EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE FOURTEEN: GENERAL LIE GROUPS

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NOTES BY DAVE ANDERSON

1

Let G be a complex semisimple linear algebraic group, with a Borel (i.e., maximal solvable) subgroup B, and maximal torus $T \subset B$. Our goal in this lecture is to describe $H^*_T(G/B)$ and $H^*_T(G/P)$, for $G \supset P \supset B$ a paraboloic subgroup.

We will need the language of roots and weights. The **roots** are the weights of T on the Lie algebra $\mathfrak{g}$ of G, so

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in R} \mathfrak{g}_\beta,$$

where $\mathfrak{h}$ is the Lie algebra of T, and $\mathfrak{g}_\beta$ is the one-dimensional subspace of weight β. The **positive roots** are those occurring in the Lie algebra $\mathfrak{b}$ of B, so

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\beta \in R^+} \mathfrak{g}_\beta.$$

Then $R = R^+ \cup R^-$, where the **negative roots** are $R^- = \{\alpha \mid -\alpha \in R^+\}$. The roots lie in M, and span the vector space $M_{\mathbb{R}}$.

The **simple roots** are the positive roots that are not positive sums of more than one positive root; there are n of them, forming a basis for $M_{\mathbb{R}}$. Let $S \subset R^+$ be the set of simple roots. Any positive root β has a unique expression

$$\beta = \sum_{\alpha \in S} n_{\beta\alpha}\alpha, \tag{1}$$

for nonnegative integers $n_{\beta\alpha}$.

Each root β corresponds to a unipotent subgroup U_β of G, whose Lie algebra is $\mathfrak{g}_\beta$. There is an isomorphism of the additive Lie group $\mathbb{G}_a \cong \mathbb{C}$ with U_β; this is T-equivariant, with multiplication by $\beta(t)$ on $\mathbb{C}$ corresponding to conjugation by t on U_β ($u \mapsto tut^{-1}$). The product of the groups U_β for $\beta \in R^+$ forms a unipotent group U, isomorphic to $\mathbb{C}^N$, with $N = \#R^+$, and $B = T \cdot U$ is a semidirect product.

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The Weyl group $W = N(T)/T$ acts on M, with the class w of $n_w \in N(T)$ taking a weight λ to the weight $w(\lambda)$ defined by

$$w(\lambda)(t) = \lambda(n_w^{-1} t n_w).$$

(One verifies this is independent of the choice of n_w.) This determines an embedding of W in the automorphism group of M, and an inner product $(\ ,\ )$ on $M_{\mathbb{R}}$ that is invariant under W.

Each root β determines an element $s_\beta \in W$, acting on $M_{\mathbb{R}}$ by the reflection

$$s_\beta(v) = v - \frac{2(v,\beta)}{(\beta,\beta)}\beta.$$

The reflections s_α, for simple roots α, generate W. The **length** $\ell(w)$ of $w \in W$ is the minimum ℓ such that w can be written as a product $s_{\alpha_1} \cdots s_{\alpha_\ell}$, for $\alpha_1, \ldots, \alpha_\ell \in S$. Such an expression $w = s_{\alpha_1} \cdots s_{\alpha_\ell}$, with $\ell = \ell(w)$, is called a **reduced decomposition**, and $\alpha_1 \cdots \alpha_\ell$ is a **reduced word** for w.

There is a unique **longest element** w_0, whose length is $N = \#R^+$. The **opposite Borel subgroup** B^- is $w_0 B w_0$. Its Lie algebra is

$$\mathfrak{b}^- = \mathfrak{h} \oplus \bigoplus_{\beta \in R^-} \mathfrak{g}_\beta.$$

2

Let $X = G/B$; this is a smooth variety of dimension N. The T-fixed points are the points $e(w) = n_w B/B$, for any lift n_w of w to $N(T)$.

The B-orbits are denoted $X^o(w) = Be(w)$, and isomorphic to an affine space of dimension $\ell(w)$. The closure $\overline{Be(w)}$ is the **Schubert variety** corresponding to w, denoted $X(w)$; these are B-invariant subvarieties of X. Each $X(w)$ is a disjoint union of orbits $X^o(v)$, and one writes $v \leq w$ for those $X^o(v)$ that occur. That is, $v \leq w$ if and only if $X(v) \subseteq X(w)$. The filtration

$$F_0 \subset F_1 \subset \cdots \subset F_N = X,$$

with $F_p = \bigcup_{\ell(w) \leq p} X(w)$, has $F_p \smallsetminus F_{p-1}$ a union of affine spaces of dimension p. It follows that the classes $[X(w)]$ of these Schubert varieties form a basis for H^*X. Since each $X(w)$ is T-invariant, the classes

$$x(w) := [X(w)]^T$$

form a basis of H_T^*X over Λ. Note that $x(w) \in H_T^{2(N-\ell(w))}X$.

Similarly, there are B^--orbits $Y^o(w) = B^- e(w)$, and $Y(w) = \overline{Y^o(w)}$. The dimension of $Y(w)$ is $N - \ell(w)$. The classes

$$y(w) := [Y(w)]^T$$

also form a basis for H_T^*X, with $y(w) \in H_T^{2\ell(w)}X$.

It is an important general fact that $v \leq w$ (i.e., $X(v) \subseteq X(w)$) if and only if $Y(v) \supseteq Y(w)$ [Che94, Prop. 5].

The tangent space to X at $e(id)$ is $\bigoplus_{\beta\in R^-}\mathfrak{g}_\beta$. Translating by w (or n_w), we have

$$T_{e(w)}X \;=\; \bigoplus_{\beta\in w(R^-)}\mathfrak{g}_\beta.$$

The Schubert varieties $X(w)$ and $Y(w)$ are nonsingular at their central points $e(w)$, with

$$T_{e(w)}X(w) \;=\; \bigoplus_{\beta\in w(R^-)\cap R^+}\mathfrak{g}_\beta, \text{ and}$$
$$T_{e(w)}Y(w) \;=\; \bigoplus_{\beta\in w(R^-)\cap R^-}\mathfrak{g}_\beta.$$

In particular, $X(w)$ and $Y(w)$ meet transversally at the point $e(w)$.

Proposition 2.1. *The bases $\{x(w)\}_{w\in W}$ and $\{y(w)\}_{w\in W}$ are Poincaré dual bases of H_T^*X. That is,*

$$\rho_*(x(u)\cdot y(v)) = \delta_{uv}\in\Lambda.$$

Proof. If $X(u)\cap Y(v)$ is not empty, then it has a T-fixed point $e(w)$. Then $X(w)\subset X(u)$ and $Y(w)\subset Y(v)$, so $v\le w\le u$. If $v\ne u$, this implies $\ell(v)<\ell(u)$, so $\operatorname{codim}X(u)+\operatorname{codim}Y(v)>N$, and $\rho_*(x(u)\cdot y(v))=0$ for dimension reasons. If $v=u$, then transversality implies $x(u)\cdot y(u)=[e(u)]^T$, so $\rho_*(x(u)\cdot y(u))=1$. Finally, if $X(u)\cap Y(v)$ is empty, then $x(u)\cdot y(v)=0$. □

Corollary 2.2. *Let $y(u)\cdot y(v)=\sum p_{uv}^w y(w)$, with $p_{uv}^w\in\Lambda$. Then p_{uv}^w is in $\mathbb{Z}_{\ge 0}[\alpha_1,\dots,\alpha_n]$, where $\alpha_1,\dots,\alpha_n$ are the simple roots.*

Proof. This is a special case of Graham's theorem, since the $y(w)$ are classes of B^--invariant subvarieties. □

3

Subgroups P with $G\supset P\supset B$ are called **parabolic subgroups**. There is a parabolic subgroup P_J for each subset of simple roots $J\subset S$. This P_J is generated by B and the groups $U_{-\beta}$ for β in the set R_J^+ of positive roots which are sums of simple roots in J; its Lie algebra is

$$\mathfrak{p}_J = \mathfrak{b}\oplus\bigoplus_{\beta\in R_J^+}\mathfrak{g}_{-\beta}.$$

The associated Weyl group is $W_P=N_P(T)/T$, the subgroup of W generated by the simple reflections s_α for $\alpha\in J$.

For each parabolic subgroup $P=P_J$, every element (coset) in W/W_P has a unique representative w of minimal (resp., maximal) length, characterized by the property that $w(\alpha)$ is a positive (resp., negative) root for all $\alpha\in J$. We often write $[w]$ for the coset wW_P/W_P determined by $w\in W$, and call

w a **minimal** (resp., **maximal**) **representative** if it has minimal (resp., maximal) length.

For $P = P_J$ paraboloic, the homogeneous variety $X_J = G/P$ has T-fixed points $e[w]$, with $[w] \in W/W_P$, where $e[w] = n_w P/P$. One has corresponding Schubert varieties $X[w] = \overline{Be[w]}$ and $Y[w] = \overline{B^- e[w]}$, with classes $x[w] = [X[w]]^T$ and $y[w] = [Y[w]]^T$ in $H_T^*(X_J)$.

Let R_J^+ be as above, and let $R_J^- = \{\beta \in R^- \mid -\beta \in R_J^+\}$. The dimension of X_J is the number of positive roots not in R_J^+.

Let $\pi = \pi_J$ be the projection from $X = G/B$ onto $X_J = G/P$; this is G-equivariant. One has corresponding descriptions of the tangent spaces:

$$
\begin{aligned}
T_{e[w]} X_J &= \bigoplus_{\beta \in w(R^- \smallsetminus R_J^-)} \mathfrak{g}_\beta, \\
T_{e[w]} X[w] &= \bigoplus_{\beta \in w(R^- \smallsetminus R_J^-) \cap R^+} \mathfrak{g}_\beta, \text{ and} \\
T_{e[w]} Y[w] &= \bigoplus_{\beta \in w(R^- \smallsetminus R_J^-) \cap R^-} \mathfrak{g}_\beta.
\end{aligned}
$$

The dimension of $X[w]$ is the minimal length of a representative of $[w]$, and the codimension of $Y[w]$ is the minimal length of a representative of $[w]$. In fact, for w minimal, $X(w) \to X[w]$ determines an isomorphism $T_{e(w)} X(w) \to T_{e[w]} X[w]$ (since $w(R_J^-) \cap R^+ = \emptyset$), so it gives an isomorphism from $X^o(w)$ to $X^o[w] = Be[w]$. In particular,

$$\pi_*(x(w)) = x[w] \text{ for } w \text{ minimal.}$$

Similarly, we have an isomorphism $T_{e(w)} Y(w) \to T_{e[w]} Y[w]$ for w maximal, so

$$\pi_*(y(w)) = y[w] \text{ for } w \text{ maximal.}$$

On the other hand, for w maximal, we have $\pi^{-1}(X[w]) = X(w)$; notice that this set-theoretic equality is also scheme-theoretic, since π is a smooth morphism and these varieties are irreducible. Hence

$$\pi^*(x[w]) = x(w) \text{ for } w \text{ maximal,}$$

and similarly,

$$\pi^*(y[w]) = y(w) \text{ for } w \text{ minimal.}$$

As in the case of G/B, the sets $\{x[w]\}$ and $\{y[w]\}$ form bases for $H_T^*(G/P)$, as $[w]$ varies in W/W_P.

Exercise 3.1. Show that these are Poincaré dual bases. (Solution: For $\eta : G/P \to pt$, and w and v maximal representatives of $[w]$ and $[v]$, we have $\eta_*(x[w] \cdot y[v]) = \eta_*(x[w] \cdot \pi_*(y(v))) = \eta_* \pi_*(\pi^* x[w] \cdot y(v)) = \rho_*(x(w) \cdot y(v)) = \delta_{wv}$.)

Let $D_J : H_T^*X \to H_T^*X$ be the Λ-linear map defined by

$$D_J = (\pi_J)^* \circ (\pi_J)_*.$$

This lowers degree by twice the codimension, i.e., by $2\#(R_J^+)$.

Lemma 3.2. (1) *If w^- is the minimal representative of its coset in W/W_P, then*

$$D_J(x(w^-)) = x(w^+),$$

where w^+ is the maximal representative of the coset. If w is not minimal in its coset, then $D_J(x(w)) = 0$.

(2) *If w^+ is the maximal representative of its coset, then*

$$D_J(y(w^+)) = y(w^-),$$

where w^- is the minimal representative. If w is not maximal in its coset, then $D_J(y(w)) = 0$.

Proof. This follows from the preceding discussion. Note that if w is not minimal, then $X(w)$ has larger dimension than its image $X[w]$, so $\pi_*[X(w)]^T = 0$, and similarly for $Y(w)$ if w is not maximal. □

We will use these operators primarily when J consists of one simple root α, so P_J is a minimal parabolic subgroup. In this case, W_J is the group of order two generated by s_α, and each coset has two representatives, w and ws_α, whose lengths differ by 1. If $\ell(ws_\alpha) = \ell(w) + 1$, then for $D_\alpha = D_{\{\alpha\}}$, we have

$$\begin{aligned} D_\alpha(x(w)) &= x(ws_\alpha), \qquad & D_\alpha(x(ws_\alpha)) &= 0; \\ D_\alpha(y(ws_\alpha)) &= y(w), \qquad & D_\alpha(y(w)) &= 0. \end{aligned}$$

Exercise 3.3. If α is a simple root, and $u, v \in W$ with $\ell(us_\alpha) > \ell(u)$ and $\ell(vs_\alpha) < \ell(v)$, then

$$D_\alpha(y(u) \cdot y(v)) = y(u) \cdot y(vs_\alpha).$$

(Solution: Write $D_\alpha = (\pi_\alpha)^* \circ (\pi_\alpha)_*$, $y(u) = \pi_\alpha^* y[u]$, $(\pi_\alpha)_* y(v) = y[v]$, and $\pi_\alpha^* y[v] = y(vs_\alpha)$, so

$$\begin{aligned} D_\alpha(y(u) \cdot y(v)) &= \pi_\alpha^*((\pi_\alpha)_*(\pi_\alpha^* y[u] \cdot y(v))) \\ &= \pi_\alpha^*(y[u] \cdot (\pi_\alpha)_* y(v)) \\ &= \pi_\alpha^*(y[u] \cdot y[v]) \\ &= \pi_\alpha^* y[u] \cdot \pi_\alpha^* y[v].) \end{aligned}$$

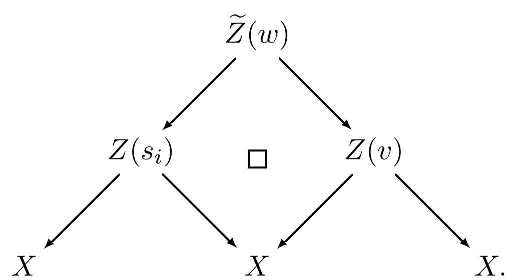
For any sequence $\alpha_1, \ldots, \alpha_\ell$ of simple roots, we have the composition

$$D_{\alpha_1} \circ \cdots \circ D_{\alpha_\ell} : H_T^*(X) \to H_T^{*-2\ell}(X).$$

This takes $y(w)$ to $y(ws_{\alpha_1} \cdots s_{\alpha_\ell})$ if $\ell(ws_{\alpha_1} \cdots s_{\alpha_\ell}) = \ell(w) - \ell$, and takes $y(w)$ to 0 otherwise; and similarly for the $x(w)$'s. Since the classes $y(w)$ form a basis for H_T^*X, it follows that if the length of $s_{\alpha_1} \cdots s_{\alpha_\ell}$ is less than ℓ, then $D_{\alpha_1} \circ \cdots \circ D_{\alpha_\ell} = 0$; on the other hand, if $s_{\alpha_1} \cdots s_{\alpha_\ell}$ is a reduced decomposition for v, then $D_{\alpha_1} \circ \cdots \circ D_{\alpha_\ell}$ depends only on v, and can be denoted D_v. Indeed, $D_v(y(w)) = y(wv)$ if $\ell(wv) = \ell(w) + \ell(v)$, and $D_v(y(w)) = 0$ otherwise. In addition, $D_u \circ D_v = D_{uv}$ if $\ell(uv) = \ell(u) + \ell(v)$, and $D_u \circ D_v = 0$ otherwise.

It follows from the preceding discussion that π^* embeds $H_T^*(G/P)$ in $H_T^*(G/B)$, taking Schubert classes to Schubert classes.

Remark 3.4. As in [Ber-Gel-Gel73] and [Dem74] (cf. [Ara89, §3.7]), the operators D_w can be defined via a correspondence: Let $Z(w) \subset X \times X$ be the closure of $G \cdot (e(id) \times e(w))$, with G acting diagonally. Then D_w is $(p_1)_* \circ (p_2)^*$, where p_1 and p_2 are the projections from $Z(w)$ to X. In particular, $Z(s_i) = X \times_{X_i} X$, where $X_i = X_{\{\alpha_i\}}$. For $w = s_i v$, with $\ell(w) = \ell(v) + 1$, we have a diagram



The map $\widetilde{Z}(w) \to Z(w)$ (by the projection p_{13}) is birational and surjective, from which it follows that $D_w = D_{s_i} \circ D_v$.

Remark 3.5. The literature also contains operators $\mathscr{L}_w : H_T^* X \to \Lambda$ (cf. [Ara86], [Ara89]). In our language, $\mathscr{L}_w(x) = \rho_*(x \cdot x(w))$. Equivalently, by Poincaré duality, $\mathscr{L}_w(x)$ is the coefficient of $y(w)$ in the expansion of x; i.e.,

$$x = \sum \mathscr{L}_w(x)\, y(w).$$

4

Let Q be the quotient field of Λ, and let $F(W, Q)$ be the Q-algebra of functions from W to Q. We know that the localization map

$$H_T^* X \to H_T^* X^T = \bigoplus_{w \in W} H_T^*(e(w))$$

embeds $H_T^* X$ in the space $F(W, \Lambda)$ of functions from W to Λ, by $x \mapsto (w \mapsto x|_w)$, and hence $H_T^* X \hookrightarrow F(W, Q)$.

For a simple root α, define the Q-linear map $A_\alpha : F(W, Q) \to F(W, Q)$ by the formula

$$(A_\alpha \psi)(w) = \frac{\psi(w s_\alpha) - \psi(w)}{w(\alpha)}.$$

Proposition 4.1. *The diagram*

$$\begin{array}{ccc} H_T^*X & \hookrightarrow & F(W,Q) \\ {\scriptstyle D_\alpha}\downarrow & & \downarrow{\scriptstyle A_\alpha} \\ H_T^*X & \hookrightarrow & F(W,Q) \end{array}$$

commutes.

Proof. Since the inclusion $H_T^*X^T \to H_T^*X$ given by the Gysin map is an isomorphism after tensoring with Q (over Λ), it suffices to show that the two paths around the diagram agree on elements of the form $x = (\iota_v)_*(1)$, where $\iota_v : \{e(v)\} \to X$ is the inclusion. Such an x localizes to the function ψ_v, defined by

$$\psi_v(v) = (\iota_v)^*(\iota_v)_*(1) = c^T_{top}(T_{e(v)}X) = \prod_{\beta \in v(R^-)} \beta,$$

and $\psi_v(w) = 0$ for $w \neq v$.

Then

$$\begin{aligned} A_\alpha(\psi_v)(v) &= -\frac{\psi_v(v)}{v(\alpha)}, \\ A_\alpha(\psi_v)(vs_\alpha) &= \frac{\psi_v(v)}{vs_\alpha(\alpha)} = -\frac{\psi_v(v)}{v(\alpha)}, \end{aligned}$$

and $A_\alpha(\psi_v)(w) = 0$ for $w \notin \{v, vs_\alpha\}$.

Going the other way around the diagram, we have $D_\alpha(x) = (\pi_\alpha)^*(\pi_\alpha)_*(\iota_v)_*(1)$. Then $D_\alpha(x)|_w = \iota_w^*(\pi_\alpha)^*(\pi_\alpha)_*(\iota_v)_*(1) = (\iota_{[w]})^*(\iota_{[v]})_*(1)$, where $\iota_{[w]} = \iota_w \circ \pi_\alpha$ is the inclusion of the point $e[w]$ in $X_\alpha = G/P_\alpha$. (Here $e(w)$ and $e[w]$ are identified.) Therefore $D_\alpha(x)|_w = 0$ if $[w] \neq [v]$, and

$$D_\alpha(x)|_{vs_\alpha} = D_\alpha(x)|_v = c^T_{top}(T_{e[v]}X_\alpha).$$

But

$$T_{e[v]}X_\alpha = \bigoplus_{\beta \in v(R^- \smallsetminus \{-\alpha\})} g_\beta,$$

so $c^T_{top}(T_{e[v]}X_\alpha)$ is equal to $c^T_{top}(T_{e(v)}X)/(-v(\alpha))$, which is the same as the value of $A_\alpha(\psi_v)$ at v. □

Corollary 4.2. *A composition $A_{\alpha_1} \circ \cdots \circ A_{\alpha_\ell}$ vanishes if the length of $s_{\alpha_1} \cdots s_{\alpha_\ell}$ is less than ℓ, and depends only on $v = s_{\alpha_1} \cdots s_{\alpha_\ell}$ if $\ell(v) = \ell$. Writing A_v for $A_{\alpha_1} \circ \cdots \circ A_{\alpha_\ell}$ for an reduced word for v, we have $A_v \circ A_w = A_{vw}$ if $\ell(vw) = \ell(v) + \ell(w)$, and $A_v \circ A_w = 0$ otherwise.*

Proof. This follows from the corresponding results for the operators D_α, and the commutativity of the diagram in the proposition. □

Lemma 4.3. (1) $y(s_\alpha)|_w = \varpi_\alpha - w(\varpi_\alpha)$.

(2) $x(w_0 s_\alpha)|_w = w_0(\varpi_\alpha) - w(\varpi_\alpha)$.

Proof. For (1), let $f_\alpha(w) = y(s_\alpha)|_w$. We know that

(i) $f_\alpha(id) = 0$,

since $e(id)$ is not contained in $Y(s_\alpha)$. Since $D_\beta(y(s_\alpha)) = 0$ for any simple root $\beta \neq \alpha$, Proposition 4.1 says that $A_\beta(f_\alpha) = 0$, i.e.,

(ii) $f_\alpha(ws_\beta) = f_\alpha(w)$ for simple roots $\beta \neq \alpha$, for any w.

Similarly, $D_\alpha(y(s_\alpha)) = y(id) = 1$ gives $A_\alpha(f_\alpha) = 1$, i.e.,

(iii) $f_\alpha(ws_\alpha) = f_\alpha(w) + w(\alpha)$ for all w.

The function $f_\alpha : W \to \Lambda$ (or $W \to Q$) is uniquely determined by properties (i), (ii), and (iii). Indeed, the difference of two such functions would take the same values at any w and ws_β, for all simple roots β, so it must be constant. (Every w has the form $w_0 s_{\beta_1} \cdots s_{\beta_\ell}$.) By (i), this constant must be zero.

Now the function $f_\alpha(w) = \varpi_\alpha - w(\varpi_\alpha)$ clearly satisfies (i), and it satisfies (ii) and (iii) since

$$\begin{aligned} f_\alpha(ws_\beta) &= \varpi_\alpha - ws_\beta(\varpi_\alpha) \\ &= \varpi_\alpha - w(\varpi_\alpha - \delta_{\alpha\beta}\beta) \\ &= f_\alpha(w) + \delta_{\alpha\beta} w(\beta). \end{aligned}$$

Similarly, to prove (2), note that the function $f_\alpha(w) = x(w_0 s_\alpha)|_w$ satisfies

(i') $f_\alpha(w_0) = 0$,

together with (ii) and (iii) (since $D_\beta(x(w_0 s_\alpha)) = \delta_{\alpha\beta} x(w_0)$, and $x(w_0) = 1$). The same argument shows there is a unique such function, and that $f_\alpha(w) = w_0(\varpi_\alpha) - w(\varpi_\alpha)$ satisfies (i'), (ii), and (iii). □

Corollary 4.4. $y(s_\alpha) - x(w_0 s_\alpha) = \varpi_\alpha - w_0(\varpi_\alpha)$.

Proof. The two sides have the same localization at every fixed point. □

Remark 4.5. In type A_n, $w_0(\varpi_i) = -\varpi_{n+1-i}$, so $y(s_\alpha) - x(w_0 s_\alpha) = \varpi_i + \varpi_{n+1-i}$.

We will give another formula for the localizations $y(s_\alpha)|_w$, and for general $y(v)|_w$, in the next chapter.

Corollary 4.6. *If $v \neq w$, there is a simple root α such that $y(s_\alpha)|_v \neq y(s_\alpha)|_w$, and $x(w_0 s_\alpha)|_v \neq x(w_0 s_\alpha)|_w$.*

Proof. Since the weights ϖ_α form a basis for $M_\mathbb{R}$, there is an α such that $v(\varpi_\alpha) \neq w(\varpi_\alpha)$. □

Exercise 4.7. For $\lambda \in M$, show that

$$c_1^T(L(\lambda)) = \sum_{\alpha \in S} a_\alpha y(s_\alpha) + \lambda,$$

where $a_\alpha = -\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$. (It suffices to do this for $\lambda = \varpi_\alpha$, where $c_1^T(L(\varpi_\alpha)) = \varpi_\alpha - y(s_\alpha)$, as is seen by evaluating both sides at $w \in W$.)

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