsymbol{R}^1\times\Omega \quad (x_0=0)\,, \end{cases}$$

where $x=(x_0, x_1, \cdots, x_n)$, $D_j=\dfrac{1}{\sqrt{-1}}\dfrac{\partial}{\partial x_j}=\dfrac{1}{i}\dfrac{\partial}{\partial x_j}$ and $D=(D_0, \cdots, D_n)$. We consider L^2-well-posedness of (P, B) in the follwing sense:

DEFINITION 1.1. *The problem (P, B) is said to be L^2-well-posed if there exist positive constants C, T such that for every $f\in H_1((-\infty, T)\times\Omega)$ with $f=0\,(x_0<0)$, $g=0$ and $h=0$, there exists a unique solution $u\in H_1((-\infty, T)\times\Omega)$ with $u=0\,(x_0<0)$ which satisfies the inequality*:

$$\int_0^T \|u\|^2_{0,\Omega}dt \leqq C\int_0^T \|f\|^2_{0,\Omega}dt\,,$$

where $H_k(G)$ is the Sobolev space with its norm $\|\cdot\|_{k,G}$.

The aim of the present article is, under somewhat strong but general restriction on the operator P, to describe, in terms of the cotangent space of $\boldsymbol{R}^1\times\partial\Omega=\Gamma$, the relations among the coefficients of boundary operator B for L^2-well-posed problem (P, B). These relations are useful for the investigation of the propagation of singularities of solutions for our problems. For example they determine whether there exist lateral waves or not. Applying the relations we prove the existance of the solution of our problem. But in contrast with the recent development of the Cauchy problems, we must essentially use the energy estimate, because of the existance of glancing rays with the associated non-vanishing reflection coefficients in these cases.

Put $\boldsymbol{C}_\pm=\{\tau\in\boldsymbol{C}^1\,;\,\mathrm{Im}\,\tau\gtrless 0\}$ respectively and let $P^0(x,\tau,\sigma,\lambda)$ be the principal part of $P(x,\tau,\sigma,\lambda)$, where (τ,σ,λ) is a covector of $x=(x_0,x'',x_n)$. By virtue of localization and certain coordinate transformations, we may restrict ourselves to the case where

$$\Omega=\boldsymbol{R}^n_+=\left\{(x'',x_n);\,x_n>0,\,x''\in\boldsymbol{R}^{n-1}\right\},\qquad n\geqq 2,$$
$$\boldsymbol{R}^1\times\Omega=\boldsymbol{R}^{n+1}_+=\left\{(x_0,x'',x_n)\,;\,x_n>0,\,(x_0,x'')\in\boldsymbol{R}^n\right\},$$
$$\Gamma=\boldsymbol{R}^n=\{x'\in\boldsymbol{R}^{n+1}\,;\,x'_n=0\}.$$

Then our assumptions are the following:

(I) α) The coefficients of P^0 and B are real, belong to $C^\infty(\boldsymbol{R}^1\times\bar\Omega)$ and are constant outside some compact set of $\boldsymbol{R}^1\times\bar\Omega$.

β) The multiplicity of the real roots $\lambda(x',\tau,\sigma)$ of the characteristic equation $\det P^0(x',\tau,\sigma,\lambda)=0$ is at most double and there is at most one real double root, for every $(x',\tau,\sigma)\in\Gamma\times(\boldsymbol{R}^n\backslash\{0\})$.

γ) (P,B) is normal, that is, (i) $\det P^0(x',0,\sigma,\lambda)\neq 0$ for every $(x',\sigma,\lambda)\in\Gamma\times(\boldsymbol{R}^n\backslash\{0\})$, (ii) rank $B(x')=m$ for every $x'\in\Gamma$.

(II) Let $R(x',\tau,\sigma)$ be the Lopatinskii determinant of (P^0,B) (for definition, see §4.)

α) If $R(x^0,\tau^0,\sigma^0)=0$ for a point $(x^0,\tau^0,\sigma^0)\in\Gamma\times(\boldsymbol{R}^n\backslash\{0\})$ such that there are no real double roots λ of $\det P^0(x^0,\tau^0,\sigma^0,\lambda)=0$, it holds for small $\gamma>0$ that

$$\left|R(x^0,\tau^0-i\gamma,\sigma^0)\right|\geq C\gamma\,,$$

with some constant $C=C(x^0,\tau^0,\sigma^0)>0$. Furthermore if there is at least one real simple root $\lambda(x^0,\tau^0,\sigma^0)$, $R(x',\tau,\sigma)$ vanishes only for $\mathrm{Im}\,\tau=0$ in some neighborhood $U(x^0,\tau^0,\sigma^0)\subset\Gamma\times\boldsymbol{C}^1\times\boldsymbol{R}^{n-1}$.

β) If $R(x^0,\tau^0,\sigma^0)=0$ for a point $(x^0,\tau^0,\sigma^0)\in\Gamma\times(\boldsymbol{R}^n\backslash\{0\})$ such that there is a real double root λ of $\det P^0(x^0,\tau^0,\sigma^0,\lambda)=0$, for small $\gamma>0$

$$\left|R(x^0,\tau^0-i\gamma,\sigma^0)\right|\geq C\gamma^{\frac{1}{2}}$$

with some constant $C=C(x^0,\tau^0,\sigma^0)>0$. Furthermore if there is at least one real simple root λ, the rank of $\underset{(x',\sigma)}{\mathrm{Hess}}\,R(x',\theta(x',\sigma),\sigma)$ at the zeros of $R(x',\theta(x'\,\sigma),\sigma)$ in some neighborhood $U(x^0,\tau^0,\sigma^0)$ is equal to

codim of $\{(x',\sigma)\,;\,R(x',\theta(x',\sigma),\sigma)=0\}$ in $\boldsymbol{R}^{2n-1}$,

where $\theta(x',\sigma)\in C^\infty$ denotes a real valued function such that $P^0(x',\tau,\sigma,\lambda)=0$ has only a real double root λ on the surface $\tau=\theta(x',\sigma)$ with $\tau^0=\theta(x^0,\sigma^0)$ (see Corollary 3.1.) Here the zero set $\{(x',\sigma)\,;\,R(x',\theta(x',\sigma),\sigma)=0\}$ in some $U(x^0,\tau^0,\sigma^0)$ is preassumed to be a regular submanifold of $\boldsymbol{R}^{2n-1}$.

γ) In the point (x^0, τ^0, σ^0) satisfying the first condition of β), if there is at least one non-real root λ of det $P^0(x^0, \tau^0, \sigma^0, \lambda)=0$, then the reflection coefficient $\widetilde{b_{\mathrm{II\,II}}}(x', \tau, \sigma)$ is real in some $U(x^0, \tau^0, \sigma^0)$ whenever $\tau, \lambda^{\pm}_{\mathrm{II}}(x', \tau, \sigma)$ are real and $R(x', \tau, \sigma)\neq 0$ (for definitions of $\lambda^{\pm}_{\mathrm{II}}(x', \tau, \sigma)$ and reflection coefficients see §3 and §4, respectively).

(III) Any constant coefficients problems $(P, B)_{x'}$ resulting from freezing the coefficients at boundary points x' are L^2-well-posed.

The main result in this article is the following

MAIN THEOREM. *Assume the conditions (I), (II) and (III), then the variable coefficients problem (P, B) is also L^2-well-posed.*

To prove Main theorem, we use the following

THEOREM 1.1. *Under the conditions (I), (II) and (III) the following a priori estimate holds with some constants $C_k, \gamma_k>0$:*

$$\|Pu\|_{k,\gamma}+|Bu|_{k+\frac{1}{2},\gamma}\geq C_k\gamma\|u\|_{k,\gamma} \tag{1.1}$$

for every $\gamma\geq\gamma_k$, $u\in H_{k,\gamma}(\boldsymbol{R}^{n+1}_+)$ and integer $k\geq 0$ (the norms used here are defined in §2.)

Our method deriving Main theorem is applicable to the case where the boundary operator $B(x')$ is complex. Main theorem is also valid if we assume the following conditions (II) β') and γ') instead of the conditions (II) β) and γ):

(II) β') If $R(x^0, \tau^0, \sigma^0)=0$ for a point $(x^0, \tau^0, \sigma^0)\in\Gamma\times(\boldsymbol{R}^n\backslash\{0\})$ such that there is a double root λ of det $P^0(x^0, \tau^0, \sigma^0, \lambda)=0$, for small $\gamma>0$

$$\left|R(x^0, \tau^0-i\gamma, \sigma^0)\right|\geq C\gamma^{\frac{1}{2}}$$

with some come constant $C=C(x^0, \tau^0, \sigma^0)>0$. Let us consider R as a function of $(x', \sqrt{\tau-\theta(x', \sigma)}, \sigma)$. Then by the implicit function theorem we can decompose R as

$$R(x', \tau, \sigma)=r\left(x', \sqrt{\tau-\theta(x', \sigma)}, \sigma\right)\left(\sqrt{\tau-\theta(x', \sigma)}-D(x', \sigma)\right)$$

where $\sqrt{1}=-1$ and $r(x^0, 0, \sigma^0)\neq 0$ and $D(x^0, \sigma^0)=0$. Now we assume that for some constant $C=C(x^0, \tau^0, \sigma^0)>0$

$$\begin{aligned} &\mathrm{Re}\ D(x', \sigma)\geq C\left(\mathrm{Im}\ D(x', \sigma)\right)^2 \qquad \text{in the case } (a),\\ &\text{or}\\ &-\mathrm{Im}\ D(x', \sigma)\geq C\left(\mathrm{Re}\ D(x', \sigma)\right)^2 \qquad \text{in the case } (b)\end{aligned} \tag{1.2}$$

in some neighborhood $U(x^0)\times U(\sigma^0)$, according to the case (a) or (b) in Lemma 3.1 respectively.

(II) γ') In the point satisfying the first condition of β'), if there is at least one real simple root λ, the zero set of Re $D(x', \sigma)$ (or Im $D(x', \sigma)$) in some $U(x^0, \sigma^0)$ is preassumed to be a regular submanifold of $\boldsymbol{R}^{2n-1}$ and the rank of $\underset{(x',\sigma)}{\text{Hess}}$ Re $D(x', \sigma)$ (or Im $D(x', \sigma)$) at the zeros of Re $D(x', \sigma)$ (or Im $D(x', \sigma)$) is equal to

$$\text{codim of } \left\{(x', \sigma)\,;\ \text{Re}\, D(x', \sigma) = 0\right\}$$
$$\text{or } \left\{(x', \sigma)\,;\ \text{Im}\, D(x', \sigma) = 0\right\} \quad \text{in } \boldsymbol{R}^{2n-1}$$

in the case (a) or (b), respectively.

Here we remark that the conditions (II) β) and γ) imply the conditions (II) β') and γ') when we are restricted to the real case (see subsection 10.2), and that the last condition of (II) β) and (II) γ') are omitted, if (6. 5) and (10. 6) are satisfied, respectively. It should also be noted that the conditions (I), (II) and (III) are invariant for certain coordinate transformations, hence Main theorem is applicable to problems defined on any smooth $\boldsymbol{R}^1 \times \Omega$.

Throughout this article we assume the condition (I). Section 2 and section 3 are a summary of elementary facts which are used in later sections. In section 4, a necessary and sufficient condition for L^2-well-posedness of the constant coefficients problem is shown in terms of 'coupling' coefficients. Using it we investigate, in section 5 and 6, the structure of variable coefficients problem under the condition (III). Section 7 is devoted to the estimates of the problems for 2×2 first order systems. The ideas used there play an essential role in the proof of Theorem 1. 1 which is shown in section 8. Section 9 is concerned with a dual problem to (P, B). In subsection 10. 1 an example showing the necessity of the condition (II) β) is presented, and in subsection 10. 2 the generalization to the case where the boundary operator $B(x')$ is complex valued is considered.

Second author lectured the outline about this theme ([12]) and first author gave the proof in detail. Authors are indebted to Dr. R. Agemi for a number of helpful discussion of these problems and for revises of the manuscript.

§ 2. Preliminaries.

2. 1. The following spaces are used in this paper. ([6], [8]) Let us define for $\gamma \neq 0$ and real numbers p, q the Hilbert spaces of functions:

$$H_{p,q;\gamma}(\boldsymbol{R}^{n+1}) = \left\{u \in \mathscr{D}'(\boldsymbol{R}_{n+1})\,;\ e^{-\gamma x_0} u \in H_{p,q}(\boldsymbol{R}_{n+1})\right\},$$
$$H_{q,\gamma}(\boldsymbol{R}^n) = \left\{v \in \mathscr{D}'(\boldsymbol{R}^n)\,;\ e^{-\gamma x_0} v \in H_q(\boldsymbol{R}^n)\right\}$$

with their norms defined by

$$\|u\|_{p,q;\gamma,R^{n+1}}=\int_{R^{n+1}}(\gamma^2+|\xi|^2)^p(\gamma^2+|\xi'|^2)^q|\widehat{e^{-\gamma x_0}u}(\xi)|^2d\xi,$$

$$|v|_{q,\gamma}=\int_{R^n}(\gamma^2+|\xi'|^2)^q|\widehat{e^{-\gamma x_0}v}(\xi')|^2d\xi'$$

and their inner products $(\cdot,\cdot)_{p,q;\gamma,R^{n+1}}$, $\langle\cdot,\cdot\rangle_{q,\gamma}$ respectively, where $H_{p,q}(\boldsymbol{R}^{n+1})$, $H_q(\boldsymbol{R}^n)$ are Sobolev spaces, $\xi=(\xi',\xi_n)=(\tau,\sigma,\lambda)\in\boldsymbol{R}^{n+1}$ is a covector of $x=(x',x_n)=(x_0,x'',x_n)\in\boldsymbol{R}^1\times\Omega$, and $\widehat{e^{-\gamma x_0}u}(\xi)$, $\widehat{e^{-\gamma x_0}v}(\xi')$ are the Fourier transforms of $e^{-\gamma x_0}u$, $e^{-\gamma x_0}v$ respectively. Moreover, let $H_{p,q;\gamma}(\boldsymbol{R}^{n+1}_+)$ be the set of all $u\in\mathscr{D}'(\boldsymbol{R}^{n+1}_+)$ such that there exist distributions $U\in H_{p,q;\gamma}(\boldsymbol{R}^{n+1})$ with $U=u$ in $\boldsymbol{R}^{n+1}_+$. The norm of u is defined by

$$\|u\|_{p,q;\gamma,R^{n+1}_+}=\inf_U\|U\|_{p,q;\gamma,R^{n+1}}.$$

From now on, for simplicity we denote by $H_{p,\gamma}(\boldsymbol{R}^{n+1}_+)$ the space $H_{p,0;\gamma}(\boldsymbol{R}^{n+1}_+)$, by $\|\cdot\|_{p,\gamma}$ the norm $\|\cdot\|_{p,0;\gamma,R^{n+1}_+}$ and by $(\cdot,\cdot)_{p,\gamma}$ the inner product respectively. Note that

$$H_{p,\gamma}(\boldsymbol{R}^{n+1}_+)=\left\{u\,;\ e^{-\gamma x_0}u\in H_p(\boldsymbol{R}^{n+1}_+)\right\}$$

and $\|u\|_{k,\gamma}$ is equivalent to

$$\left(\sum_{j+l=k}\int_0^\infty\left|D_n^ju(\cdot,x_n)\right|^2_{l,\gamma}dx_n\right)^{\frac{1}{2}}$$

if $k\geq 0$ is an integer.

2.2. We consider the following pseudo-differential operator with parameters $\gamma>0$ and x_n. Let $a(x',x_n,\xi',\gamma)\in C^\infty(\overline{\boldsymbol{R}^{n+1}}\times((\boldsymbol{R}^n\times\overline{\boldsymbol{R}^1_+})\backslash\{0\})$ be a function which is independent of x outside some compact set of $\overline{\boldsymbol{R}^{n+1}_+}$. That $a(x',x_n,\xi',\gamma)\in S^k_+$ (k: real) means that for every α, β there exists a constant $C_{\alpha,\beta}>0$ such that

$$\left|D^\alpha_{x'}D^\beta_{\xi'}a(x',x_n,\xi',\gamma)\right|\leq C_{\alpha,\beta}\left(\gamma^2+|\xi'|^2\right)^{\frac{1}{2}(k-|\beta|)}$$

for every $(x,\xi',\gamma)\in\overline{\boldsymbol{R}^{n+1}_+}\times\boldsymbol{R}^n\times\boldsymbol{R}^1_+$. Matrix functions in S^k_+ are defined similarly. With a matrix function $A\in S^k_+$ called a symbol we associate a pseudo-differential operator $A(x,D',\gamma)$ defined for any vector $v\in H_{s,\gamma}(\boldsymbol{R}^n)$, which means that the components of v belong to $H_{s,\gamma}(\boldsymbol{R}^n)$, by the formula

$$A(x,D',\gamma)\,v(x')$$
$$=(2\pi)^{-n}e^{\gamma x_0}\int_{R^n}e^{i\eta x_0+i\sigma x''}A(x,\eta,\sigma,\gamma)\,\hat{v}(\tau,\sigma)\,d\eta d\sigma$$

where $\tau=\eta-i\gamma$ and $\xi'=(\eta,\sigma)$. Note that if

$$A(x, \eta, \sigma, \gamma) = A(x, \eta - i\gamma, \sigma) = A(x, \tau, \sigma)$$

then it holds that if $A(x, \tau, \sigma)$ is analytic with respect to τ,

$$A(x, D', \gamma)\, v(x') = A(x, D')\, v(x')$$

where $v \in H_{s,\gamma}(\boldsymbol{R}^n)$ and $A(x, D')$ is a usual pseudo-differential operator.

The basic properties of usual pseudo-differential operators hold analogously for this case:

LEMMA 2.1. ([9], [5])

α) *For every real s there exist positive constants γ_s and C_s independent of v such that for every $\gamma \geq \gamma_s$ it holds that*

(i) *for real k, $A \in S^k_+$ and $v \in H_{s+k,\gamma}(\boldsymbol{R}^n)$*

$$\left| A(x, D', \gamma) v \right|_{s,\gamma} \leq C_s |v|_{s+k,\gamma},$$

(ii) *for real k, $A \in S^k_+$, $A^*(x, \xi', \gamma)$ the adjoint matrix, $A^\#$ the formal adjoint of $A(x, D', \gamma)$ with respect to $\langle \cdot, \cdot \rangle_{0,\gamma}$ and $v \in H_{s+k,\gamma}(\boldsymbol{R}^n)$*

$$\left| \left(A^*(x, D', \gamma) - A^\#(x, D', \gamma) \right) v \right|_{s+1,\gamma} \leq C_s |v|_{s+k,\gamma},$$

(iii) *for $A_i \in S^{k_i}_+$, k_i: real, $i = 1, 2$, $A_3(x, \xi', \gamma) = A_1 A_2 \in S^{k_1+k_2}_+$ and $v \in H_{s+k_1+k_2,\gamma}(\boldsymbol{R}^n)$*

$$\left| \left(A_3(x, D', \gamma) - A_1(x, D', \gamma) \cdot A_2(x, D', \gamma) \right) v \right|_{s+1,\gamma} \leq C_s |v|_{s+k_1+k_2,\gamma}.$$

(iv) *Let $A(x, \xi', \gamma) = A^*(x, \xi', \gamma) \in S^k_+$, k: real. Then there exist $C, \gamma_0 > 0$ such that for every $\gamma \geq \gamma_0$ and $v \in H_{k,\gamma}(\boldsymbol{R}^n)$*

$$\operatorname{Im} \langle A(x, D', \gamma)\, v, v \rangle_{0,\gamma} \leq C |v|^2_{\frac{k-1}{2},\gamma}.$$

β) *Let $A \in S^0_+$ and $A(x, \xi', \gamma) = A^*(x, \xi', \gamma) \geq 0$ for $\gamma \geq \gamma_0 > 0$. Then there exist C, $\gamma_1 > 0$ such that for $\gamma \geq \gamma_1$*

$$\operatorname{Re} \langle A(x, D', \gamma)\, v, v \rangle_{0,\gamma} \geq -C |v|^2_{-\frac{1}{2},\gamma}.$$

This is a Gårding's sharp form. From this the following hold.

(i) *Let $A \in S^0_+$ and $A(x, \xi', \gamma) = A^* \geq C$ with $C > 0$. Then there exists a $\gamma_1 > 0$ such that for $\gamma \geq \gamma_1$*

$$\operatorname{Re} \langle A(x, D', \gamma)\, v, v \rangle_{0,\gamma} \geq 2^{-1} C |v|^2_{0,\gamma}.$$

(ii) *Let $A \in S^0_+$ and $|\det A(x, \xi', \gamma)| \geq C$ with $C > 0$. Then for every real s there exist C_s, $\gamma_s > 0$ such that for $\gamma \geq \gamma_s$*

$$\left| A(x, D', \gamma)\, v \right|_{s,\gamma} \geq C_s |v|_{s,\gamma}.$$

(iii) *Let* $A(x, \xi', \gamma) \in S^0_+$ *and* $A^{-1}(x, \xi', \gamma) \in S^0_+$. *Then for every real* s *there exist* C_s, $\gamma_s > 0$ *such that for* $\gamma \geq \gamma_s$ $A(x, D', \gamma)$ *has an inverse* $A(x, D', \gamma)^{-1} = A^{-1}(x, D', \gamma) + T$ *in* $H_{s,\gamma}$ *which satisfies*

$$|A^{-1}x|_{s,\gamma} \leq C_s |v|_{s,\gamma},$$
$$|Tv|_{s,\gamma} \leq C_s |v|_{s-1,\gamma}$$

for every $v \in H_{s,\gamma}(\boldsymbol{R}^n)$.

(iv) *Let* $A \in S^1_+$ *and* $A(x, \xi', \gamma) = A^* \geq C_0 \gamma$ *for* $\gamma \geq \gamma_0$, *where* C_0, $\gamma_0 > 0$. *Then there exist* C_1, $\gamma_1 > 0$ *such that for* $\gamma \geq \gamma_1$

$$\mathrm{Re}\, \langle A(x, D', \gamma)\, v, v \rangle_{0,\gamma} \geq C_1 \gamma\, |v|^2_{0,\gamma}.$$

(v) *Let* $A \in S^1_+$ *and* $A(x, \xi', \gamma) = A_1(x, \xi', \gamma) + iA_2(x, \xi', \gamma)$, *where for some* C_0, $\gamma_0 > 0$, $A_1(x, \xi', \gamma) = A_1^* \geq C_0 \gamma$ *and* $A_2(x, \xi', \gamma) = A_2^*$ *for every* $\gamma \geq \gamma_0$. *Then there exist* C_1, $\gamma_1 > 0$ *such that for* $\gamma \geq \gamma_1$

$$\mathrm{Re}\, \langle A(x, D', \gamma)\, v, v \rangle_{0,\gamma} \geq C_1 \gamma\, |v|^2_{0,\gamma}.$$

(vi) *Let* $a(x, \xi', \gamma) \in S^0_+$ *be a scalar function such that* $|a(x, \xi', \gamma)| \leq C_0 - \varepsilon$ *for* $\gamma \geq \gamma_0 > 0$, *where* $\varepsilon > 0$. *Then there exists a constant* $\gamma_1 > 0$ *such that for* $\gamma \geq \gamma_1$

$$\left| a(x, D', \gamma)\, v \right|_{0,\gamma} \leq C_0 |v|_{0,\gamma}.$$

In the present paper, our process of proofs of Main theorem are carried somewhat long and classically, in order to use the Gårging's sharp form directly and decompose the original problem into boundary value problems well defined over $\boldsymbol{R}^{n+1}_+$ and $\boldsymbol{R}^n$ in which we are interested.

Furthermore for the sake of simplicity of description we denote by $a(x, \eta - i\gamma, \sigma)$ the symbols $a(x, \eta, \sigma, \gamma)$ and by operators $a(x, D')$ on $H_{\cdot,\gamma}$ the corresponding pseudo-differential operators $a(x, D', \gamma)$ respectivevely.

2. 3. We reduce the global estimate (1. 1) into the micro-local one as usual ([5]). Since lower order terms of the operator P do not affect the validity of the estimate (1. 1), we have only to consider the boundary value problem:

$$(P^0, B) \begin{cases} P^0(x, D)\, u(x) = \big(ED_n - A(x, D')\big) u = f, & \text{in } \boldsymbol{R}^{n+1}_+, \\ B(x')\, u(x', 0) = g(x'), & \text{in } \boldsymbol{R}^n. \end{cases}$$

where the symbol of $A(x, D')$ is $A(x, \tau, \sigma) = A_0(x)\tau + \sum_{i=1}^{n-1} A_i(x)\, \sigma_i$ and E is the $2m \times 2m$ identity matrix.

First, let $K \subset \boldsymbol{R}^{n+1}$ be the compact set outside which $A_i(x)$ $i = 0, \cdots, n-1$ and $B(x')$ are constant. Let us take a finite number of points

$\{x^j\}_{j=1,\cdots,j_0-1}\subset\Gamma$ such that the spheres $S_{2\varepsilon_0}(x^j)=\{x\in \boldsymbol{R}^{n+1};\ |x-x^j|<2\varepsilon_0\}$, $j=1,\cdots,j_0-1$ are a covering of $K\cap\Gamma$, where $\varepsilon_0>0$ is sufficiently small. Take a finite partition of unity $\{\phi^j\}_{j=0,\cdots,j_0}$ such that

$$\begin{aligned}
&\operatorname{supp}\phi^j\subset S_{2\varepsilon_0}(x^j) && j=1,\cdots,j_0-1,\\
&\sum_{j=0}^{j_0}\phi^j(x)=1 && \text{on } \overline{\boldsymbol{R}^{n+1}_+},\\
&\phi^j\in C^\infty(\overline{\boldsymbol{R}^{n+1}_+}) && j=0,\cdots,j_0,\\
&\phi^{j_0}(x)=\phi^{j_0}(x_n)=\begin{cases}1, & \text{for } x_n\geq\frac{3}{2}\varepsilon_0,\\ 0, & \text{for } x_n\leq\varepsilon_0,\end{cases}\\
&\phi^0(x)=0 \quad \text{for} \quad x_n\geq\tfrac{3}{2}\varepsilon_0,\\
&A_i(x),\ i=0,\cdots,n-1 \text{ and } B(x') \text{ are constant in } \operatorname{supp}\phi^0(x).
\end{aligned}$$

For $1\leq j\leq j_0-1$ we construct the corresponding operators $B^j(x')$ and $P^j(x, D)=ED_n-A^j(x, D')$ where the symbol of $A^j(x, D')$ is $A^j(x,\tau,\sigma)=A_0^j(x)\tau+\sum_{i=1}^{n-1}A_i^j(x)\sigma_i$, which have the following properties:

$$\begin{aligned}
&\text{(i)} && A^j(x,\tau,\sigma)\in S^1_+ \text{ and } B^j(x')\in S^0_+,\\
(2.1)\quad &\text{(ii)} && A^j(x,\tau,\sigma)=A(x,\tau,\sigma) \text{ and } B^j(x')=B(x') \text{ for every } x\in S_{2\varepsilon_0}(x^j),\\
&\text{(iii)} && A^j(x,\tau,\sigma)=A(x^j,\tau,\sigma) \text{ and } B^j(x')=B(x^j) \text{ for every } x\notin S_{\frac{5}{2}\varepsilon_0}(x^j).
\end{aligned}$$

The construction of P^j, B^j is done as follows: Choose a function $\beta_1(t)\in C^\infty(\overline{\boldsymbol{R}^1_+})$ such that equals 1 for $t\leq 1$, 0 for $t\geq\frac{5}{4}$, $0<\beta_1(t)<1$ for $1<t<\frac{5}{4}$. For $x\in\overline{\boldsymbol{R}^{n+1}_+}$ put

$$\widetilde{x^j}(x)=x^j+\beta_1(|x-x^j|/2\varepsilon_0)(x-x^j)$$

and let

$$A^j(x,\tau,\sigma)=A\left(\widetilde{x^j}(x),\tau,\sigma\right),$$
$$B^j(x')=B\left(\widetilde{x^j}(x')\right).$$

Then it is seen that these operators satisfy the required properties (2.1).

Now we have the following

LEMMA 2.2. *Suppose the condition (III) and that there are constants, C, $\gamma_0>0$ such that for every $\gamma\geq\gamma_0$ and $u\in H_{1,\gamma}(\boldsymbol{R}^{n+1}_+)$ the inequalities*

$$(2.2)\qquad \|P^j\phi^j u\|_{0,\gamma}+|B^j\phi^j u|_{\frac{1}{2},\gamma}\geq C\gamma\|\phi^j u\|_{0,\gamma},\qquad j=1,\cdots,j_0-1,$$

hold. Then the estimate (1.1) *holds for $k=0$.*

PROOF. The strict hyperbolicity of $P^0(x, D)$ implies that there exist constants γ_0, $C>0$ such that for every $\gamma\geq\gamma_0$ and $u\in H_{1,\gamma}(\boldsymbol{R}^{n+1}_+)$

(2. 2. 1) $$\|P^0(x, D)\phi^{j_0}u\|_{0,\gamma} \geq C\gamma\|\phi^{j_0}u\|_{0,\gamma}.$$

From property (2. 1) (ii), it follows that for $j=1, \cdots, j_0-1$

$$P^j(\phi^j u) = P^0(\phi^j u) = \phi^j P^0 u + [P^0, \phi^j]u \quad in\ \boldsymbol{R}_+^{n+1},$$
$$B^j(\phi^j u) = B(\phi^j u) = \phi^j Bu \quad \text{on } \Gamma.$$

Furthermore from the condition (III), (1. 1) holds for constant coefficients problems. Hence it follows from (2. 2) and (2. 2. 1) that

$$C\gamma\|\phi^j u\|_{0,\gamma} \leq \|\phi^j P^0 u + [P^0, \phi^j]u\|_{0,\gamma} + |\phi^j Bu|_{\frac{1}{2},\gamma}$$
$$\leq \|P^0 u\|_{0,\gamma} + |Bu|_{\frac{1}{2},\gamma} + \|u\|_{0,\gamma}$$

for $j=0, \cdots, j_0$. Summing up these, we have

$$C\gamma\|u\|_{0,\gamma} = C\gamma\|\sum_{j=0}^{j_0}\phi^j u\|_{0,\gamma}$$
$$\leq C'(\|P^0 u\|_{0,\gamma} + |Bu|_{\frac{1}{2},\gamma} + \|u\|_{0,\gamma}).$$

If γ is taken sufficiently large we obtain (1. 1).

Secondly, let $\Sigma_- = \{\tau', \sigma') \in \boldsymbol{C}_- \times \boldsymbol{R}^{n-1};\ |\tau'|^2 + |\sigma'|^2 = 1\}$ and $\overline{\Sigma_-}$ be its closure. Let us take a finite number of points $\{(\tau_k', \sigma_k')\}_{k=1,\cdots,k_1} \subset \overline{\Sigma_-}$ such that $\mathrm{Im}\ \tau_k' = 0$ for $k \leq k_0$, $\mathrm{Im}\ \tau_k' \leq -3\varepsilon_0$ for $k_0+1 \leq k \leq k_1$ and $\{S_{2\varepsilon_0}(\tau_k', \sigma_k')\}_{k=1,\cdots,k_1}$ is a covering of $\overline{\Sigma_-}$, where $S_\varepsilon(\tau_k', \sigma_k') = \{(\tau', \sigma') \in \overline{\Sigma_-};\ |\tau' - \tau_k'|^2 + |\sigma' - \sigma_k'|^2 < \varepsilon^2\}$. Let $\{\psi_k'(\tau', \sigma')\}$ be C^∞-functions on $\overline{\Sigma_-}$ such that

$$\sum_{k=1}^{k_1}\left(\psi_k'(\tau', \sigma')\right)^2 = 1 \qquad \text{on } \overline{\Sigma_-}$$

and $\mathrm{supp}\ \psi_k' \subset S_{2\varepsilon_0}(\tau_k', \sigma_k')$. Put

$$\psi_k(\tau, \sigma) = \psi_k'(\Lambda_\gamma^{-1}\tau, \Lambda_\gamma^{-1}\sigma) \quad \text{for} \quad (\tau, \sigma) \in \overline{\boldsymbol{C}_-} \times \boldsymbol{R}^{n-1}$$

where $\Lambda_\gamma^2 = |\tau|^2 + |\sigma|^2 = \gamma^2 + |\xi'|^2$. Then we have $\psi_k \in S_+^0$.

For each $j=1, \cdots, j_0-1$, it is possible to construct the corresponding operators $P_k^j(x, D) = ED_n - A_k^j(x, D')$ for $k=1, \cdots, k_1$, such that the symbols $A_k^j(x, \tau, \sigma)$ of $A_k^j(x, D')$ fulfill the following properties:

(2. 3) (i) $A_k^j(x, \tau, \sigma) \in S_+^1$,

(ii) $A_k^j(x, \tau, \sigma) = A^j(x, \tau, \sigma)$ for $(x, \tau\Lambda_\gamma^{-1}, \sigma\Lambda_\gamma^{-1}) \in \overline{\boldsymbol{R}_+^{n+1}} \times S_{2\varepsilon_0}(\tau_k', \sigma_k')$, and hence $A_k^j(x, \tau, \sigma) = A(x, \tau, \sigma)$ for $(x, \tau\Lambda_\gamma^{-1}, \sigma\Lambda_\gamma^{-1}) \in S_{2\varepsilon_0}(x^j) \times S_{2\varepsilon_0}(\tau_k', \sigma_k')$.

The way of construction of $A_k^j(x, \tau, \sigma)$ is as follows:

Case 1. $k \geq k_0 + 1$.

For $(\tau', \sigma') \in \overline{\Sigma_-}$ put

(2. 4) $$\tilde{\tau}_k(\tau', \sigma') = \Big[\tau'_k + \beta_1\Big((|\tau'-\tau'_k|^2 + |\sigma'-\sigma'_k|^2)^{\frac{1}{2}}/2\varepsilon_0\Big)(\tau'-\tau'_k)\Big]\cdot\delta^{-1}$$
$$\tilde{\sigma}_k(\tau', \sigma') = \Big[\sigma'_k + \beta_1\Big((|\tau'-\tau'_k|^2 + |\sigma'-\sigma'_k|^2)^{\frac{1}{2}}/2\varepsilon_0\Big)(\sigma'-\sigma'_k)\Big]\cdot\delta^{-1}$$

where $\delta(=\delta(\tau', \sigma'))$ is chosen such that $(\tilde{\tau}_k, \tilde{\sigma}_k) \in \overline{\Sigma_-}$. Put

(2. 5) $$A^j_k(x, \tau, \sigma) = A^j\Big(x, \tilde{\tau}_k(\tau\Lambda_\gamma^{-1}, \sigma\Lambda_\gamma^{-1}), \tilde{\sigma}_k(\tau\Lambda_\gamma^{-1}, \sigma\Lambda_\gamma^{-1})\Big)\cdot\Lambda_\gamma$$

for $(\tau, \sigma) \in \overline{\boldsymbol{C}_-} \times \boldsymbol{R}^{n-1}$. Then A^j_k fulfill (2. 3), by virture of the homogeneity of A^j of order 1 in (τ, σ). Remark that

(2. 4)′ (i) $(\tilde{\tau}_k, \tilde{\sigma}_k)$ mapps Σ_- into $S_{\frac{5}{2}\varepsilon_0}(\tau'_k, \sigma'_k)$,

(ii) $S_{\frac{5}{2}\varepsilon_0}(\tau'_k, \sigma'_k) \subset \Big\{(\tau', \sigma') \in \Sigma_-;\ \mathrm{Im}\ \tau' \leq -\frac{1}{2}\varepsilon_0\Big\}$,

(iii) $(\tilde{\tau}_k, \tilde{\sigma}_k) = (\tau', \sigma')$ for every $(\tau', \sigma') \in S_{2\varepsilon_0}(\tau'_k, \sigma'_k)$.

Case 2. $k \leq k_0$.

Let $\beta_2(t) \in C^\infty(\overline{\boldsymbol{R}^1_+})$ be a function such that equals to t for $t \leq 1$, 1 for $t \geq 2$ and $1 \leq \beta_2(t) \leq 2$ for $1 < t < 2$. For $(\tau', \sigma') = (\eta' - i\gamma', \sigma') \in \overline{\Sigma_-}$ put

(2. 6) $$\rho_k = \Big(|\eta'-\tau'_k|^2 + |\sigma'-\sigma'_k|^2\Big)^{\frac{1}{2}},$$
$$\tilde{\sigma}_k = \Big[\sigma'_k + \beta_1(\rho_k/2\varepsilon_0)(\sigma'-\sigma'_k)\Big]\delta^{-1},$$
$$\tilde{\eta}_k = \Big[\tau'_k + \beta_1(\rho_k/2\varepsilon_0)(\eta'-\tau'_k)\Big]\delta^{-1},$$
$$\tilde{\gamma}_k = 2\varepsilon_0\beta_2(\gamma'/2\varepsilon_0),$$
$$\tilde{\tau}_k = \tilde{\eta}_k - i\tilde{\gamma}_k$$

where δ is chosen to be $(\tilde{\tau}_k(\tau', \sigma'), \tilde{\sigma}_k(\tau', \sigma')) \in \overline{\Sigma_-}$. Define A^j_k also by (2. 5). Then A^j_k fulfill (2. 3). Remark that

(2. 6)′ (i) $(\tilde{\tau}_k, \tilde{\sigma}_k)$ mapps $\overline{\Sigma_-}$ into $S_{8\varepsilon_0}(\tau'_k, \sigma'_k)$,

(ii) $\tilde{\gamma}_k \leq 4\varepsilon_0$,

(iii) $\tilde{\gamma}_k = \gamma'$ for $\gamma' \leq 2\varepsilon_0$ and $\tilde{\gamma}_k \geq 2\varepsilon_0$ for $\gamma' \geq 2\varepsilon_0$,

(iv) $(\tilde{\tau}_k, \tilde{\sigma}_k) = (\tau', \sigma')$ for every $(\tau', \sigma') \in S_{2\varepsilon_0}(\tau'_k, \sigma'_k)$.

The reasons of different constructions of $(\tilde{\tau}_k, \tilde{\sigma}_k)$ according to k are elucidated in §5 and §7.

By Corollary 2. 3 of [8] (or c. f. (8. 2)) we obtain the following

LEMMA 2. 3. *Suppose that there exist constants, C, $\gamma_0 > 0$ such that for every $\gamma \geq \gamma_0$ and $u \in H_{1,\gamma}(\boldsymbol{R}^{n+1}_+)$ the inequalities*

(2. 7) $$\Big\|P^j_k\phi_k(D')u\Big\|_{0,\gamma} + \Big|B^j\phi_k(D')u\Big|_{\frac{1}{2},\gamma} \geq C\gamma\Big\|\phi_k(D')u\Big\|_{0,\gamma}$$

hold for $k=1, \cdots, k_1$. *Then the estimates* (2.2) *are valid.*

From Lemma 2.2 and 2.3 it is seen that in order to prove (1.1) for $k=0$ we have only to obtain the estimate (2.7) for the localized operators P_k^j and B^j. Hereafter let $(x^0, \tau^0, \sigma^0) \in \Gamma \times \overline{\Sigma_-}$ be one of the points $\{(x^j, \tau_k', \sigma_k')\}$ and $U(x^0) \times U(\tau^0, \sigma^0)$ be its sufficiently small neighborhood. If $a(x, \tau, \sigma)$ is a smooth $(C^\infty(U(x^0) \times U(\tau^0, \sigma^0)))$ function of homogeneous degree k in (τ, σ), then we denote by $a(x, D')$ the operator constructed as above, whose symbol $\tilde{a}(x, \tau, \sigma)$ is $a(\tilde{x}, \tilde{\tau}, \tilde{\sigma}) \cdot \Lambda_\gamma^k = a(\tilde{x}(x), \tilde{\tau}(\tau\Lambda_\gamma^{-1}, \sigma\Lambda_\gamma^{-1}), \tilde{\sigma}(\tau\Lambda_\gamma^{-1}, \sigma\Lambda_\gamma^{-1})) \cdot \Lambda_\gamma^k \in S_+^k$.

§ 3. Decompositions of the problem (*P*, *B*).

In order to decompose the operators P_k^j, B^j into more simple ones, we consider the transformations of the operators P_k^j, B^j by a non-singular and smooth $2m \times 2m$ matrix. We denote by $\lambda(x, \tau, \sigma)$ the eigenvalues of $A(x, \tau, \sigma)$ and by $\lambda_i^\pm(x, \tau, \sigma)$ $(i=1, \cdots, m)$ the ones with positive and negative imaginary parts for $(\tau, \sigma) \in \boldsymbol{C}_- \times \boldsymbol{R}^{n-1}$, respectively.

Let us fix a point $(x^0, \tau^0, \sigma^0) \in \Gamma \times \overline{\Sigma_-}$. Because of the condition (I), in a neighborhood $U(x^0) \times U(\tau^0, \sigma^0)$ we can rearrange the eigenvalues $\{\lambda_i^\pm(x, \tau, \sigma)\}$ into the following three sets $\mathrm{I} = \mathrm{I}_+ \cup \mathrm{I}_-$, II and $\mathrm{III} = \mathrm{III}_+ \cup \mathrm{III}_-$:

$\mathrm{I}_\pm$: $\lambda_i^\pm (i=1, \cdots, l-1)$ are real and simple at (x^0, τ^0, σ^0),

II : λ_l^+, λ_l^- are real and double at (x^0, τ^0, σ^0),

$\mathrm{III}_\pm$: $\lambda_i^\pm (i \geq l+1)$ are not real at (x^0, τ^0, σ^0).

Hereafter we will also identify $\mathrm{I}_\pm$, II and $\mathrm{III}_\pm$ to the corresponding sets of indeces $\subset \{1, 2, \cdots, m\}$. It should be noted that some of the sets I, II and III may be empty according to the point $(x^0, \tau^0, \sigma^0) \in \Gamma \times \overline{\Sigma_-}$ being considered. If the set II is empty, we denote by l the number of elements of the set $\mathrm{I}_+(\mathrm{I}_-)$.

As is shown below, we can take a smooth and non-singular matrix $S(x, \sigma, \tau)$ defined in $U(x^0) \times U(\sigma^0, \tau^0)$ which is homogeneous of degree 0 in (σ, τ) and satisfies

$$S^{-1} P^0(x, \tau, \sigma, \lambda) S = E\lambda - M(x, \tau, \sigma),$$

where

$$M = \begin{pmatrix} \lambda_{\mathrm{I}}^+ & & & & 0 \\ & \lambda_{\mathrm{I}}^- & & & \\ & & M_{\mathrm{II}} & & \\ & & & M_{\mathrm{III}}^+ & \\ 0 & & & & M_{\mathrm{III}}^- \end{pmatrix},$$

$$\lambda_{\mathrm{I}}^{\pm}=\begin{pmatrix}\lambda_{\mathrm{I}}^{\pm} & & 0\\ & \ddots & \\ 0 & & \lambda_{l-1}^{\pm}\end{pmatrix},$$

M_{II} is a 2×2 matrix defined below in Lemma 3. 2 and $M_{\mathrm{III}}^{\pm}$ are $(m-l)\times m-l)$ matrices whose eigenvalues are $\{\lambda_i^{\pm}\}_{i\in\mathrm{III}_{\pm}}$, respectively.

Here we define the above matrix S. Let $\{h_{\mathrm{I}}^{\pm},\cdots,h_{l-1}^{\pm}\}=h_{\mathrm{I}}^{\pm}(x,\tau,\sigma)$ be the eigenvectors of $A(x,\tau,\sigma)$ which are homogeneous of degree 0 in (τ,σ) and correspond to the eigenvalues $\{\lambda_{\mathrm{I}}^{\pm},\cdots,\lambda_{l-1}^{\pm}\}$, respectively. Then the smooth $2m\times(l-1)$ matrices $h_{\mathrm{I}}^{\pm}$ are analytic in (τ,σ).

As for the set III_+, first take all the generalized eigenvectors $h_{\mathrm{I}}^{+},\cdots,h_{p_j}^{+}$ corresponding to an eigenvalue $\lambda_j^{+}(x^0,\tau^0,\sigma^0)$ with $j\in\mathrm{III}_+$. Put

$$h_{jk}^{+}(x,\tau,\sigma)=\frac{1}{2\pi i}\oint_{c_j^{+}}\left(E\lambda-A(x,\tau,\sigma)\right)^{-1}d\lambda\cdot h_k^{+},\quad k=1,\cdots,p_j$$

where c_j^{+} is a small circle enclosing only $\lambda_j^{+}(x^0,\tau^0,\sigma^0)$. Rearrange the above linearly independent vectors $\{h_{jk}^{+}\}_{j,k}$ and let them be $\{h_{l+1}^{+},\cdots,h_m^{+}\}=h_{\mathrm{III}}^{+}(x,\tau,\sigma)$. By the same method as above we can choose vectors $\{h_{l+1}^{-},\cdots,h_m^{-}\}=h_{\mathrm{III}}^{-}$ for III_-. Then the smooth $2m\times(m-l)$ matrices $h_{\mathrm{III}}^{\pm}(x,\tau,\sigma)$ are homogeneous of degree 0 and analytic in (τ,σ).

For the set II, we take the smooth $2m$-vectors (i.e. column vectors with $2m$ components) h_l', h_l'' such that $h_{\mathrm{II}}'(x,\tau,\sigma)=h_l'$ and $h_{\mathrm{II}}''(x,\tau,\sigma)=h_l''$ are homogeneous of degree 0 and analytic in (τ,σ), which is defined below in Lemma 3. 2.

Put $S(x,\tau,\sigma)=(h_{\mathrm{I}}^{+},h_{\mathrm{I}}^{-},h_{\mathrm{II}}',h_{\mathrm{II}}'',h_{\mathrm{III}}^{+},h_{\mathrm{III}}^{-})$. Then $S(x,\tau,\sigma)$ has the required properties, by virture of the linear independence of the column vectors.

Now we describe on the set II in detail.

LEMMA 3. 1. *Let $(x^0,\tau^0,\sigma^0)\in\Gamma\times(\overline{\Sigma_-}-\Sigma_-)$ and λ^0 be a double root in λ of det $P^0(x^0,\tau^0,\sigma^0,\lambda)=0$. Then there exist a neighborhood $U(x^0)\times U(\tau^0,\sigma^0)$ and functions $\lambda_{\mathrm{II}}^{\pm}(x,\tau,\sigma)$ continuous in $U(x^0)\times(U(\tau^0,\sigma^0)\cap(\boldsymbol{C}_-\times\boldsymbol{R}^{n-1}))$ such that*

(i) $\mathrm{Im}\,\lambda_{\mathrm{II}}^{\pm}(x,\tau,\sigma)\gtrless 0$ *if* $\mathrm{Im}\,\tau<0$

respectively, and

(ii) $\det P^0(x,\tau,\sigma,\lambda_{\mathrm{II}}^{\pm}(x,\tau,\sigma))=0$ *if* $\mathrm{Im}\,\tau\leqq 0$.

Furthermore, they are represented by

(a) $\lambda_{\mathrm{II}}^{\pm}(x,\tau,\sigma)=\lambda_1(x,\mu,\sigma)\pm\sqrt{\mu}\,\lambda_2(x,\mu,\sigma)$ *or*

(b) $\lambda_{\mathrm{II}}^{\pm}(x,\tau,\sigma)=\lambda_1(x,\mu,\sigma)\mp i\sqrt{\mu}\,\lambda_2'(x,\mu,\sigma)$

according as the normal surface cut by $x=x^0$ and $\sigma=\sigma^0$ is convex or concave with respect to τ at (τ^0,λ^0), respectively. Here

$$\mu=\mu(x, \tau, \sigma)=\tau-\theta(x, \sigma),$$

$\theta(x, \sigma)$ is S^1 in (x, σ), analytic and real for σ contained in a conical neighborhood of (x^0, σ^0),

$$\theta(x^0, \sigma^0)=\tau^0,$$

and

$\lambda_1 \in S^1$, λ_2 and $\lambda_2' \in S^{\frac{1}{2}}$ in (x, μ, σ), analytic and real for (μ, σ) contained in a conical neighborhood of (x^0, O, σ^0) whenever μ is real,

$$\lambda_1(x^0, O, \sigma^0)=\lambda_0,$$
$$\lambda_2(x^0, O, \sigma^0)>O,$$
$$\lambda_2'(x^0, O, \sigma^0)>O$$

and $\sqrt{\mu}$ denotes a branch with positive imaginary part when $\operatorname{Im}\mu<O$ *(i.e., $\sqrt{1}=-1$).*

Corollary 3.1. *In the small neighborhood of (x^0, τ^0, σ^0) $\lambda_{\mathrm{II}}^+(x, \tau, \sigma)=\lambda_{\mathrm{II}}^-(x, \tau, \sigma)$ is equivalent to $\mu=O$, that is, $\tau=\theta(x, \sigma)$.*

Proof of Lemma 3.1. The strict hyperbolicity implies that

$$\det P^0(x, \tau, \sigma, \lambda)=A_0(x)\prod_{j=1}^{2m}\left(\tau-\tau_j(x, \sigma, \lambda)\right)$$

where τ_j is C^∞ in $(x, \sigma, \lambda)\in \boldsymbol{R}^{2n+1}$, analytic and real for $(\sigma, \lambda)\in \boldsymbol{R}^n$ and $\tau_j\neq \tau_k$ for every $j\neq k$ and $(x, \sigma, \lambda)\in \boldsymbol{R}^{2n+1}$. Without loss of generality we may assume

$$\tau_0=\tau_1(x^0, \sigma^0, \lambda^0).$$

Since λ_0 is a double root it holds that

$$\frac{\partial\tau_1}{\partial\lambda}(x^0, \sigma^0, \lambda^0)=O \quad \text{and} \quad \frac{\partial^2\tau_1}{\partial\lambda^2}(x^0, \sigma^0, \lambda^0)\neq O.$$

Therefore from the implicit function theorem there exists a function $\lambda(x, \sigma)$ which is C^∞ in (x, σ) and real analytic in σ such that

$$\frac{\partial\tau_1}{\partial\lambda}\left(x, \sigma, \lambda(x, \sigma)\right)=O \quad \text{and} \quad \lambda(x^0, \sigma^0)=\lambda^0.$$

Hence by Taylor's expansion we have

$$\tau_1(x, \sigma, \lambda)=\tau_1\left(x, \sigma, \lambda(x, \sigma)\right) + \frac{1}{2}\cdot\frac{\partial^2\tau_1}{\partial\lambda^2}\left(x, \sigma, \lambda(x, \sigma)\right)\left(\lambda-\lambda(x, \sigma)\right)^2+\cdots.$$

Since the equation in $(y-\lambda(x, \sigma))$:

$$\tau-\tau_1\big(x, \sigma, \lambda(x, \sigma)\big)$$
$$=\frac{1}{2}\cdot\frac{\partial^2\tau_1}{\partial\lambda^2}\big(x, \sigma, \lambda(x, \sigma)\big)\big(y-\lambda(x, \sigma)\big)^2+\cdots$$

has real coefficients, it has solutions of the following type:

$$y-\lambda(x, \sigma)=\zeta^{\frac{1}{2}}+a_1(x, \sigma)\zeta^{\frac{2}{2}}+a_2(x, \sigma)\zeta^{\frac{3}{2}}+\cdots$$

where $a_i(x, \sigma)$ is C^∞ in (x, σ) and real analytic in σ,

$$\zeta=\Big(\tau-\tau_1\big(x, \sigma, \lambda(x, \sigma)\big)\Big)\, 2\Big(\frac{\partial^2\tau_1}{\partial\lambda^2}\big(x, \sigma, \lambda(x, \sigma)\big)\Big)^{-1}.$$

Put $\mu=\tau-\tau_1(x, \sigma, \lambda(x, \sigma))$ and let $\zeta^{\frac{1}{2}}$ be the positive square root of ζ when $\zeta>O$ and let $\sqrt{\zeta}$ be the negative one and its extension when $\mathrm{Im}\,\zeta\leqq O$. Then note that

(a) if $\frac{\partial^2\tau_1}{\partial\lambda^2}(x^0, \sigma^0, \lambda^0)>O$

$$\pm\sqrt{\zeta}=\pm\sqrt{\mu}\Big\{2\Big(\frac{\partial^2\tau_1}{\partial\lambda^2}\big(x, \sigma, \lambda(x, \sigma)\big)\Big)^{-1}\Big\}^{\frac{1}{2}},$$

(b) if $\frac{\partial^2\tau_1}{\partial\lambda^2}(x^0, \sigma^0, \lambda^0)<O$

$$\pm\sqrt{\zeta}=\mp i\sqrt{\mu}\Big\{-2\Big(\frac{\partial^2\tau_1}{\partial\lambda^2}\big(x, \sigma, \lambda(x, \sigma)\big)\Big)^{-1}\Big\}^{\frac{1}{2}}.$$

For $\mathrm{Im}\,\tau<O$ put

$$y^{\pm}=\lambda(x, \sigma)+(\pm\sqrt{\zeta})+a_1(x, \sigma)(\pm\sqrt{\zeta})^2+\cdots.$$

Then

$$\mathrm{Im}\, y^{\pm}=\mathrm{Im}\,(\pm\sqrt{\zeta})+a_1(x, \sigma)\ \mathrm{Im}\,(\pm\sqrt{\zeta})^2+\cdots\gtrless O$$

for small $|\zeta|$, respectively. Hence we see

$$\lambda^{\pm}(x, \tau, \sigma)\equiv y^{\pm}=\big(\lambda(x, \sigma)+a_1(x, \sigma)\zeta+\cdots\big)$$
$$\pm\sqrt{\zeta}\big(1+a_2(x, \sigma)\zeta+\cdots\big)$$

fulfill (i) and (ii). Setting

$$\theta(x, \sigma)=\tau_1\big(x, \sigma, \lambda(x, \sigma)\big),$$

we obtain the desired representation of $\lambda_{\mathrm{II}}^{\pm}$ with

$$\lambda_2(x^0, O, \sigma^0)=\Big\{2\Big(\frac{\partial^2\tau_1}{\partial\lambda^2}(x^0, \sigma^0, \lambda^0)\Big)^{-1}\Big\}^{\frac{1}{2}}>O$$

or

$$\lambda_2'(x^0, O, \sigma^0) = \left\{-2\left(\frac{\partial^2 \tau_1}{\partial \lambda^2}(x^0, \sigma^0, \lambda^0)\right)^{-1}\right\}^{\frac{1}{2}} > O$$

according to the case (a) or (b) respectively. Thus the proof is completed.

Hereafter, we will consider mainly only the case (a) of Lemma 3.1, because in the case (b) we can treat by the analogous way. Furthermore let $c(x, \tau, \sigma)$ be a function defined in $U(x^0) \times U(\tau^0, \sigma^0)$, then we denote $c(x, \mu+\theta(x, \sigma), \sigma)$ also by $c(x, \mu, \sigma)$ with some definition domain $U(x^0) \times U(O, \sigma^0)$.

LEMMA 3.2. *Let* $(x^0, \tau^0, \sigma^0) \in \Gamma \times (\bar{\Sigma}_- - \Sigma_-)$. *Then there exist a neighborhood* $U(x^0) \times U(\tau^0, \sigma^0)$ *and* $2m$*-vectors* $h'_{\mathrm{II}}(x, \tau, \sigma)$, $h''_{\mathrm{II}}(x, \tau, \sigma)$ *and a* 2×2 *matrix* $M_{\mathrm{II}}(x, \tau, \sigma)$ *which are smooth in* (x, τ, σ) *and analytic in* (τ, σ) *such that for every* $(x, \tau, \sigma) \in U(x^0) \times U(\tau^0, \sigma^0)$ *the following hold*:

(i) $h'_{\mathrm{II}}(x, \theta(x, \sigma), \sigma)$ *and* $h''_{\mathrm{II}}(x, \theta(x, \sigma), \sigma)$ *are the eigenvector and the generalized eigenvector of* $M(x, \theta(x, \sigma), \sigma)$ *corresponding to* $\lambda^+_{\mathrm{II}}(x, \theta(x, \sigma), \sigma) = \lambda^-_{\mathrm{II}}(x, \theta(x, \sigma), \sigma)$, *respectively.*

(ii) $A(h'_{\mathrm{II}}, h''_{\mathrm{II}}) = (h'_{\mathrm{II}}, h''_{\mathrm{II}}) M_{\mathrm{II}}$.

(iii)

$$M_{\mathrm{II}}(x, \mu, \sigma) = \begin{pmatrix} \lambda_1(x, O, \sigma) & \Lambda_0(O, \sigma) \\ O & \lambda_1(x, O, \sigma) \end{pmatrix} + \mu \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} (x, \mu, \sigma) \tag{3.1}$$

where p_{ij} *are smooth in* (x, μ, σ), *analytic and real for real* (μ, σ) *and for* $\mu = O$

$$\Lambda_\gamma p_{21} = \lambda_2(x, \mu, \sigma)^2 > O \quad \text{in the case } (a), \text{ or} \tag{3.2}$$
$$\Lambda_\gamma p_{21} = -\lambda_2'(x, \mu, \sigma)^2 < O \quad \text{in the case } (b).$$

Furthermore, let $\lambda_1(x, \mu, \sigma) = \lambda_1(x, O, \sigma) + \lambda_1^{(1)}(x, \mu, \sigma)\mu$, *then*

$$\lambda_1^{(1)}(x, \mu, \sigma) = 2^{-1}(p_{11} + p_{22}). \tag{3.3}$$

PROOF. Since det $(E\lambda - A(x, \tau, \sigma))$ is x_0-strictly hyperbolic, there exist a real eigenvector $h_0(x, \theta(x, \sigma), \sigma)$ and a real generalized eigenvector $h_1(x, \theta(x, \sigma), \sigma)$ corresponding to the real double eigenvalue $\lambda_1(x, O, \sigma) = \lambda^{\pm}_{\mathrm{II}}(x, \theta(x, \sigma), \sigma)$ of $A(x, \theta(x, \sigma), \sigma)$. Note that h_0, h_1 are S^0 in (x, σ) and analytic in σ, because so is $\lambda_1(x, O, \sigma)$. Put

$$h''(x, \tau, \sigma) = \frac{1}{2\pi i}\oint_{c_{\mathrm{II}}} \left(E\lambda - A(x, \tau, \sigma)\right)^{-1} h_1\left(x, \theta(x, \sigma), \sigma\right) d\lambda$$

where c_{II} is a small circle enclosing only the eigenvalues $\lambda^{\pm}_{\mathrm{II}}(x, \tau, \sigma)$. Then it holds that

$$h''_{\mathrm{II}}\bigl(x, \theta(x, \sigma), \sigma\bigr) = h_1\bigl(x, \theta(x, \sigma), \sigma\bigr).$$

Hence if we set

$$\Lambda_\gamma h'_{\mathrm{II}}(x, \tau, \sigma) = \Bigl(\lambda^{\pm}_{\mathrm{II}}\bigl(x, \theta(x, \sigma), \sigma\bigr) - A(x, \tau, \sigma)\Bigr)\, h''_{\mathrm{II}}(x, \tau, \sigma),$$

(i) is fulfilled. Here we remark that $\{h'_{\mathrm{II}}, h''_{\mathrm{II}}\}$ are linearly independent vectors which are smooth in (x, τ, σ) and analytic in (τ, σ). Since $h'_{\mathrm{II}}(x, \tau, \sigma)$ and $h''_{\mathrm{II}}(x, \tau, \sigma)$ are invariant by the projection operator $\frac{1}{2\pi i}\oint_{C_{\mathrm{II}}}\bigl(E\lambda - A(x, \tau, \sigma)\bigr)^{-1} d\lambda$, we see that (ii) is valid.

From Lemma 3.1 $\lambda^{\pm}_{\mathrm{II}}(x, \tau, \sigma)$ is real if $\tau \geq \theta(x, \sigma)$. Hence $\{h'_{\mathrm{II}}, h''_{\mathrm{II}}\}$ are real if $\tau \geq \theta(x, \sigma)$. This together with the analyticity in τ of $\{h'_{\mathrm{II}}, h''_{\mathrm{II}}\}$, implies that they are always real when τ is real and $(x, \tau, \sigma) \in U(x^0) \times U(\tau^0, \sigma^0)$. Therefore it follows from (ii) that M_{II} is real. Since (i) and (ii) implies that defining $\Lambda_0^{(0)} \equiv \Lambda_0(O, \sigma) = (\theta(x, \sigma)^2 + |\sigma|^2)^{\frac{1}{2}}$

$$M_{\mathrm{II}} = \begin{pmatrix} \lambda_1(x, O, \sigma) & \Lambda_0^{(0)} \\ O & \lambda_1(x, O, \sigma) \end{pmatrix}$$

on $\mu = \tau - \theta(x, \sigma) = O$, we see that (3.1) is valid. Hence noting that $\lambda^{\pm}_{\mathrm{II}}$ are eigenvalues of M_{II} we have

$$\det\begin{pmatrix} \lambda - \lambda_1(x, O, \sigma) - p_{11}\mu, & -\Lambda_0^{(0)} - p_{12}\mu \\ -p_{21}\mu, & \lambda - \lambda_1(x, O, \sigma) - p_{22}\mu \end{pmatrix}$$
$$= \bigl(\lambda - \lambda^{+}_{\mathrm{II}}(x, \mu, \sigma)\bigr)\bigl(\lambda - \lambda^{-}_{\mathrm{II}}(x, \mu, \sigma)\bigr).$$

Put $\lambda = \lambda_1(x, O, \sigma)$ in the above equation. Then (3.2) follows from comparing the coefficient of μ. Write $\lambda - \lambda_1(x, O, \sigma) = \lambda - \lambda_1(x, \mu, \sigma) + \mu\, \lambda_1^{(1)}(x, \mu, \sigma)$ in the left hand side. Then (3.3) follows from comparing the coefficient of $(\lambda - \lambda_1(x, \mu, \sigma))$.

Now let $\tilde{S}(x, \tau, \sigma) = S(\tilde{x}(x), \tilde{\tau}(\tau\Lambda_\gamma^{-1}, \sigma\Lambda_\gamma^{-1}), \tilde{\sigma}(\tau\Lambda_\gamma^{-1}, \sigma\Lambda_\gamma^{-1}))$ be the extention of $S(x, \tau, \sigma)$ such that $\tilde{S}$ is non-singular and belongs to S^0_+, which is obtained by the method of §2. With fixed j and k put

$$P_1(x, \tau, \sigma, \lambda) = \tilde{S}^{-1}(x, \tau, \sigma)\, P^j_k(x, \tau, \sigma, \lambda)\, \tilde{S}(x, \tau, \sigma),$$
$$B_1(x', \tau, \sigma) = B_j(x')\, \tilde{S}(x', O, \tau, \sigma),$$
$$F(x) = S^{-1}(x, D')\, f(x),$$
$$U(x) = S^{-1}(x, D')\, u(x),$$

where $S^{-1}(x, D')$ is the operator with its symbol $\tilde{S}^{-1}(x, \tau, \sigma) \in S^0_+$. Then the problem (P^0, B) is reduced to the problem

$$(P_1, B_1) \begin{cases} P_1(x, D)\, U(x) = F(x) & \text{in } \boldsymbol{R}_+^{n+1}, \\ B_1(x', D)\, U(x', 0) = g(x') & \text{on } \boldsymbol{R}^n. \end{cases}$$

Noting that $\tilde{S}, \tilde{S}^{-1} \in S_+^0$, we have from corollary 2.3 of [8] and Lemma 2.1 (α) the following

LEMMA 3.3. *Let $\psi(\tau, \sigma)$ be ψ_k in Lemma 2.3. If there exist constants $C_1, \gamma_1 > 0$ such that for every $\gamma \geq \gamma_1$ and $U \in H_{1,\gamma}(R_+^{n+1})$*

$$(3.4) \qquad \left\| P_1 \psi(D')\, U \right\|_{0,\gamma} + \left| B_1 \psi(D')\, U \right|_{\frac{1}{2},\gamma} \geq C_1 \gamma \left\| \psi(D')\, U \right\|_{0,\gamma},$$

then there exist C, $\gamma_0 > 0$ such that (2.7) *holds.*

Let $\lambda_{\rm I}^{\pm}(x, D')$, $M_{\rm II}(x, D')$ and $M_{\rm III}^{\pm}(x, D')$ be the operators of order 1 with their symbols $\tilde{\lambda}_{\rm I}^{\pm}(x, \tau, \sigma)$, $\widetilde{\boldsymbol{M}}_{\rm II}(x, \tau, \sigma)$ and $\widetilde{\boldsymbol{M}}_{\rm III}^{\pm}(x, \tau, \sigma)$, respectively. Then we see from (P_1, B_1)

$$P_1(x, D)\, U = \left[E D_n - \begin{pmatrix} \lambda_{\rm I}^{+}(x, D') & & & & 0 \\ & \lambda_{\rm I}^{-}(x, D') & & & \\ & & M_{\rm II}(x, D') & & \\ & & & M_{\rm III}^{+}(x, D') & \\ 0 & & & & M_{\rm III}^{-}(x, D') \end{pmatrix} \right] U = F.$$

Hence putting

$$U(x) = {}^t({}^t u_{\rm I}^{+}, {}^t u_{\rm I}^{-}, u'_{\rm II}, u''_{\rm II}, {}^t u_{\rm III}^{+}, {}^t u_{\rm III}^{-})(x)$$
$$F(x) = {}^t({}^t f_{\rm I}^{+}, {}^t f_{\rm I}^{-}, f'_{\rm II}, f''_{\rm II}, {}^t f_{\rm III}^{+}, {}^t f_{\rm III}^{-})(x),$$

we have for $x \in R_+^{n+1}$

$$(P_{\rm I}^{\pm}) \qquad P_{\rm I}^{\pm}(x, D)\, u_{\rm I}^{\pm} = \left(E_{\rm I} D_n - \lambda_{\rm I}^{\pm}(x, D') \right) u_{\rm I}^{\pm}(x) = f_{\rm I}^{\pm}(x),$$

$$(P_{\rm II}) \qquad P_{\rm II}(x, D)\, u_{\rm II} = \left(E_{\rm II} D_n - M_{\rm II}(x, D') \right) \begin{pmatrix} u'_{\rm II} \\ u''_{\rm II} \end{pmatrix} = \begin{pmatrix} f'_{\rm II}(x) \\ f''_{\rm II}(x) \end{pmatrix},$$

$$(P_{\rm III}^{\pm}) \qquad P_{\rm III}^{\pm}(x, D)\, u_{\rm III}^{\pm} = \left(E_{\rm III} D_n - M_{\rm III}^{\pm}(x, D') \right) u_{\rm III}^{\pm} = f_{\rm III}^{\pm}(x),$$

furthermore for $x' \in \boldsymbol{R}^n$

$$(B_1) \qquad B_1 U = B^J(x')\, S(x', 0, D')\, U(x', 0) = g(x'),$$

where $E_{\rm I}, E_{\rm II}, E_{\rm III}$ are the unit matrices of order $l-1$, 2 and $m-l$, respectively. For each above problem we show a priori estimates in §5 and §7. Using them we show in §8 that the assumptions of Lemma 3.3 are fulfilled.

§ 4. Lopatinskii determinant, coupling coefficients and reflection coefficients.

4.1. Let $(x^0, \tau^0, \sigma^0) \in \Gamma \times \overline{\Sigma_-}$ and $S(x, \tau, \sigma)$ be the matrix defined it some $U(x^0) \times U(\tau^0, \sigma^0)$ which is defined in § 3. Put

$$(V_{\mathrm{I}}^+, V_{\mathrm{I}}^-, V_{\mathrm{II}}', V_{\mathrm{II}}'', V_{\mathrm{III}}^+, V_{\mathrm{III}}^-)(x', \tau, \sigma) = B(x') S(x', 0, \tau, \sigma)$$

where $V_{\mathrm{I}}^{\pm}$ are $m \times (l-1)$ matrices, V_{II}', V_{II}'' are m-vectors and $V_{\mathrm{III}}^{\pm}$ are $m \times (m-l)$ matrices.

Let $S_{\mathrm{II}}(x, \mu, \sigma) = \begin{pmatrix} 1 & 0 \\ s_{21}(x, \mu, \sigma) & 1 \end{pmatrix}$, where $s_{21} = (\lambda_{\mathrm{II}}^+(x, \mu, \sigma) - p_{11}(x, \mu, \sigma)\mu - \lambda_1(x, 0, \sigma))(\Lambda_0^{(0)} + p_{12}(x, \mu, \sigma)\mu)^{-1}$. Then S_{II} is continuous in $U(x^0) \times (U(\tau^0, \sigma^0) \cap (\bar{C}_- \times \boldsymbol{R}^{n-1}))$, because so is λ_{II}^+. Furthermore the first column of S_{II} is an eigenvector of M_{II} corresponding to λ_{II}^+, hence we see that

$$S_{\mathrm{II}}^{-1} M_{\mathrm{II}} S_{\mathrm{II}} = \begin{pmatrix} \lambda_{\mathrm{II}}^+ & \alpha \\ 0 & \lambda_{\mathrm{II}}^- \end{pmatrix}(x, \mu, \sigma)$$

with some $\alpha(x, \mu, \sigma)$ which is real for $\mu \geq 0$ and equal to $\Lambda_0^{(0)}$ for $\mu = 0$. Put

$$S_1(x, \mu, \sigma) = \begin{pmatrix} E_{\mathrm{I}} & & 0 \\ & S_{\mathrm{II}} & \\ 0 & & E_{\mathrm{III}} \end{pmatrix}$$

where E_{I}, E_{III} are the $2(l-1) \times 2(l-1)$, $2(m-l) \times 2(m-l)$ identity matrices, respectively. Then we have from Lemma 3.2

$$(4.1) \qquad (SS_1)^{-1} P(SS_1) = E\lambda - \begin{pmatrix} \lambda_{\mathrm{I}}^+ & & & & 0 \\ & \lambda_{\mathrm{I}}^- & & & \\ & & \begin{matrix} \lambda_{\mathrm{II}}^+ & \alpha \\ 0 & \lambda_{\mathrm{II}}^- \end{matrix} & & \\ & & & M_{\mathrm{III}}^+ & \\ 0 & & & & M_{\mathrm{III}}^- \end{pmatrix},$$

This means that the $(2l-1)$ th column of SS_1 is the eigenvector h_{II}^+ which corresponds to λ_{II}^+ and is homogeneous of degree 0 in (τ, σ). Hence putting $V_{\mathrm{II}}^+ = Bh_{\mathrm{II}}^+$ we have

$$BSS_1 = (V_{\mathrm{I}}^+, V_{\mathrm{I}}^-, V_{\mathrm{II}}^+, V_{\mathrm{II}}'', V_{\mathrm{III}}^+, V_{\mathrm{III}}^-).$$

DEFINITION 4.1. *Let $B^+(x', \tau, \sigma) = (V_{\mathrm{I}}^+, V_{\mathrm{II}}^+, V_{\mathrm{III}}^+)$ and $R(x', \tau, \sigma) = \det B^+$, then $B^+(x', \tau, \sigma)$ and $R(x', \tau, \sigma)$ are called Lopatinskii matrix and*

Lopatinskii determinant of the problem (P, B) *for* $(x', \tau, \sigma) \in U(x^0) \times (U(\tau^0, \sigma^0) \cap (\bar{C}_- \times R^{n-1}))$ *respectively.*

REMARK 4.1. $B^+(x', \tau, \sigma)$ and $R(x', \tau, \sigma)$ are S_+^0 in $U(x^0) \times (U(\tau^0, \sigma^0) \cap (C_- \times R^{n-1}))$, analytic in $(\tau, \sigma) \in C_- \times R^{n-1}$ and continuous in $U(x^0) \times (U(\tau^0, \sigma^0) \cap (\bar{C}_- \times R^n))$. Moreover, if the set II is empty they are S_+^0 in $U(x^0) \times (U(\tau^0, \sigma^0) \cap (\bar{C}_- \times R^{n-1}))$ and analytic in $(\tau, \sigma) \in U(\tau^0, \sigma^0) \cap (\bar{C}_- \times R^{n-1})$.

DEFINITION 4.2. *When* $R \neq 0$ *let us define a matrix*

$$(b_{ij}) = (V_{\mathrm{I}}^+, V_{\mathrm{II}}^+, V_{\mathrm{III}}^+)^{-1} (V_{\mathrm{I}}^-, V_{\mathrm{II}}'', V_{\mathrm{III}}^-)$$

and call the elements b_{ij} *coupling coefficients with respect to a matrix* (SS_1).

Let

$$T_{\mathrm{II}}(x, \mu, \sigma) = \begin{pmatrix} 1 & \alpha^{-1}(\lambda_{\mathrm{II}}^- - \lambda_{\mathrm{II}}^+) \\ O & 1 \end{pmatrix} = (T_{\mathrm{II}}^+, T_{\mathrm{II}}^-).$$

Then it holds that

$$T_{\mathrm{II}}^{-1} S_{\mathrm{II}}^{-1} M_{\mathrm{II}} S_{\mathrm{II}} T_{\mathrm{II}} = T_{\mathrm{II}}^{-1} \begin{pmatrix} \lambda_{\mathrm{II}}^+ & \alpha \\ O & \lambda_{\mathrm{II}}^- \end{pmatrix} T_{\mathrm{II}} = \begin{pmatrix} \lambda_{\mathrm{II}}^+ & O \\ O & \lambda_{\mathrm{II}}^- \end{pmatrix}.$$

Hence putting

$$S_2(x, \mu, \sigma) = \begin{pmatrix} E_{\mathrm{I}} & & 0 \\ & T_{\mathrm{II}} & \\ 0 & & E_{\mathrm{III}} \end{pmatrix},$$

we have that $(SS_1S_2)^{-1} P(SS_1S_2)$ is the matrix (4.1) with $\alpha = O$.

This means that the $2l$-th column of SS_1S_2 is the eigenvector h_{II}^- which corresponds to λ_{II}^- and is homogeneous of degree O in (τ, σ). Hence if we put $V_{\mathrm{II}}^- = Bh_{\mathrm{II}}^-$ it holds that

$$BSS_1S_2 = (V_{\mathrm{I}}^+, V_{\mathrm{I}}^-, V_{\mathrm{II}}^+, V_{\mathrm{II}}^-, V_{\mathrm{III}}^+, V_{\mathrm{III}}^-).$$

Let $(\widetilde{b_{ij}}) = (V_{\mathrm{I}}^+, V_{\mathrm{II}}^+, V_{\mathrm{III}}^+)^{-1} (V_{\mathrm{I}}^-, V_{\mathrm{II}}^-, V_{\mathrm{III}}^-)$ be the coupling coefficients with respect to a matrix SS_1S_2. Then we have the

DEFINITION 4.3. $\widetilde{b_{ij}}$ *is called (generalized) reflection coefficients with respect to a matrix* SS_1S_2, *by which* A *is similar to a generalized diagonal matrix such as* (4.1) *with* $\alpha = O$ ([3]).

The reflection coefficients depend, in general, on matrices S, S_1 and S_2. But since for fixed (x, τ, σ) the column vectors of SS_1S_2 may be considered as eigenvectors or generalised those of $A(x, \tau, \sigma)$, for another (generalized) eigenvectors $\{h_k^\pm\}$ and corresponding $\{V_k^\pm\}$, there exist non-zero numbers $\{a_k^\pm\}$ and a non-singular matrix T such that

$$V_j^{\pm}=a_j^{\pm}\widetilde{V}_j^{\pm} \quad \text{for} \quad j\in \mathrm{I}\cup\mathrm{II} \quad \text{and}$$
$$(V_{l+1}^{+}, \cdots, V_m^{+})=(\widetilde{V}_{l+1}^{+}, \cdots, \widetilde{V}_m^{+})\, T,$$

where $a_j^{\pm}$ are real if $V_j^{\pm}$ and $\widetilde{V}_j^{\pm}$ are real. Therefore the invariance of the reflection coefficients in the following sense is derived from the definition.

LEMMA 4.1. *For fixed $i, j\in I\cup II$ and (x', τ, σ), $\widetilde{b_{ij}}(x', \tau, \sigma)$ depends only on the vectors h_i^+ and h_j^-, where $SS_1S_2=(h_{\mathrm{I}}^+, h_{\mathrm{I}}^-, h_{\mathrm{II}}^+, h_{\mathrm{II}}^-, h_{\mathrm{III}}^+, h_{\mathrm{III}}^-)$. In particular, if for some point (x', τ, σ) the vectors $\{h_1^{\pm}, \cdots, h_l^{\pm}\}$ are restricted to be real, then $\widetilde{b_{ij}}(x', \tau, \sigma)$ is determined except real factor for every $i, j\in I\cup II$.*

REMARK 4.2. From the same consideration as in Lemma 4.1 we see that the zeroes of Lopatinskii determinant are not depend on the choice of S and S_1.

Hereafter, taking account of Lemma 4.1 we choose real vectors $\{h_{\mathrm{I}}^{\pm}\}$ and consider only real $h_{\mathrm{II}}^{\pm}$ whenever $\lambda_{\mathrm{II}}^{\pm}$ are real, i.e., $\mu\geqq 0$.

4.2. In this subsection we give necessary conditions for the condition (III) in terms of coupling coefficients. Note that these conditions are also sufficient for the constant coefficients problem $(P, B)_{x^0}$ to be L^2-well-posed.

THEOREM 4.1. *The condition (III) is equivalent to the following conditions (α) and (β):*

α) *For any $(x^0, \tau^0, \sigma^0)\in \Gamma\times(\overline{\Sigma_-}-\Sigma_-)$ there exist a positive constant C and a neighborhood $U(\tau^0, \sigma^0)$ such that for every $(\tau, \sigma)\in U(\tau^0, \sigma^0)\cap\Sigma_-$*

(i) *for $i, j=1, \cdots, l$*

$$\left|b_{ij}(x^0, \tau, \sigma)\right|\leq C\gamma^{-1}\left|\mathrm{Im}\, \lambda_i^+(x^0, \tau, \sigma)\cdot\mathrm{Im}\, \lambda_j^-(x^0, \tau, \sigma)\right|^{\frac{1}{2}},$$

(ii) *for $i=1, \cdots, l$ and $j\in III_-$*

$$\left|b_{ij}(x^0, \tau, \sigma)\right|\leq C\gamma^{-1}\left|\mathrm{Im}\, \lambda_i^+(x^0, \tau, \sigma)\right|^{\frac{1}{2}},$$

(iii) *for $i\in III_+$ and $j=1, \cdots, l$*

$$\left|b_{ij}(x^0, \tau, \sigma)\right|\leq C\gamma^{-1}\left|\mathrm{Im}\, \lambda_j^-(x^0, \tau, \sigma)\right|^{\frac{1}{2}},$$

(iv) *for $i\in III_+$ and $j\in III_-$*

$$\left|b_{ij}(x^0, \tau, \sigma)\right|\leq C\gamma^{-1}.$$

β) *for every $(x', \tau, \sigma)\in\Gamma\times \boldsymbol{C}_-\times\boldsymbol{R}^{n-1}$*

$$R(x', \tau, \sigma)\neq 0\,.$$

To show our assertion, first of all we recall a characterization of L^2-

well posedness for constant coefficients case [3], [4]. Let us consider the constant coefficients problem resulting from freezing the coefficients of $P(x, D)$ and $B(x')$ at boundary points $x=x^0$:

$$(P, B)_{x^0}\begin{cases} P(D)\,u(x)=(ED_n-\sum_{j=0}^{n-1}A_jD_j-C)\,u(x)=f(x) \\ \qquad\qquad\qquad\qquad \text{for } x\in \boldsymbol{R}_+^{n+1}(x_0>0), \\ Bu(x',0)=0 \qquad \text{for } x'\in \boldsymbol{R}^n(x_0>0), \\ u(0,x'',x_n)=0 \qquad \text{for } (x'',x_n)\in \boldsymbol{R}_+^n(x_0=0), \end{cases}$$

and the associated problem by Fourier-Laplace transformation with respect to (x'', x_0):

$$(P^0, B)_{x^0}\begin{cases} P^0(\tau,\sigma,D_n)\,\hat{u}(\tau,\sigma,x_n) \\ \quad =(ED_n-A_0\tau-\sum_{j=1}^{n-1}A_j\sigma_j)\,\hat{u} \\ \quad =\hat{f}(x_n) \qquad \text{for } x_n>0, \\ B\hat{u}(\tau,\sigma,0)=0. \end{cases}$$

Then the problem $(P, B)_{x^0}$ is L^2-well-posed, if and only if $(P^0, B)_{x^0}$ is so, that is, there exists a constant $C>0$ such that for every $(\tau,\sigma)\in \boldsymbol{C}_-\times\boldsymbol{R}^{n-1}$ and $\hat{f}(x_n)\in H_1(x_n>0)$, $(P^0, B)_{x^0}$ has a unique solution $\hat{u}(\tau,\sigma,x_n)\in H_1(x_n>0)$ which satisfies

$$\|\hat{u}(\tau,\sigma,\cdot)\|\leq C\gamma^{-1}\|\hat{f}(\cdot)\| \tag{4.2}$$

where

$$\|\hat{u}(\tau,\sigma,\cdot)\|^2=\int_0^\infty |\hat{u}(\tau,\sigma,x_n)|^2\,dx_n$$

Next to obtain its more concrete characterization, let $(\tau^0,\sigma^0)\in\overline{\Sigma_-}$ be an arbitrary point. For $(\tau,\sigma)\in U(\tau^0,\sigma^0)\cap\Sigma_-$ and the solution $\hat{u}(\tau,\sigma,x_n)$ of $(P^0, B)_{x^0}$ set

$$W(\tau,\sigma,x_n)=(SS_1)^{-1}(\tau,\sigma)\,\hat{u}(\tau,\sigma,x_n),$$

where

$$(SS_1)(\tau,\sigma)=(h_{\mathrm{I}}^+, h_{\mathrm{I}}^-, h_{\mathrm{II}}^+, h_{\mathrm{II}}'', h_{\mathrm{III}}^+, h_{\mathrm{III}}^-)(\tau,\sigma),$$
$$\det SS_1(\tau,\sigma)\neq 0 \quad \text{for} \quad (\tau,\sigma)\in U(\tau^0,\sigma^0)\cap\overline{\Sigma_-}.$$

If we put

$$W(\tau,\sigma,x_n)={}^t({}^tW_{\mathrm{I}}^+, {}^tW_{\mathrm{I}}^-, W_{\mathrm{II}}^+, W_{\mathrm{II}}'', {}^tW_{\mathrm{III}}^+, {}^tW_{\mathrm{III}}^-),$$
$$F(x_n)=(SS_1)^{-1}\hat{f}={}^t({}^tf_{\mathrm{I}}^+, {}^tf_{\mathrm{I}}^-, f_{\mathrm{II}}^+, f_{\mathrm{II}}'', {}^tf_{\mathrm{III}}^+, {}^tf_{\mathrm{III}}^-),$$

$(P^0, B)_{x^0}$ is reduced to problem as follows; for $(\tau,\sigma)\in U(\tau^0,\sigma^0)\cap\Sigma_-$

$$
(4.3)\quad \begin{cases} P_2W = \left[ED_n - \begin{pmatrix} \lambda_I^+ & & & & & 0 \\ & \lambda_I^- & & & & \\ & & \begin{array}{cc} \lambda_{II}^+ & \alpha \\ O & \lambda_{II}^- \end{array} & & & \\ & & & & M_{III}^+ & \\ 0 & & & & & M_{III}^- \end{pmatrix}(\tau, \sigma)\right] W(\tau, \sigma, x_n) \\ \quad = F(x_n), \qquad x_n > O, \\ B_2W = (BSS_1)W \\ \quad = (V_I^+, V_{II}^+, V_{III}^+)\cdot {}^t({}^tW_I^+, W_{II}^+, {}^tW_{III}^+)(\tau, \sigma, O) \\ \qquad + (V_I^-, V_{II}', V_{III}^-)\cdot {}^t({}^tW_I^-, W_{II}'', {}^tW_{III}^-)(\tau, \sigma, O) = O. \end{cases}
$$

Setting

$$
\begin{aligned} W(\tau, \sigma, x_n) &= {}^t({}^tW_{1,I}^+, {}^tW_{1,I}^-, W_{1,II}^+, W_{1,II}'', {}^tW_{1,III}^+, {}^tW_{1,III}^-) \\ &\quad + {}^t({}^tW_{2,I}^+, O, W_{2,II}^+, O, {}^tW_{2,III}^+, O) \\ &= (W_1 + W_2)(\tau, \sigma, x_n) \end{aligned}
$$

and

$$
\begin{aligned} &W_I^+(\tau, \sigma, x_n) = {}^t({}^tW_{1,I}^+, W_{1,II}^+, {}^tW_{1,III}), \\ &W_I^-(\tau, \sigma, x_n) = {}^t({}^tW_{1,I}^-, W_{1,II}'', {}^tW_{1,III}^-), \\ &W_2^+(\tau, \sigma, x_n) = {}^t({}^tW_{2,I}^+, W_{2,II}^+, {}^tW_{2,III}^+), \quad W_2^-(\tau, \sigma, x_n) = O, \end{aligned}
$$

we consider the following two problems:

$$
(4.4)\qquad \begin{cases} P_2(\tau, \sigma, D_n)\, W_1(\tau, \sigma, x_n) = F(x_n), \\ x_n > O, \quad W_I^+(\tau, \sigma, O) = O, \end{cases}
$$

and

$$
(4.5)\qquad \begin{cases} P_2(\tau, \sigma, D_n)\, W_2(\tau, \sigma, x_n) = O, \qquad x_n > O, \\ W_2^+(\tau, \sigma, O) + (b_{ij})(\tau, \sigma)\, W_I^-(\tau, \sigma, O) = O. \end{cases}
$$

Then we see from the uniqueness of the solution of (4.3) that $W_1 + W_2$ is the solution W of (4.3) for (τ, σ) satisfying $\det(V_I^+, V_{II}^+, V_{III}^+)(\tau, \sigma) = R \neq O$.

From (4.4) we have

$$
(4.6)\qquad \left[ED_n - \begin{pmatrix} \lambda_I^- & & 0 \\ & \lambda_{II}^- & \\ 0 & & M_{III}^- \end{pmatrix}(\tau, \sigma)\right] W_I^-(\tau, \sigma, x_n) = \begin{pmatrix} f_I^- \\ f_{II}'' \\ f_{III}^- \end{pmatrix}.
$$

Since $W_2^-(\tau, \sigma, x_n)=O$ we have from (4.5)

$$(4.7)\quad \begin{cases} \left[ED_n - \begin{pmatrix} \lambda_{\mathrm{I}}^+ & & 0 \\ & \lambda_{\mathrm{II}}^+ & \\ O & & M_{\mathrm{III}}^+ \end{pmatrix}\right] W_2^+(\tau, \sigma, x_n) = O\,, \qquad x_n > O\,, \\ W_2^+ + (b_{ij}) W_1^-)(\tau, \sigma, O) = O\,. \end{cases}$$

From (4.6) the problem (4.7) has a unique tempered solution for $\mathrm{Im}\,\tau < O$:

$$(4.8)\quad \begin{aligned} &W_2^+(\tau, \sigma, x_n) \\ &\quad = -\exp(iM^+(\tau, \sigma)x_n)\cdot(b_{ij})(\tau, \sigma)\cdot W_1^-(\tau, \sigma, O) \\ &\quad = -\exp(iM^+(\tau, \sigma)x_n)\cdot(b_{ij})(\tau, \sigma) \\ &\qquad \cdot \int_0^\infty \exp(-iM^-(\tau, \sigma)x_n) f''(x_n)\, dx_n\,, \end{aligned}$$

where

$$M^{\pm}(\tau, \sigma) = \begin{pmatrix} \lambda_{\mathrm{I}}^{\pm} & & O \\ & \lambda_{\mathrm{II}}^{\pm} & \\ O & & M_{\mathrm{III}}^{\pm} \end{pmatrix}, \qquad f'' = \begin{pmatrix} f_{\mathrm{I}}^- \\ f_{\mathrm{II}}'' \\ f_{\mathrm{III}}^- \end{pmatrix}.$$

On the other hand, let us put

$$\widetilde{W}_{1,\mathrm{II}} = \begin{pmatrix} \widetilde{W}_{1,\mathrm{II}}^+ \\ \widetilde{W}_{1,\mathrm{II}}'' \end{pmatrix} = S_{\mathrm{II}} \begin{pmatrix} W_{1,\mathrm{II}}^+ \\ W_{1,\mathrm{II}}'' \end{pmatrix} = \begin{pmatrix} W_{1,\mathrm{II}}^+ \\ s_{21} W_{1,\mathrm{II}}^+ + W_{1,\mathrm{II}}'' \end{pmatrix},$$

$$g_{\mathrm{II}} = S_{\mathrm{II}}^{-1} f_{\mathrm{II}}, \qquad f_{\mathrm{II}} = {}^t(f_{\mathrm{II}}^+, f_{\mathrm{II}}'').$$

Then using the relation $S_{\mathrm{II}} \begin{pmatrix} \lambda_{\mathrm{II}}^+ & \alpha \\ O & \lambda_{\mathrm{II}}^- \end{pmatrix} S_{\mathrm{II}}^{-1} = M_{\mathrm{II}}$, we see that the problem (4.4) becomes for $x_n > O$,

$$\begin{aligned} &(P_{\mathrm{I}}^+)_{x^0} && P_{\mathrm{I}}^+ W_{1,\mathrm{I}}^+ = f_{\mathrm{I}}^+, \\ &(P_{\mathrm{I}}^-)_{x^0} && P_{\mathrm{I}}^- W_{1,\mathrm{I}}^- = f_{\mathrm{I}}^-, \\ &(P_{\mathrm{II}})_{x^0} && P_{\mathrm{II}} \cdot {}^t(\widetilde{W}_{1,\mathrm{II}}^+, \widetilde{W}_{1,\mathrm{II}}'') = g_{\mathrm{II}}, \\ &(P_{\mathrm{III}}^+)_{x^0} && P_{\mathrm{III}}^+ W_{1,\mathrm{III}}^+ = f_{\mathrm{III}}^+, \\ &(P_{\mathrm{III}}^-)_{x^0} && P_{\mathrm{III}}^- W_{1,\mathrm{III}}^- = f_{\mathrm{III}}^-, \end{aligned}$$

and

$$W_{1,\mathrm{I}}^+(\tau, \sigma, O) = \widetilde{W}_{1,\mathrm{II}}^+(\tau, \sigma, O) = W_{1,\mathrm{III}}^+(\tau, \sigma, O) = O\,.$$

Hence from Corollary 5.1 and 7.1 (ii) described later we have

$$\|f_{\mathrm{I}}^{+}\| \geq C\gamma \| W_{1,\mathrm{I}}^{+}\|, \qquad \|f_{\mathrm{I}}^{-}\| \geq C\gamma \| W_{1,\mathrm{I}}^{-}\|,$$
$$\|g_{\mathrm{II}}\| \geq C\gamma \| \widetilde{W}_{1,\mathrm{II}}\|, \text{ i. e., } \|f_{\mathrm{II}}\| \geq C\gamma \left(\| W_{1,\mathrm{II}}^{+}\| + \| W_{1,\mathrm{II}}''\|\right),$$
$$\|f_{\mathrm{III}}^{+}\| \geq C\gamma \| W_{1,\mathrm{III}}^{+}\|, \qquad \|f_{\mathrm{III}}^{-}\| \geq C\gamma \| W_{1,\mathrm{III}}^{-}\|,$$

and hence

$$\left\|W_1(\tau, \sigma, \cdot)\right\| \leq C\gamma^{-1}\left\|F(\cdot)\right\| \leq C\gamma^{-1}\left\|\hat{f}(\cdot)\right\|$$

for $(\tau, \sigma) \in U(\tau^0, \sigma^0) \cap \Sigma_-$. Therefore from (4. 2) and $W_{\mathrm{I}} + W_2 = SS_{\mathrm{I}}^{-1}\hat{u}$ it follows that for $(\tau, \sigma) \in U(\tau^0, \sigma^0) \cap \Sigma_-$ such that $R(\tau, \sigma) \neq O$,

$$\left\|W_2(\tau, \sigma, \cdot)\right\| \leq C\gamma^{-1}\left\|\hat{f}(\cdot)\right\|. \tag{4. 9}$$

We see from the proof that (4. 9) is also sufficient for L^2-well-posedness of $(P, B)_{x^0}$.

Put

$$G_c(x, y\,;\, \tau, \sigma) = e^{iM^+(\tau,\sigma)x}\,(b_{ij})\,(\tau, \sigma)\, e^{-iM^-(\tau,\sigma)y}$$

and note that

$$\left\|f''(\cdot)\right\|^2 \leq C\left\|\hat{f}(\cdot)\right\|^2$$
$$= C\left(\left\|f''(\cdot)\right\|^2 + \left\|f_{\mathrm{I}}^{+}(\cdot)\right\|^2 + \left\|f_{\mathrm{II}}^{+}(\cdot)\right|^2 + \left\|f_{\mathrm{III}}^{+}(\cdot)\right\|^2\right).$$

Then, from (4. 8) and (4. 9) we obtain the following

LEMMA 4. 2. *The problem $(P, B)_{x^0}$ is L^2-well-posed if and only if the following $\alpha)'$ and $\beta)'$ are fulfilled*:

$\alpha)'$ *For every $(x^0, \tau^0, \sigma^0) \in \Gamma \times (\overline{\Sigma_-} - \Sigma_-)$ there exist a constant $C > 0$ and a neighborhood $U(\tau^0, \sigma^0)$ such that for every $(\tau, \sigma) \in U(\tau^0, \sigma^0) \cap \Sigma_-$*

$$\int_0^\infty \left|\int_0^\infty G_c(x, y\,;\, \tau, \sigma)\, f''(y)\, dy\right|^2 dx \leq C^2\gamma^{-2}\int_0^\infty \left|f''(x)\right|^2 dx,$$

i. e.,

$$\left\|G_c(x, y\,;\, \tau, \sigma)\right\|_{\mathscr{L}(L^2, L^2)} \leq C\gamma^{-1}. \tag{4. 10}$$

$\beta)'$ *It holds for every $(\tau, \sigma) \in \boldsymbol{C}_- \times \boldsymbol{R}^{n-1}$ that*

$$R(\tau, \sigma) \neq O.$$

Thus, in order to prove Theorem 4. 1 we have only to show the equivalence of the conditions $\alpha)$ and $\alpha)'$. We use the same technique as in [10].

First note that it holds that

$$\|G_c\|_{\mathscr{L}(L^2 \times L^2, C^1)} \leq \|G_c\|_{\mathscr{L}(L^2, L^2)} \leq \left\|G_c(x, y)\right\|_{x,y}. \tag{4. 11}$$

Put

$$N_{\pm}=\begin{Bmatrix}|\operatorname{Im}\lambda_{\mathrm{I}}^{\pm}|^{-\frac{1}{2}} & & 0\\ & |\operatorname{Im}\lambda_{\mathrm{II}}^{\pm}|^{-\frac{1}{2}} & \\ 0 & & E_{\mathrm{III}}^{\pm}\end{Bmatrix},$$

$$L_{+}(x)=N_{+}^{-1}\exp\left(i^{t}(M^{+})x\right),$$

$$L_{-}(x)=N_{-}^{-1}\exp(-iM^{-}x),$$

$$S_{+}=\int_{0}^{\infty}\overline{L_{+}(x)}\cdot{}^{t}L_{+}(x)\,dx,$$

$$S_{-}=\int_{0}^{\infty}L_{-}(y)\cdot\overline{{}^{t}L_{-}(y)}\,dy.$$

Then we have

$$\begin{aligned}&S_{+}N_{+}(b_{ij})N_{-}S_{-}\\&\quad=\int_{0}^{\infty}\int_{0}^{\infty}\overline{L_{+}(x)}\cdot{}^{t}L_{+}(x)\,(N_{+}(b_{ij})N_{-})\,L_{-}(y)\cdot\overline{{}^{t}L_{-}(y)}\,dx\,dy\\&\quad=\int_{0}^{\infty}\overline{L_{+}(x)}\,G_{c}(x,y)\cdot\overline{{}^{t}L_{-}(y)}\,dx\,dy.\end{aligned}$$

Hence

$$\begin{aligned}&\left|S_{+}N_{+}(b_{ij})N_{-}S_{-}\right|\\&\quad\leq\sup_{f,g\in L^{2}}\left|\int_{0}^{\infty}\int_{0}^{\infty}\left\langle f(x),\,G_{c}(x,y)\,g(y)\right\rangle dx\,dy\right|\\&\quad\leq C\|G_{c}\|_{\mathcal{L}(L^{2}\times L^{2},C^{1})},\end{aligned}\tag{4.12}$$

where $|\cdot|$ denotes matrix norm and $C>0$. On the other hand it holds that

$$\begin{aligned}S_{+}&=N_{+}^{-1}\int_{0}^{\infty}\exp\left\{i\left((M^{+})-\left((M^{+})\right)^{*}\right)x\right\}dx\,N_{+}^{-1}\\&=N_{+}^{-1}\begin{pmatrix}(2\operatorname{Im}\lambda_{\mathrm{I}}^{+})^{-1} & & 0\\ & (2\operatorname{Im}\lambda_{\mathrm{II}}^{+})^{-1} & \\ 0 & & \int_{0}^{\infty}\exp\left(-2\operatorname{Im}M_{\mathrm{III}}^{+}\cdot x\right)dx\end{pmatrix}N_{+}^{-1}.\end{aligned}$$

Since we may assume that $\operatorname{Im}{}^{t}(M_{\mathrm{III}}^{+})\geq C>0$, there exist constants $C_1, C_2>0$ such that for $(\tau,\sigma)\in U(\tau^{0},\sigma^{0})\cap\Sigma_{-}$

$$\begin{aligned}&C_{1}\leq\bar{S}_{+}=S_{+}\leq C_{2},\\&C_{1}\leq S_{-}\leq C_{2},\end{aligned}\tag{4.13}$$

where the lower inequalities follow from the same calculation. Combining (4.12) and (4.13) we obtain

(4. 14) $$\left|N_+(b_{ij})N_-\right| \leq C\|G_c\|_{\mathscr{L}(L^2\times L^2, C^1)}.$$

On the other hand we have the estimates:

$$\begin{aligned}
&\left\|G_c(x, y)\right\|_{x,y} \\
&= \int\left|\exp(iM^+x)(b_{ij})\exp(-iM^-y)\right|^2 dx\,dy \\
&\leq \int\left|\exp(iM^+x)N_+^{-1}\right|^2 dx\cdot\int\left|N_-^{-1}\exp(-iM^-y)\right|^2 dy\cdot\left|N_+(b_{ij})N_-\right|^2 \\
&\leq C\left|N_+(b_{ij})N_-\right|^2,
\end{aligned}$$

where the last inequality holds for every $(\tau, \sigma)\in U(\tau^0, \sigma^0)\cap\Sigma_-$ with some $C>0$. Combining this with (4. 11) and (4. 14) we see that there exist constants $C_1, C_2>0$ such that for every $(\tau, \sigma)\in U(\tau^0, \sigma^0)\cap\Sigma_-$

$$C_1\left|N_+(b_{ij})N_-\right| \leq \|G_c\|_{\mathscr{L}(L^2, L^2)} \leq C_2\left|N_+(b_{ij})N_-\right|.$$

Hence (4. 10) is equivalent to

$$\left|N_+(b_{ij})N_-\right| \leq C\gamma^{-1},$$

which is equivalent to α). Thus the theorem is proved.

§ 5. A Priori estimates for the case where the set II is empty.

In this section we discuss the decomposed problem (P_1, B_1) in § 3 in the case where the set II is empty for the point $(x^0, \tau^0, \sigma^0)\in\Gamma\times\boldsymbol{R}^n$ being considered. In subsection 5. 1 a priori estimates for the problem $(P_{\mathrm{I}}^{\pm})$ and $(P_{\mathrm{III}}^{\pm})$ are given and subsection 5. 2 is devoted to that for the boundary condition (B_1).

5. 1. For the sake of completeness of our proof of (3. 4) we prove the following usual

LEMMA 5. 1. *There exist constants $C, \gamma_0>0$ such that for every $\gamma\geq\gamma_0$ it holds that*

$$\begin{aligned}
&(P_{\mathrm{I}}^+) && \|P_{\mathrm{I}}^+u_{\mathrm{I}}^+\|_{0,\gamma}+\gamma^{\frac{1}{2}}|u_{\mathrm{I}}^+|_{0,\gamma}\geq C\gamma\|u_{\mathrm{I}}^+\|_{0,\gamma}, \\
&(P_{\mathrm{I}}^-) && \|P_{\mathrm{I}}^-u_{\mathrm{I}}^-\|_{0,\gamma}\geq C\left(\gamma^{\frac{1}{2}}|u_{\mathrm{I}}^-|_{0,\gamma}+\gamma\|u_{\mathrm{I}}^-\|_{0,\gamma}\right), \\
&(P_{\mathrm{III}}^+) && \|P_{\mathrm{III}}^+u_{\mathrm{III}}^+\|_{0,\gamma}+\gamma|u_{\mathrm{III}}^+|_{-\frac{1}{2},\gamma}\geq C\gamma\|u_{\mathrm{III}}^+\|_{0,\gamma}, \\
&(P_{\mathrm{III}}^-) && \|P_{\mathrm{III}}^-u_{\mathrm{III}}^-\|_{0,\gamma}\geq C\left(|u_{\mathrm{III}}^-|_{\frac{1}{2},\gamma}+\|u_{\mathrm{III}}^-\|_{1,\gamma}\right),
\end{aligned}$$

for every $u_{\mathrm{I}}^{\pm}, u_{\mathrm{III}}^{\pm}\in H_{1,\gamma}(\boldsymbol{R}_+^{n+1})$.

Now we define the norms $\|u(\cdot)\|$ and $|u|$ for $u(x_u)\in H_1(\boldsymbol{R}^1_+)$ by

$$\|u(\cdot)\|^2=\int_0^\infty\left|u(x_n)\right|^2dx_n,\quad |u|=\left|u(0)\right|$$

with their inner products denoted by $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$, respectively. Then we have the following estimates for constant coefficients problems, which are direct consequences of the proof of Lemma 5.1. and will be used in the proofs of Theorem 4.1 and Lemma 6.3.

COROLLARY 5.1. *There exist constants $C,\gamma_0>0$ and a neighborhood $U(\tau^0,\sigma^0)$ such that for every $(\tau\Lambda_\gamma^{-1},\sigma\Lambda_\gamma^{-1})\in U(\tau^0,\sigma^0)\cap\Sigma_-$ and $\gamma\geq\gamma_0$*

$$(P_{\mathrm{I}}^+)_{x^0}\quad \left\|P_{\mathrm{I}}^+(x^0,\tau,\sigma,D_n)\hat{u}_{\mathrm{I}}^+(\cdot)\right\|+\gamma^{\frac{1}{2}}|\hat{u}_{\mathrm{I}}^+|\geq C\gamma\left\|\hat{u}_{\mathrm{I}}^+(\cdot)\right\|,$$

$$(P_{\mathrm{I}}^-)_{x^0}\quad \left\|P_{\mathrm{I}}^-(x^0,\tau,\sigma,D_n)\hat{u}_{\mathrm{I}}^-(\cdot)\right\|\geq C\left(\gamma^{\frac{1}{2}}|\hat{u}_{\mathrm{I}}^-|+\gamma\left\|\hat{u}_{\mathrm{I}}^-(\cdot)\right\|\right),$$

$$(P_{\mathrm{III}}^+)_{x^0}\quad \left\|P_{\mathrm{III}}^+(x^0,\tau,\sigma,D_n)\hat{u}_{\mathrm{III}}^+(\cdot)\right\|+\gamma\Lambda_\gamma^{-\frac{1}{2}}|u_{\mathrm{III}}^+|\geq C\gamma\left\|u_{\mathrm{III}}^+(\cdot)\right\|,$$

$$(P_{\mathrm{III}}^-)_{x^0}\quad \left\|P_{\mathrm{III}}^-(x^0,\tau,\sigma,D_n)\hat{u}_{\mathrm{III}}^-(\cdot)\right\|\geq C\left(\Lambda_\gamma^{\frac{1}{2}}|u_{\mathrm{III}}^-|+\Lambda_\gamma\left\|u_{\mathrm{III}}^-(\cdot)\right\|\right),$$

for every $\hat{u}_{\mathrm{I}}^\pm(x_n)$, $\hat{u}_{\mathrm{III}}^\pm(x_n)\in H_1(\boldsymbol{R}^1_+)$.

PROOF OF LEMMA 5.1. First we prove the estimates $(P_{\mathrm{I}}^\pm)$. Let us consider the bilinear form for $j\in I$ and $u=u_j^\pm$:

$$2\,\mathrm{Re}\left(\left(D_n-\lambda_j^\pm(x,D')\right)u,\ \mp iu\right)_{0,\gamma}$$
$$=\mathrm{Re}\,\langle\mp u,u\rangle_{0,\gamma}+2\,\mathrm{Re}\left(\mp i\lambda_j^\pm(x,D')\,u,u\right)_{0,\gamma}.$$

There exists a constant $C_1>0$ such that for any $(x,\tau',\sigma')\in U(x^0)\times(U(\tau^0,\sigma^0)\cap\overline{\Sigma_-})$

$$\mp\mathrm{Re}\,i\lambda_j^\pm(x,\tau',\sigma')\geq C_1\gamma'.$$

Hence from (2.6)′ this is also valid for the symbol of $\mp i\lambda_j^\pm(x,D')$:

$$\begin{aligned}&\mp\mathrm{Re}\,i\lambda_j^\pm(\tilde{x},\tilde{\eta}-i\tilde{\gamma},\tilde{\sigma})\cdot\Lambda_\gamma\geq C_1\tilde{\gamma}\Lambda_\gamma\\&\quad\begin{cases}=C_1(\gamma\Lambda_\gamma^{-1})\,\Lambda_\gamma=C_1\gamma, & \text{for } \gamma\Lambda_\gamma^{-1}\leq 2\varepsilon_0,\\ \geq C_12\varepsilon_0\Lambda_\gamma\geq 2\varepsilon_0C_1\gamma, & \text{for } \gamma\Lambda_\gamma^{-1}\geq 2\varepsilon_0,\end{cases}\\&\quad\geq C\gamma\end{aligned}$$

for every $(x,\tau,\sigma)\in\overline{\boldsymbol{R}^{n+1}_+\times\boldsymbol{C}_-\times\boldsymbol{R}^{n-1}}$. Using Lemma 2.1 β) (v) we have

$$\left\|\left(D_n-\lambda_j^\pm(x,D')\right)u\right\|_{0,\gamma}\cdot\|u\|_{0,\gamma}\geq C\left(\mp|u|_{0,\gamma}^2+\gamma\|u\|_{0,\gamma}^2\right)$$

for large γ. Accordingly for any $\delta>0$ we have

$$\frac{1}{\delta\gamma}\left\|\left(D_n-\lambda_i^{\pm}(x,D')\right)u\right\|_{0,\gamma}^2+\delta\gamma\|u\|_{0,\gamma}^2$$
$$\geq C\left(\mp|u|_{0,\gamma}^2+\gamma\|u\|_{0,\gamma}^2\right).$$

Choosing δ sufficiently small we obtain the inequalities $(P_{\mathrm{I}}^{\pm})$.

Next we prove the estimates $(P_{\mathrm{III}}^{\pm})$. Consider the bilinear form for $u=u_{\mathrm{III}}^{\pm}$:

$$2\,\mathrm{Re}\,(P_{\mathrm{III}}^{\pm}(x,D)\,u,\ \mp i\Lambda u)_{0,\gamma}$$
$$=\mp\,\mathrm{Re}\,\langle u,\Lambda u\rangle_{0,\gamma}+2\,\mathrm{Re}\left(\mp iM_{\mathrm{III}}^{\pm}(x,D')\,u,\ \Lambda u\right)_{0,\gamma}$$

where $\Lambda^s u=(2\pi)^{-n}\int_{R^n}e^{i\xi'x'}|\xi'|^s\,\hat{u}(\xi',x_n)\,d\xi'$ and the symbol $\widetilde{\mp iM_{\mathrm{III}}^{\pm}}(x,\tau,\sigma)$ of $\mp iM_{\mathrm{III}}^{\pm}(x,D')$ is

$$\mp iM_{\mathrm{III}}^{\pm}(\tilde{x},\tilde{\tau},\tilde{\sigma})\cdot\Lambda_\gamma\,.$$

Since $\mathrm{Im}\,\tau^0\leq-3\varepsilon_0$ or $\mathrm{Im}\,\tau^0=0$, taking sufficiently small $U(\tau_0,\sigma_0)$ we may assume

$$\pm i\left(M_{\mathrm{III}}^{\pm}-(M_{\mathrm{III}}^{\pm})^*\right)(x,\tau',\sigma')\geq C\qquad\text{with}\quad C>0\,,$$

for every $(x,\tau',\sigma')\in U(x^0)\times(U(\tau^0,\sigma^0)\cap\overline{\Sigma_-})$.

If we are considering such a point $(x^0,\tau^0,\sigma^0)\in\Gamma\times\overline{\Sigma_-}$ that $\mathrm{Im}\,\tau_0\leq-3\varepsilon_0$, then the Hermite part of $\mp iM_{\mathrm{III}}^{\pm}(\tilde{x},\tilde{\tau},\tilde{\sigma})$ is

$$\mp i\left(M_{\mathrm{III}}^{\pm}-(M_{\mathrm{III}}^{\pm})^*\right)\left(\tilde{x}(x),\tilde{\tau}(\tau\Lambda_\gamma^{-1},\sigma\Lambda_\gamma^{-1}),\tilde{\sigma}(\tau\Lambda_\gamma^{-1},\sigma\Lambda_\gamma^{-1})\right)\geq C\tag{5.1.1}$$

for every $(x,\tau,\sigma)\in\overline{\boldsymbol{R}_+^{n+1}}\times\overline{\boldsymbol{C}_+}\times\boldsymbol{R}^{n-1}$, because from (2.4)$'$ we have $(\tilde{\tau},\tilde{\sigma})\in S_{\frac{5}{2}\varepsilon_0}(\tau^0,\sigma^0)\subset U(\tau^0,\sigma^0)$ and then $\mathrm{Im}\,\tilde{\tau}\leq-\frac{1}{2}\varepsilon_0$, which is one reason for different constructions of A_k^j in §2.

If a point $(x^0,\tau^0,\sigma^0)\in\Gamma\times\overline{\Sigma_+}$ with $\mathrm{Im}\,\tau^0=0$ is being considered, then (5.1.1) is also valid, because from (2.6$'$) we have $(\tilde{\tau},\tilde{\sigma})\in S_{8\varepsilon_0}(\tau^0,\sigma^0)\subset U(\tau^0,\sigma^0)$ for small ε_0.

Since $\mp iM_{\mathrm{III}}^{\pm}(\tilde{x},\tilde{\tau},\tilde{\sigma})\in S_+^0$ we obtain from Lemma 2.1 β) (i)

$$\frac{1}{\delta}\|P_{\mathrm{III}}^{\pm}u\|_{0,\gamma}^2\geq\mp|u|_{\frac{1}{2},\gamma}^2+C\|\Lambda u\|_{0,\gamma}^2\qquad\text{with}\quad C>0$$

if $\delta>0$ is chosen sufficiently small. On the other hand

$$D_n u=P_{\mathrm{III}}^{\pm}u+M_{\mathrm{III}}^{\pm}u\,.$$

Hence it holds that

$$\|D_n u\|_{0,\gamma}^2\leq C\left(\|P_{\mathrm{III}}^{\pm}u\|_{0,\gamma}^2+\|\Lambda u\|_{0,\gamma}^2\right).$$

Thus we obtain

$$\|P^{+}_{\mathrm{III}}u\|^2_{0,\gamma}+|u|^2_{\frac{1}{2},\gamma}\geq C\|u\|^2_{1,\gamma}$$

and

$$\|P^{-}_{\mathrm{III}}u\|^2_{0,\gamma}\geq C\left(\|u\|^2_{1,\gamma}+|u|^2_{\frac{1}{2},\gamma}\right).$$

This proves (P^{-}_{III}).

Let us show the inequality (P^{+}_{III}). Consider the bilinear form

$$\begin{aligned}
&2\mathrm{Re}\left(P^{\pm}_{\mathrm{III}}(x, D)\,u,\ \mp i\Lambda^{-1}u\right)_{0,\gamma}\\
&\quad=\mathrm{Re}\,\langle u,\ \mp\Lambda^{-1}u\rangle_{0,\gamma}+2\mathrm{Re}\left(\mp iM^{\pm}_{\mathrm{III}}(x, D')\,u,\ \Lambda^{-1}u\right)_{0,\gamma}\\
&\quad=\mathrm{Re}\,\langle u,\ \mp\Lambda^{-1}u\rangle_{0,\gamma}+2\mathrm{Re}\left(\mp iM^{\pm}_{\mathrm{III}}(x, D')\,\Lambda^{-1}u,\ u\right)_{0,\gamma}\\
&\quad+2\mathrm{Re}\left([\mp iM^{\pm}_{\mathrm{III}},\ \Lambda^{-1}]\,u,\ u\right)_{0,\gamma}.
\end{aligned}$$

Then by the same way as in the proof of (P^{-}_{III}), we see that

$$\begin{aligned}
&\delta^{-1}\gamma^{-2}\|P^{\pm}_{\mathrm{III}}u\|^2_{0,\gamma}+\delta\gamma^2\|\Lambda^{-1}u\|^2_{0,\gamma}\\
&\qquad\geq\mp|u|^2_{-\frac{1}{2},\gamma}+C\|u\|^2_{0,\gamma}\qquad\text{with}\quad C<0.
\end{aligned}$$

Since $\gamma^2\|\Lambda^{-1}u\|^2_{0,\gamma}\leq\|u\|^2_{0,\gamma}$ we obtain (P^{+}_{III}) and complete the proof of Lemma 5.1.

5.2. Let us consider a priori estimates for the boundary condition (B_1) in the case where the set II is empty. If we use the notations in §4 (B_1) is rewritten by

$$\begin{aligned}
(B_1)\qquad&(V^{+}_{\mathrm{I}},\ V^{-}_{\mathrm{I}},\ V^{+}_{\mathrm{III}},\ V^{-}_{\mathrm{III}})(x', D')\cdot{}^t({}^tu^{+}_{\mathrm{I}},\ {}^tu^{-}_{\mathrm{I}},\ {}^tu^{+}_{\mathrm{III}},\ {}^tu^{-}_{\mathrm{III}})(x', 0)\\
&=g(x')\qquad\text{in}\quad \boldsymbol{R}^n,
\end{aligned}$$

where the symbol of $(V^{+}_{\mathrm{I}},\ V^{-}_{\mathrm{I}},\ V^{+}_{\mathrm{III}},\ V^{-}_{\mathrm{III}})(x', D')$ is $(V^{+}_{\mathrm{I}},\ V^{-}_{\mathrm{I}},\ V^{+}_{\mathrm{III}},\ V^{-}_{\mathrm{III}})(\tilde{x}, \tilde{\tau}, \tilde{\sigma})$. Note that in this case it holds that

$$\begin{aligned}
&B^{+}(x', \tau, \sigma)=(V^{+}_{\mathrm{I}},\ V^{+}_{\mathrm{III}})\\
&R(x', \tau, \sigma)=\det(V^{+}_{\mathrm{I}},\ V^{+}_{\mathrm{III}}),
\end{aligned}$$

hence $B^{+}(x', \tau, \sigma)$ and $R(x', \tau, \sigma)$ are in $S^0_{+}(U(x^0)\times U(\tau^0, \sigma^0))$ and analytic in $(\tau, \sigma)\in U(\tau^0, \sigma^0)\cap(\bar{\boldsymbol{C}}_{-}\times\boldsymbol{R}^{n-1})$. Then an a priori estimate for (B_1) is given by the following

THEOREM 5.1. *Assume the conditions (II) α) and (III). If the set II is empty, there exist constants $C, \gamma_0>0$ such that for every $\gamma\geq\gamma_0$ and $v^{+}(x')={}^t({}^tv^{+}_{\mathrm{I}},\ {}^tv^{+}_{\mathrm{III}})\in H_{\frac{1}{2},\gamma}(\boldsymbol{R}^n)$*

$$\begin{aligned}
(B^{+})\qquad&\left|B^{+}(x', D')\,v^{+}(x')\right|_{\frac{1}{2},\gamma}\\
&\geq C\left(\gamma^{\frac{1}{2}}|v^{+}_{\mathrm{I}}|_{0,\gamma}+\gamma\,|v^{+}_{\mathrm{III}}|_{-\frac{1}{2},\gamma}\right).
\end{aligned}$$

PROOF OF THEOREM 5. 1. If $R(x^0, \tau^0, \sigma^0) \neq 0$ the theorem is a direct consequence of Lemma 2. 1 β) (ii). Hence we may assume from the condition (III) and Theorem 4. 1 that $R(x^0, \tau^0, \sigma^0)=0$ for $(x^0, \tau^0, \sigma^0) \in \Gamma \times \boldsymbol{R}^n$. Since $R(x^0, \tau, \sigma)$ is analytic in τ, the condition (II) α) implies

$$C\gamma \leq \left|R(x^0, \tau^0 - i\gamma, \sigma^0)\right| \leq C'\gamma, \tag{5. 1}$$

with some $C, C'>0$. Hence we have

$$\frac{\partial R}{\partial \tau}(x^0, \tau^0, \sigma^0) \neq 0. \tag{5. 2}$$

Since $R(x', \tau, \sigma)$ is C^∞ in (x', τ, σ), analytic in (τ, σ) and homogenous of degree 0 in (τ, σ), we have from the implicit function theorem

$$R(x', \tau, \sigma) = \left(\tau - \nu(x', \sigma)\right) \cdot c_0(x', \tau, \sigma) \tag{5. 3}$$

in some $U(x^0) \times U(\tau^0, \sigma^0)$, where $\nu = \nu_1 + i\nu_2$ and c_0 are smooth, $c_0(x^0, \tau^0, \sigma^0) \neq 0$, $\nu_1(x', \sigma)$ and $\nu_2(x', \sigma)$ are homogeneous of degree 1 in σ and $c_0(x', \tau, \sigma)$ is homogeneous of degree -1 in (τ, σ). From Theorem 4. 1 it holds that $R(x', \tau, \sigma) \neq 0$ for $\mathrm{Im}\, \tau < 0$. Hence we have

$$\nu_2(x', \sigma) \geq 0. \tag{5. 4}$$

To complete the proof of the theorem we require the following two lemmas.

LEMMA 5. 2. *Under the same conditions as in Theorem* 5. 1, *there exists a neighborhood* $U(x^0) \times U(\tau^0, \sigma^0)$ *such that the following hold*:

(i) *There exist indeces* $j \in III_+$ *and* $k \in III_-$ *such that for any* $(x, \tau, \sigma) \in U(x^0) \times U(\tau^0, \sigma^0)$ *the vectors* $\{V_1^+, V_{l+1}^+, \cdots, V_{j-1}^+, V_k^-, V_{j+1}^+, \cdots, V_m^+\}$ *are linearly independent.*

Let $j \in III_+$ *in* (i) *be* $l+1$, *for simplicity, and denote by* $L(\cdot)$ *the linear space spaned by the vectors in the parenthesis. Then we have for* $(\tau, \sigma) \in U(x^0) \times U(\tau^0, \sigma^0)$ *such that* $\tau = \nu(x^0, \sigma)$

(ii) $V_{l+1}^+(x^0, \tau, \sigma) \in L(V_{l+2}^+, \cdots, V_m^+)$, *and*

(iii) $V_j^-(x^0, \tau, \sigma) \in L(V_1^+, V_{l+2}^+, \cdots, V_m^+)$ *for every* $j \in I_-$.

PROOF OF LEMMA 5. 2.

(i) First of all we remark that

$$\sum_{j=1}^{m} \det\left(V_1^+, \cdots, \frac{\partial V_j^+}{\partial \tau}, \cdots, V_m^+\right)(x^0, \tau^0, \sigma^0) \neq 0, \tag{5. 2. 1}$$

for every $j, k \in \mathrm{I}_+ \cup \mathrm{III}_+$

(5. 2. 2) $$\det(V_1^+, \cdots, \overset{j}{\check{V}_k^+}, \cdots, V_m^+)(x^0, \tau, \sigma) = O \qquad \text{on} \quad \tau = \nu(x^0, \sigma),$$

(5. 2. 3) $$\operatorname{rank}(V_{\mathrm{I}}^+, V_{\mathrm{III}}^+, V_{\mathrm{I}}^-, V_{\mathrm{III}}^-)(x', \tau, \sigma) = m \qquad \text{in some} \quad U(x^0) \times U(\tau^0, \sigma^0),$$

which follow from (5. 2), (5. 3) and (I) γ) (ii), respectively.

Suppose that there exists an index $j \in \mathrm{I}_+$ such that

(5. 2. 4) $$\det\left(V_1^+, \cdots, \frac{\partial V_j^+}{\partial \tau}, \cdots, V_m^+\right)(x^0, \tau, \sigma) \neq O.$$

Then we first show it contradicts (5. 2. 3) under the condition (III).

From Theorem 4. 1 there exists a neighborhood $U_1(\tau^0, \sigma^0)$ such that for $(\tau', \sigma') \in U_1(\tau^0, \sigma^0) \cap \Sigma_-$

(5. 2. 5) $$\left|\det(V_1^+, \cdots, \overset{j}{\check{V}_k^-}, \cdots, V_m^+)(x^0, \tau', \sigma')\right| \leq C(x^0, \tau^0, \sigma^0) \cdot \left|\operatorname{Im} \lambda_j^+(x^0, \tau', \sigma')\right|^{\frac{1}{2}} \cdot \left|R(x^0, \tau', \sigma')\right| \cdot (\gamma')^{-1}$$

for $j \in \mathrm{I}_+$ and $k \in \mathrm{I}_- \cup \mathrm{III}_-$. On the other hand, there exists a neighborhood $U(\tau^0, \sigma^0)$ such that for any $(\tau, \sigma) \in U(\tau^0, \sigma^0) \cap (\boldsymbol{C}_- \times \boldsymbol{R}^{n-1})$

$$(\tau \Lambda_\gamma^{-1}, \sigma \Lambda_\gamma^{-1}) \in U_1(\tau^0, \sigma^0) \cap \Sigma_-.$$

Since $V_i^\pm$ and R are homogeneous of order O and $\lambda_j^\pm$ homogeneous of order 1, we have from (5. 2. 5)

$$\left|\det(V_1^+, \cdots, \overset{j}{\check{V}_k^-}, \cdots, V_m^+)(x^0, \tau, \sigma)\right| \leq C(x^0, \tau^0, \sigma^0) \left|\operatorname{Im} \lambda_j^+(x^0, \tau, \sigma)\right|^{\frac{1}{2}} \times \left|R(x^0, \tau, \sigma)\right| \cdot \gamma^{-1} \cdot \left(|\tau|^2 + |\sigma|^2\right)^{\frac{1}{2}}$$

for $(\tau, \sigma) \in U(\tau^0, \sigma^0) \cap (\boldsymbol{C}_- \times \boldsymbol{R}^{n-1})$. Note that $|\tau|^2 + |\sigma|^2 \leq C$ in $U(\tau^0, \sigma^0)$. Then it is seen that for every $(\eta, \sigma) \in U(\tau^0, \sigma^0) \cap \boldsymbol{R}^n$ and small $\gamma > O$

(5. 2. 6) $$\left|\det(V_1^+, \cdots, \overset{j}{\check{V}_k^-}, \cdots, V_m^+)(x^0, \eta - i\gamma, \sigma)\right| \leq C \cdot \left|\operatorname{Im} \lambda_j^+(x^0, \eta - i\gamma, \sigma)\right|^{\frac{1}{2}} \cdot \left|R(x^0, \eta - i\gamma, \sigma)\right| \gamma^{-1}$$

with some $C > O$. Since λ_j^+ is a simple root we have

(5. 2. 7) $$\left|\operatorname{Im}(\lambda_j^+(x^0, \eta - i\gamma, \sigma)\right| \leq C\gamma \quad \text{for} \quad j \in \mathrm{I}_+$$

with some $C > O$. Take (τ^0, σ^0) as (η, σ) in (5. 2. 6), (5. 2. 7) and make $\gamma > O$ converge to zero. Then it follows from (5. 1) that

(5. 2. 8) $$\det(V_1^+, \cdots, \overset{j}{\check{V}_k^-}, \cdots, V_m^+)(x^0, \tau^0, \sigma^0) = O$$

for every $j \in \mathrm{I}_+$ and $k \in \mathrm{I}_- \cup \mathrm{III}_-$.

On the other hand (5.2.4) implies

$$\{V_1^+, \cdots, V_{j-1}^+, V_{j+1}^+, \cdots, V_m^+\}(x^0, \tau^0, \sigma^0)$$

are linearly independent. Hence from (5.2.8) there exist numbers $\{a_i^k\}$ such that

$$V_k^-(x^0, \tau^0, \sigma^0) = \sum_{i \neq j}^{m} a_i^k V_i^+ \quad \text{for every } k \in \mathrm{I}_- \cup \mathrm{III}_-.$$

This together with (5.2.2) contradicts (5.2.3). Hence we see from (5.2.1) that there exists an index $j \in \mathrm{III}_+$ satistying (5.2.4).

Secondly, by the same method deriving (5.2.8) from (5.2.5), we see from Theorem 4.1 (iii) that (5.2.8) is valid for every $j \in \mathrm{III}_+$ and $k \in \mathrm{I}_-$. This together with (5.2.2) and (5.2.3) implies (i).

(ii) If the set I is empty, the lemma follows immediately, because from (i) $\{V_{l+2}^+, \cdots, V_m^+\}(x', \tau, \sigma)$ are linearly independent and from (5.3) $\{V_{l+1}^+, \cdots, V_m^+\}(x', \tau, \sigma)$ are linearly dependent on $\tau = \nu(x^0, \sigma)$. Hence we assume that the set I is not empty and show that $\{V_{l+1}^+, \cdots, V_m^+\}$ are linearly dependent on $\tau = \nu(x^0, \sigma)$.

Here we use again (5.2.6). Take $(\eta, \sigma) \in U(\tau^0, \sigma^0) \cap \boldsymbol{R}^n$ such that $\eta = \nu_1(x^0, \sigma)$, then it follows from (5.3) and the condition (II) α) that

$$(5.2.9) \qquad \left|R(x^0, \eta - i\gamma, \sigma)\right| \leq C\left|\nu_1(x^0, \sigma) - i\gamma - \nu(x^0, \sigma)\right| = C\gamma.$$

Making $\gamma > 0$ converge to zero we obtain from (5.2.6), (5.2.7) and (5.2.9)

$$(5.2.10) \qquad \det(V_1^+, \cdots, \overset{j}{\check{V}_k^-}, \cdots, V_m^+)(x^0, \eta, \sigma) = 0$$

for $j \in \mathrm{I}_+$, $k \in \mathrm{I}_- \cup \mathrm{III}_-$ and $(\eta, \sigma) \in U(\tau^0, \sigma^0) \cap \boldsymbol{R}^n$ such that $\eta = \nu_1(x^0, \sigma) = \nu(x^0, \sigma)$.

In view of (5.2.3) we see from (5.2.10) and (5.2.2) that for every $\{v_i\} \subset \boldsymbol{C}^m$

$$\det(v_1, \cdots, v_l, V_{\mathrm{III}}^+) = 0 \qquad \text{on} \quad \tau = \nu(x^0, \sigma) \in \boldsymbol{R}^1.$$

This means that $\{V_{\mathrm{III}}^+\}$ are linearly dependent on $\tau = \nu(x^0, \sigma)$ and proves (ii).

(iii) As in (5.2.10) we see from Theorem 4.1 (iii)

$$(5.2.11) \qquad \det(V_{\mathrm{I}}^+, V_i^-, V_{l+2}^+, \cdots, V_m^+)(x^0, \tau, \sigma) = 0, \qquad i \in \mathrm{I}_-$$

for $(\tau, \sigma) \in U(\tau^0, \sigma^0) \cap \boldsymbol{R}^n$ such that $\tau = \nu(x^0, \sigma)$. Since from (i) $\{V_{\mathrm{I}}^+, V_{l+2}^+, \cdots, V_m^+\}$ are linearly independent, the statement (iii) follows from (5.2.11) directly.

LEMMA 5.3. *Under the same conditions as in Theorem* 5.1 *the fol-*

lowing hold:

(i) *There exist smooth scalar functions* $a_i(x', \tau, \sigma)$ *and a smooth vector function* $a(x', \tau, \sigma)$ *which are homogeneous of order 0 and* -1 *in* (τ, σ), *respectively, and satisfy*

$$V_{l+1}^{+} - a_{l+2} V_{l+2}^{+} - \cdots - a_m V_m^{+} = \big(\tau - \nu(x', \sigma)\big) \cdot a$$

in $U(x^0) \times U(\tau^0, \sigma^0)$.

(ii) *There exists a smooth* $m \times l$ *matrix* $C_{\mathrm{I}}(x', \tau, \sigma)$ *of homogeneous order 0 in* (τ, σ) *such that*

$$V_{\mathrm{I}}^{-}(x', \tau, \sigma) = B^{+} \cdot C_{\mathrm{I}} \quad in \quad U(x^0) \times U(\tau^0, \sigma^0).$$

PROOF OF LEMMA 5.3. Before proving it should be noted that Lemma 5.2 (ii), (iii) are valid if $\tau = \nu(x^0, \sigma)$ is replaced $\tau = \nu(x', \sigma)$, because in their proofs we can take x^0 as an arbitrary point such that the condition (II) α) holds for some (τ^0, σ^0).

(i) From Lemma 5.2 (i) and (ii) it holds that

$$\text{(5.3.1)} \qquad \operatorname{rank}(V_{l+1}^{+}, V_{l+2}^{+}, \cdots, V_m^{+}) = m - l - 1 \qquad \text{on } \tau = \nu(x', \sigma)$$

and

$$\text{(5.3.2)} \qquad \operatorname{rank}(V_{l+2}^{+}, \cdots, V_m^{+}) = m - l - 1 \qquad \text{in } U(x^0) \times U(\tau^0, \sigma^0).$$

To determine $a_j(x', \tau, \sigma)$ we consider the system of equations:

$$\text{(5.3.3)} \qquad (V_{l+2}^{+}, \cdots, V_m^{+}) \begin{pmatrix} a_{l+2} \\ \vdots \\ a_m \end{pmatrix} = V_{l+1}^{+}(x', \tau, \sigma).$$

Since there is such a minor of order $m-l-1$ of the matrix $(V_{l+2}^{+}, \cdots, V_m^{+})$ (x^0, τ^0, σ^0) that is non-singular, let it be the upper part of that matrix and write it by

$$C(x', \tau, \sigma) = \begin{pmatrix} v_{l+2,1}^{+}, & \cdots, & v_{m,1}^{+} \\ \vdots & & \vdots \\ v_{l+2,m-l-1}^{+}, & \cdots, & v_{m,m-l-1}^{+} \end{pmatrix},$$

where $\det C \neq 0$ in some $U(x^0) \times U(\tau^0, \sigma^0)$. Put

$$\begin{pmatrix} a_{l+2} \\ \vdots \\ a_m \end{pmatrix} = C^{-1} \begin{pmatrix} v_{l+1,1}^{+} \\ \vdots \\ v_{l+1,m-l-1}^{+} \end{pmatrix} (x', \tau, \sigma).$$

Then $\{a_i\}$ are C^∞ in (x', τ, σ), analytic and homogeneous of degree 0, with respect to (τ, σ), and fulfill the upper $m-l-1$ equations of (5.3.3).

Moreover, we see that they are also the solution of the system when $\tau = \nu(x', \sigma)$ by virture of (5.3.1) and (5.3.2).

Put

$$f(x', \tau, \sigma) = V^+_{l+1} - a_{l+2} V^+_{l+2} - \cdots - a_m V^+_m = {}^t(f_1, \cdots, f_m).$$

Then f is C^∞ in $(x' \tau, \sigma)$, analytic and homogeneous of degree 0 in (τ, σ) and

$$f(x', \tau, \sigma) = 0 \quad \text{on} \quad U(x^0) \times U(\tau^0, \sigma^0) \cap \{\tau = \nu(x', \sigma)\}.$$

Apply Wierstrass preperation theorem to f. Then there exist integers $m_j \geq 1$ and smooth functions $g_i(x', \tau, \sigma)$ which are analytic and homogeneous of order $-m_j$ in (τ, σ) and satisfy for $j = 1, \cdots, m$

$$f_j(x', \tau, \sigma) = \big(\tau - \nu(x', \sigma)\big)^{m_j} \cdot g_j(x', \tau, \sigma), \qquad \text{in some} \quad U(x^0) \times U(\tau^0, \sigma^0).$$

If we put

$$a(x', \tau, \sigma) = {}^t\Big((\tau-\nu)^{m_1 - 1} g_1, \cdots, (\tau-\nu)^{m_m - 1} g_m\Big),$$

the statement (i) is proved.

(ii) By the same way as in the proof of (i) it follows from Lemma 5.2 (iii) that there exist smooth scalar functions $a_{ij}(x', \tau, \sigma)$ and smooth vector functions $d_i(x', \tau, \sigma)$ which are homogeneous of order 0 and -1 in (τ, σ) respectively, and satisfy

$$\text{(5.3.4)} \qquad V^-_i - (a_{i,1} V^+_1 + \cdots + a_{i,l} V^+_l + a_{i,l+2} V^+_{l+2} + \cdots + a_{i,m} V^+_m = \big(\tau - \nu(x', \sigma)\big) d_i \qquad \text{in some} \quad U(x^0) \times U(\tau^0, \sigma^0)$$

for $i = 1, \cdots, l$. Put

$$D_1(x', \tau, \sigma) = \begin{pmatrix} a_{1,1} & \cdots\cdots & a_{m,1} \\ \vdots & & \vdots \\ a_{1,l} & & a_{m,l} \\ 0 & \cdots\cdots & 0 \\ a_{1,l+2} & & a_{m,l+2} \\ \vdots & & \vdots \\ a_{1,m} & \cdots\cdots & a_{m,m} \end{pmatrix}, \quad D_2 = (d_1, \cdots, d_l).$$

Then we have from (5.3.4)

$$V^-_{\mathrm{I}} = B^+ D_1 + \big(\tau - \nu(x', \sigma)\big) D_2,$$

where D_1, D_2 are homogeneous of order 0, -1 in (τ, σ) respectively. For any non-singular $m \times m$ matrix C_1 it holds that

$$\begin{aligned} &(\tau - \nu) D_2 \\ &\quad = C_1 (\tau - \nu) C_1^{-1} D_2 \end{aligned}$$

$$= C_1\begin{pmatrix} E_{\mathrm{I}}^+ & & 0 \\ & (\tau-\nu)\Lambda_r^{-1} & \\ 0 & & E' \end{pmatrix}\begin{pmatrix} E_{\mathrm{I}}^+ & & 0 \\ & (\tau-\nu)\Lambda_r^{-1} & \\ 0 & & E' \end{pmatrix}^{-1}(\tau-\nu)C_1^{-1}D_2$$

$$= C_1\begin{pmatrix} E_{\mathrm{I}}^+ & & 0 \\ & (\tau-\nu)\Lambda_r^{-1} & \\ 0 & & E' \end{pmatrix}\begin{pmatrix} (\tau-\nu)E_{\mathrm{I}}^+ & & 0 \\ & \Lambda_r & \\ 0 & & (\tau-\nu)E' \end{pmatrix}C_1^{-1}D_2$$

where E_{I}^+, E' are the $l\times l$, $(m-l-1)\times(m-l-1)$ identity matrices respectively.

We will choose C_1 in order to show that $V_{\mathrm{I}}^-\in$ range B^+. Take $a_i(x', \tau, \sigma)$ in (i) and put

$$C_0(x', \tau, \sigma)=\left(\begin{array}{c|cccc} E_{\mathrm{I}}^+ & & & 0 & \\ \hline & 1 & & & 0 \\ & -a_{l+2} & \ddots & & \\ 0 & \vdots & & \ddots & \\ & -a_m & 0 & & 1 \end{array}\right).$$

Then with $a(x', \tau, \sigma)$ in (i) it holds that

$$\begin{aligned} & B^+C_0(x', \tau, \sigma) \\ & \quad =(V_{\mathrm{I}}^+, (\tau-\nu)a, V_{l+2}^+, \cdots, V_m^+) \\ & \quad =(V_{\mathrm{I}}^+, a\Lambda_r, V_{l+2}^+, \cdots, V_m^+)\begin{pmatrix} E_{\mathrm{I}}^+ & & 0 \\ & \big(\tau-\nu(x', \sigma)\big)\Lambda_r^{-1} & \\ 0 & & E' \end{pmatrix}. \end{aligned} \tag{5.3.5}$$

Since $\det C_0(x', \tau, \sigma)\equiv 1$ we have

$$\begin{aligned} R=\det B^+ &= \det B^+C_0 \\ &=(\tau-\nu)\det(V_{\mathrm{I}}^+, a, V_{l+2}^+, \cdots, V_m^+), \end{aligned}$$

where from (5.2)

$$\det(V_{\mathrm{I}}^+, a, V_{l+2}^+, \cdots, V_m^+)\neq O.$$

Hence if we put

$$C_1=(V_{\mathrm{I}}^+, a\Lambda_r, V_{l+2}^+, \cdots, V_m^+)$$

then $C_1(x', \tau, \sigma)$ is homogeneous of order O in (τ, σ) and $\det C_1\neq O$. Thus we obtain from (5.3.5)

$$V_{\mathrm{I}}^-=B^+D_1+B^+C_0\begin{pmatrix} (\tau-\nu)E_{\mathrm{I}}^+ & & 0 \\ & \Lambda_r & \\ 0 & & (\tau-\nu)E' \end{pmatrix}C_1^{-1}D_2.$$

$$\left|q(x', D') v_{l+1}^{+}\right|_{\frac{1}{2},\gamma} \geq C\gamma\, |v_{l+1}^{+}|_{-\frac{1}{2},\gamma}$$

with some $C>0$. Hence it holds for large γ that

$$|B^{+}\cdot C_0 v^{+}|_{\frac{1}{2},\gamma} \geq C\left(\gamma^{\frac{1}{2}}\, |v_{\mathrm{I}}^{+}|_{0,\gamma} + \gamma\, |v_{\mathrm{III}}^{+}|_{-\frac{1}{2},\gamma}\right).$$

On the other hand

$$\begin{aligned}|B^{+}\cdot C_0 v^{+}|_{\frac{1}{2},\gamma} &\leq |C_0\cdot B^{+} v^{+}|_{\frac{1}{2},\gamma} + \left|[B^{+}, C_0]\, v^{+}\right|_{\frac{1}{2},\gamma} \\ &\leq C\left(|B^{+} v^{+}|_{\frac{1}{2},\gamma} + |v^{+}|_{-\frac{1}{2},\gamma}\right).\end{aligned}$$

Thus Theorem 5. 1 is proved.

§ 6. The structure of $(\boldsymbol{P}_1, \boldsymbol{B}_1)$ for the case where the set II is not empty.

This section is devoted to the case where the conditions (II) β) and γ) are applied. We only consider the point $(x^0, \tau^0, \sigma^0) \in \Gamma \times \boldsymbol{R}^n$ and its neighborhood $U(x^0) \times (U(\tau^0, \sigma^0) \cap (\bar{\boldsymbol{C}}_- \times \boldsymbol{R}^{n-1}))$ for which the set II is not empty and $R(x^0, \tau^0, \sigma^0) = 0$. Note that in this case the boundary condition (B_1) may be replaced by

$$(B_1) \qquad (V_{\mathrm{I}}^{+}, V_{\mathrm{I}}^{-}, V_{\mathrm{II}}', V_{\mathrm{II}}'', V_{\mathrm{III}}^{+}, V_{\mathrm{III}}^{-})(x', D') \cdot {}^t({}^t u_{\mathrm{I}}^{+}, {}^t u_{\mathrm{I}}^{-}, u_{\mathrm{II}}', u_{\mathrm{II}}'', {}^t u_{\mathrm{III}}^{+}, {}^t u_{\mathrm{III}}^{-})(x', 0) = g(x'), \qquad \text{in } \boldsymbol{R}^n$$

and it holds that

$$\begin{aligned}B^{+}(x', \tau, \sigma) &= (V_{\mathrm{I}}^{+}, V_{\mathrm{II}}^{+}, V_{\mathrm{III}}^{+}),\\ R(x', \tau, \sigma) &= \det(V_{\mathrm{I}}^{+}, V_{\mathrm{II}}^{+}, V_{\mathrm{III}}^{+}),\end{aligned}$$

hence B^{+} and R are S_{+}^{0} in $U(x^0) \times (U(\tau^0, \sigma^0) \cap (\boldsymbol{C}_- \times \boldsymbol{R}^{n-1}))$, analytic in $(\tau, \sigma) \in U(\tau^0, \sigma^0) \cap (\boldsymbol{C}_- \times \boldsymbol{R}^{n-1})$ and continuous in $U(x^0) \times (U(\tau^0, \sigma^0) \cap (\bar{\boldsymbol{C}}_- \times \boldsymbol{R}^{n-1}))$. We give some lemmas on the decomposed problem (P_1, B_1) in § 3 of $2m \times 2m$ system, which will be used in § 8 for the proof of the estimate (1. 1).

For the functions defined in § 3 and § 4 let us also use the following notations for simplicity in symbolic caluculations:

$$\begin{aligned}B'(x', \mu, \sigma) &= (V_{\mathrm{I}}^{+}, V_{\mathrm{II}}', V_{\mathrm{III}}^{+})(x', \tau, \sigma),\\ B''(x', \mu, \sigma) &= (V_{\mathrm{I}}^{+}, V_{\mathrm{II}}'', V_{\mathrm{III}}^{+})(x', \tau, \sigma),\end{aligned}$$

and

$$(6.1) \qquad s_{21}(x, \mu, \sigma) = s_1(x, \mu, \sigma) + \sqrt{\mu}\, s_2(x, \mu, \sigma),$$

where

Let

$$C_{\mathrm{I}}(x', \tau, \sigma) = D_1 + C_0 \begin{pmatrix} (\tau-\nu) E_{\mathrm{I}}^+ & & 0 \\ & \Lambda_\gamma & \\ 0 & & (\tau-\nu) E' \end{pmatrix} C_{\mathrm{I}}^{-1} D_2 .$$

Then C_{I} is homogeneous of order 0 in (τ, σ) and

$$V_{\mathrm{I}}^-(x', \tau, \sigma) = B^+ \cdot C_{\mathrm{I}} \quad \text{in} \quad U(x^0) \times U(\tau^0, \sigma^0).$$

Thus the proof of the lemma is completed.

Now we must go back to the proof of Theorem 5. 1. Since Im $\tau^0 = 0$ we take $(\tilde{\tau}_k, \tilde{\sigma}_k)$ by (2. 6) as the extention and denote them by $(\tilde{\tau}, \tilde{\sigma})$. Take $C_0(x', \tau, \sigma)$ and $C_1(x', \tau, \sigma)$ in the proof of Lemma 5. 3 (ii) and put

$$\begin{aligned} \tilde{C}_0(x', \tau, \sigma) &= C_0(\tilde{x}, \tilde{\tau}, \tilde{\sigma}) \\ \tilde{C}_1(x', \tau, \sigma) &= C_1(\tilde{x}, \tilde{\tau}, \tilde{\sigma}) = (V_{\mathrm{I}}^+, a, V_{l+2}^+, \cdots, V_m^+)(\tilde{x}, \tilde{\tau}, \tilde{\sigma}) \\ \tilde{C}_2(x', \tau, \sigma) &= \begin{pmatrix} E_{\mathrm{I}}^+ & & 0 \\ & \tilde{\tau} - \nu(\tilde{x}, \tilde{\sigma}) & \\ 0 & & E' \end{pmatrix} = \begin{pmatrix} E_{\mathrm{I}}^+ & & 0 \\ & \tilde{q}(x', \tau, \sigma) & \\ 0 & & E' \end{pmatrix}. \end{aligned}$$

Then we have with some $C > 0$

$$\left| \det \tilde{C}_1(x', \tau, \sigma) \right| \geq C \tag{5. 5}$$

for every $(x', \tau, \sigma) \in \Gamma \times \bar{\boldsymbol{C}}_- \times \boldsymbol{R}^{n-1}$,

and it follows from (5. 3. 5) that

$$\tilde{B}^+ \tilde{C}_0(x', \tau, \sigma) = \tilde{C}_1 \tilde{C}_2(x', \tau, \sigma). \tag{5. 6}$$

Furthermore, we see from (2. 6)′ and (5. 4) that

$$\begin{aligned} -\mathrm{Im}\, \tilde{q}(x', \tau, \sigma) &= \tilde{\gamma} + \nu_2(\tilde{x}, \tilde{\sigma}) \geq \tilde{\gamma} \\ &= \tilde{\gamma}_k(\tau \Lambda_\gamma^{-1}, \sigma \Lambda_\gamma^{-1}) \geq C \gamma \Lambda_\gamma^{-1} \end{aligned} \tag{5. 7}$$

with some $C > 0$. Let B^+, C^0, C_1, C_2, q be the operators with their symbol $\tilde{B}^+, \tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \tilde{q} \in S_+^0$ respectively. Then from (5. 5), (5. 6) and Lemma 2. 1 α) and β) (ii) it holds for large γ that

$$\begin{aligned} &\left| B^+(x', D') \cdot C_0(x', D') v^+ \right|_{\frac{1}{2}, \gamma} \\ &\geq C \left| C_2(x', D') v^+ \right|_{\frac{1}{2}, \gamma} \\ &\geq C \Big(|v_I^+|_{\frac{1}{2}, \gamma} + \left| q(x', D') v_{l+1}^+ \right|_{\frac{1}{2}, \gamma} + \sum_{j=l+2}^{m} |v_j^+|_{\frac{1}{2}, \gamma} \Big). \end{aligned}$$

If we consider the bilinear form $-\mathrm{Im} \langle q(x', D') v, v \rangle_{0, \gamma}$, we see from (5. 7) and Lemma 2. 1 α) (iv) and β) (iv) that for large γ

$$s_1=\Big(-p_{11}\mu+\lambda_1(x,\mu,\sigma)-\lambda_1(x,0,\sigma)\Big)(\Lambda_0^{(0)}+p_{12}\mu)^{-1}$$
$$=\mu s_1^{(1)}(x,\mu,\sigma),$$
$$s_2=\lambda_2(x,\mu,\sigma)(\Lambda_0^{(0)}+p_{12}\mu)^{-1}$$

and

s_1, s_2 are real for real μ. (see Lemma 3. 1 and subsection 4. 1). Here note that B' and B'' are smooth and it holds that in $U(x^0)\times(U(0,\sigma^0)\cap R^n)$

(6. 2) $$\left|s_1^{(1)}(x,\mu,\sigma)\right|\le C,\quad\text{and}\quad s_2(x,\mu,\sigma)>0$$

with some $C>0$. Here after we consider only τ, μ with $\operatorname{Im}\tau=\operatorname{Im}\mu\leqq 0$. Then we have the following

LEMMA 6. 1.

α) *For the functions in* §4 *the relations* (i), (ii) *and* (iii) *follow directly from their definitions*:

(i)
$$h'_{\text{II}}=h^+_{\text{II}}-s_{21}h''_{\text{II}},\ h^-_{\text{II}}=h^+_{\text{II}}+\alpha^{-1}(\lambda^-_{\text{II}}-\lambda^+_{\text{II}})h''_{\text{II}},$$

hence
$$V'_{\text{II}}=V^+_{\text{II}}-s_{21}V''_{\text{II}},\ V^-_{\text{II}}=V^+_{\text{II}}+\alpha^{-1}(\lambda^-_{\text{II}}-\lambda^+_{\text{II}})V''_{\text{II}},$$

(ii)
$$b_{\text{II II}}=R^{-1}\cdot\det B''=\alpha(\widetilde{b}_{\text{II II}}-1)(\lambda^-_{\text{II}}-\lambda^+_{\text{II}})^{-1},$$
$$1-s_{21}b_{\text{II II}}=R^{-1}\cdot\det B',$$
$$b_{ij}=\widetilde{b}_{ij}\ \textit{for every } i \textit{ and every }\ j\notin\text{II},$$

(iii)
$$\det\Big(V^+_{\text{I}},(1-s_{21}b_{\text{II II}})V''_{\text{II}}-b_{\text{II II}}V'_{\text{II}},V^+_{\text{III}}\Big)=0.$$

β) *The condition (II) β) implies*

(i)
$$\det B'(x^0,0,\sigma^0)=0\quad\text{and}\quad\det B''(x^0,0,\sigma^0)\neq 0.$$

Now let $Q(x',\mu,\sigma)\in S^0_+$ be defined as follows:
$$Q(x',\mu,\sigma)=(\det B')(\det B'')^{-1}\qquad in\quad U(x^0)\times U(0,\sigma^0).$$

Then it holds that

(ii)
$$Q(x',\mu,\sigma)=(1-s_{21}b_{\text{II II}})(b_{\text{II II}})^{-1}=R(\det B'')^{-1}-s_{21}$$
$$=(\lambda^-_{\text{II}}-\lambda^+_{\text{II}})\Big(\alpha(\widetilde{b}_{\text{II II}}-1)\Big)^{-1}-s_{21}$$

(iii)

$$\det(V_{\mathrm{I}}^{+}, QV_{\mathrm{II}}''-V_{\mathrm{II}}', V_{\mathrm{III}}^{+})=0 \qquad in \quad U(x^0)\times U(0, \sigma^0).$$

PROOF. α) (i) follows from the fact that $(h_{\mathrm{II}}^{+}, h_{\mathrm{II}}'')=(h_{\mathrm{II}}', h_{\mathrm{II}}'')S_{\mathrm{II}}$ and $(h_{\mathrm{II}}^{+}, h_{\mathrm{II}}^{-})=(h_{\mathrm{II}}', h_{\mathrm{II}}'')S_{\mathrm{II}}T_{\mathrm{II}}$. α) (i) and α) (ii) implies α) (ii) and α) (iii), respectively. From α) and β) (i) it follows directly that β) (ii) and (iii) are valid. Therefore we only prove β) (i). From α) (i) and (6.1) we have

$$\begin{aligned} R(x^0, \tau^0-i\gamma, \sigma^0) &= \det(V_{\mathrm{I}}^{+}, V_{\mathrm{II}}'+s_{21}V_{\mathrm{II}}'', V_{\mathrm{III}}^{+}) \\ &= \det B'+s_{21}\det B'' \\ &= (\det B'+s_1\det B'')(x^0, -i\gamma, \sigma^0) \\ &\quad +\sqrt{-i\gamma}(s_2\det B'')(x^0, -i\gamma, \sigma^0). \end{aligned}$$

Hence it follows from $R(x^0, \tau^0, \sigma^0)=0$ that

$$(\det B'+s_1\det B'')(x^0, 0, \sigma^0)=0,$$

which implies $\det B'(x^0, 0, \sigma^0)=0$. Since $(\det B'+s_1\det B'')(x^0, \mu, \sigma^0)$ is analytic in μ, it follows from the condition (II) β) that

$$(s_2\cdot\det B'')(x^0, 0, \sigma^0)\neq 0,$$

which together with (6.2) implies β) (i).

LEMMA 6.2. *Assume the condition (II) β). Let the equality (B_1) be satisfied for $U\in H_{\frac{1}{2},\gamma}(\boldsymbol{R}_{+}^{n+1})$. Then there exist a constant $C>0$ and operators $C_1, C_2\in S_{+}^{-1}$ and $k_{JK}\in S_{+}^{0}$ independent of U such that for large γ*

$$\begin{aligned} &|g-V_{\mathrm{III}}^{-}u_{\mathrm{III}}^{-}-C_1u_{\mathrm{I}}^{-}-C_2u_{\mathrm{II}}'|_{\frac{1}{2},\gamma} \\ &\geq C\Big(|u_{\mathrm{I}}^{+}+k_{\mathrm{I\,I}}u_{\mathrm{I}}^{-}+k_{\mathrm{I\,II}}u_{\mathrm{II}}'|_{\frac{1}{2},\gamma}+|u_{\mathrm{II}}''+k_{\mathrm{II\,I}}u_{\mathrm{I}}^{-}+Qu_{\mathrm{II}}'|_{\frac{1}{2},\gamma} \\ &\quad +|u_{\mathrm{III}}^{+}+k_{\mathrm{III\,I}}u_{\mathrm{I}}^{-}+k_{\mathrm{III\,II}}u_{\mathrm{II}}'|_{\frac{1}{2},\gamma}\Big). \end{aligned}$$

Moreover, in $U(x^0)\times(U(\tau^0, \sigma^0)\cap(\bar{C}_{-}\times\boldsymbol{R}^{n-1}))$, the symbol of (k_{JK}) is analytic in (τ', σ') and has the form:

$$(6.3)\qquad \begin{pmatrix} k_{\mathrm{I\,I}} & k_{\mathrm{I\,II}} & k_{\mathrm{I\,III}} \\ k_{\mathrm{II\,I}} & k_{\mathrm{II\,II}} & k_{\mathrm{II\,III}} \\ k_{\mathrm{III\,I}} & k_{\mathrm{III\,II}} & k_{\mathrm{III\,III}} \end{pmatrix} = (V_{\mathrm{I}}^{+}, V_{\mathrm{II}}'', V_{\mathrm{III}}^{+})^{-1}(V_{\mathrm{I}}^{-}, V_{\mathrm{II}}', V_{\mathrm{III}}^{-})(x', \tau', \sigma')$$

where k_{JK} are $m_J\times m_K$ matrices with $m_J=l-1$, 1 or $(m-l)$ for $J=I, II$ or III, respectively.

In the constant coefficients case the analogous inequality holds, which follows directly from the proof of the above lemma:

COROLLARY 6.1. *Assume the same condition as in Lemma 6.2.*

Let

$$(B_1)_{x^0} \qquad \hat{g}(\tau, \sigma) = (V_{\mathrm{I}}^+, V'_{\mathrm{II}}, V_{\mathrm{III}}^+)\cdot {}^t({}^t\hat{u}_{\mathrm{I}}^+, \hat{u}'_{\mathrm{II}}, {}^t\hat{u}_{\mathrm{III}}^+)(\tau, \sigma, 0) + (V_{\mathrm{I}}^-, V''_{\mathrm{II}}, V_{\mathrm{III}}^-)\cdot {}^t({}^t\hat{u}_{\mathrm{I}}^-, \hat{u}''_{\mathrm{II}}, {}^t\hat{u}_{\mathrm{III}}^-)(\tau, \sigma, 0).$$

Then there exist a constant $C>0$ independent of $\hat{u}$ such that

$$|\hat{g} - V_{\mathrm{III}}^- \hat{u}_{\mathrm{III}}^-| \geq C\Big(|\hat{u}_{\mathrm{I}}^+ + k_{\mathrm{I}\,\mathrm{I}}\hat{u}_{\mathrm{I}}^- + k_{\mathrm{I}\,\mathrm{II}}\hat{u}'_{\mathrm{II}}| + |\hat{u}''_{\mathrm{II}} + k_{\mathrm{II}\,\mathrm{I}}\hat{u}_{\mathrm{I}}^- + Q\hat{u}'_{\mathrm{II}}| + |\hat{u}_{\mathrm{III}}^+ + k_{\mathrm{III}\,\mathrm{I}}\hat{u}_{\mathrm{I}}^- + k_{\mathrm{III}\,\mathrm{II}}\hat{u}'_{\mathrm{II}}|\Big)$$

in $U(\tau^0, \sigma^0)\cap(\bar{C}_-\times \boldsymbol{R}^{n-1})$, where (k_{JK}) are functions defined by (6. 3).

PROOF OF LEMMA 6. 2. Define k_{JK} by (6. 3). Then it holds that

$$(6.2.1) \qquad \begin{aligned} V_{\mathrm{I}}^-(x', \tau, \sigma) &= V_{\mathrm{I}}^+ k_{\mathrm{I}\,\mathrm{I}} + V''_{\mathrm{II}} k_{\mathrm{II}\,\mathrm{I}} + V_{\mathrm{III}}^+ k_{\mathrm{III}\,\mathrm{I}}, \\ V'_{\mathrm{II}}(x', \tau, \sigma) &= V_{\mathrm{I}}^+ k_{\mathrm{I}\,\mathrm{II}} + Q V''_{\mathrm{II}} + V_{\mathrm{III}}^+ k_{\mathrm{III}\,\mathrm{II}}. \end{aligned}$$

Here we used the relation $k_{\mathrm{II}\,\mathrm{II}}=Q$. Hence we have from (B_1)

$$\begin{aligned} g(x') = {} & V_{\mathrm{I}}^+(x', D') u_{\mathrm{I}}^+(x', 0) + V''_{\mathrm{II}}(x', D') u''_{\mathrm{II}}(x', 0) \\ & + V_{\mathrm{III}}^+(x', D') u_{\mathrm{III}}^+(x', 0) \\ & + (V_{\mathrm{I}}^+\cdot k_{\mathrm{I}\,\mathrm{I}} + V''_{\mathrm{II}}\cdot k_{\mathrm{II}\,\mathrm{I}} + V_{\mathrm{III}}^+\cdot k_{\mathrm{III}\,\mathrm{I}}) u_{\mathrm{I}}^-(x', 0) \\ & + (V_{\mathrm{I}}^+\cdot k_{\mathrm{I}\,\mathrm{II}} + V''_{\mathrm{II}}\cdot Q + V_{\mathrm{III}}^+\cdot k_{\mathrm{III}\,\mathrm{II}}) u'_{\mathrm{II}}(x', 0) \\ & + V_{\mathrm{III}}^-(x', D') (u_{\mathrm{III}}^-(x', 0) \\ & + C_1(x', D') u_{\mathrm{I}}^-(x', 0) + C_2(x', D') u'_{\mathrm{II}}(x', 0), \end{aligned}$$

where C_1 and $C_2 \in S_+^{-1}$ arise from commutators. Hence we obtain

$$\begin{aligned} & g - V_{\mathrm{III}}^- u_{\mathrm{III}}^- - C_1 u_{\mathrm{I}}^- - C_2 u'_{\mathrm{II}} \\ & = (V_{\mathrm{I}}^+, V''_{\mathrm{II}}, V_{\mathrm{III}}^+)(x', D') \begin{pmatrix} u_{\mathrm{I}}^+ + k_{\mathrm{I}\,\mathrm{I}} u_{\mathrm{I}}^- + k_{\mathrm{I}\,\mathrm{II}} u'_{\mathrm{II}} \\ u''_{\mathrm{II}} + k_{\mathrm{II}\,\mathrm{I}} u_{\mathrm{I}}^- + Q u'_{\mathrm{II}} \\ u_{\mathrm{III}}^+ + k_{\mathrm{III}\,\mathrm{I}} u_{\mathrm{I}}^- + k_{\mathrm{III}\,\mathrm{II}} u'_{\mathrm{II}} \end{pmatrix}. \end{aligned}$$

Since $\det \widetilde{B}''(x^0, \tau^0, \sigma^0)\neq 0$ by Lemma 6. 1 β) (i), the lemma follows from Lemma 2. 1 β) (ii).

LEMMA 6. 3. *Consider the constant coefficients problem $(P_{\mathrm{II}}, Q)_{x^0}$ of 2×2 system in a sufficiently small neighborhood $(U(\tau^0, \sigma^0)\cap(\Sigma_-)$:*

$$(P_{\mathrm{II}}, Q)_{x^0} \begin{cases} P_{\mathrm{II}}(x^0, \tau, \sigma, D_n)\, \hat{u}_{\mathrm{II}}(\tau, \sigma, x_n) \\ \quad = \big(E_{\mathrm{II}} D_n - M_{\mathrm{II}}(x^0, \tau, \sigma)\big)\cdot {}^t(\hat{u}'_{\mathrm{II}}, \hat{u}''_{\mathrm{II}})(\tau, \sigma, x_n) \\ \quad = \hat{f}_{\mathrm{II}}(x_n), \quad \textit{for} \quad x_n>0, \\ Q(x^0, \tau, \sigma)\, \hat{u}'_{\mathrm{II}}(\tau, \sigma, 0) + \hat{u}''_{\mathrm{II}}(\tau, \sigma, 0) = 0. \end{cases}$$

If the conditions (II) β) and (III) are assumed, then $(P_{II}, Q)_{x^0}$ *is* L^2*-well-posed, that is, for any* $\hat{f}_{II}(x_n) \in H_1(x_n>0)$ *and any* $(\tau, \sigma) \in U(x^0) \times (U(\tau^0, \sigma^0) \cap \Sigma_-)$*, the problem* $(P_{II}, Q)_{x^0}$ *has a unique solution* $\hat{u}_{II}(\tau, \sigma, x_n) \in H_1(x_n>0)$ *satisfying*

$$\|\hat{u}_{II}(\tau, \sigma, \cdot)\| \leq C\gamma^{-1} \|\hat{f}_{II}(\cdot)\| \tag{6.4}$$

for some $C = C(x^0, \tau^0, \sigma^0) > 0$.

PROOF. Recall the proof of Theorem 4.1. First we obtain from the condition (III) the L^2-well-posedness of the problem $(P^0, B)_{x^0}$. Transform it by a non-singular matrix

$$S(\tau, \sigma) = (h_I^+, h_I^-, h'_{II}, h''_{II}, h^+_{III}, h^-_{III}).$$

Then we have the following constant coefficients problem equivalent to $(P^0, B)_{x^0}$:

$$(P_1, B_1)_{x^0} \begin{cases} P_1 \hat{U} = \left[ED_n - \begin{pmatrix} \lambda_I^+ & & & & 0 \\ & \lambda_I^- & & & \\ & & M_{II} & & \\ & & & M^+_{III} & \\ 0 & & & & M^-_{III} \end{pmatrix} (\tau, \sigma) \right] \hat{U} = \hat{F}, \quad \text{for} \quad x_n > 0, \\ B_1 \hat{U} = BS\hat{U} \\ \quad = (V_I^+, V'_{II}, V^+_{III})\,{}^t(\hat{u}_I^+, \hat{u}'_{II}, {}^t\hat{u}^+_{III})(\tau, \sigma, 0) \\ \qquad + (V_I^-, V''_{II}, V^-_{III})\,{}^t({}^t\hat{u}_I^-, \hat{u}''_{II}, {}^t\hat{u}^-_{III})(\tau, \sigma, 0) \\ \quad = 0, \end{cases}$$

where

$$\hat{U}(\tau, \sigma, x_n) = S^{-1}\hat{u}(\tau, \sigma, x_n) = {}^t({}^t\hat{u}_I^+, {}^t\hat{u}_I^-, \hat{u}'_{II}, \hat{u}''_{II}, {}^t\hat{u}^+_{III}, {}^t\hat{u}^-_{III})$$

and

$$\hat{F}(x_n) = S^{-1}\hat{f}(x_n) = {}^t({}^t\hat{f}_I^+, {}^t\hat{f}_I^-, \hat{f}'_{II}, \hat{f}''_{II}, {}^t\hat{f}^+_{III}, {}^t\hat{f}^-_{III}).$$

1) Existance of a solution of $(P_{II}, Q)_{x^0}$. From Corollary 5.1 the solution $\hat{U}$ of $(P_1, B_1)_{x^0}$ for $\hat{F}(x_n) = {}^t(0, 0, \hat{f}'_I, \hat{f}''_{II}, 0, 0)$ fufills

$$\hat{u}_I^-(\tau, \sigma, x_n) = \hat{u}^-_{III}(\tau, \sigma, x_n) = 0.$$

Hence from Corollary 6.1 it holds that

$$\begin{cases} \hat{u}_I^+(\tau, \sigma, 0) + k_{I\,II}\hat{u}'_{II}(\tau, \sigma, 0) = 0, \\ \hat{u}''_{II}(\tau, \sigma, 0) + Q\hat{u}'_{II}(\tau, \sigma, 0) = 0, \\ \hat{u}^+_{III}(\tau, \sigma, 0) + k_{III\,II}\hat{u}'_{II}(\tau, \sigma, 0) = 0. \end{cases} \tag{6.3.1}$$

Therefore the ${}^t(\hat{u}'_{\mathrm{II}}, \hat{u}''_{\mathrm{II}})(\tau, \sigma, x_n)$ is a solution of $(P_{\mathrm{II}}, Q)_{x^0}$ for $\hat{f}_{\mathrm{II}}(x_n) = {}^t(\hat{f}'_{\mathrm{II}}, \hat{f}''_{\mathrm{II}})$.

2) The estimate (6. 4) for the above solution follows from the L^2-well-posedness of $(P_1, B_1)_{x^0}$:

$$\left\|\hat{U}(\tau, \sigma, \cdot)\right\| \leq C\gamma^{-1}\left\|\hat{F}(x_n)\right\|.$$

3) Uniqueness of the solution of $(P_{\mathrm{II}}, Q)_{x^0}$. Let $\hat{u}_{\mathrm{II}} = {}^t(\hat{u}'_{\mathrm{II}}, \hat{u}''_{\mathrm{II}})$ be the solution of $(P_{\mathrm{II}}, Q)_{x^0}$ corresponding to $\hat{f}_{\mathrm{II}} = O$. Take the bounded solutions $\hat{u}^+_{\mathrm{I}}, \hat{u}^+_{\mathrm{III}}$ of the equations

$$\begin{cases} \left(E^+_{\mathrm{I}} D_n - \lambda^+_{\mathrm{I}}(\tau, \sigma)\right) \hat{u}^+_{\mathrm{I}}(\tau, \sigma, x_n) = O & \text{for} \quad x_n > O, \\ \hat{u}^+_{\mathrm{I}}(\tau, \sigma, O) = -k_{\mathrm{I\,II}} \hat{u}'_{\mathrm{II}}(\tau, \sigma, O), \end{cases}$$

and

$$\begin{cases} \left(E^+_{\mathrm{III}} D_n - M^+_{\mathrm{III}}(\tau, \sigma)\right) \hat{u}^+_{\mathrm{III}}(\tau, \sigma, x_n) = O & \text{for} \quad x_n > O, \\ \hat{u}^+_{\mathrm{III}}(\tau, \sigma, O) = -k_{\mathrm{III\,II}} \hat{u}'_{\mathrm{II}}(\tau, \sigma, O). \end{cases}$$

Put

$$\hat{U} = {}^t({}^t\hat{u}^+_{\mathrm{I}}, O, \hat{u}'_{\mathrm{II}}, \hat{u}''_{\mathrm{II}}, {}^t\hat{u}^+_{\mathrm{III}}, O).$$

Then $P_1\hat{U} = O$ and (6. 3. 1) is fulfilled. Hence it follows from (6. 2. 1) that

$$\begin{aligned} B_1\hat{U} &= (V^+_{\mathrm{I}}, V''_{\mathrm{II}}, V^+_{\mathrm{III}}) \cdot {}^t({}^t\hat{u}^+_{\mathrm{I}}, \hat{u}''_{\mathrm{II}}, {}^t\hat{u}^+_{\mathrm{III}}) + V'_{\mathrm{II}}\hat{u}'_{\mathrm{II}} \\ &= (V^+_{\mathrm{I}}, V''_{\mathrm{II}}, V^+_{\mathrm{III}}) \cdot \begin{pmatrix} \hat{u}^+_{\mathrm{I}} + k_{\mathrm{I\,II}}\hat{u}'_{\mathrm{II}} \\ \hat{u}''_{\mathrm{II}} + Q\hat{u}'_{\mathrm{II}} \\ \hat{u}^+_{\mathrm{III}} + k_{\mathrm{III\,II}} u'_{\mathrm{II}} \end{pmatrix} = O. \end{aligned}$$

Therefore $\hat{U}$ is a solution of $(P_1, B_1)_{x^0}$ with $\hat{F} = O$. From the L^2-well-posedness of $(P_1, B_1)_{x^0}$ we obtain $\hat{U} = O$ and hence $\hat{u}'_{\mathrm{II}} = \hat{u}''_{\mathrm{II}} = O$. This proves the lemma.

LEMMA 6. 4. *Assume the conditions* (*II*) β) and γ).

(i) *Then the function* $Q(x', \mu, \sigma)$ *defined in Lemma* 6. 1 β) (i) *takes real values in* $U(x^0) \times U(O, \sigma^0)$ *for real* μ.

(ii) *Suppose, in addition, the condition* (*III*). *Then for* $(x', \tau, \sigma) \in U(x^0) \times U(\tau^0, \sigma^0)$, $R(x', \tau, \sigma) = O$ *for* $\mathrm{Im}\,\tau \leq O$ *is equivalent to* $\tau = \theta(x', \sigma)$ *and* $Q(x', O, \sigma) = O$.

PROOF.

(i) Since, in the case (a), the choice of the vectors $\{h^\pm_{\mathrm{I}}, h^\pm_{\mathrm{II}}\}$ are restricted to be real for $\mu \geq O$, Lemma 4. 1 and the condition (II) γ) imply that

$$\widetilde{b_{\text{II II}}}(x', \tau, \sigma) = \det B(h_{\text{I}}^{+}, h_{\text{II}}^{-}, h_{\text{III}}^{+})\cdot\left(\det B(h_{\text{I}}^{+}, h_{\text{II}}^{+}, h_{\text{III}}^{+})\right)^{-1}$$

is real for $\mu \geq 0$ and $R(x', \tau, \sigma) \neq 0$. Here note that $h_{\text{II}}^{\pm}$ are the eigenvectors which were defined in §4 by

$$(h_{\text{II}}^{+}, h_{\text{II}}^{-}) = (h'_{\text{II}}, h''_{\text{II}})\, S_{\text{II}}\, T_{\text{II}}$$

and hence are real for $\mu \geq 0$. From Lemma 6.1 β) (ii) it holds for Im $\mu \leq 0$ that

$$Q(x', \mu, \sigma) = R(\det B'')^{-1} - s_{21} = (\lambda_{\text{II}}^{-} - \lambda_{\text{II}}^{+})\left(\alpha(\widetilde{b_{\text{II II}}} - 1)\right)^{-1} - s_{21}\,. \tag{6.4.1}$$

If $R(x', \tau, \sigma) = 0$ and $\mu \geq 0$ then Q is real at such points, because s_{21} is real for $\mu \geq 0$ by virture of (6.1). If $R(x', \tau, \sigma) \neq 0$, $\mu \geq 0$ and $\widetilde{b_{\text{II II}}} \neq 1$ Q is also real, because $\lambda_{\text{II}}^{\pm}$, α, $\widetilde{b_{\text{II II}}}$ and s_{21} are so. From Lemma 6.1 α) (ii) we have

$$\widetilde{b_{\text{II II}}}(x^0, \tau^0 - i\gamma, \sigma^0) - 1 = -2\sqrt{-i\gamma}\,(\lambda_2 \cdot \det B'' \cdot \alpha^{-1})\,(x^0, \tau^0 - i\gamma, \sigma^0)\cdot R^{-1}(x^0, \tau^0 - i\gamma, \sigma^0)\,.$$

Since $(\lambda_2 \cdot \det B'' \cdot \alpha^{-1})\,(x^0, \tau^0, \sigma^0) \neq 0$ the condition (II) β) and $R(x^0, \tau^0, \sigma^0) = 0$ imply $\widetilde{b_{\text{II II}}}(x^0, \tau^0, \sigma^0) \neq 1$. Thus we see that $Q(x', \mu, \sigma)$ takes real values whenever $\mu \geq 0$. Since Q is analytic in μ at $\mu = 0$ Q always takes real values for real μ.

(ii) Suppose $\tau = \theta(x', \sigma)$ and $Q(x', 0, \sigma) = 0$. Then from (6.1) and (6.4.1) we have $R(x', \theta(x', \sigma), \sigma) = 0$. Hence we have only to prove the converse. Suppose that $R(x', \tau, \sigma) = 0$ and Im $\tau \leq 0$, then from Theorem 4.1 Im $\tau =$ Im $\mu = 0$. First if $\mu < 0$, it follows from (6.1) that Im $s_{21} \neq 0$ and from (i) that Q is real. This contradicts (6.4.1). We shall secondly show that $R(x' \tau, \sigma) = 0$ for $\mu > 0$ is also impossible under the condition (III). Let $R_{\text{II}}(x^0, \tau, \sigma)$ be a Lopatinskii determinant of $(P_{\text{II}}, Q)_{x^0}$. From Lemma 6.3 and Theorem 3 of [11] we easily see the condition (III) implies that for every fixed $x' \in U(x^0)$

$$R_{\text{II}}(x', \tau, \sigma) \neq 0 \quad \text{for} \quad \tau - \theta(x', \tau) > 0 \tag{6.4.2}$$

(for the case (*a*) in Lemma 3.1.)

On the other hand we see in §4 that

$$M_{\text{II}} S_{\text{II}} = \begin{pmatrix} \lambda_{\text{II}}^{+} & \alpha \\ 0 & \lambda_{\text{II}}^{-} \end{pmatrix}.$$

Hence letting S_{II}^{1} be the first column of S_{II} we obtain

$$(6.4.3)\qquad \begin{aligned} R_{II}(x', \tau, \sigma) &= (Q, 1)S_{II}^1 \\ &= Q + s_{21} = R(\det B'')^{-1} \end{aligned}$$

where the last equality follows from (6.4.1). Thus (6.4.2) and (6.4.3) imply that

$$R(x', \tau, \sigma) \neq 0 \quad \text{for} \quad \mu = \tau - \theta(x', \sigma) > 0$$

from which we obtain $\mu = 0$.

If $\mu = 0$ and $Q(x', 0, \sigma) \neq 0$ then we have from (6.1) and (6.4.1)

$$R\big(x', \theta(x', \sigma), \sigma\big) = Q(x', 0, \sigma)(\det B'')(x', 0, \sigma) \neq 0 .$$

This is a contradiction, hence $Q(x', 0, \sigma) = 0$ holds. Thus the proof of (ii) is completed.

LEMMA 6.5. *Assume the conditions (II) β), γ) and (III). Then there exists a neighborhood $U(x^0, \sigma^0)$ such that for every $(x', \sigma) \in U(x^0, \sigma^0)$*

$$-Q(x', 0, \sigma) \geq 0 , \qquad \textit{in the case (a).}$$

COROLLARY 6.2. *Under the same conditions as in Lemma 6.5 we have*

$$\operatorname*{grad}_{(x',\sigma)} Q(x', 0, \sigma) = 0 \quad \text{on} \quad \left\{(x', \sigma) \in \boldsymbol{R}^{2n-1} ; \ Q(x', 0, \sigma) = 0\right\} .$$

PROOF OF LEMMA 6.5. The condition (III) together with Lemma 6.3 implies (6.4.2). On the other hand we have from (6.4.3) and (6.1)

$$R_{II}(x', \tau, \sigma) = Q + \mu s_1^{(1)} + \sqrt{\mu}\, s_2 .$$

Regarding our convention $\sqrt{1} = -1$ we consider the equation $R_{II} = 0$. Then we see

$$(6.5.1)\qquad \sqrt{\mu}\, s_2(x', 0, \sigma) = -Q(x', 0, \sigma) + r(x', \sqrt{\mu}, \sigma)\, \mu$$

where $r(x', \sqrt{\mu}, \sigma)$ is an analytic function in $\sqrt{\mu}$. Now suppose that there exists a sequence (x^n, σ^n) such that

$$\begin{aligned} &(x^n, \sigma^n) \to (x^0, \sigma^0), \quad \text{as} \quad n \to \infty , \\ &-Q(x^n, 0, \sigma^n) < 0 . \end{aligned}$$

Then there exists a sequence $\{\sqrt{\mu^n}\}$ with $\mu^n > 0$ such that (6.5.1) holds for (x^n, μ^n, σ^n), because $s_2(x^n, 0, \sigma^n) > 0$ from (6.2) and $Q(x^0, 0, \sigma^0) = R(x^0, 0, \sigma^0) = 0$ from Lemma 6.1 β) (ii). Putting $\tau^n = \mu^n + \theta(x^n, \sigma^n)$ we have a sequence $\{\tau^n\}$ such that

$$\mu^n > 0 \quad \text{and} \quad R_{II}(x^n, \tau^n, \sigma^n) = 0 .$$

This contradicts (6. 4. 2) and proves the lemma.

PROOF OF COROLLARY 6. 2. Assume that

$$\operatorname*{grad}_{(x',\sigma)} \; Q(x^0, O, \sigma^0) \neq O$$

and for simplicity let

$$\frac{\partial Q}{\partial x_0}(x^0, O, \sigma^0) \neq O\,.$$

Then from the implicit function theorem we have

$$Q(x', O, \sigma) = \left(x_0 - c_1(x'', \sigma)\right) c_2(x', \sigma)$$

where c_1 is C^∞ in (x'', σ), $c_2(x^0, \sigma^0) \neq O$ and c_1, c_2 are real, because $Q(x', O, \sigma)$ is real. This contradicts Lemma 6. 5, hence $\operatorname*{grad}_{(x',\sigma)} Q(x^0, O, \sigma^0) = O$. Furthermore the same argument is valid for the points (x', σ) such that $Q(x', O, \sigma) = O$. Hence the corollary is proved.

REMARK 6. 1. In the case (*b*) in Lemma 3. 1 the discussions of Lemma 6. 4 and 6. 5 are also valid. In this case the conclusion of Lemma 6. 5 becomes $-Q(x, O, \sigma) \leq O$.

REMARK 6. 2. If the coefficients of B are real and the set III is empty, the condition (II) γ) is fulfilled automatically. Because it then holds that

$$\widetilde{b_{\mathrm{II\,II}}} = \det B(h_{\mathrm{I}}^+, h_{\mathrm{II}}^-) \left(\det B(h_{\mathrm{I}}^+, h_{\mathrm{II}}^+)\right)^{-1}.$$

LEMMA 6. 6. *Assume the conditions (II) β) and (III). Then it holds that*

(i) $V_{\mathrm{I}}^-(x', \tau, \sigma) \in L(V_{\mathrm{I}}^+, V_{\mathrm{III}}^+)$ *and*

(ii) $V_{\mathrm{II}}'(x', \tau, \sigma) \in L(V_{\mathrm{III}}^+)$

for such points $(x', \eta, \sigma) \in U(x^0) \times (U(\tau^0, \sigma^0) \cap \boldsymbol{R}^n)$ *that* $\eta = \theta(x', \sigma)$ *and* $Q(x', O, \sigma) = O$.

PROOF. Let (x', η, σ) be the fixed real point satisfying the conditions in Lemma 6. 6.

(i) Then from Theorem 4. 1 it holds that for some $C(x', \tau^0, \sigma^0) > O$

$$\begin{aligned} &\left|\det(V_{\mathrm{I}}^+, V_k^-, V_{\mathrm{III}}^+)(x', \eta - i\gamma, \sigma)\right| \\ &\quad \leq C(x', \tau^0, \sigma^0)\left|\operatorname{Im} \lambda_{\mathrm{II}}^+(x', \eta - i\gamma, \sigma)\right|^{\frac{1}{2}} \\ &\quad \cdot \left|R(x', \eta - i\gamma, \sigma)\right| \cdot \gamma^{-\frac{1}{2}} \end{aligned} \tag{6. 6. 1}$$

for $k \in \mathrm{I}_-$. From Lemma 6. 1 β) (ii) and (6. 1) we have

$$R(x', \tau, \sigma) = \left(Q + \sqrt{\mu}\,(\sqrt{\mu}\, s_{\mathrm{I}}^{(1)} + s_2)\right) \det B''\,.$$

Since $Q(x', 0, \sigma)=0$ we see that

$$Q(x', \mu, \sigma) = \mu c_0(x', \mu, \sigma)$$

with some function. c_0. Hence we have for such a point

$$R(x', \tau, \sigma) = \sqrt{\mu}(\sqrt{\mu}\, c_0 + \sqrt{\mu}\, s_1^{(1)} + s_2)\cdot \det B'',$$

hence

$$\left|R(x', \tau, \sigma)\right| \leq C\left|\tau - \theta(x', \sigma)\right|^{\frac{1}{2}} = C\gamma^{\frac{1}{2}}, \tag{6.6.2}$$

where $C>0$ and $\tau=\eta-i\gamma$. On the other hand

$$\mathrm{Im}\, \lambda_{\mathrm{II}}^{+}(x', \eta-i\gamma, \sigma) \leq C\gamma^{\frac{1}{2}} \tag{6.6.3}$$

with some $C>0$. Hence it follows from (6.6.1), (6.6.2) and (6.6.3) that

$$\det(V_{\mathrm{I}}^{+}, V_{k}^{-}, V_{\mathrm{III}}^{+})(x', \eta, \sigma) = 0, \quad k \in \mathrm{I}_{-},$$

which implies our assersion (i).

(ii) From Theorem 4.1 it holds that

$$\begin{aligned}\Big|\det(V_{\mathrm{I}}^{+}, \cdots, \overset{j}{\check{V}''_{\mathrm{II}}}, \cdots, V_{l-1}^{+}, V'_{\mathrm{II}}, V_{\mathrm{III}}^{+})(x', \eta-i\gamma, \sigma)\Big| \\ \leq C(x', \tau^0, \sigma^0)\left|\mathrm{Im}\, \lambda_{\mathrm{II}}^{-}(x', \eta-i\gamma, \sigma)\right|^{\frac{1}{2}} \\ \cdot\left|R(x', \eta-i\gamma, \sigma)\right| \cdot \gamma^{-\frac{1}{2}},\end{aligned}$$

for $j\in \mathrm{I}_+$. Hence the same argument as in (i) leads us to

$$\det(V_{\mathrm{I}}^{+}, \cdots, \overset{j}{\check{V}''_{\mathrm{II}}}, \cdots, V_{l-1}^{+}, V'_{\mathrm{II}}, V_{\mathrm{III}}^{+})(x', \eta, \sigma) = 0$$

for $j\in \mathrm{I}_+$. On the other hand for such points (x', η, σ) we have

$$\det(V_{\mathrm{I}}^{+}, \cdots, \overset{j}{\check{V}_{i}^{+}}, \cdots, V_{l-1}^{+}, V'_{\mathrm{II}}, V_{\mathrm{III}}^{+}) = 0$$

for $j\in \mathrm{I}_+$ and $i \in \mathrm{I}_+ \cup \mathrm{III}_+$, because $R(x', \eta, \sigma)=0$. Furthermore $\{V_{\mathrm{I}}^{+}, V''_{\mathrm{II}}, V_{\mathrm{III}}^{+}\}$ are linearly independent in $U(x^0)\times U(\tau^0, \sigma^0)$, because of $\det B''(x^0, \tau^0, \sigma^0) \neq 0$. Hence we obtain for any vectors $\{v_i\}\subset \boldsymbol{R}^n$

$$\det(v_1, \cdots, v_{l-1}, V'_{\mathrm{II}}, V_{\mathrm{III}}^{+}) = 0\,.$$

This implies that for such points

$$V'_{\mathrm{II}} \in L(V_{\mathrm{III}}^{+}),$$

by virture of the linear independence of $\{V_{\mathrm{III}}^{+}\}$ and completes the proof of the lemma.

LEMMA 6.7. *Assume the conditions (II) β), γ) and (III). Then* k_{III} and k_{III} in (6.3) are functions in

$C^{\infty}(U(x^0)\times U(0,\sigma^0))$ and analytic in (μ,σ) (Im $\mu\leq 0$), and for some $C>0$ and for every (x',σ) belonging to some neighborhood $U(x^0,\sigma^0)$

$$
|k_{\mathrm{II\,I}}(x',0,\sigma)|^2, \quad \text{and} \quad |k_{\mathrm{I\,II}}(x',0,\sigma)|^2\leq C|Q(x',0,\sigma)|. \tag{6.5}
$$

PROOF. From (6.2.1) and Lemma 6.6 we have for every (x',μ,σ) such that $\mu=\tau-\theta(x',\sigma)=0$ and $Q(x',0,\sigma)=0$

$$
k_{\mathrm{I\,II}}(x',\mu,\sigma)=k_{\mathrm{II\,I}}(x',\mu,\sigma)=0\,.
$$

Hence it holds for every (x',σ) satisfying $Q(x',0,\sigma)=0$ that

$$
k_{\mathrm{I\,II}}(x',0,\sigma)=k_{\mathrm{II\,I}}(x',0,\sigma)=0\,. \tag{6.7.1}
$$

On the other hand, from Lemma 6.1 β) we have that for $(x',\sigma)\in U(x^0,\sigma^0)$

$$
R\big(x',\theta(x',\sigma),\sigma\big)=Q(x',0,\sigma)\cdot\det B''(x',0,\sigma)\,.
$$

Therefore from the condition (II) β), Lemma 6.5 and its corollary, we see that there exists a coordinate system $\{y_1,\cdots,y_N,z_1,\cdots,z_{2n-1-N}\}$ of $\boldsymbol{R}^{2n-1}$ at (x^0,σ^0) such that

$$
\begin{aligned}
Q(x',0,\sigma)&=F(Y,Z)\\
&=\sum_{i,j=1}^{N}\frac{\partial^2 F(0,Z)}{\partial y_i\partial y_j}y_iy_j+G(Y,Z)
\end{aligned}
$$

where $Y=Y(x',\sigma)=(y_1,\cdots,y_N)$, $Z=Z(x',\sigma)=(z_1,\cdots,z_{2n-1-N})$, $Y(x^0,\sigma^0)=0$, $Z(x^0,\sigma^0)=0$,

$$
\frac{\partial^2 F(0,Z)}{\partial y_i\partial y_j}<0 \quad\text{and}\quad |G(Y,Z)|\leq C|Y|^3
$$

with some $C>0$.
Theorefore $Q(x',0,\sigma)=0$ on $Y=0$, where (6.7.1) is valid.
Furthermore from the above form of $Q(x',0,\sigma)$ we see that it satisfies (6.5).
Thus we see our assertions.

§7. Problems for 2×2 system of first order.

This section is devoted to problems for 2×2 system which play an essential role for the proof of the estimate (1.1), in the case where the set II is not empty. In subsection 7.1 a priori estimates for the decomposed problem (P_{II}) in §3 are given by the modification of Kreiss' consideration

[7]. It should be noted that the symmetrizer $R_0(x, \tau, \sigma)$ used for the operator P_{II} in a neighborhood $U(x^0)\times U(\tau^0, \sigma^0)$ of the point (x^0, τ^0, σ^0) is constructed along the 'surface' $\tau=\theta(x, \sigma)$. In subsection 7. 2 we consider a boundary value problem (P_{II}, Q) for 2×2 system, whose main purpose is to see how the essential part of the proof of (1. 1) is analyzed. It will also be seen why the function Q was considered in §6. The point (x^0, τ^0, σ^0) considered in this section is only the one where the set II is not empty (and hence $\sigma^0\neq O$).

7. 1. Let $M_{\mathrm{II}}(x, \mu, \sigma)$ be the matrix defined in Lemma 3. 2 and as in §6 put

$$\tau=\eta-i\gamma,$$
$$\mu=\tau-\theta(x, \sigma)=\varepsilon-i\gamma \quad \text{and}$$
$$\Lambda_\gamma=(|\tau|^2+|\sigma|^2)^{\frac{1}{2}}.$$

Expand M_{II} in $U(x^0)\times U(O, \sigma^0)$ around the surface $\tau=\theta(x, \sigma)$, i.e., $\mu=O$. Then it is seen from Lemma 3. 2 that

$$\begin{aligned} &M_{\mathrm{II}}(x, \mu, \sigma)\\ &=M_{\mathrm{II}}(x, \varepsilon, \sigma)+M_{\mathrm{II}}(x, \varepsilon-i\gamma, \sigma)-M_{\mathrm{II}}(x, \varepsilon, \sigma)\\ &=\overline{M_{\mathrm{II}}}(x, \tau, \sigma)+\varepsilon E(x, \varepsilon, \sigma)+(-i\gamma)H(x, \varepsilon, \sigma)+O(\gamma^2\Lambda_\gamma^{-1}), \end{aligned} \tag{7. 1}$$

where E, H are real valued functions $\in S_+^0$ in (ε, σ) and

$$\overline{M_{\mathrm{II}}}(x, \tau, \sigma)=\begin{pmatrix}\lambda_1(x, O, \sigma), & \Lambda_\gamma\\ O & , \lambda_1(x, O, \sigma)\end{pmatrix},$$

$$E(x, \varepsilon, \sigma)=\begin{pmatrix}e_{11} & e_{12}\\ e_{21} & e_{22}\end{pmatrix},\quad H(x, \varepsilon, \sigma)=\begin{pmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix}$$

and consider the characteristic polynomial

$$\det\left(E_{\mathrm{II}}\lambda-M_{\mathrm{II}}(x, \mu, \sigma)\right)=\left(\lambda-\lambda_{\mathrm{II}}^+(x, \mu, \sigma)\right)\left(\lambda-\lambda_{\mathrm{II}}^-(x, \mu, \sigma)\right),$$

we see from putting $\lambda=\lambda_1(x, O, \sigma)$ in the above equation that

$$\begin{aligned} &(\Lambda_\gamma e_{21})(x, O, \sigma)=(\Lambda_\gamma h_{21})(x, O, \sigma)=\lambda_2(x, O, \sigma)^2>O \qquad \text{in the case } (a),\\ &\text{or}\\ &(\Lambda_\gamma e_{21})(x, O, \sigma)=(\Lambda_\gamma h_{21})(x, O, \sigma)=-\lambda_2'(x, O, \sigma)^2<O \qquad \text{in the case } (b) \end{aligned} \tag{7. 2}$$

with the notations in Lemma 3. 1.

As in [7] let us define in some $U(x^0)\times(U(\tau^0, \sigma^0)\cap(\bar{C}_-\times R^{n-1}))$ a skew symmetric 2×2 matrix R_0 of order O by

$$R_0(x, \tau, \sigma)=i\left(D(x, \tau, \sigma)-i\gamma' F+\varepsilon' B(x, \tau, \sigma)\right),$$

where

$$D(x,\tau,\sigma)=\begin{pmatrix} O & d_1 \\ d^1 & d_2(x,\tau,\sigma)\end{pmatrix},$$

$$F=\begin{pmatrix} O & -f \\ f & O\end{pmatrix},\quad B(x,\tau,\sigma)=\begin{pmatrix} b(x,\tau,\sigma) & O \\ O & O\end{pmatrix},$$

d_1, d_2, f and b are real and in S_+^0.

Choose b so that

(7. 3) $$(D+\varepsilon' B)(C+\varepsilon' E) \quad \text{is a self-adjoint matrix,}$$

where $C=\begin{pmatrix} O & 1 \\ O & O\end{pmatrix}$ and E is the matrix defined above. Then we obtain that for $(\tau,\sigma)\in\overline{\Sigma_-}$

(7. 4) $$b(x,\tau,\sigma)=(1+\varepsilon e_{12})^{-1}(d_1 e_{11}-d_1 e_{22}+d_2 e_{21}).$$

Let $(\tilde{\tau},\tilde{\sigma})$ be one of $(\tilde{\tau}_k,\tilde{\sigma}_k)$ defined by (2. 6) and put

$$\tilde{R}_0(x,\tau,\sigma)=R_0\Big(\tilde{x}(x),\tilde{\tau}(\tau\Lambda_\gamma^{-1},\sigma\Lambda_\gamma^{-1}),\tilde{\sigma}(\tau\Lambda_\gamma^{-1},\sigma\Lambda_\gamma^{-1})\Big)\in S_+^0 .$$

Hereafter, in this section we consider only such functions $u(x)={}^t(u_1,u_2)\in H_{1,\gamma}(\boldsymbol{R}_+^{n+1})$ that $u=\psi(D')\cdot{}^t(v_1,v_2)$, where ${}^t(v_1,v_2)\in H_{1,\gamma}(\boldsymbol{R}_+^{n+1})$ and $\psi(\tau,\sigma)=\psi'(\tau\Lambda_\gamma^{-1},\sigma\Lambda_\gamma^{-1})\in S_+^0$ with supp $\psi'\subset S_{2\varepsilon_0}(\tau^0,\sigma^0)$. Then we have the following

LEMMA 7. 1. ([7]). *One can choose R_0 (i.e., d_1, d_2, f) $\in S_+^0$ so that there exist constants $C, C', \gamma_0>0$ such that for every $\gamma\geq\gamma_0$ and $u(x)$*

(P_{II}) $$\Big\|P_{\text{II}}(x,D)\,u\Big\|_{0,\gamma}^2\geq C\Big(\gamma^2(1\cdot\|u_1\|_{0,\gamma}^2+C'\|u_2\|_{0,\gamma}^2)$$
$$+\gamma\,\text{Im}\,\langle R_0(x',D')\,u,u\rangle_{0,\gamma}\Big),$$

where C' can be taken however large if f is taken large enough, and the symbol of $R_0(x,D')$ is $\tilde{R}_0(x,\tau,\sigma)$ and

$$\begin{aligned}\text{Im}\,&\langle R_0(x',D')\,u,u\rangle_{0,\gamma}\\ &=2\,\text{Re}\,\langle d_1u_2,u_1\rangle_{0,\gamma}+\text{Re}\,\langle d_2u_2,u_2\rangle_{0,\gamma}\\ &\quad-2\gamma\,\text{Im}\,\langle f\Lambda^{-1}u_2,u_1\rangle_{0,\gamma}+\text{Re}\,\langle(\varepsilon b)\,u_1,u_1\rangle_{0,\gamma}.\end{aligned}$$

In order to avoid the ambiguity in the use of this lemma in what follows, we sketch the

PROOF. Estimating the commutator, we have

$$\begin{aligned}2\,\text{Re}\,&(P_{\text{II}}u,R_0u)_{0,\gamma}\\ &\geq\text{Im}\,\langle R_0(x',D')\,u,u\rangle_{0,\gamma}+\Big(\big((R_0M_{\text{II}})+(R_0M_{\text{II}})^*\big)\,u,u\Big)_{0,\gamma}\\ &\quad-C\Big(\|u\|_{0,\gamma}+|u|_{-\frac{1}{2},\gamma}\Big)\end{aligned}$$

with $C>0$ independent of γ. The symbol of $((R_0M_{II})+(R_0M_{II})^*)(x, D')$ is

$$R_0\left(\tilde{x}(x), \tilde{\tau}, \tilde{\sigma}\right) M_{II}\left(\tilde{x}(x), \tilde{\tau}, \tilde{\sigma}\right)\cdot \Lambda_\gamma$$
$$-M_{II}^*\left(\tilde{x}(x), \tilde{\tau}, \tilde{\sigma}\right) R_0\left(\tilde{x}(x), \tilde{\tau}, \tilde{\sigma}\right)\cdot \Lambda_\gamma .$$

Using (7. 1) and (7. 3) to calculate the symbol, we obtain for any $(x, \tau', \sigma')\in U(x^0)\times(U(\tau^0, \sigma^0)\cap(\overline{\Sigma_-}))$ and for any constant vector $u={}^t(u_1, u_2)$

$$\begin{aligned}&\left\langle(R_0M_{II}-M_{II}^*R_0)(x, \tau', \sigma')\, u, u\right\rangle\\ &\quad\geq -C_1\gamma'(\varepsilon'+\gamma')\,|u|^2\\ &\quad\quad+\gamma' \operatorname{Re}\Big\{\langle d_1h_{21}u_1, u_1\rangle+\langle d_1h_{22}u_2, u_1\rangle\\ &\quad\quad+\left\langle(d_1h_{11}+d_2h_{21})\,u_1, u_2\right\rangle+\left\langle(d_1h_{12}+d_2h_{22}+f)\,u_2, u_2\right\rangle\Big\}\end{aligned}$$

where $C_1=C_1(D, F)>0$ does not depend on (x, τ', σ'). For any $\delta>0$ and $(x, \tau', \sigma')\in U(x^0)\times U(\tau^0, \sigma^0)\cap\overline{\Sigma_-})$ we have

$$\begin{aligned}&\left|\langle d_1h_{22}u_2, u_1\rangle+\langle d_1h_{11}+d_2h_{21})\,u_1, u_2\rangle\right|\\ &\quad\leq C_2\left(\delta\,|u_1|^2+\delta^{-1}\,|u_2|^2\right),\end{aligned}$$

with some $C_2=C_2(D)>0$. Choose $U(\tau^0, \sigma^0)$ sufficiently small, and $d_1=\text{const}$ and $f=\text{const}$ such that for $(x, \tau', \sigma')\in U(x^0)\times(U(\tau^0, \sigma^0)\cap\overline{\Sigma_-}$

(7. 1. 1) $$d_1h_{21}(x, \varepsilon', \sigma')\geq 2\,,$$

(7. 1. 2) $$d_1h_{12}+d_2h_{22}+f-\delta^{-1}C_2>0\,.$$

(We see from (7. 1. 2) that we can choose d_2 arbitrarily if choosing $f>0$ large enough.) Since $(\tilde{x}, \tilde{\tau}, \tilde{\sigma})\in S_{\frac{5}{2}\varepsilon_0}(x^0)\times S_{8\varepsilon_0}(\tau^0, \sigma^0)$ we see from (2. 6)' that for every

$$\begin{aligned}&(x, \tau, \sigma)\in\overline{\boldsymbol{R}_+^{n+1}}\times\overline{\boldsymbol{C}_-}\times\boldsymbol{R}^{n-1}\\ &\quad\left\langle(R_0M_{II}-M_{II}^*R_0)(\tilde{x}, \tilde{\tau}, \tilde{\sigma})\,\Lambda_\gamma u, u\right\rangle\\ &\quad\quad\geq -\tilde{\gamma}\Lambda_\gamma(\tilde{\varepsilon}+\tilde{\gamma})\,C_1\,|u|^2+\tilde{\gamma}\Lambda_\gamma\left(\frac{3}{2}\,|u_1|^2+(C'+1)\,|u_2|^2\right)\\ &\quad\quad\geq\left(\frac{3}{2}-12\varepsilon_0C_1\right)\tilde{\gamma}\Lambda_\gamma\,|u_1|^2+(C'+1-12\varepsilon_0C_1)\,\tilde{\gamma}\Lambda_\gamma\,|u_2|^2\,,\end{aligned}$$

where $C'>0$ is a constant such that

(7. 1. 3) $$C'+1\geq d_1h_{11}+d_2h_{22}+f-\delta^{-1}C_2\,.$$

Since supp $\psi'\subset S_{2\varepsilon_0}(\tau^0, \sigma^0)$ it follows from (2. 6)' that $\tilde{\gamma}\Lambda_\gamma\psi'=\gamma\psi'$. Hence we see from Lemma 2. 1 β) (v) that for large γ

$$\mathrm{Re}\Big(\big((R_0M_{\mathrm{II}})+(R_0M_{\mathrm{II}})^*\big)(x, D')\,u, u\Big)_{0,\gamma}$$
$$\geq \gamma\Big(1\cdot\|u_1\|_{0,\gamma}^2+C'\,\|u_2\|_{0,\gamma}^2\Big)$$

with some $C'>0$. Hence from (7.10) we obtain

$$(\delta\gamma)^{-1}\|P_{\mathrm{II}}u\|_{0,\gamma}^2+\delta\gamma\,\|u\|_{0,\gamma}^2\geq\gamma\Big(1\cdot\|u_1\|_{0,\gamma}^2+C'\|u_2\|_{0,\gamma}^2\Big)$$
$$+\mathrm{Im}\Big\langle R_0(x', D')\,u, u\Big\rangle_{0,\gamma}.$$

Choosing $\delta>0$ small we complete the proof.

In the above proof, choose R_0 with d_1, f satisfying (7.1.1), (7.1.2) and $d_2=\mathrm{const}>0$ sufficiently large. Then we obtain the following

COROLLARY 7.1.

(i) *There exists a constant* γ_0 *such that for every* $\gamma\geq\gamma_0$ *and* $C_1>0$ *there exist constants* $C_2, C_3>0$ *satisfying*

$$(P_{\mathrm{II}})' \qquad C_2\|P_{\mathrm{II}}u\|_{0,\gamma}^2+C_1\gamma\,|u_1|_{0,\gamma}^2\geq C_3\gamma^2\|u\|_{0,\gamma}^2+1\cdot\gamma\,|u_2|_{0,\gamma}^2$$

for every $u(x)$.

(ii) *There exist a constant* γ_0 *and a neighborhood* $U(\tau^0, \sigma^0)$ *such that for every* $\gamma\geq\gamma_0$, $(\tau\Lambda_\gamma^{-1}, \sigma\Lambda_\gamma^{-1})\in U(\tau^0, \sigma^0)\cap\overline{\Sigma_-}$, $C_1>0$ *and* $\hat{u}={}^t(\hat{u}_1, \hat{u}_2)(x_n)\in H_1(\boldsymbol{R}_+^1)$ *there exist constants* $C_2, C_3>0$ *satisfying*

$$(P_{\mathrm{II}})'_{x^0} \qquad C_2\Big\|P_{\mathrm{II}}(x^0, \tau, \sigma, D_n)\,\hat{u}(\cdot)\Big\|^2+C_1\gamma\,|\hat{u}_1|^2$$
$$\geq C_3\gamma^2\Big\|\hat{u}(\cdot)\Big\|^2+1\cdot\gamma\,|\hat{u}_2|^2.$$

7.2. Let $Q(x', \mu, \sigma)$ be the function defined in Lemma 6.1 β) (i) and assume the conditions (II) β), γ). Then it follows from Lemmas 6.4 (i), 6.1 β) (i) and 6.5 (or Remark 6.1) that for $(x, \mu, \sigma)\in U(x^0)\times U(0, \sigma^0)$

(7.5) $\qquad Q(x', \mu, \sigma)$ is real valued for real μ,

(7.6) $\qquad Q(x^0, 0, \sigma^0)=0$,

(7.7) $\qquad -Q(x', 0, \sigma)\geq 0 \qquad$ in the case (a),

or $\qquad \leq 0 \qquad$ in the case (b).

Let $Q(x', D')$ be the operator whose symbol is the extension $\tilde{Q}(x', \tau, \sigma)\in S_+^0$ of $Q(x', \tau-\theta(x', \sigma), \sigma)$. We consider the following boundary value problem (P_{II}, Q) for 2×2 system:

$$(P_{\mathrm{II}}, Q)\begin{cases} P_{\mathrm{II}}(x, D)\,u=\big(D_n-M_{\mathrm{II}}(x, D')\big)\cdot{}^t(u_1, u_2) & \\ \qquad =f(x)={}^t(f_1, f_2) & \text{in } \boldsymbol{R}_+^{n+1}, \quad (7.8)\\ Q(x', D')\,u_1(x', 0)+u_2(x', 0)=g(x') & \text{on } \boldsymbol{R}^n. \quad (7.9)\end{cases}$$

THEOREM 7.1. *Assume the conditions* (7.5), (7.6) *and* (7.7). *Then there exist constants* $C, \gamma_0>0$ *such that for every* $\gamma\geq\gamma_0$ *and* $u(x)$ *the following a priori estimate for the problem* (P_{II}, Q) *holds*:

$$(P_{II}, Q) \qquad \|P_{II}u\|_{0,\gamma}+|Qu_1+u_2|_{\frac{1}{2},\gamma}\geq C\gamma\|u\|_{0,\gamma}.$$

To prove this we need the following lemmas. The technique used in the proof is also used in §8 for the proof of (1.1).

LEMMA 7.2. *Let* (7.8) *be satisfied for* $u(x)$. *Then for every* $\delta>0$ *one can choose* ε_0 (*or the size of supp* ψ') *and a constant* $\gamma_0>0$ *such that for every* $\gamma\geq\gamma_0$ *and* $u(x)$

$$(7.10) \qquad |\Lambda^{-\frac{1}{2}}u_1|_{0,\gamma}^2\leq\delta\|u_1\|_{0,\gamma}^2+\delta^{-1}\|u_2\|_{0,\gamma}^2+\delta^{-1}\|\Lambda^{-1}f_1\|_{0,\gamma}^2$$

consequently, let $\gamma_0(D')$ *be the operator with its symbol* $\tilde{\gamma}_0(\tau\Lambda_\gamma^{-1}, \sigma\Lambda_\gamma^{-1})\in S_+^0$, *then*

$$(7.11) \qquad \left\langle\gamma_0(D')u_1, u_1\right\rangle_{0,\gamma}\leq\gamma\delta\|u_1\|_{0,\gamma}^2+\gamma\delta^{-1}\|u_2\|_{0,\gamma}^2 +\gamma\delta^{-1}\|\Lambda^{-1}f_1\|_{0,\gamma}^2.$$

LEMMA 7.3. *Let* $R_0(x', D')$ *be the one in Lemma* 7.1 *and assume* (7.9). *Then there exist constants* $C>0, \gamma_0>0$ *such that for any* $\delta>0, \gamma\geq\gamma_0$ *and* $u(x)$

$$\begin{aligned}\operatorname{Im}&\left\langle R_0(x', D')u, u\right\rangle_{0,\gamma}\\ &\geq -C\Big((\delta\gamma)^{-1}|g|_{\frac{1}{2},\gamma}^2+\delta\gamma|u_1|_{-\frac{1}{2},\gamma}^2+|u_1|_{-\frac{1}{2},\gamma}^2\Big)\\ &\quad+\operatorname{Re}\left\langle R_1(x', D')u_1, u_1\right\rangle_{0,\gamma},\end{aligned}$$

where the symbol of $R_1\in S_+^0$ *is represented in* $U(x^0)\times(U(\tau^0, \sigma^0)\cap\overline{\Sigma_-})$ *by*

$$(7.12) \qquad \begin{aligned}R_1(x', \tau', \sigma')&=-2d_1\operatorname{Re}Q+d_2|Q|^2+\varepsilon'b+2\gamma'f\operatorname{Im}Q\\ &=d_1\Big(-2\operatorname{Re}Q+(1+\varepsilon'e_{12})^{-1}(e_{11}-e_{22})\varepsilon'\Big)\\ &\quad+d_2\Big(|Q|^2+(1+\varepsilon'e_{12})^{-1}e_{21}\varepsilon'\Big)+2\gamma'f\operatorname{Im}Q\end{aligned}$$

(Q *is allowed to take complex values.*)

LEMMA 7.4. *Assume the same conditions as in Theorem* 7.1 *and let* (7.9) *be satisfied for* $u(x)$. *Then one can choose* $R_1\in S_+^0$ (*especially* d_1 *and* d_2) *so that there exist constants* $C, C'', \gamma_0>0$ *such that for every* $\gamma\geq\gamma_0$ *and* $u(x)$

$$\operatorname{Re}\left\langle R_1(x', D')u_1, u_1\right\rangle_{0,\gamma}\geq -C|u_1|_{-\frac{1}{2},\gamma}^2-C''\left\langle\gamma_0(D')u_1, u_1\right\rangle_{0,\gamma}.$$

Here C'' *depends only on* (x^0, τ^0, σ^0) *and* f, d_1, d_2 *but not on the size of the support of* ψ'.

Using these lemmas we first prove Theorem 7.1. The proofs of these lemmas are given after that.

PROOF OF THEOREM 7.1. First choose d_1 so that it satisfies (7.1.1) and (7.4.10) and the expression (7.4.7) are nonnegative. Secondly choose $d_2(x, \varepsilon', \sigma')=d_2^{(0)}(x', \sigma')+\varepsilon' d_2^{(1)}$ satisfying (7.4.5) and (7.4.6) with $C_0>1$. We fix d_1 and d_2. Then Lemma 7.3 and 7.4 are valid for every f with their constants depending on f. Furthermore if f is chosen so that it satisfies (7.1.2), Lemma 7.1 is valid and we obtain

$$\begin{aligned}&\|P_{\rm II}u\|_{0,\gamma}^2\\ &\quad\geq C\Big(\gamma^2\big(\|u_1\|_{0,\gamma}^2+C'\|u_2\|_{0,\gamma}^2\big)+\gamma\,\mathrm{Im}\,\langle R_0u, u\rangle_{0,\gamma}\Big)\\ &\quad\geq C\Big(\gamma^2\big(\|u_1\|_{0,\gamma}^2+C'\|u_2\|_{0,\gamma}^2\big)-C''\gamma\big\langle\gamma_0(D')\,u_1, u_1\big\rangle_{0,\gamma}\\ &\qquad-C_1\big(\delta_1^{-1}|g|_{\frac{1}{2},\gamma}^2+\delta_1\gamma^2|u_1|_{-\frac{1}{2},\gamma}^2+\gamma|u_1|_{-\frac{1}{2},\gamma}^2\big)\\ &\qquad-C_2\gamma|u_1|_{-\frac{1}{2},\gamma}^2\Big),\end{aligned}$$

whepe C_1, C_2, C' and C'' depend on f and $\delta_1>O$ can be taken arbitrarily. Hence it follows from (7.10) with fixed δ that

$$\begin{aligned}&C\big(\|P_{\rm II}u\|_{0,\gamma}^2+C_1\delta_1^{-1}|g|_{\frac{1}{2},\gamma}^2\big)\\ &\quad\geq\gamma^2(1-\delta_1C_3-C_4\gamma^{-1})\,\|u_1\|_{0,\gamma}^2\\ &\qquad+\gamma^2(C'-\delta_1C_3-C_4\gamma^{-1})\,\|u_2\|_{0,\gamma}^2\\ &\qquad-C''\gamma\big\langle\gamma_0(D')\,u_1, u_1\big\rangle_{0,\gamma},\end{aligned}$$

where C_3 and C_4 depend on f. Using (7.11) we obtain for any $\delta>O$ depending on ε_0

$$\begin{aligned}&C\big(\|P_{\rm II}u\|_{0,\gamma}^2+|g|_{\frac{1}{2},\gamma}^2\big)\\ &\quad\geq\gamma^2(1-\delta_1C_3-C_4\gamma^{-1}-\delta C'')\,\|u_1\|_{0,\gamma}^2\\ &\qquad+\gamma^2(C'-\delta_1C_3-C_4\gamma^{-1}-\delta^{-1}C'')\,\|u_2\|_{0,\gamma}^2\\ &\qquad\equiv\gamma^2\big(C_5\|u_1\|_{0,\gamma}^2+C_6\|u_2\|_{0,\gamma}^2\big).\end{aligned}$$

Since d_1 and d_2 are fixed we see from (7.1.3) and (7.4.11) that in order to make C_5 and C_6 positive it suffices to choose f, δ and δ_1 so that

$$1-\delta_1C_3-C_4\gamma^{-1}-\delta\,|2f\,\mathrm{Im}\,Q-2d_2\,\mathrm{Im}\,Q^{(1)}|>O\,,$$

and

$$f-\delta_1C_3-C_4\gamma^{-1}-\delta^{-1}|2f\,\mathrm{Im}\,Q-2d_2\,\mathrm{Im}\,Q^{(1)}|$$

is sufficiently large.

The process of the choice is as follows:

(1) Choose δ (hence ε_0) such that $\delta\cdot|2d_2\,\mathrm{Im}\,Q^{(1)}|<\frac{1}{4}$ and fix the δ.

(2) Choose f so that $f-\delta^{-1}|2d_2 \operatorname{Im} Q^{(1)}|$ is sufficiently large.

(3) Since $Q(x^0, 0, \sigma^0)=0$ we can choose such a small ε_0 that in $S_{8\varepsilon_0}(\tau^0, \sigma^0)$

$$\delta\cdot|2f \operatorname{Im} Q|<\frac{1}{4},$$
$$f-\delta^{-1}|2f \operatorname{Im} Q|-\delta^{-1}|2d_2 \operatorname{Im} Q^{(1)}| \qquad \text{remains large.}$$

(4) Choose δ_1 and γ_0 so that

$$\delta_1 C_3+C_4\gamma_0^{-1}<\frac{1}{4}.$$

Thus we obtain C_5 and C_6 larger than $\frac{1}{4}$ and complete the proof.

PROOF OF LEMMA 7.2. From integration by parts it holds that

$$|\Lambda^{-\frac{1}{2}}u_1|_{0,\gamma}^2=\langle\Lambda^{-1}u_1, u_1\rangle_{0,\gamma}=2\operatorname{Im}(\Lambda^{-1}D_n u_1, u_1)_{0,\gamma}.$$

It follows from Lemma 3.2 and (7.8) that

$$D_n u_1=\lambda_1^{(0)}u_1+\Lambda_0^{(0)}u_2+(\varepsilon'-i\gamma')p_{11}\Lambda u_1+(\varepsilon'-i\gamma')p_{12}\Lambda u_2+f_1,$$

where $\lambda_1^{(0)}$, ε', γ', p_{11} and p_{12} are of order 0 and the symbol of $\lambda_1^{(0)}$ is $\lambda_1(\tilde{x}, 0, \tilde{\sigma})$ and real. Hence we see from Lemma 2.1 α) (iv) that

$$\begin{aligned}|\Lambda^{-\frac{1}{2}}u_1|_{0,\gamma}^2\le 2\operatorname{Im}\Big\{&(\Lambda_0^{(0)}\Lambda^{-1}u_2, u_1)_{0,\gamma}+(\Lambda^{-1}f_1, u_1)_{0,\gamma}\\&+\big((\varepsilon'-i\gamma')p_{11}u_1, u_1\big)_{0,\gamma}+\big((\varepsilon'-i\gamma')p_{12}u_2, u_1\big)_{0,\gamma}\Big\}\\&+C\gamma^{-1}\|u\|_{0,\gamma}^2,\end{aligned}$$

where the last term of the right arises from commutators and $C>0$ depend only on M_{II}. Since $(\tilde{x}, \tilde{\tau}, \tilde{\sigma})\in S_{\frac{5}{2}\varepsilon_0}(x^0)\times S_{8\varepsilon_0}(\tau^0, \sigma^0)$ we see from (2.6)′ that

$$\big|\varepsilon'(\tilde{x}, \tilde{\tau}, \tilde{\sigma})\big|\le 8\varepsilon_0 \quad\text{and}\quad \gamma'(\tilde{\tau}, \tilde{\sigma})\le 4\varepsilon_0.$$

Consequently the first inequality follows from Lemma 2.1 β) (vi) if ε_0 is taken sufficiently small. The second one follows from the first one because of $\gamma_0(D')\cdot\phi(D')=\gamma_0(D')\cdot\Lambda\cdot\Lambda^{-1}\cdot\phi(D')=\gamma\Lambda^{-1}\phi(D')$.

PROOF OF LEMMA 7.3. From (7.9) we have

$$u_2(x', 0)=g(x')-Q(x', D')\,u_1(x', 0).$$

Hence it follows from the definition of R_0 that

$$\begin{aligned}&\operatorname{Im}\big\langle R_0(x', D')\,u, u\big\rangle_{0,\gamma}\\&\quad=2d_1\operatorname{Re}\langle g-Qu_1, u_1\rangle_{0,\gamma}+\operatorname{Re}\big\langle d_2\cdot(g-Qu_1), g-Qu_1\big\rangle_{0,\gamma}\\&\qquad-2\gamma f\operatorname{Im}\big\langle\Lambda^{-1}\cdot(g-Qu_1), u_1\big\rangle_{0,\gamma}+\operatorname{Re}\big\langle(\varepsilon b)\,u_1, u_1\big\rangle_{0,\gamma}\end{aligned}$$

$$= 2d_1 \operatorname{Re} \langle g, u_1\rangle_{0,\gamma} + \operatorname{Re} \langle d_2 g, g\rangle_{0,\gamma} - 2\gamma f \operatorname{Im} \langle \Lambda^{-1} g, u_1\rangle_{0,\gamma}$$
$$- \operatorname{Re} \langle d_2 \cdot Q u_1, g\rangle_{0,\gamma} - \operatorname{Re} \langle d_2 g, Q u_1\rangle_{0,\gamma}$$
$$- 2d_1 \operatorname{Re} \langle Q u_1, u_1\rangle_{0,\gamma} + \operatorname{Re} \langle d_2 \cdot Q u_1, Q u_1\rangle_{0,\gamma}$$
$$+ 2\gamma f \operatorname{Im} \langle \Lambda^{-1} \cdot Q u_1, u_1\rangle_{0,\gamma} + \operatorname{Re} \left\langle (\varepsilon b)\, u_1, u_1\right\rangle_{0,\gamma}$$

Noting that $d_2, Q \in S^0_+$, we have the following estimates with respect to the first five terms of the right hand side of the above equality:

$$\left|\langle g, u_1\rangle_{0,\gamma}\right| = \left|\langle \Lambda^{\frac{1}{2}} g, \Lambda^{-\frac{1}{2}} u_1\rangle_{0,\gamma}\right|$$
$$\leq |g|_{\frac{1}{2},\gamma} |u_1|_{-\frac{1}{2},\gamma} \leq \delta\gamma\, |u_1|^2_{-\frac{1}{2},\gamma} + (\delta\gamma)^{-1} |g|^2_{\frac{1}{2},\gamma},$$
$$\left|\langle d_2 g, g\rangle_{0,\gamma}\right| \leq C |g|^2_{0,\gamma} \leq C\gamma^{-1} |g|^2_{\frac{1}{2},\gamma},$$
$$\left|\gamma \langle \Lambda^{-1} g, u_1\rangle_{0,\gamma}\right| \leq \gamma\, |g|_{-\frac{1}{2},\gamma} |u_1|_{-\frac{1}{2},\gamma}$$
$$\leq \delta\gamma\, |u_1|^2_{-\frac{1}{2},\gamma} + (\delta\gamma)^{-1} |g|^2_{\frac{1}{2},\gamma},$$
$$\left|\langle d_2 \cdot Q u_1, g\rangle_{0,\gamma}\right|,\ \left|\langle d_2 g, Q u_1\rangle_{0,\gamma}\right| \leq C |g|_{\frac{1}{2},\gamma} |u_1|_{-\frac{1}{2},\gamma}$$
$$\leq \delta\gamma\, |u_1|^2_{-\frac{1}{2},\gamma} + (\delta\gamma)^{-1} |g|^2_{\frac{1}{2},\gamma}.$$

Hence we obtain for large γ

$$\operatorname{Im} \left\langle R_0(x', D')\, u, u\right\rangle_{0,\gamma}$$
$$\geq -C(d_1, d_2, f) \cdot \left(\delta\gamma\, |u_1|^2_{-\frac{1}{2},\gamma} + (\delta\gamma)^{-1} |g|^2_{\frac{1}{2},\gamma}\right)$$
$$- 2d_1 \operatorname{Re} \langle Q u_1, u_1\rangle_{0,\gamma} + \operatorname{Re} \left\langle (d_2 Q^* Q)\, u_1, u_1\right\rangle_{0,\gamma}$$
$$+ 2\gamma f \operatorname{Im} \left\langle (Q\Lambda^{-1})\, u_1, u_1\right\rangle_{0,\gamma} + \operatorname{Re} \left\langle (\varepsilon b)\, u_1, u_1\right\rangle_{0,\gamma}$$
$$- C(d_2, f, Q)\, |u_1|^2_{-\frac{1}{2},\gamma}$$

where the last term described above arises from the commutators. Thus the lemma follows from (7.4).

Proof of Lemma 7.4. For the sake of the generalization (see subsection 10.2), assume that Q is complex valued. From Taylor's expansion we have

$$Q(x', \mu', \sigma') = Q^{(0)}(x', \sigma') + \mu' Q^{(1)}(x', \sigma') + (\mu')^2 Q^{(2)}(x', \mu', \sigma'),$$
$$e_{ij}(x', \varepsilon', \sigma') = e^{(0)}_{ij}(x', \sigma') + \varepsilon' e^{(1)}_{ij}(x', \varepsilon', \sigma'), \qquad (i, j = 1, 2),$$
$$\left(1 + \varepsilon' e_{12}(x', \varepsilon', \sigma')\right)^{-1} = 1 - \varepsilon' e^{(0)}_{12}(x', \sigma') + O\left((\varepsilon')^2\right),$$

where $Q^{(0)} = Q(x', 0, \sigma')$, $Q^{(1)} = \frac{\partial Q}{\partial \mu}(x', 0, \sigma')$ and $e^{(0)}_{ij} = e_{ij}(x', 0, \sigma')$. Since μ'

$=\varepsilon'-i\gamma'$, from the definition (7.12) of R_1 we see that for $(x', \tau', \sigma')\in U(x^0)\times(U(\tau^0, \sigma^0)\cap\overline{\Sigma_-})$

$$
\begin{aligned}
R_1(x', \mu', \sigma') = {} & 2\gamma' f\,\mathrm{Im}\,Q \\
& -2d_1\Big(\mathrm{Re}\,Q^{(0)}+\varepsilon'\,\mathrm{Re}\,Q^{(1)}+\gamma'\,\mathrm{Im}\,Q^{(1)} \\
& \qquad +\big((\varepsilon')^2-(\gamma')^2\big)\,\mathrm{Re}\,Q^{(2)}+2\varepsilon'\gamma'\,\mathrm{Im}\,Q^{(2)}\Big) \\
& +d_2\Big\{|Q^{(0)}|^2+\big((\varepsilon')^2+(\gamma')^2\big)\,|Q^{(1)}|^2+\big((\varepsilon')^2+(\gamma')^2\big)^2\,|Q^{(2)}|^2 \\
& \qquad +2\,\mathrm{Re}\,(\varepsilon'+i\gamma')\,\overline{Q^{(1)}}\,Q^{(0)} \\
& \qquad +2\,\mathrm{Re}\,\big((\varepsilon')^2-(\gamma')^2+i2\varepsilon'\gamma'\big)\,\overline{Q^{(2)}}\,Q^{(0)} \\
& \qquad +2\,\mathrm{Re}\,(\varepsilon'-i\gamma')\big((\varepsilon')^2-(\gamma')^2+i2\varepsilon'\gamma'\big)\,\overline{Q^{(2)}}Q^{(1)}\Big\} \\
& +\varepsilon'\Big(1-\varepsilon' e_{12}^{(0)}+O\big((\varepsilon')^2\big)\Big)\Big\{d_1(e_{11}^{(0)}-e_{22}^{(0)})+d_2e_{21}^{(0)} \\
& \qquad +d_1\varepsilon'(e_{11}^{(1)}-e_{22}^{(1)})+d_2\varepsilon' e_{21}^{(1)}\Big\}.
\end{aligned}
$$

Put $d_2(x, \varepsilon'-i\gamma', \sigma')=d_2^{(0)}(x', \sigma')+\varepsilon' d_2^{(1)}$ with some real $d_2^{(0)}$ and $d_2^{(1)}(=\mathrm{const})$. Then the above equals to

$$
\begin{aligned}
& =\gamma' C_1+\Big(-2d_1\,\mathrm{Re}\,Q^{(0)}+d_2^{(0)}\,|Q^{(0)}|^2\Big) \\
& \quad +\varepsilon'\Big(d_2^{(1)}\,|Q^{(0)}|^2-d_1\big(2\,\mathrm{Re}\,Q^{(1)}-(e_{11}^{(0)}-e_{22}^{(0)})\big) \\
& \qquad +d_2^{(0)}(e_{21}^{(0)}+2\,\mathrm{Re}\,\overline{Q^{(1)}}\,Q^{(0)})\Big) \qquad (7.4.1) \\
& \quad +(\varepsilon')^2\Big(d_2^{(1)}(e_{21}^{(0)}+2\,\mathrm{Re}\,\overline{Q^{(1)}}\,Q^{(0)})+k_1d_1+k_2d_2^{(0)}\Big) \\
& \qquad +O\big(|\varepsilon'|^3\big),
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 = {} & 2f\,\mathrm{Im}\,Q-2d_1(\mathrm{Im}\,Q^{(1)}+2\varepsilon'\,\mathrm{Im}\,Q^{(2)}-\gamma'\,\mathrm{Re}\,Q^{(2)}) \\
& +d_2\Big\{-2\,\mathrm{Im}\,\overline{Q^{(1)}}\,Q^{(0)}+2\,\mathrm{Re}\,(-\gamma'+i2\varepsilon')\,\overline{Q^{(2)}}\,Q^{(0)} \\
& +\gamma'\big(2(\varepsilon')^2+(\gamma')^2\big)\,|Q^{(2)}|^2+\gamma'\,|Q^{(1)}|^2 \\
& +2\,\mathrm{Re}\,\big(-i(\overline{\mu'})^2-\varepsilon'\gamma'+i2(\varepsilon')^2\big)\,\overline{Q^{(2)}}\,Q^{(1)}\Big\},
\end{aligned}
$$

and k_1, k_2 do not depend on d_1, d_2 and f:

$$
\begin{aligned}
k_1 &= -2\,\mathrm{Re}\,Q^{(2)}-e_{12}^{(0)}(e_{11}^{(0)}-e_{22}^{(0)})+e_{11}^{(1)}-e_{22}^{(1)}, \\
k_2 &= |Q^{(1)}|^2+2\,\mathrm{Re}\,\overline{Q^{(2)}}\,Q^{(0)}-e_{12}^{(0)}e_{21}^{(0)}+e_{21}^{(1)}.
\end{aligned}
$$

Now we shall determine $d_1, d_2^{(0)}$ and $d_2^{(1)}$ as follows:

$$-2d_1\,\mathrm{Re}\,Q^{(0)}+d_2^{(0)}\,|Q^{(0)}|^2\geq 0, \qquad (7.4.2)$$

(7.4.3) the coefficient of ε' is zero,

(7.4.4) the coefficient of $(\varepsilon')^2 \equiv C_0 > 1$.

From (7.4.3) we obtain

$$d_2^{(0)} = -k_3^{-1}\left(d_2^{(1)}|Q^{(0)}|^2 - d_1\left(2\operatorname{Re} Q^{(1)} - (e_{11}^{(0)} - e_{22}^{(0)})\right)\right), \tag{7.4.5}$$

where $k_3 \equiv e_{21}^{(0)} + 2\operatorname{Re}\overline{Q^{(1)}}Q^{(0)}$ does not vanish because of $e_{21}^{(0)} \neq 0$ ((7.2)) and $Q^{(0)}(x^0, \sigma^0) = 0$. Insert this into C_0, then

$$\begin{aligned} C_0 = {} & d_2^{(1)}\left(k_3 - k_2 k_3^{-1}|Q^{(0)}|^2\right) \\ & + d_1\left(k_1 + k_2 k_3^{-1}\left(2\operatorname{Re} Q^{(1)} - (e_{11}^{(0)} - e_{22}^{(0)})\right)\right). \end{aligned} \tag{7.4.6}$$

Let us show that it is possible to choose d_1 so that (7.4.2) is fulfilled. From (7.4.5) and (7.4.6) we have for some fixed $C_0 > 1$

$$\begin{aligned} & -2d_1 \operatorname{Re} Q^{(0)} + d_2^{(0)}|Q^{(0)}|^2 \\ & \quad = d_1\left(-2\operatorname{Re} Q^{(0)} + k_3^{-1}|Q^{(0)}|^2\left(2\operatorname{Re} Q^{(1)} - (e_{11}^{(0)} - e_{22}^{(0)})\right)\right) \\ & -k_3^{-1}|Q^{(0)}|^4 d_2^{(1)} \\ & \quad = k_3^{-2}\left(1 - k_3^{-2}k_2|Q^{(0)}|^2\right)^{-1}\left(k_4 d_1 - C_0|Q^{(0)}|^4\right), \end{aligned} \tag{7.4.7}$$

where

$$\begin{aligned} k_4 = {} & -2k_3^2 \operatorname{Re} Q^{(0)} + |Q^{(0)}|^2\left(2\operatorname{Re} Q^{(1)} - (e_{11}^{(0)} - e_{22}^{(0)})\right)k_3 \\ & + |Q^{(0)}|^2\left(2k_2 \operatorname{Re} Q^{(0)} + k_1|Q^{(0)}|^2\right). \end{aligned}$$

The condition (7.5) implies that for any small $\delta > 0$ there exists a neighborhood $U(x^0) \times U(\sigma^0)$ such that

$$\delta|\operatorname{Re} Q^{(0)}| \geq |Q^{(0)}|^2 \geq |Q^{(0)}|^4 \qquad \text{in} \quad U(x^0) \times U(\sigma^0). \tag{7.4.8}$$

Hence it holds for some $C > 0$ that

$$C_0|Q^{(0)}|^4 \leq C|k_4| \qquad \text{in} \quad U(x^0) \times U(\sigma^0). \tag{7.4.9}$$

Note that for d_1 satifying (7.1.1) we have from (7.2) and (7.7)

$$k_4 d_1 \geq -C d_1 \operatorname{Re} Q^{(0)} \geq 0 \tag{7.4.10}$$

with some $C > 0$, because of $e_{12}^{(0)} = h_{21}(x, 0, \sigma)$. Then it is seen from (7.4.9) and (7.4.10) there exists a real number d_1 (satisfying (7.1.1)) such that

$$k_4 d_1 - C_0|Q^{(0)}|^4 \geq 0 \qquad \text{in} \quad U(x^0) \times U(\sigma^0).$$

Since $k_3^{-2}(1 - k_3^{-2}k_2|Q^{(0)}|^2)^{-1}$ in (7.4.7) is positive, we thus obtain d_1 satisfying (7.4.2.)

Since $k_3 \neq O$ and $Q(x^0, O, \sigma^0)=O$, a real number $d_2^{(1)}$ is determined by (7. 4. 6) so that (7. 4. 4) is valid. Take $d_2^{(0)}$ satisfying (7. 4. 5), then (7. 4. 2), (7. 4. 3) and (7. 4. 4) hold. Furthermore $-2d_1 \operatorname{Re} Q^{(0)} + d_2^{(0)} |Q^{(0)}|^2$ does not depend on ε'. Thus we see from (7. 4. 1) that

$$R_1(x', \mu', \sigma') - \gamma' C_1 \geq O$$

in a sufficiently small neighborhood $U(x^0) \times U(O, \sigma^0)$.

Finally choose ε_0 so that $S_{8\varepsilon_0}(\tau^0, \sigma^0) \subset U(\tau^0, \sigma^0)$. Then for the symbol of $R_1(x' D')$, $R_1(\tilde{x}(x'), \tilde{\tau}(\tau\Lambda_\gamma^{-1}, \sigma\Lambda_\gamma^{-1}), \tilde{\sigma}(\tau\Lambda_\gamma^{-1}, \sigma\Lambda_\gamma^{-1}))$, it holds that for every $(x', \tau, \sigma) \in \boldsymbol{R}^n \times \boldsymbol{C}_- \times \boldsymbol{R}^{n-1}$

$$R_1(\tilde{x}, \tilde{\tau}, \tilde{\sigma}) + C'' \tilde{\gamma} \geq O ,$$

where $C'' > O$ is a constant such that for all $(x', \tau', \sigma') \in S_{\frac{5}{2}\varepsilon_0}(x^0) \times S_{8\varepsilon_0}(\tau^0, \sigma^0)$

$$C'' \geq |C_1| = \left| C_1(x', \tau,' \sigma' ; f, d_1, d_2) \right| . \tag{7. 4. 11}$$

Hence the assertion of Lemma 7. 4 follows from Lemma 2. 1 β).

REMARK 7. 1. It is seen from the above proof and (7. 12) that the assertion of Theorem 7. 1 is also valid if it satisfies (7. 1. 1) and

$$d_1\left(-2\operatorname{Re} Q + (1+\varepsilon' e_{12})^{-1}(e_{11}-e_{22})\varepsilon'\right) + d_2\left(|Q|^2 + (1+\varepsilon' e_{12})^{-1} e_{21}\varepsilon'\right) \geq O$$

for any $(x', \varepsilon', \gamma')$ in a real neighborhood U of (x^0, O, σ^0) such that $(\varepsilon' + \theta(x', \gamma'))^2 + |\sigma'|^2 \leq 1$.

Furthermore using $\begin{pmatrix} 1 & k \\ O & 1 \end{pmatrix}^{-1} M_{\text{II}} \begin{pmatrix} 1 & k \\ O & 1 \end{pmatrix}$ with some $k \in S_+^0$ over U, we see that the above condition is equivalent to that on U

$\operatorname{Re} Q \leq O$ if $|Q|^2 + (1+\varepsilon' e_{12})^{-1} e_{21}\, \varepsilon' = O$ in the case (a), which follow from that $\operatorname{Re} D(x', \sigma') \geq O$ on U.

§ 8. The proof of Theorem 1. 1

In § 6 and the latter part of § 5 we have tried to analyse how the L^2-well-posedness of the freezing problem dominates the relations among the coefficients of the boundary operator. In this section we prove (1. 1) on the base of that analysis in three subsections. In order to prove (1. 1) for $k=O$ we have only to show (3. 4) whether the set II is empty or not. In subsection 8. 1 we prove (3. 4) by the consideration in § 5, for the case where the set II is empty. In subsection 8. 2 we do so by that in § 6 together with the methods used in § 7, for the case where the set II is not empty. Note that the dependence of C_1 in (3. 4) on support of ψ does not affect the validity of Lemma 3. 3. The proof of (1. 1) for $k \geq 1$ is given in subsection 8. 3.

8. 1. We prove (3. 4) in the case where the set II is empty, using the

estimates $(P_{\mathrm{I}}^{\pm})$, $(P_{\mathrm{III}}^{\pm})$ and (B^{+}) in §5.

From $(P_{\mathrm{I}}^{\pm})$ and $(P_{\mathrm{III}}^{\pm})$ it holds for $U\in H_{1,\gamma}(\boldsymbol{R}_{+}^{n+1})$ that

$$
\begin{aligned}
&C\gamma\|U\|_{0,\gamma} \\
&\leq C\gamma\left(\|u_{\mathrm{I}}^{+}\|_{0,\gamma}+\|u_{\mathrm{I}}^{-}\|_{0,\gamma}+\|u_{\mathrm{III}}^{+}\|_{0,\gamma}+\|u_{\mathrm{III}}^{-}\|_{0,\gamma}\right) \\
&\leq \|P_{\mathrm{I}}^{+}u_{\mathrm{I}}^{+}\|_{0,\gamma}+\|P_{\mathrm{III}}^{+}u_{\mathrm{III}}^{+}\|_{0,\gamma}+\|P_{\mathrm{I}}^{-}u_{\mathrm{I}}^{-}\|_{0,\gamma}+\|P_{\mathrm{III}}^{-}u_{\mathrm{III}}^{-}\|_{0,\gamma} \\
&+\gamma^{\frac{1}{2}}|u_{\mathrm{I}}^{+}|_{0,\gamma}+\gamma|u_{\mathrm{III}}^{+}|_{-\frac{1}{2},\gamma}.
\end{aligned}
\tag{8.1}
$$

On the other hand it follows from (B_1) and Lemma 5.3 (ii) that

$$
\begin{aligned}
g= &V_{\mathrm{I}}^{+}(x',D')u_{\mathrm{I}}^{+}(x',0)+V_{\mathrm{III}}^{+}(x',D')u_{\mathrm{III}}^{+}(x',0) \\
&+B^{+}(x',D')\cdot C_{\mathrm{I}}(x',D')u_{\mathrm{I}}^{-}(x',0) \\
&+K_{\mathrm{I}}(x',D')u_{\mathrm{I}}^{-}(x',0)+V_{\mathrm{III}}^{-}(x',D')u_{\mathrm{III}}^{-}(x',0),
\end{aligned}
$$

that is,

$$
\begin{aligned}
&(V_{\mathrm{I}}^{+},V_{\mathrm{III}}^{+})\left[\begin{pmatrix}u_{\mathrm{I}}^{+}(0)\\ u_{\mathrm{III}}^{+}(0)\end{pmatrix}+C_{\mathrm{I}}u_{\mathrm{I}}^{-}(0)\right] \\
&=g-V_{\mathrm{III}}^{-}u_{\mathrm{III}}^{-}(0)-K_{\mathrm{I}}u_{\mathrm{I}}^{-}(0),
\end{aligned}
$$

where

$$
\begin{aligned}
&K_{\mathrm{I}}(x',D') \\
&=(B^{+}C_{\mathrm{I}})(x',D')-B^{+}(x',D')\cdot C_{\mathrm{I}}(x',D')\in S_{+}^{-1},
\end{aligned}
$$

$$
C_{\mathrm{I}}(x',D')=\begin{pmatrix}C_{\mathrm{I}}^{1}\\ C_{\mathrm{I}}^{3}\end{pmatrix}\in S_{+}^{0}\quad\text{with}\quad\begin{cases}C_{\mathrm{I}}^{1}: & l\times l \text{ matrix}\\ C_{\mathrm{I}}^{3}: & (m-l)\times l\text{ matrix}.\end{cases}
$$

Using the estimates (B^{+}) and (P_{III}^{-}) we have

$$
\begin{aligned}
&C(\gamma^{\frac{1}{2}}|u_{\mathrm{I}}^{+}+C_{\mathrm{I}}^{1}u_{\mathrm{I}}^{-}|_{0,\gamma}+\gamma|u_{\mathrm{III}}^{+}+C_{\mathrm{I}}^{3}u_{\mathrm{I}}^{-}|_{-\frac{1}{2},\gamma}) \\
&\leq |g|_{\frac{1}{2},\gamma}+|V_{\mathrm{III}}^{-}u_{\mathrm{III}}^{-}|_{\frac{1}{2},\gamma}+|K_{\mathrm{I}}u_{\mathrm{I}}^{-}|_{\frac{1}{2},\gamma} \\
&\leq |g|_{\frac{1}{2},\gamma}+\|P_{\mathrm{III}}^{-}u_{\mathrm{III}}^{-}\|_{0,\gamma}+|u_{\mathrm{I}}^{-}|_{-\frac{1}{2},\gamma}.
\end{aligned}
$$

Hence for large γ it follows from the estimate (P_{I}^{-}) that

$$
\begin{aligned}
&C\left(\gamma^{\frac{1}{2}}|u_{\mathrm{I}}^{+}|_{0,\gamma}+\gamma|u_{\mathrm{III}}^{+}|_{-\frac{1}{2},\gamma}\right) \\
&\leq |g|_{\frac{1}{2},\gamma}+\|P_{\mathrm{III}}^{-}u_{\mathrm{III}}^{-}\|_{0,\gamma}+\gamma^{\frac{1}{2}}|u_{\mathrm{I}}^{-}|_{0,\gamma} \\
&\leq |g|_{\frac{1}{2},\gamma}+\|P_{\mathrm{I}}^{-}u_{\mathrm{I}}^{-}\|_{0,\gamma}+\|P_{\mathrm{III}}^{-}u_{\mathrm{III}}^{-}\|_{0,\gamma}.
\end{aligned}
$$

Combining this with (8.1) we obtain (3.4).

8.2. In this subsection we prove (3.4) in the case where the set II is not empty. If $R(x^0,\tau^0,\sigma^0)\neq 0$, the proof is the same as that in [7], that is, (3.4) is obtained by using the estimates $(P_{\mathrm{I}}^{\pm})$, $(P_{\mathrm{III}}^{\pm})$, $(P_{\mathrm{II}})'$ and the relation (B_1) together with Lemma 2.1 β) (ii) and taking the constant C_1 in $(P_{\text{II}})'$ suitably. Therefore we prove it in the case where $R(x^0, \tau^0, \sigma^0)=O$, by using the estimates $(P_{\text{I}}^{\pm})$, $(P_{\text{III}}^{\pm})$, (P_{II}) and Lemmas in §6 (especially Lemma 6.2 and 6.7).

From Lemma 6.2 it holds that for small $\delta_1>0$

$$|g-V_{\text{III}}^{-}u_{\text{III}}^{-}-C_1u_{\text{I}}^{-}-C_2u'_{\text{II}}|^2_{\frac{1}{2},\gamma}$$
$$\geq C\left\{\gamma\left(|g_1|^2_{0,\gamma}+|g_2|^2_{0,\gamma}\right)+\delta_1\gamma^2|g_3|^2_{-\frac{1}{2},\gamma}\right\},$$

where $C_1, C_2\in S_+^{-1}$ and

$$|g_1|^2_{0,\gamma}\geq|u_{\text{I}}^{+}|^2_{0,\gamma}-|k_{\text{I}\,\text{I}}u_{\text{I}}^{-}|^2_{0,\gamma}-|k_{\text{I}\,\text{II}}u'_{\text{II}}|^2_{0,\gamma},$$
$$|g_2|^2_{0,\gamma}=|u''_{\text{II}}+Qu'_{\text{II}}+k_{\text{II}\,\text{I}}u_{\text{I}}^{-}|^2_{0,\gamma},$$
$$|g_3|^2_{-\frac{1}{2},\gamma}\geq|u_{\text{III}}^{+}|^2_{-\frac{1}{2},\gamma}-|k_{\text{III}\,\text{I}}u_{\text{I}}^{-}|^2_{-\frac{1}{2},\gamma}-|k_{\text{III}\,\text{II}}u'_{\text{II}}|^2_{-\frac{1}{2},\gamma}.$$

Since the symbols of $k_{JK}\in S_+^0$,

$$C\left(|g|^2_{\frac{1}{2},\gamma}+|u_{\text{III}}^{-}|^2_{\frac{1}{2},\gamma}\right)$$
$$\geq\gamma|u_{\text{I}}^{+}|^2_{0,\gamma}+\delta_1\gamma^2|u_{\text{III}}^{+}|^2_{-\frac{1}{2},\gamma}$$
$$-C\left(\gamma|u_{\text{I}}^{-}|^2_{0,\gamma}+\delta_1\gamma^2|u_{\text{I}}^{-}|^2_{-\frac{1}{2},\gamma}+\delta_1\gamma^2|u'_{\text{II}}|^2_{-\frac{1}{2},\gamma}\right)$$
$$-\gamma|k_{\text{I}\,\text{II}}u'_{\text{II}}|^2_{0,\gamma}.$$

Using the estimates $(P_{\text{I}}^{\pm})$, $(P_{\text{III}}^{\pm})$ we have for large γ

$$C\left(\|P_{\text{I}}^{+}u_{\text{I}}^{+}\|^2_{0,\gamma}+\|P_{\text{I}}^{-}u_{\text{I}}^{-}\|^2_{0,\gamma}+\|P_{\text{III}}^{+}u_{\text{III}}^{+}\|^2_{0,\gamma}+\|P_{\text{III}}^{-}u_{\text{III}}^{-}\|^2_{0,\gamma}+|g|^2_{\frac{1}{2},\gamma}\right)$$
$$\geq\gamma^2\left(\|u_{\text{I}}^{+}\|^2_{0,\gamma}+\|u_{\text{I}}^{-}\|^2_{0,\gamma}+\delta_1\|u_{\text{III}}^{+}\|^2_{0,\gamma}+\|u_{\text{III}}^{-}\|^2_{0,\gamma}\right)$$
$$-\gamma|k_{\text{I}\,\text{II}}u'_{\text{II}}|^2_{0,\gamma}-C\delta_1\gamma^2|u'_{\text{II}}|^2_{-\frac{1}{2},\gamma}.$$

From Lemma 7.2 we have

$$|u'_{\text{II}}|^2_{-\frac{1}{2},\gamma}\leq C^2\left(\gamma^{-2}\|f'_{\text{II}}\|^2_{0,\gamma}+\|u_{\text{II}}\|^2_{0,\gamma}\right). \tag{8.2}$$

Since δ_1 is small it follows from the estimate (P_{II}) that

$$\begin{aligned}&\|P_1U\|^2_{0,\gamma}+|g|^2_{\frac{1}{2},\gamma}\\ &\quad\geq C\gamma^2\|U\|^2_{0,\gamma}\\ &\qquad+C\gamma\left\{\text{Im}\,\langle R_0u_{\text{II}}, u_{\text{II}}\rangle_{0,\gamma}-|k_{\text{I}\,\text{II}}u'_{\text{II}}|^2_{0,\gamma}\right\}.\end{aligned} \tag{8.3}$$

On the other hand from the boundary condition (B_1) we have

$$(V_{\text{I}}^{+}, V''_{\text{II}}, V_{\text{II}}^{+})(x', D')\cdot{}^t({}^tu_{\text{I}}^{+}, u''_{\text{II}}, {}^tu_{\text{III}}^{+})(x')$$
$$=g(x')-(V_{\text{I}}^{-}, V'_{\text{II}}, V_{\text{III}}^{-})(x', D')\cdot{}^t({}^tu_{\text{I}}^{-}, u'_{\text{II}}, {}^tu_{\text{III}}^{-})(x').$$

Since $\det(V_{\text{I}}^{+}, V''_{\text{II}}, V_{\text{II}}^{+})(x^0, \tau^0, \sigma^0)\neq O$ it follows from Lemma 2.1 β) (iii) that

$$
\begin{aligned}
&{}^t({}^tu_{\mathrm{I}}^+,\ u_{\mathrm{II}}'',\ {}^tu_{\mathrm{III}}^+)\\
&\qquad = C'g-\Big((V_{\mathrm{I}}^+,\ V_{\mathrm{II}}'',\ V_{\mathrm{III}}^+)^{-1}(x',D')+T'(x',D')\Big)\\
&\qquad\qquad \cdot(V_{\mathrm{I}}^-,\ V_{\mathrm{II}}',\ V_{\mathrm{III}}^-)(x',D')\cdot{}^t({}^tu_{\mathrm{I}}^-,\ u_{\mathrm{II}}',\ {}^tu_{\mathrm{III}}^-)
\end{aligned}
$$

where $C'\in S_+^0$ and $T'\in S_+^{-1}$. Hence from Lemma 6.2 we have

$$
u_{\mathrm{II}}''(x')=C_1g-Tu'-k_{\mathrm{II\,I}}u_{\mathrm{I}}^- -k_{\mathrm{II\,II}}u_{\mathrm{II}}'-k_{\mathrm{II\,III}}u_{\mathrm{III}}^-, \tag{8.4}
$$

where

$$
\begin{aligned}
&T=(T_{\mathrm{I}},\ T_{\mathrm{II}},\ T_{\mathrm{III}})\in S_+^{-1},\quad C_1\in S_+^0,\\
&u'={}^t({}^tu_{\mathrm{I}}^-,\ u_{\mathrm{II}}',\ {}^tu_{\mathrm{III}}^-),\\
&\widetilde{k_{\mathrm{II\,II}}}(x',\tau,\sigma)=\tilde{Q}(x',\tau,\sigma).
\end{aligned}
$$

As in the proof of Lemma 7.3, let us delete u_{II}'' by using (8.4). Then we have

$$
\begin{aligned}
&\mathrm{Im}\,\langle R_0u_{\mathrm{II}},\ u_{\mathrm{II}}\rangle_{0,\gamma}\\
&\quad =2d_1\,\mathrm{Re}\,\langle C_1g-Tu'-k_{\mathrm{II\,I}}u_{\mathrm{I}}^- -Qu_{\mathrm{II}}'-k_{\mathrm{II\,III}}u_{\mathrm{III}}^-,\ u_{\mathrm{II}}'\rangle_{0,\gamma}\\
&\qquad +\mathrm{Re}\,\langle d_2\cdot(C_1g-Tu'-k_{\mathrm{II\,I}}u_{\mathrm{I}}^- -Qu_{\mathrm{II}}'-k_{\mathrm{II\,III}}u_{\mathrm{III}}^-)\\
&\qquad\qquad ,(C_1g-Tu'-k_{\mathrm{II\,I}}u_{\mathrm{I}}^- -Qu_{\mathrm{II}}'-k_{\mathrm{II\,III}}u_{\mathrm{III}}^-)\rangle_{0,\gamma}\\
&\qquad -2\gamma f\,\mathrm{Im}\,\langle \Lambda^{-1}\cdot(C_1g-Tu'-k_{\mathrm{II\,I}}u_{\mathrm{I}}^- -Qu_{\mathrm{II}}'-k_{\mathrm{II\,III}}u_{\mathrm{III}}^-),\ u_{\mathrm{II}}'\rangle_{0,\gamma}\\
&\qquad +\mathrm{Re}\,\langle(\varepsilon b)\,u_{\mathrm{II}}',\ u_{\mathrm{II}}'\rangle_{0,\gamma}.
\end{aligned}
$$

Since $C_1,\ k_{JK}\in S_+^0$ and $T\in S_+^{-1}$, the estimates of the right hand terms of the above are as follows: for some constant $C>0$

$$
\begin{aligned}
&\Big|2d_1\langle C_1g,\ u_{\mathrm{II}}'\rangle_{0,\gamma}\Big|\le C\Big(\delta\gamma\,|u_{\mathrm{II}}'|^2_{-\frac{1}{2},\gamma}+(\delta\gamma)^{-1}|g|^2_{\frac{1}{2},\gamma}\Big),\\
&\Big|2d_1\langle Tu',\ u_{\mathrm{II}}'\rangle_{0,\gamma}\Big|\le C\,|u'|_{-\frac{1}{2},\gamma}\cdot|u_{\mathrm{II}}'|_{-\frac{1}{2},\gamma}\\
&\qquad\qquad \le C\Big(|u_{\mathrm{I}}^-|_{-\frac{1}{2},\gamma}+|u_{\mathrm{II}}'|_{-\frac{1}{2},\gamma}+|u_{\mathrm{III}}^-|_{-\frac{1}{2},\gamma}\Big)|u_{\mathrm{II}}'|_{-\frac{1}{2},\gamma}\\
&\qquad\qquad \le C\Big(|u_{\mathrm{I}}^-|^2_{-\frac{1}{2},\gamma}+|u_{\mathrm{II}}'|^2_{-\frac{1}{2},\gamma}+|u_{\mathrm{III}}^-|^2_{-\frac{1}{2},\gamma}\Big),\\
&\Big|2d_1\langle k_{\mathrm{II\,III}}u_{\mathrm{III}}^-,\ u_{\mathrm{II}}'\rangle_{0,\gamma}\Big|\le C\,|u_{\mathrm{III}}^-|_{\frac{1}{2},\gamma}\cdot|u_{\mathrm{II}}'|_{-\frac{1}{2},\gamma}\\
&\qquad\qquad \le C\Big(\delta\gamma\,|u_{\mathrm{II}}'|^2_{-\frac{1}{2},\gamma}+(\delta\gamma)^{-1}|u_{\mathrm{III}}^-|^2_{\frac{1}{2},\gamma}\Big),\\
&2d_1\,\mathrm{Re}\,\langle k_{\mathrm{II\,I}}u_{\mathrm{I}}^-,\ u_{\mathrm{II}}'\rangle_{0,\gamma}\\
&\quad =2d_1\,\mathrm{Re}\,\langle u_{\mathrm{I}}^-,\ k_{\mathrm{II\,I}}^{\#}u_{\mathrm{II}}'\rangle_{0,\gamma}\\
&\quad \le 2d_1\,\mathrm{Re}\,\langle u_{\mathrm{I}}^-,\ k_{\mathrm{II\,I}}^{*}u_{\mathrm{II}}'\rangle_{0,\gamma}+C\,|u_{\mathrm{I}}^-|_{0,\gamma}\cdot|u_{\mathrm{II}}'|_{-1,\gamma}\\
&\quad \le 1\cdot|k_{\mathrm{II\,I}}^{*}u_{\mathrm{II}}'|^2_{0,\gamma}+C\Big(|u_{\mathrm{I}}^-|^2_{0,\gamma}+|u_{\mathrm{II}}'|^2_{-1,\gamma}\Big),
\end{aligned}
$$

$$\left|\langle d_2\cdot k_{\mathrm{II\,I}}u_{\mathrm{I}}^-,\ k_{\mathrm{II\,III}}u_{\mathrm{III}}^-\rangle_{0,\gamma}\right|\le C\left(|u_{\mathrm{I}}^-|_{0,\gamma}^2+|u_{\mathrm{III}}^-|_{0,\gamma}^2\right),$$

$$\begin{aligned}\left|\langle d_2\cdot k_{\mathrm{II\,I}}u_{\mathrm{I}}^-,\ Qu'_{\mathrm{II}}\rangle_{0,\gamma}\right|&\le |d_2\cdot k_{\mathrm{II\,I}}u_{\mathrm{I}}^-|_{0,\gamma}^2+1\cdot|Qu'_{\mathrm{II}}|_{0,\gamma}^2\\&\le C|u_{\mathrm{I}}^-|_{0,\gamma}^2+1\cdot|Qu'_{\mathrm{II}}|_{0,\gamma}^2,\end{aligned}$$

$$\begin{aligned}\left|2\gamma f\langle\Lambda^{-1}\cdot C_1 g,\ u'_{\mathrm{II}}\rangle_{0,\gamma}\right|&\le C\gamma\left((\delta\gamma)^{-1}|g|_{0,\gamma}^2+\delta\gamma\,|u'_{\mathrm{II}}|_{-1,\gamma}^2\right)\\&\le C\left(\delta^{-1}|g|_{0,\gamma}^2+\delta\gamma\,|u'_{\mathrm{II}}|_{-\frac12,\gamma}^2\right),\end{aligned}$$

$$\begin{aligned}&\left|2\gamma f\langle\Lambda^{-1}\cdot Tu',\ u'_{\mathrm{II}}\rangle_{0,\gamma}\right|\\&\quad\le C\gamma\left\{(\delta\gamma)^{-1}|Tu'|_{0,\gamma}^2+\delta\gamma\,|u'_{\mathrm{II}}|_{-1,\gamma}^2\right\}\\&\quad\le C\left\{\delta^{-1}\left(|u_{\mathrm{I}}^-|_{-1,\gamma}^2+|u'_{\mathrm{II}}|_{-1,\gamma}^2+|u_{\mathrm{III}}^-|_{-1,\gamma}^2\right)+\delta\gamma\,|u'_{\mathrm{II}}|_{-\frac12,\gamma}^2\right\},\end{aligned}$$

$$\begin{aligned}\left|2\gamma f\langle\Lambda^{-1}\cdot k_{\mathrm{II\,I}}u_{\mathrm{I}}^-,\ u'_{\mathrm{II}}\rangle_{0,\gamma}\right|&\le C\gamma\left((\delta\gamma)^{-1}|u_{\mathrm{I}}^-|_{0,\gamma}^2+\delta\gamma\,|u'_{\mathrm{II}}|_{-1,\gamma}^2\right)\\&\le C\left(\delta^{-1}|u_{\mathrm{I}}^-|_{0,\gamma}^2+\delta\gamma\,|u'_{\mathrm{II}}|_{-\frac12,\gamma}^2\right),\end{aligned}$$

$$\begin{aligned}\left|2\gamma f\langle\Lambda^{-1}\cdot k_{\mathrm{II\,III}},\ u_{\mathrm{III}}^-,\ u'_{\mathrm{II}}\rangle_{0,\gamma}\right|&\le C\gamma\left((\delta\gamma)^{-1}|u_{\mathrm{III}}^-|_{0,\gamma}^2+\delta\gamma\,|u'_{\mathrm{II}}|_{-1,\gamma}^2\right)\\&\le C\left(\delta^{-1}|u_{\mathrm{III}}^-|_{0,\gamma}^2+\delta\gamma\,|u'_{\mathrm{II}}|_{-\frac12,\gamma}^2\right).\end{aligned}$$

The other terms are estimated by the same way as above. Thus we obtain for large γ

$$\begin{aligned}&\mathrm{Im}\,\langle R_0u_{\mathrm{II}},\ u_{\mathrm{II}}\rangle_{0,\gamma}\\&\quad\ge -C(d_2,Q)\left((\delta\gamma)^{-1}|g|_{\frac12,\gamma}^2+\delta^{-1}|u_{\mathrm{I}}^-|_{0,\gamma}^2+(\delta\gamma)^{-1}|u_{\mathrm{III}}^-|_{\frac12,\gamma}^2+\delta\gamma\,|u'_{\mathrm{II}}|_{-\frac12,\gamma}^2\right)\\&\qquad-|k_{\mathrm{II\,I}}^*u'_{\mathrm{II}}|_{0,\gamma}^2-2d_1\,\mathrm{Re}\,\langle Qu'_{\mathrm{II}},\ u'_{\mathrm{II}}\rangle_{0,\gamma}\\&\qquad-2|Qu'_{\mathrm{II}}|_{0,\gamma}^2+\mathrm{Re}\left\langle(d_2Q^*Q)\,u'_{\mathrm{II}},\ u'_{\mathrm{II}}\right\rangle_{0,\gamma}\\&\qquad+2\gamma f\,\mathrm{Im}\left\langle(Q\Lambda^{-1})\,u'_{\mathrm{II}},\ u'_{\mathrm{II}}\right\rangle_{0,\gamma}+\mathrm{Re}\left\langle(\varepsilon b)\,u'_{\mathrm{II}},\ u'_{\mathrm{II}}\right\rangle_{0,\gamma}\\&\qquad-C(d_2,f,Q)\,|u'_{\mathrm{II}}|_{-\frac12,\gamma}^2,\end{aligned}$$

where the last term of the right arises from the commutators.

From (8. 3) and the estimates (P_{I}^-) and (P_{III}^-), using the above inquality, we see that

$$\begin{aligned}\|P_1U\|_{0,\gamma}^2+|g|_{\frac12,\gamma}^2&\ge C\gamma^2\|U\|_{0,\gamma}^2\\&\quad+C_1\gamma\Big\{-|k_{\mathrm{I\,II}}u'_{\mathrm{II}}|_{0,\gamma}^2-|k_{\mathrm{II\,I}}^*u'_{\mathrm{II}}|_{0,\gamma}^2-2|Qu'_{\mathrm{II}}|_{0,\gamma}^2\\&\quad-2d_1\,\mathrm{Re}\,\langle Qu'_{\mathrm{II}},\ u'_{\mathrm{II}}\rangle_{0,\gamma}+\mathrm{Re}\left\langle(d_2Q^*Q)\,u'_{\mathrm{II}},\ u'_{\mathrm{II}}\right\rangle_{0,\gamma}\\&\quad+\mathrm{Re}\left\langle(\varepsilon b)\,u'_{\mathrm{II}},\ u'_{\mathrm{II}}\right\rangle_{0,\gamma}+2\gamma f\,\mathrm{Im}\left\langle(Q\Lambda^{-1})\,u'_{\mathrm{II}},\ u'_{\mathrm{II}}\right\rangle_{0,\gamma}\Big\}\\&=C\gamma^2\|U\|_{0,\gamma}^2+C_1\gamma\Big\{-|k_{\mathrm{I\,II}}u'_{\mathrm{II}}|_{0,\gamma}^2-|k_{\mathrm{II\,I}}^*u'_{\mathrm{II}}|_{0,\gamma}^2\\&\quad-2|Qu'_{\mathrm{II}}|_{0,\gamma}^2+\langle R_1u'_{\mathrm{II}},\ u'_{\mathrm{II}}\rangle_{0,\gamma}\Big\},\end{aligned}\tag{8. 5}$$

with $R_1 \in S_+^0$ in Lemma 7.3.

From Lemma 6.7 we have for $(x', \tau', \sigma') \in U(x^0) \times (U(\tau^0, \sigma^0) \cap \overline{\Sigma_-})$

$$\widetilde{k_{\mathrm{I\,II}}}(x', \tau' \sigma') = \mu' k^{(1)}_{\mathrm{I\,II}}(x', \mu' \sigma') + k^{(0)}_{\mathrm{I\,II}}(x', \sigma'),$$
$$\widetilde{k_{\mathrm{II\,I}}}(x', \tau' \sigma') = \mu' k^{(1)}_{\mathrm{II\,I}}(x', \mu' \sigma') + k^{(0)}_{\mathrm{II\,I}}(x', \sigma'),$$

where (6.5) is valid for $k^{(0)}_{\mathrm{I\,II}}$ and $k^{(0)}_{\mathrm{II\,I}}$. Furthermore note that

$$|\mu'|^2 = (\varepsilon')^2 + (\gamma')^2,$$

and that the Q satisfies the conditions (7.5), (7.6) and (7.7) in Theorem 7.1 by virture of Lemmas 6.4 (i), 6.1 β) (i) and 6.5. Then by the same method as in the proof of Lemma 7.4, changing (7.4.2) slightly one can choose d_1 (in this case d_1 must be chosen so that $d_1 e^{(0)}_{21}(x_0, \sigma_0)$ is sufficiently large) and $d_2(x, \tau', \sigma') = d^{(0)}_2(x', \sigma') + \varepsilon' d^{(1)}_2$ so that the bracket of the second term of the right hand of (8.5):

$$\{\quad\} \geq -C|u'_{\mathrm{II}}|^2_{-\frac{1}{2},\gamma} - C''\left\langle \gamma_0(D') u'_{\mathrm{II}}, u'_{\mathrm{II}} \right\rangle_{0,\gamma}$$

for every $u_{\mathrm{II}} = \phi(D') v_{\mathrm{II}} \in H_{1,\gamma}(\boldsymbol{R}^{n+1}_+)$ with $v_{\mathrm{II}} \in H_{1,\gamma}(\boldsymbol{R}^{n+1}_+)$, where C and C'' are some constants corresponding to those in Lemma 7.4.

Thus by the same way as in the proof of Theorem 7.1 we obtain for $U = {}^t({}^t u^+_{\mathrm{I}}, {}^t u^-_{\mathrm{I}}, \phi(D') v_{\mathrm{II}}, {}^t u^+_{\mathrm{III}}, {}^t u^-_{\mathrm{III}}) \in H_{1,\gamma}(\boldsymbol{R}^{n+1}_+)$

$$\|P_1 U\|^2_{0,\gamma} + |g|^2_{\frac{1}{2},\gamma} \geq C_1 \gamma^2 \|U\|^2_{0,\gamma}$$

for large $\gamma > O$, where C_1 is a positive constant depending only on the support of ϕ, which implies (3.4). Thus Theorem 1.1 is proved for $k=O$.

8.3. To obtain (1.1) for $k \geq 1$, apply the operator Λ^k to the equations of (P^0, B) and put $u_k = \Lambda^k u, f_k = \Lambda^k f$ and $g_k = \Lambda^k g$. We obtain

$$\text{(8.6)} \qquad \begin{aligned} D_n u_k &= A u_k + T_1 \Lambda^{-k} u_k + f_k, \\ B u_k &= T_2 \Lambda^{-k} u_k(x', O) + g_k, \end{aligned}$$

where by virture of Lemma 2.1 α) (iii)

$$\|T_1 \Lambda^{-k} u_k\|_{0,\gamma} \leq C \|u_k\|_{0,\gamma}$$
$$|T_2 \Lambda^{-k} u_k|_{\frac{1}{2},\gamma} \leq C |u_k|_{-\frac{1}{2},\gamma}.$$

Hence from (1.1) for $k=O$ we obtain

$$C\gamma \|u_k\|_{0,\gamma} \leq \|u_k\|_{0,\gamma} + |u_k|_{-\frac{1}{2},\gamma} + \|f_k\|_{0,\gamma} + |g_k|_{\frac{1}{2},\gamma}.$$

On the other hand it follows from the same method as in the proof of Lemma 7.2 that

$$|u_k|_{-\frac{1}{2},\gamma} \leq C\left(\|u_k\|_{0,\gamma} + \gamma^{-1} \|f_k\|_{0,\gamma}\right).$$

Thus we obtain for large γ

$$C\gamma\|\Lambda^k u\|_{0,\gamma}\leq\|\Lambda^k f\|_{0,\gamma}+|\Lambda^k g|_{\frac{1}{2},\gamma}.$$

Use this for $k=1$. Then we have

$$\begin{aligned}\gamma\|D_n u\|_{0,\gamma}&\leq\gamma\Big(\|Au\|_{0,\gamma}+\|f\|_{0,\gamma}\Big)\\&\leq\gamma\Big(C\|\Lambda u\|_{0,\gamma}+\|f\|_{0,\gamma}\Big)\\&\leq C\Big(\|f\|_{1,\gamma}+|g|_{\frac{3}{2},\gamma}\Big).\end{aligned}$$

From this the theorem follows for $k=1$:

$$C\gamma\|u\|_{1,\gamma}\leq\|f\|_{1,\gamma}+|g|_{\frac{3}{2},\gamma}.$$

To obtain (1. 1) for $k=2$, we differentiate (8. 6) with respect to x_n and use the fact that $D_n A(x, D')=AD_n+A_1$, where A_1 is the pseudo-differential operator with symbol $(D_n A)(x, \tau, \sigma)$. Thus (1. 1) holds for any integer $k\geq 0$ and the proof of Theorem 1. 1 is completed.

§ 9. Adjoint problems.

Let $b'_i(x')={}^t(b'_{i,1},\cdots,b'_{i,2m})(x')\,(i=1,\cdots,m)$ be a certain real base of the space of null $B(x')$ whose elements are $C^\infty(\boldsymbol{R}^n)$ and constant outside some compact set of $\boldsymbol{R}^n$(here we may assume the existance of such a base.) Set

$$T(x')=(b_1,\cdots,b_m,\ b'_1,\cdots,b'_m),$$

where $b_i(x')={}^t(b_{i,1},\cdots,b_{i,2m})$ and $B^*(x')=(b_1,\cdots,b_m)$. Then it holds that

$$B(x')\,T=\Big(B(b_1,\cdots,b_m),\,O,\cdots,O\Big)=\Big(BB^*,\,O\Big).$$

Since rank $B=m$, det $BB^*\neq O$. Hence we have

$$(BB^*)^{-1}BT=(E_1,\,O).$$

where E_1 is the $m\times m$ identity matrix. Therefore the problem (P^0, B) is equivalent to

$$(\tilde{P}^0,\,E_1)\ \begin{cases}\tilde{P}^0\tilde{u}=\Big(ED_n-\sum\limits_{j=0}^{n-1}(T^{-1}A_jT)\,D_j\Big)\tilde{u}=T^{-1}f & \text{in}\quad \boldsymbol{R}^{n+1}_+,\\ (E_1,\,O)\,\tilde{u}=(BB^*)^{-1}g \quad \text{on}\quad \boldsymbol{R}^n,\end{cases}$$

where $\tilde{u}(x)=T^{-1}(x')\,u(x)$. Let $\tilde{P}^\#$ be the formal adjoint of $\tilde{P}^0$. Then we have the Green's formula:

$$(9.1)\qquad (\tilde{P}^0\tilde{u},\,\tilde{v})_{L^2(\boldsymbol{R}^{n+1}_+)}-(\tilde{u},\,\tilde{P}^\#\tilde{v})_{L^2(\boldsymbol{R}^{n+1}_+)}=i\langle\tilde{u},\,\tilde{v}\rangle_{L^2(\boldsymbol{R}^n)}$$

for $\tilde{u},\tilde{v}\in C_0^\infty(\overline{\boldsymbol{R}^{n+1}_+})$. Let us consider the adjoint boundary value problem:

$$(\tilde{P}', E_2)\begin{cases}\tilde{P}'\tilde{v}=\tilde{\phi} & \text{in } \boldsymbol{R}_+^{n+1},\\ (O, E_2)\,\tilde{v}=\psi & \text{on } \boldsymbol{R}^n,\end{cases}$$

where $\tilde{P}'$ is the principal part of $\tilde{P}^{\#}$ and $E_2=E_1$. Then this is equivalent to

$$(P', B')\begin{cases}P'(x, D)\,v=(ED_n-\sum_{j=0}^{n-1}A_j^*(x)\,D_j)\,v\\ \qquad\qquad =\left(ED_n-{}^tA\,(x, D')\right)v=\phi & \text{in } \boldsymbol{R}_+^{n+1},\\ B'(x')\,v=\psi \quad \text{on } \boldsymbol{R}^n,\end{cases}$$

where we put

$$\begin{aligned}&v=(T^*)^{-1}\tilde{v},\quad \phi=(T^*)^{-1}\tilde{\phi},\\ &B'(x')=(O, E_2)\,T^*=(b_1', \cdots, b_m')^*.\end{aligned}\tag{9.2}$$

Now we have the following

THEOREM 9.1. *Under the conditions* (*I*), (*II*) *and* (*III*), *the following a priori estimate holds with some constants* $C_k, \gamma_k>0$:

$$\|P'v\|_{k,-\gamma}+|B'v|_{k+\frac{1}{2},-\gamma}\geq C_k\gamma\,\|v\|_{k,-\gamma}\tag{9.3}$$

for every $\gamma\geq\gamma_k$, $v\in H_{k,-\gamma}(\boldsymbol{R}_+^{n+1})$ *and integer* $k\geq 0$.

PROOF of THEOREM 9.1. Putting $x_0=-x_0$ in (P', B') we consider the following problem:

$$(P^{(*)}, B^{(*)})\begin{cases}P^{(*)}(x, D)\,v=\phi & \text{in } \boldsymbol{R}_+^{n+1},\\ B^{(*)}(x')\,v(x', 0)=\psi & \text{on } \boldsymbol{R}^n,\end{cases}$$

where we put

$$\begin{cases}P^{(*)}(x_0, x'', x_n;\ D_0, D'', D_n)=P'(-x_0, x'', x_n;\ -D_0, D'', D_n)\\ \qquad\qquad =ED_n-{}^tA(-x_0, x'', x_n;\ -D_0, D''),\\ B^{(*)}(x_0, x'')=B'(-x_0, x'').\end{cases}\tag{9.4}$$

From (9.4) and the proof of Theorem 1.1 it suffices to show that each of the assumptions (I), (II) and (III) for (P^0, B) implies the same for $(P^{(*)}$, $B^{(*)})$ respectively. Note that the characteristic equation is

$$\begin{aligned}&\det P^{(*)}(x_0, x'', x_n;\ \tau, \sigma, \lambda)\\ &=\det\left(E\lambda-{}^tA(-x_0, x'', x_n;\ -\tau, \sigma)\right)\\ &=\det\left(E\lambda-A(-x_0, x'', x_n;\ -\tau, \sigma)\right)\\ &=\det\Big(E\lambda-A_0(-x_0, x'', x_n)(-\eta+i\gamma)\\ &\qquad -\sum_{j=1}^{n-1}A_j(-x_0, x'', x_n)\,\sigma_j\Big)\\ &=0.\end{aligned}\tag{9.5}$$

Then it is easy to show that the condition (I) is fulfilled for $(P^{(*)}, B^{(*)})$. Since (P^0, B), $(\tilde{P}^0, E_1)$ are equivalent and (P', B'), $(\tilde{P}', E_2)$ are so, the condition (III) for (P^0, B) together with (9.1) and (9.4) implies the same for $(P^{(*)}, B^{(*)})$ (Theorem 2 of [8].) Therefore we have only to show that the condition (II) is fulfilled for $(P^{(*)}, B^{(*)})$.

First we seek the Lopatinskii determinant and coupling coefficients for $(P^{(*)}, B^{(*)})$. Put

$$\begin{aligned} &S_0(x, \tau, \sigma) = (h_{\mathrm{I}}^+, h_{\mathrm{II}}^+, h_{\mathrm{III}}^+, h_{\mathrm{I}}^-, h_{\mathrm{II}}'', h_{\mathrm{III}}^-)\,, \\ &(B_+, B_2)(x', \tau, \sigma) = B(x')\, S_0(x', 0, \tau, \sigma)\,, \end{aligned} \tag{9.6}$$

where S_0 is a $2m \times 2m$ matrix and B_+, B_2 are $m \times m$ matrices. Then it holds that

$$(S_0^{-1} A S_0)(x, \tau, \sigma) = \begin{pmatrix} \lambda_{\mathrm{I}}^+ & & & & & 0 \\ & \lambda_{\mathrm{II}}^+ & \cdots\cdots & & \alpha & \\ & & M_{\mathrm{III}}^+ & 0 & \vdots & \\ & & & \lambda_{\mathrm{I}}^- & \vdots & \\ & & & & \lambda_{\mathrm{II}}^- & \\ 0 & & & & & M_{\mathrm{III}}^- \end{pmatrix},$$

$$(b_{ij})(x', \tau, \sigma) = (B_+)^{-1} B_2\,.$$

Hence we have

$$\left(S_0^* A^* (S_0^*)^{-1}\right)(x, \tau, \sigma) = \begin{pmatrix} \overline{\lambda_{\mathrm{I}}^+} & & & & & 0 \\ & \overline{\lambda_{\mathrm{II}}^+} & & & & \\ & \vdots & (M_{\mathrm{III}}^+)^* & & & \\ & \vdots & 0 & \overline{\lambda_{\mathrm{I}}^-} & & \\ & \bar{\alpha} & \cdots\cdots & & \overline{\lambda_{\mathrm{II}}^-} & \\ 0 & & & & & (M_{\mathrm{III}}^-)^* \end{pmatrix}.$$

Put

$$(B_2', B_+')(x', \tau, \sigma) = B'(x')\, (S_0^*)^{-1}(x', 0, \tau, \sigma) \tag{9.7}$$

where B_+', B_2' are $m \times m$ matrices. Then we see that the Lopatinskii determinant and coupling coefficients of $(P^{(*)}, B^{(*)})$ are

$$\begin{aligned} &R^{(*)}(x_0, x'', \tau, \sigma) = \det B_+'(-x_0, x'', -\bar{\tau}, \sigma)\,, \\ &(b_{ij}^{(*)})(x_0, x'', \tau, \sigma) = \left((B_+')^{-1} B_2'\right)(-x_0, x'', -\bar{\tau}, \sigma) \end{aligned} \tag{9.8}$$

respectively. Now we show the following two lemmas.

LEMMA 9.1. *For the quantities defined above the following hold:*

(i) $\overline{R^{(*)}(x_0, x'', \tau, \sigma)} \cdot \det B_2(-x_0, x'', -\bar{\tau}, \sigma)$
$= -R(-x_0, x'', -\bar{\tau}, \sigma) \cdot \det \overline{B_2'(-x_0, x'', -\bar{\tau}, \sigma)}$,

(ii) $(b_{ij}^{(*)})(x_0, x'', \tau, \sigma) = -\overline{(b_{ji})(-x_0, x'', -\bar{\tau}, \sigma)}$.

PROOF. From (9. 2) we have

$$B(B')^* = O.$$

Hence it follows from (9. 6) and (9. 7) that

$$\begin{aligned} O &= \left(BS_0S_0^{-1}(B')^*\right)(-x_0, x'', O, -\bar{\tau}, \sigma) \\ &= (B_+, B_2)\left(B'(S_0^*)^{-1}\right)^* = (B_+, B_2)\begin{pmatrix}(B_2')^* \\ (B_+')^*\end{pmatrix} \\ &= B_+(B_2')^* + B_2(B_+')^*. \end{aligned}$$

This implies (i). Furthermore we have

$$\begin{aligned} (b_{ij})(-x_0, x'', -\bar{\tau}, \sigma) &= (B_+)^{-1}B_2(-x_0, x'', -\bar{\tau}, \sigma) \\ &= -\left((B_+')^{-1} \cdot B_2'\right)^*(-x_0, x'', -\bar{\tau}, \sigma) \\ &= -(b_{ij}^{(*)})^*(x_0, x'', \tau, \sigma). \end{aligned}$$

This shows (ii) and completes the proof of the lemma.

LEMMA 9. 2. *The zeroes of* $R^{(*)}(x_0, x'', \tau, \sigma)$ *and* $R(-x_0, x'', -\bar{\tau}, \sigma)$ *coincide in* $\Gamma \times \bar{C}_- \times R^{n-1}$.

PROOF. Let us consider the following constant coefficients problems with parameters $(\tau, \sigma) \in C_- \times R^{n-1}$:

$$(\tilde{P}^0, E_1)_{x^0} \quad \begin{cases} \left(ED_n - \tilde{A}(x^0, \tau, \sigma)\right)\tilde{u} = O & \text{in } x_n > O, \\ (E_1, O)\,\tilde{u} = \tilde{g} & \text{on } x_n = O, \end{cases}$$

$$(\tilde{P}', E_2)_{x^0} \quad \begin{cases} \left(ED_n - \widetilde{{}^tA}(x^0, \bar{\tau}, \sigma)\right)\tilde{v} = O & \text{in } x_n > O, \\ (O, E_2)\,\tilde{v} = \phi & \text{on } x_n = O, \end{cases}$$

where

$$\begin{aligned} \tilde{A}(x, \tau, \sigma) &= T^{-1}(x')\,A(x, \tau, \sigma)\,T(x'), \\ \widetilde{{}^tA}(x, \bar{\tau}, \sigma) &= T^*(x')\,{}^tA(x, \bar{\tau}, \sigma)\left(T^*(x')\right)^{-1} \\ &= T^*(x')\,A^*(x, \tau, \sigma)\left(T^*(x')\right)^{-1}. \end{aligned}$$

Let $\tilde{u}$ and $\tilde{v}$ be exponentially decaying solutions of $(\tilde{P}^0, E_1)_{x^0}$ and $(\tilde{P}', E_2)_{x^0}$, respectively. Then it holds that

$$\begin{aligned} O &= \left((ED_n - \tilde{A})\,\tilde{u}, \tilde{v}\right) - \left(\tilde{u}, (ED_n - \widetilde{{}^tA})\,\tilde{v}\right) \\ &= i\left\langle \tilde{u}(O), \tilde{v}(O)\right\rangle = i\left(\left\langle \tilde{g}, \tilde{v}_1(O)\right\rangle + \left\langle \tilde{u}_2(O), \phi\right\rangle\right). \end{aligned} \tag{9. 2. 1}$$

On the other hand we see that the fact

$$(9.2.2) \qquad R(x^0, \tau^0, \sigma^0) \neq 0 \quad \text{for some} \quad (\tau^0, \sigma^0) \in \bar{C}_- \times \boldsymbol{R}^{n-1}$$

is equivalent to that there exist a constant C_0 and a neighborhood $U(\tau^0, \sigma^0)$ such that for every $(\tau, \sigma) \in U(\tau^0, \sigma^0) \cap (\boldsymbol{C}_- \times \boldsymbol{R}^{n-1})$, $\tilde{g}$ and exponetially decaying solution $\tilde{u}$ of $(\tilde{P}^0, E_1)_{x^0}$

$$\left|\tilde{u}_2(\tau, \sigma, 0)\right| \leq C_0 |\tilde{g}|.$$

Hence if (9.2.2) is valid we have from (9.2.1)

$$\left|\tilde{v}_1(\bar{\tau}, \sigma, 0)\right| = \sup_{\tilde{g}} \frac{\left|\left\langle \tilde{g}, \tilde{v}_1(0)\right\rangle\right|}{|\tilde{g}|}$$
$$= \sup_{\tilde{g}} \frac{\left|\left\langle \tilde{u}_2(0), \phi\right\rangle\right|}{|\tilde{g}|} \leq C_0 |\phi|,$$

for every $\tau \in U(\tau^0, \sigma^0) \cap (\boldsymbol{C}_- \times \boldsymbol{R}^n)$ and exponentially decaying solution $\tilde{v}$ of $(\tilde{P}', E_2)_{x^0}$. This implies $R'(x^0, \tau^0, \sigma^0) \neq 0$, where $R'(x', \bar{\tau}, \sigma)$ is a Lopatinskii determinant of $(\tilde{P}', E_2)_{x^0}$ or $(P', B')_{x^0}$ for $(x', \bar{\tau}, \sigma) \in \Gamma \times \bar{C}_+ \times \boldsymbol{R}^{n-1}$. Since $(x^0, \tau^0, \sigma^0) \in \Gamma \times \bar{C}_- \times \boldsymbol{R}^{n-1}$ is arbitrary we see that $R(x', \tau, \sigma) \neq 0$ implies $R'(x', \bar{\tau}, \sigma) \neq 0$. The converse is also valid by the same discussion as above. In view of (9.4) and

$$R'(-x_0, x'', -\tau, \sigma) = R^{(*)}(x_0, x'', \tau, \sigma),$$

we obtain the lemma.

Now we show the condition (II) for $(P^{(*)}, B^{(*)})$. If the characteristic equation (9.5) has a double root λ at $(x_0, x'', x_n; \tau^0, \sigma^0) \in \Gamma \times \boldsymbol{R}^n$, then in a neighborhood of the point (9.5) has a real double root λ only on the surface

$$(9.9) \qquad \tau = -\theta(-x_0, x'', x_n; \sigma).$$

If the set II for $P^{(*)}$ is empty for the point $(x_0, x'', x_n; \tau^0, \sigma^0) \in \Gamma \times \boldsymbol{R}^n$, then that for P^0 is so for the point $(-x_0, x'', x_n; -\tau^0, \sigma^0)$.

(II) α): Let $(x^0, \tau^0, \sigma^0) \in \Gamma \times \boldsymbol{R}^n$ be a point for which the set II for $P^{(*)}$ is empty. From Lemma 9.1 we have for every $j, k \leq m$

$$(9.10) \qquad \begin{aligned} &R^{(*)}(x_0, x'', \tau, \sigma) \cdot \overline{\det(V_1^+, \cdots, \overset{\overset{k}{\vee}}{V_j^-}, \cdots, V_m^+)(-x_0, x'', -\bar{\tau}, \sigma)} \\ &= -\overline{R(-x_0, x'', -\bar{\tau}, \sigma)} \cdot \det(W_1^+, \cdots, \overset{\overset{j}{\vee}}{W_k^-}, \cdots, W_m^+)(x_0, x'', \tau, \sigma) \end{aligned}$$

where

$$(V_1^+, \cdots, V_m^+, V_1^-, \cdots, V_m^-)(-x_0, x'', -\bar{\tau}, \sigma)$$
$$= (BS)(-x_0, x'', -\bar{\tau}, \sigma),$$

$$(W_1^-, \cdots, W_m^-, W_1^+, \cdots, W_m^+)(x_0, x'', \tau, \sigma)$$
$$= \left(B'(S^*)^{-1}\right)(-x_0, x'', -\bar{\tau}, \sigma).$$

Assume that for all $j, k \leq m$

$$\det(W_1^+, \cdots, \overset{\overset{j}{\vee}}{W_k^-}, \cdots, W_m^+)(x^0, \tau^0, \sigma^0) = 0.$$

Note that it holds that for all $j, i \leq m$

$$\det(W_1^+, \cdots, \overset{\overset{j}{\vee}}{W_i^+}, \cdots, W_m^+)(x^0, \tau^0, \sigma^0) = 0.$$

Since

$$\text{rank}(W_1^+, \cdots, W_m^+, W_1^-, \cdots, W_m^-)(x^0, \tau^0, \sigma^0) = m,$$

we obtain for every $v_i \in R^m$

$$\det(v_1, \cdots, v_m) = 0.$$

This is a contradiction. Hence there exist indeces $j_0, k_0 (\leq m)$ such that

$$\det(W_1^+, \cdots, \overset{\overset{j_0}{\vee}}{W_{k_0}^-}, \cdots, W_m^+)(x^0, \tau^0, \sigma^0) \neq 0.$$

Since $R(-x_0, x'', -\bar{\tau}, \sigma)$ is not identically zero, (9. 10) for (j_0, k_0) implies that $\det(V_1^+, \cdots, \overset{\overset{k_0}{\vee}}{V_{j_0}^-}, \cdots, V_m^+)$ is not identically zero. Hence we obtain for small $\gamma > 0$

$$\left|R^{(*)}(x_0, x'', \tau^0 - i\gamma, \sigma^0)\right| \geq C\left|R(-x_0, x'', -\tau^0 - i\gamma, \sigma^0)\right| \geq C\gamma$$

with some $C = C(x_0, x'', \tau^0, \sigma^0) > 0$, where the last inequality follows from the condition (II) α) for (P^0, B).

(II) β): Let $(x^0, \tau^0, \sigma^0) \in \Gamma \times \boldsymbol{R}^n$ be a point for which the set II for $P^{(*)}$ is not empty. By the same method as in the proof of (II) α) we have from (II) β) for (P^0, B) the order relation:

$$\left|R^{(*)}(x^0, \tau^0 - i\gamma, \sigma^0)\right| \geq C\gamma^{\frac{1}{2}}.$$

Hence it follows from Lemma 6. 1 β) (i) that

$$\det(B^{(*)})''(x^0, \tau^0, \sigma^0) \neq 0.$$

On the other hand we see from Lemma 9. 1

$$b_{\text{II II}}^{(*)}(x_0, x'', \tau, \sigma) = \frac{\det(B^{(*)})''}{R^{(*)}} = -\overline{\left(\frac{\det B''}{R}\right)}.$$

Since from Lemma 6. 1 β) (i)

$$\det B''(-x^0, -\tau^0, \sigma^0) \neq 0,$$

we obtain

$$R^{(*)}(x_0, x'', \tau, \sigma) = -C(x', \tau, \sigma)\cdot\overline{R(-x_0, x'', -\bar{\tau}, \sigma)}, \tag{9.11}$$

where $C(x', \tau, \sigma)=\det(B^{(*)})''(\det B'')^{-1}\neq 0$. Hence from (9.9) we see that the second part of the condition (II) β) are valid for $(P^{(*)}, B^{(*)})$.

(II) γ): We can define the reflection coefficients $\widetilde{b_{ij}^{(*)}}$ for $(P^{(*)}, B^{(*)})$ by the same way as in (9.8). Then we see that

$$(\widetilde{b_{ij}^{(*)}})(x_0, x'', \tau, \sigma) = -\overline{(\widetilde{b_{ji}})(-x_0, x'', -\bar{\tau}, \sigma)}$$

as in Lemma 9.1. Hence if $\widetilde{b_{\mathrm{II}\,\mathrm{II}}}$ is real then $\widetilde{b_{\mathrm{II}\,\mathrm{II}}^{(*)}}$ is real. This implies (II) γ). Thus Theorem 9.1 is proved.

From (9.1), Theorem 1.1 and Theorem 9.1 we obtain the L^2-well-posedness of (P, B) in half space which is described in the following form (Theorem 1 and Remarks 1 of [8] applied to first order system.):

THEOREM 9.2. *Assume the conditions (I), (II), and (III) for half space* $\boldsymbol{R}^1\times\Omega=\boldsymbol{R}_+^{n+1}$ *and* $\Gamma=\boldsymbol{R}^n$. *Then for any integer* $k, s\,(k\geq 0)$ *there exist positive constants* $\gamma_{k,s}$ *and* $C_{k,s}$ *such that for any* $\gamma\geq\gamma_{k,s}$ *and for any* $f\in H_{k,s;\gamma}(\boldsymbol{R}_+^{n+1})$, $g\in H_{k+s+\frac{1}{2},\gamma}(\boldsymbol{R}^n)$ *the problem* (P, B) *has a unique solution* $u\in H_{1+k,s-1;\gamma}(\boldsymbol{R}_+^{n+1})$ *which satisfies*

$$\|u\|_{1+k,s-1;\gamma}\leq C_{k,s}\gamma^{-1}\Big(\|f\|_{k,s;\gamma}+|g|_{k+s+\frac{1}{2},\gamma}\Big).$$

Furthermore if $s\geq 0$ *and* $f=g=0$ *for* $x_0\leq T$ *then* $u=0$ *for* $x_0\leq T$.

§ 10. Remarks on the conditions (II) β) and (II) γ).

10.1. We remark in this subsection that the condition (II) β) is necessary for the case where (P, B) is a second order problem with real constant coefficients. This interesting fact is proved by R. Agemi using [2].

Let us consider the following second order problem of real constant coefficients:

$$(\square, B)\quad\begin{cases}\Big(\dfrac{\partial^2}{\partial t^2}-\Delta\Big)u=\Big(D_n^2-\Big(D_0^2-\sum\limits_{j=1}^{n-1}D_j^2\Big)\Big)u(x)=f & \text{in } \boldsymbol{R}_+^{n+1},\\ Bu|_\Gamma=\Big(D_n-\sum\limits_{j=1}^{n-1}b_jD_j-cD_0\Big)u(x', 0)=g(x') & \text{on } \Gamma=\boldsymbol{R}^n,\end{cases}$$

where $D_0=-i\dfrac{\partial}{\partial t}$. By virture of the transformation:

$$v_1(x)=\Lambda u \quad\text{and}\quad v_2(x)=D_nu,$$

we have the following equivalent 2×2 system $(\square, Q)$:

$$(\square, Q)\ \begin{cases} P(D_n)\hat{v}=\left[E_{\text{II}}D_n-\Lambda_\gamma\begin{pmatrix}0 & 1\\ -(-\tau^2+|\sigma|^2)\Lambda_\gamma^{-2} & 0\end{pmatrix}\right]\begin{pmatrix}\hat{v}_1\\ \hat{v}_2\end{pmatrix}(x_n)\\ \qquad =\begin{pmatrix}0\\ \hat{f}\end{pmatrix}(x_n), \quad \text{for } x_n>O,\\ Q\hat{v}_1(O)+\hat{v}_2(O)=\hat{g}, \end{cases}$$

where $Q(\tau, \sigma)=-\left(\sum_{j=1}^{n-1}b_j\sigma_j+c\tau\right)\Lambda_\gamma^{-1}$. Lopatinskii determinant R of the problem $(\square, Q)$ is

$$R(\tau, \sigma)=(Q, 1)\cdot{}^t(\Lambda_\gamma, \lambda_{\text{II}}^+)\cdot\Lambda_\gamma^{-1} \tag{10. 1}$$
$$=\left(\lambda_{\text{II}}^+-\left(\sum_{j=1}^{n-1}b_j\sigma_j+c\tau\right)\right)\Lambda_\gamma^{-1}.$$

where

$$\lambda_{\text{II}}^{\pm}(\tau, \sigma)=\pm\sqrt{\tau^2-|\sigma|^2} \qquad (\sqrt{1}=-1.)$$

Apply $\square$ to Lemma 3. 1 then we have

$$\mu=\mu(\tau, \sigma)=\tau-|\sigma| \qquad \text{(the case } (a)),$$

hence we have $\theta(\sigma)=|\sigma|$. Since the problem $(\square, B)$ felfills the condition (I), we have the following

LEMMA 10. 1. *Assume the condition (III). Let* $R(\tau^0, \sigma^0)=O$ *where* $\mu(\tau^0, \sigma^0)=\tau^0-|\sigma^0|=O$ *and* $\sigma^0\neq O$. *Then it holds that*

$$c^2=\sum_{j=1}^{n-1}b_j^2. \tag{10. 2}$$

PROOF. Since we have from (10. 1)

$$R(\tau^0, \sigma^0)=R\left(|\sigma^0|, \sigma^0\right) \tag{10. 1. 1}$$
$$=-\left(\sum_{j=1}^{n-1}b_j\sigma_j^0+c|\sigma^0|\right)\Lambda_\gamma^{-1}=O,$$

we see from Corollary 6. 2 and (6. 7. 2) that

$$\underset{\sigma}{\text{grad}}\, R\left(\theta(\sigma^0), \sigma^0\right)=\underset{\sigma}{\text{grad}}\, R\left(|\sigma^0|, \sigma^0\right)$$
$$=-\left(\left(b_1+c\frac{\sigma_1^0}{|\sigma^0|}\right), \cdots, \left(b_{n-1}+c\frac{\sigma_{n-1}^0}{|\sigma^0|}\right)\right)\Lambda_\gamma^{-1} \tag{10. 1. 2}$$
$$=O.$$

Hence if $c=O$ we have $b_j=O$ for all $j=1, \cdots, n-1$. If $c\neq O$, it follows from (10. 1. 1) and (10. 1. 2) that

$$c^2=\sum_{j=1}^{n-1}b_j^2.$$

This proves the lemma.

Next we show that the condition (II) β) is necessary for the problem $(\Box, Q)$ to be L^2-well-posed:

THEOREM 10.1. *Under the same assumptions as in Lemma* 10.1, *we have*

$$\operatorname{codim}\left\{(x', \sigma)\in \boldsymbol{R}^n\times\boldsymbol{R}^{n-1};\ R\left(\theta(\sigma), \sigma\right)=0\right\}$$
$$=\operatorname{rank}\operatorname*{Hess}_{(x',\sigma)} R\left(\theta(\sigma^0), \sigma^0\right)$$
$$\begin{cases} = n-2, & \text{for } c\neq 0,\\ = 0, & \text{for } c=0.\end{cases}$$

PROOF. From (10.1.2) we have

$$\frac{\partial^2 R}{\partial\sigma_j\partial\sigma_k}\left(\theta(\sigma), \sigma\right)=-c\frac{\partial}{\partial\sigma_j}\left(\frac{\sigma_k}{|\sigma|}\right)\Lambda_r^{-1}$$
$$=\begin{cases} c\sigma_k\left(\dfrac{+\sigma_j}{|\sigma|^3}\right)\Lambda_r^{-1} & \text{for } j\neq k,\\ c\left(\dfrac{-1}{|\sigma|}+\dfrac{\sigma_j\sigma_k}{|\sigma|^3}\right)\Lambda_r^{-1} & \text{for } j=k.\end{cases}$$

Hence

$$\operatorname*{Hess}_{\sigma} R\left(\theta(\sigma), \sigma\right)$$
$$=\frac{+c}{|\sigma|^3}\begin{pmatrix} \sigma_1^2-|\sigma|^2 & \sigma_1\sigma_2\cdots & \sigma_1\sigma_{n-1}\\ \vdots & & \vdots\\ \sigma_2\cdot\sigma_1 & \sigma_2^2-|\sigma|^2 & \vdots\\ \vdots & & \ddots\\ \sigma_{n-1}\sigma_1 & \cdots\cdots & \sigma_{n-1}^2-|\sigma|^2\end{pmatrix}\Lambda_r^{-1}$$

If $c=0$

$$\operatorname{rank}\operatorname*{Hess}_{(x',\sigma)}\left(\theta(\sigma), \sigma\right)=0.$$

If $c\neq 0$ this matrix has the eigenvalues

$$\underbrace{-c|\sigma|^{-1}, \cdots, -c|\sigma|^{-1}}_{n-2}, 0 \qquad \text{(see § 4 in [2].)}$$

Hence

$$\operatorname{rank}\operatorname*{Hess}_{(x',\sigma)} R\left(\theta(\sigma^0), \sigma^0\right)=n-2.$$

Put

$$M_R=\Big\{(x',\sigma)\in \boldsymbol{R}^n\times \boldsymbol{R}^{n-1}\,;\ R\big(\theta(\sigma),\sigma\big)=0\Big\}$$
$$=\boldsymbol{R}^n\times\Big\{\sigma\in\boldsymbol{R}^{n-1}\,;\ \sum_{j=1}^{n-1}b_j\sigma_j+c|\sigma|=0\Big\}.$$

If $c=0$ we have from Lemma 10.1 $b_i=0$ and hence $M_R=R^{2n-1}$. Let $c\neq 0$. It follows from $R(\theta(\sigma),\sigma)=0$ that

$$c^2|\sigma|^2=\Big(\sum_{j=1}^{n-1}b_j\sigma_j\Big)^2.$$

Hence from (10.2) we have

$$|\sigma|^2\Big(\sum_{j=1}^{n-1}b_j^2\Big)=\Big(\sum_{j=1}^{n-1}b_j\sigma_j\Big)^2,$$

i. e.

$$|\sigma|\cdot|b|=\big|\langle\sigma,b\rangle\big|,$$

where $b={}^t(b_1,\cdots,b_{n-1})$. Lec $c>0$ for simplicity, then it follows from (10.1.1) that

$$M_R=\boldsymbol{R}^n\times\Big\{\sigma\in\boldsymbol{R}^{n-1}\,;\ |\sigma|\cdot|b|=\big|\langle\sigma,b\rangle\big|,\ \sum b_j\sigma_j<0\Big\}.$$

Hence dim $M_R=n+1$ (also, in the case $c<0$) for fixed b. Therefor we obtain codim $M_R=n-2=\text{rank Hess}_\sigma\, R(\theta(\sigma^0),\sigma^0)$, for $c\neq 0$.

Thus the proof is completed.

10. 2. This subsection is devoted to the generalization of our Main theorem to the case where the boundary operator $B(x')$ is complex. Since the condition (II) γ) has an essential meaning in the case where B is real, we consider the conditions (II) β') and γ') described in §1 instead of (II) β) and γ). Throughout this subsection, we assume the condition (I) with complex valued B and assume all symbols are homogeneous of order 0.

From (II) β') and (6.4.3) it follows that for $(\tau,\sigma)\in\overline{\Sigma_-}\cap U(\tau^0,\sigma^0)$, $x'\in U(X^0)$

$$R_{\text{II}}(x',\tau,\sigma)=\big(r_1(x',\mu,\sigma)+\sqrt{\mu}\,r_2(x',\mu,\sigma)\big)\big(\sqrt{\mu}-D(x',\sigma)\big), \tag{10.3}$$

where r_1 and r_2 are smooth and $r_1(x^0,0,\sigma^0)\neq 0$. Furthermore let Q be the function defined in Lemma 6.1 β) (i). Then it follows from (6.4.3) and (6.1) that R_{II} is real if Q is real and $\sqrt{\mu}$ is real or pure imaginary in the case (a) or (b), respectively. Applying implicit function theorem to (10.3) we see from Lemma 6.4 (i) that (II) β) and γ) imply that D is real or pure imaginary in the case (a) or (b), respectively. This shows that (II) β') is fulfilled. Since it follows from (6.4.3) that (II) β) and β') imply (II) γ'), we see that (II) β) implies (II) β') and γ').

Hereafter we consider (τ,σ) as vectors $\in\overline{\Sigma_-}$.

LEMMA 10. 2. *Using the notations in Lemma* 3. 1 *and* (7 .1) Q *is written in the form*: except terms with respect to γ

$$Q(x', \mu, \sigma) = \left(2^{-1}(1+e_{12}\varepsilon)^{-1}(e_{11}-e_{22})+r_2\right)\varepsilon$$

$$\begin{cases} -\left((1+e_{12}\varepsilon)^{-1}\lambda_2+r_2D\right)D & \textit{in the case } (a), \\ or & \\ -\left(-i(1+e_{12}\varepsilon)^{-1}\lambda_2'+r_2D\right)D & \textit{in the case } (b) \end{cases}$$

respectively.

PROOF. It follows from (6. 1), (6. 4. 3) and (10. 3) that

$$\begin{aligned} Q &= (r_1+\sqrt{\mu}\,r_2)(\sqrt{\mu}-D)-(\mu s_1^{(1)}+\sqrt{\mu}\,s_2) \\ &= -r_1D+\mu r_2-\mu s_1^{(1)}+(r_1-r_2D-s_2)\sqrt{\mu}. \end{aligned}$$

Since Q is analytic in μ we have

$$r_1-r_2D-s_2=0,$$

hence

$$\begin{aligned} Q &= -r_1D+\mu r_2-\mu s_1^{(1)} \\ &= (r_2-s_1^{(1)})\mu-(r_2D+s_2)D. \end{aligned}$$

This together with (3. 1), (3. 3), (6. 1) and (7. 1) implies the lemma.

Put $Q^{(0)}=Q(x', 0, \sigma)$ and $Q^{(1)}=\dfrac{\partial Q}{\partial\mu}(x', 0, \sigma)$ as in the proof of Lemma 7. 4. Then we see from Lemma 10. 2 that

$$Q^{(0)} \begin{cases} = -\left(\lambda_2^{(0)}(x', \sigma)+r_2^{(0)}(x', \sigma)\,D(x', \sigma)\right)Dx', \sigma) \\ \text{or} \\ = -\left(-i(\lambda_2')^{(0)}(x', \sigma)+r_2^{(0)}(x', \sigma)\,D(x', \sigma)\right)D(x', \sigma), \end{cases}$$

$$\begin{aligned} Q^{(1)} &= \left(2^{-1}(e_{11}^{(0)}-e_{22}^{(0)})+r_2^{(0)}\right)(x', \sigma) \\ &\begin{cases} +(\lambda_2^{(0)}e_{12}^{(0)}-\lambda_2^{(1)}-r_2^{(1)}D)(x', \sigma)\cdot D(x', \sigma) \\ \text{or} \\ +\left(-i(\lambda_2')^{(0)}e_{12}^{(0)}+i(\lambda_2')^{(1)}-r_2^{(1)}D\right)(x', \sigma)\cdot D(x', \sigma) \end{cases} \end{aligned} \tag{10. 4}$$

with the same kind of notations as $Q^{(0)}$ and $Q^{(1)}$, in the case (*a*) and (*b*) respectively. Now Theorem 7. 1 is generalized in the following

THEOREM 10. 2. *Assume the conditions* (*II*) α), β') *and* (*III*). *Then the conclusion of Theorem* 7. 1 *is valid.*

PROOF. We have only to show that Lemma 7. 4 is valid under the conditions of this theorem. We see from the proof of Lemma 7. 4 that it suffices to show that the expression (7. 4. 7) is non-negative. Let us consider the coefficient k_4 of d_1 in the right of (7. 4. 7):

$$\begin{aligned}(10.5)\qquad &\Big\{-2k_3 \operatorname{Re} Q^{(0)} + (\operatorname{Im} Q^{(0)})^2\big(2\operatorname{Re} Q^{(1)} - (e_{11}^{(0)} - e_{22}^{(0)})\big) \\ &+ (\operatorname{Re} Q^{(0)})^2\big(2\operatorname{Re} Q^{(1)} - (e_{11}^{(0)} - e_{22}^{(0)})\big)\Big\} k_3 + O\big(|Q^{(0)}|^3\big).\end{aligned}$$

Use (10. 4), the relation $k_3 = e_{21}^{(0)} + 2\operatorname{Re}\overline{Q^{(1)}}Q^{(0)}$, then the above equals in the case (a) to

$$\begin{aligned}&= k_3\Big\{2(\lambda_2^{(0)}\operatorname{Re} D + \operatorname{Re} r_2^{(0)} D^2)(e_{21}^{(0)} + 2\operatorname{Re}\overline{Q^{(1)}}Q^{(0)}) \\ &+ 2(\lambda_2^{(0)}\operatorname{Im} D + \operatorname{Im} r_2^{(0)} D^2)^2 \operatorname{Re}\big(r_2^{(0)} + D(\lambda_2^{(0)} e_{12}^{(0)} - \lambda_2^{(1)} - r_2^{(1)} D)\big)\Big\} \\ &+ O\big((\operatorname{Re} D)^2\big) + O\big(|D|^3\big).\end{aligned}$$

Since $e_{21}^{(0)} = (\lambda_2^{(0)})^2$ it holds that

$$\begin{aligned}&e_{21}^{(0)}\operatorname{Re} r_2^{(0)} D^2 + (\lambda_2^{(0)}\operatorname{Im} D + \operatorname{Im} r_2^{(0)} D^2)^2 \operatorname{Re} r_2^{(0)} \\ &= (e_{21}^{(0)}\operatorname{Re} r_2^{(0)}\operatorname{Re} D - 2e_{21}^{(0)}\operatorname{Im} r_2^{(0)}\operatorname{Im} D)\operatorname{Re} D + O\big(|D|^3\big).\end{aligned}$$

Hence (10. 5) equals to

$$= 2k_3^2\lambda_2^{(0)}\operatorname{Re} D + O\big(|D|\big)\operatorname{Re} D + O\big(|D|^3\big).$$

In the case (b), we see by the same way that (10. 5) equals to

$$= 2k_3^2(\lambda_2')^{(0)}\operatorname{Im} D + O\big(|D|\big)\operatorname{Im} D + O\big(|D|^3\big).$$

Therefore it is non-negative (or non-positive in the case (b)) in a sufficiently small $U(x^0)\times U(\sigma^0)$, because of the condition (II) β'). Moreover from (10. 4) we have $|Q^0|^4 = O(|D|^4)$. Consequently, we see that there exist real d_1 satisfying (7. 1. 1) and a neighborhood $U(x^0)\times U(\sigma^0)$ such that (7. 4. 7) is non-negative. Thus the theorem is proved.

Using the above theorem let us generalize Theorem 1. 1:

THEOREM 10. 3. *Assume the conditions (II) α), β'), γ') and (III). Then the conclusion of Theorem* 1. 1 *is valid for complex valued* $B(x')$.

PROOF. Since $\operatorname{Im}\sqrt{\mu} \geq 0$ and $\operatorname{Re}\sqrt{\mu} \leq 0$, it follows from the condition (II) β') that $R(x', \mu, \sigma) = 0$ for $\operatorname{Im}\mu \leq 0$ is equivalent to $\mu = 0$ and $D(x', \sigma) = 0$. Hence we see from the proof of Lemma 6. 6 that Lemma 6. 6 is valid for such points $(x', \eta, \sigma) \in U(x^0)\times(U(\tau^0, \sigma^0)\cap \boldsymbol{R}^n)$ that $\eta = \theta(x', \sigma)$ and $D(x', \sigma) = 0$. On the other hand, we see from (1. 2) that in some $U(x^0)\times U(\sigma^0)$ of a point (x^0, σ^0) in Lemma 10.2

$$\operatorname{Re} D(x', \sigma) \geq 0$$

or

$$-\operatorname{Im} D(x', \sigma) \geq 0$$

in the case (*a*) and (*b*) respectively.

Hence we obtain

$$\operatorname*{grad}_{(x',\sigma)} \operatorname{Re} D(x', \sigma) = 0$$

or

$$\operatorname*{grad}_{(x',\sigma)} \operatorname{Im} D(x', \sigma) = 0$$

whenever

$$\operatorname{Re} D(x', \sigma) = 0$$

or

$$\operatorname{Im} D(x', \sigma) = 0$$

respectively. Consequently it is seen from the condition (II) γ') that the inequalities (6. 5) are replaced by

$$(10.6)\qquad \begin{gathered} \left|k_{\mathrm{II\,I}}^{(0)}(x', \sigma)\right|^2, \left|k_{\mathrm{I\,II}}^{(0)}(x', \sigma)\right|^2 \leq C \operatorname{Re} D(x', \sigma) \\ \text{or} \\ \left|k_{\mathrm{II\,I}}^{(0)}(x', \sigma)\right|^2, \left|k_{\mathrm{I\,II}}^{(0)}(x', \sigma)\right|^2 \leq -C \operatorname{Im} D(x', \sigma) \end{gathered}$$

in the case (*a*) and (*b*) respectively.

In the proof of (1. 1) in subsection 8. 2, use these inequalities instead of (6. 5). Then we see from the proofs of Theorems 7. 1 and 10. 2 that (1. 1) is valid for our case.

Thus we can generalize our Main theorem:

THEOREM 10. 4. *Assume the conditions (II) α), β'), γ') and (III). Then the conclusion of Main theorem is valid for complex valued $B(x')$.*

PROOF. Follow the proof of Theorem 9. 1 and note the relation (9. 11) especially. Then it is seen that the conditions (II) β') and γ') are preserved for the problem $(P^{(*)}, B^{(*)})$. Therefore the assertions of Theorems 9. 1 and 9. 2 are valid under the conditions of this theorem.

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