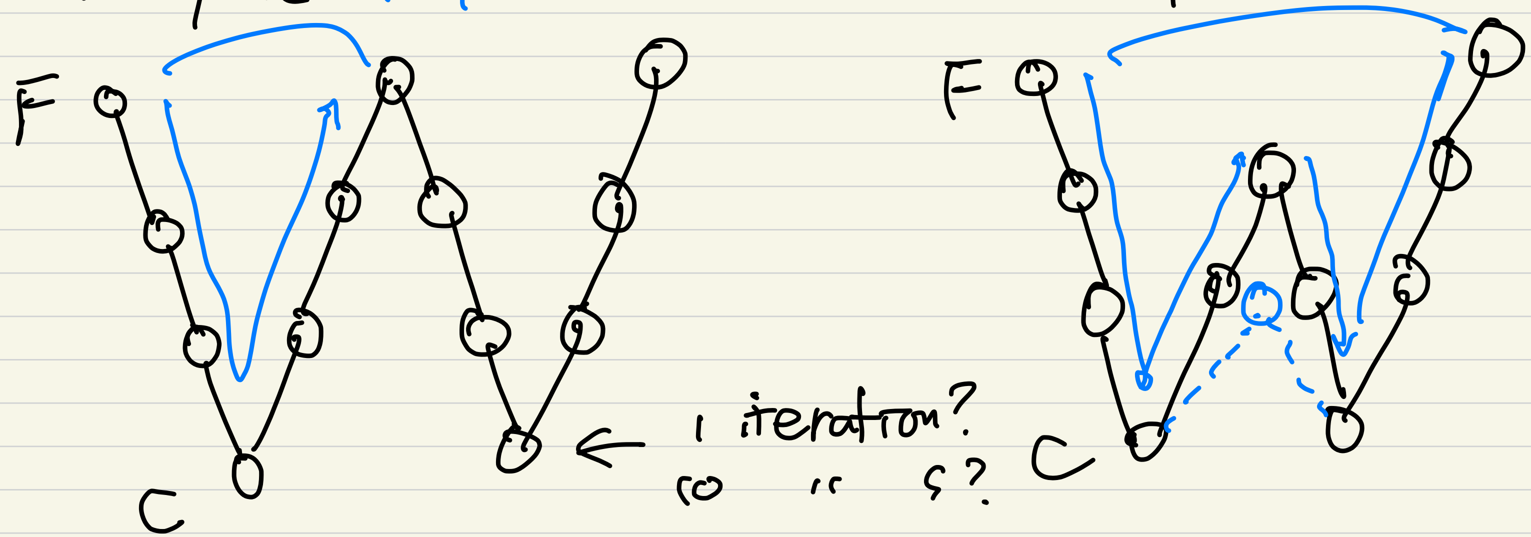
- HW 4 due date Dec. 14 → Dec. 21 (6 pm)
- Final exam: Dec. 14 (Monday) 9:30 – 11:30 am

  (student id starting w/ 2020 → 301-204
  others → 301 → 105)

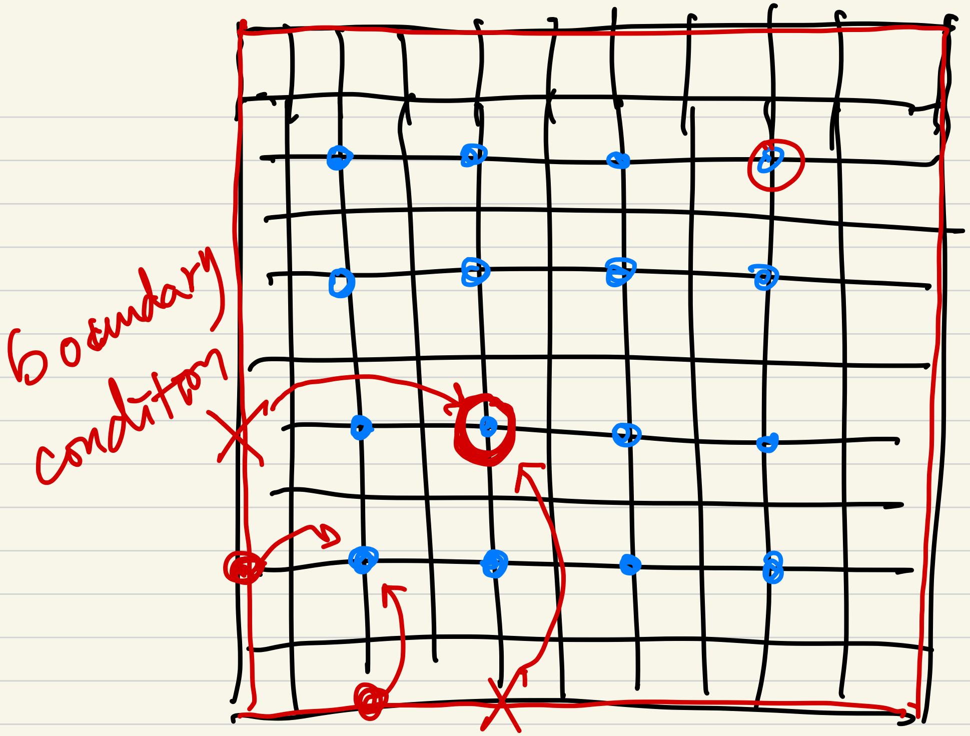
  Those who have high body temperature before exam should tell TA about the situation via e-mail.

* V cycle (1 cycle 1 iteration) W cycle (1 cycle 1 iteration)



- single grid transfers the information to an adjacent grid per iteration,

while multigrid " " " to all grids per iteration



• : fine grid

• : second-level

• : third-level

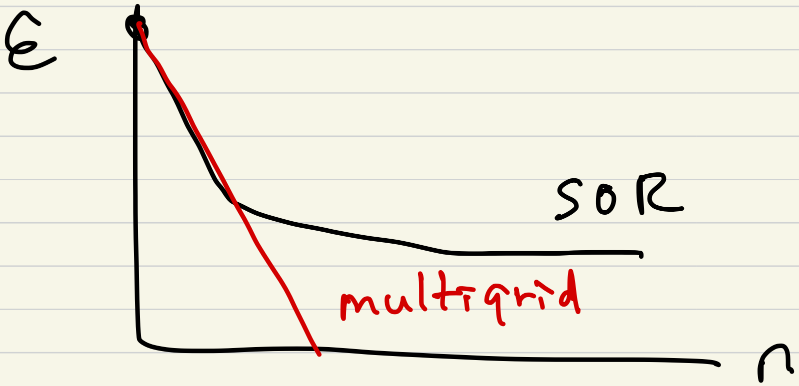
$A_1 \phi^{n+1} = r^{n+1}$

As $N \uparrow$, $\lambda \to 1$ $\Rightarrow$ slow convergence.

in multigrid method, $N \to N/2 \to N/4 \to N/8 \to N/16 \cdots$

$\Downarrow$

$\lambda \downarrow$ $(\lambda \ll 1)$



- Convergence rate depends on smoother, cycle, ...

iteration

- Number of cycle for convergence is in principle indep. of N.

⑤ Hyperbolic PDE

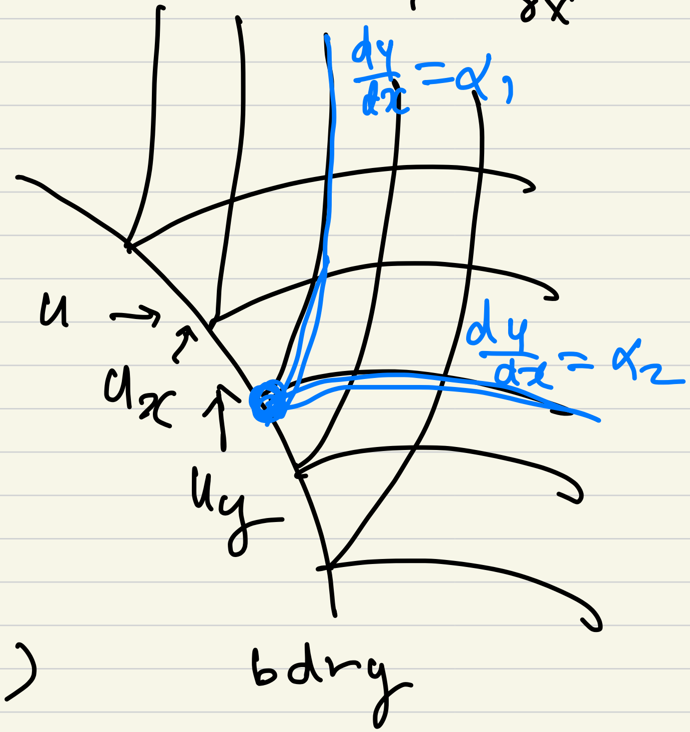
$$a \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} = F \quad (b^2 - ac > 0)$$

$a, b, c = f(x, y, u, u_x, u_y)$

$u_x = \frac{\partial u}{\partial x}$

characteristic lines

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \alpha_1, \alpha_2$$



We need to know locations of char. lines (x, y) and variations of u_x and u_y on char. lines.

Along any diff'l line element (dx, dy)

$$d\left(\frac{\partial u}{\partial x}\right) = du_x = \frac{\partial^2 u}{\partial x^2} dx + \frac{\partial^2 u}{\partial x \partial y} dy \quad -①$$

$$d\left(\frac{\partial u}{\partial y}\right) = du_y = \frac{\partial^2 u}{\partial x \partial y} dx + \frac{\partial^2 u}{\partial y^2} dy \quad -②$$

$$\text{GE}: \quad a\,u_{xx} + 2b\,u_{xy} + c\,u_{yy} = F$$

$$①: \quad dx\,u_{xx} + dy\,u_{xy} = du_x$$

$$②: \quad dx\,u_{xy} + dy\,u_{yy} = du_y$$

$$\begin{vmatrix} a & 2b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0 \rightarrow$$

eq. for char. lines

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \alpha_1, \alpha_2$$

For the existence of u_{xx}, u_{xy} & u_{yy}, $\begin{vmatrix} a & F & c \\ dx & du_x & 0 \\ 0 & du_y & dy \end{vmatrix} = 0.$

$$\rightarrow dy\,(a\,du_x - F\,dx) - du_y\,(-c\,dx) = 0$$

$$\rightarrow \frac{dy}{dx}\,a\,du_x + c\,du_y - F\,dy = 0$$

Char. lines $\dfrac{dy}{dx} = \alpha_1, \alpha_2$ — 2 eqs.

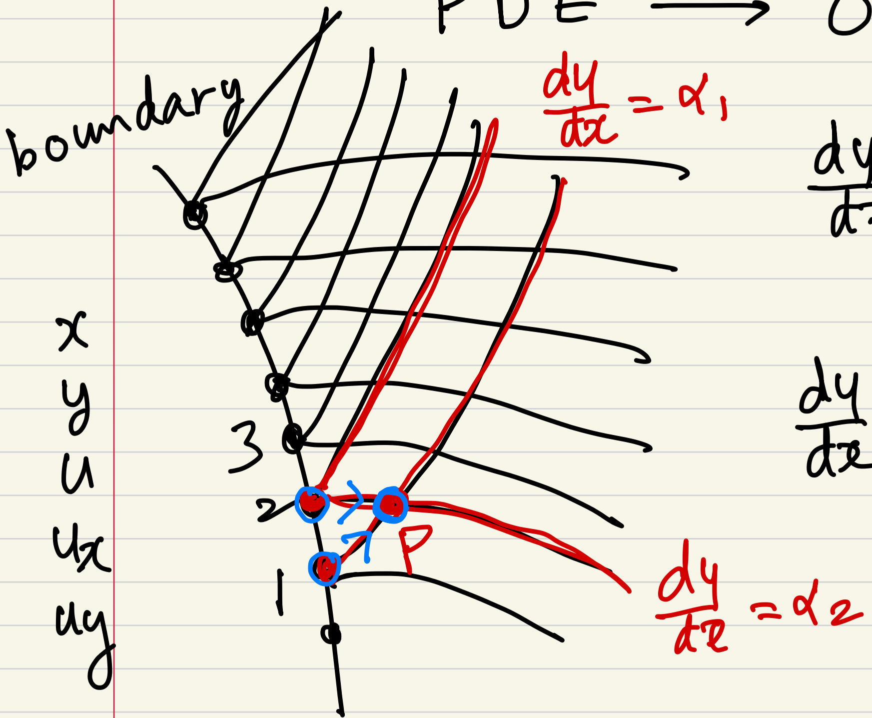
$$\alpha_1\,a\,du_x + c\,du_y - F\,dy = 0 \quad \text{for } \frac{dy}{dx} = \alpha_1 \quad \text{— 1 eq.}$$

$$\alpha_2\,a\,du_x + c\,du_y - F\,dy = 0 \quad \text{for } {''} = \alpha_2 \quad \text{— 1 eq.}$$

from chain rule, $du = u_x\,dx + u_y\,dy$ — 1 eq.

$\Rightarrow$ 5 eqs. for 5 unknowns $(\underbrace{x, y}_{\text{char. lines}}, u, u_x, u_y)$

PDE $\longrightarrow$ ODEs : method of characteristics (MOC)



$$\frac{dy}{dx} = \alpha_1 : \quad \frac{y(P) - y(1)}{x(P) - x(1)} = \frac{1}{2}\left[\alpha_1(P) + \alpha_1(1)\right]$$

$$\frac{dy}{dx} = \alpha_2 : \quad \frac{y(P) - y(2)}{x(P) - x(2)} = \frac{1}{2}\left[\alpha_2(P) + \alpha_2(2)\right]$$

- $\frac{1}{2}\left[\alpha_1(P)a(P) + \alpha_1(1)a(1)\right]\left(u_x(P) - u_x(1)\right) + \frac{1}{2}\left[c(P) + c(1)\right]\left(u_y(P) - u_y(1)\right) - \frac{1}{2}\left[F(P) + F(1)\right]\left(y(P) - y(1)\right) = 0$

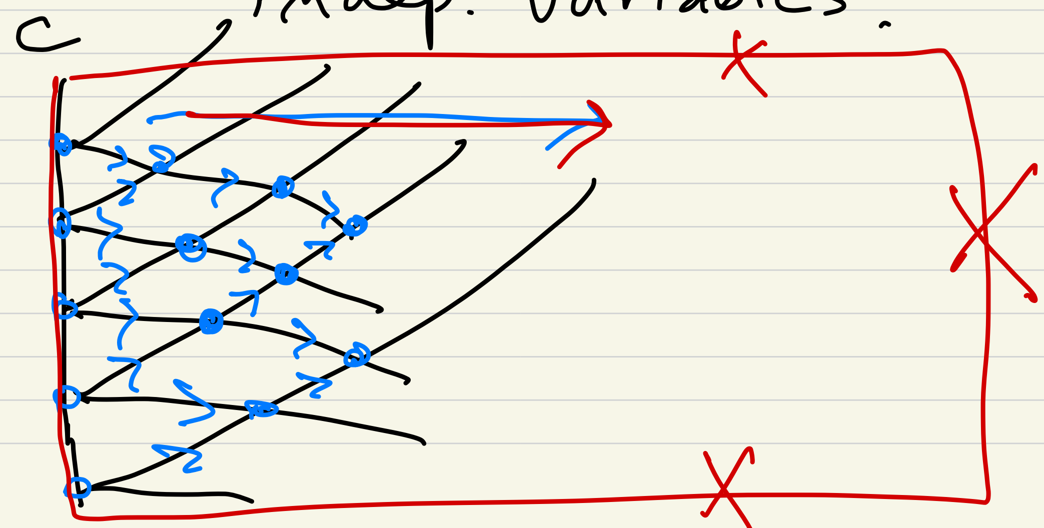
- $\frac{1}{2}\left[\alpha_2(P)a(P) + \alpha_2(2)a(2)\right]\left(u_x(P) - u_x(2)\right) + \frac{1}{2}\left[c(P) + c(2)\right]\left(u_y(P) - u_y(2)\right) - \frac{1}{2}\left(F(P) + F(2)\right)\left(y(P) - y(2)\right) = 0$

$$\bullet\ u(P) - u(1) = \frac{1}{2}\left[u_x(P) + u_x(1)\right](x(P) - x(1)) + \frac{1}{2}\left[u_y(P) + u_y(1)\right](y(P) - y(1))$$

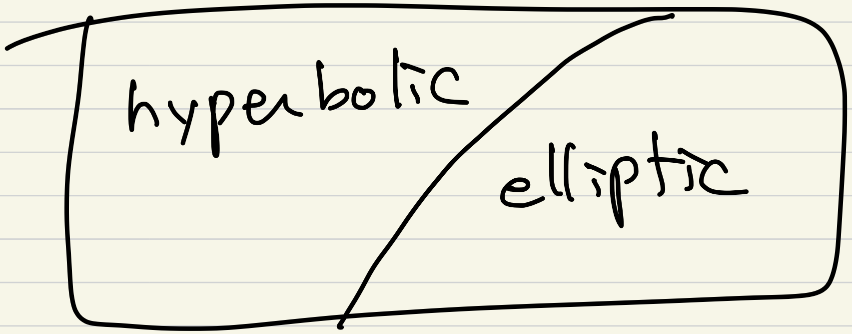
$\Rightarrow$ In general, these are nonlinear $\rightarrow$ iterative method.

* Advantages of MOC

- important properties of exact sol. are preserved in numerical sol.
- method is easily adapted to the computation of problems that contain discontinuities.
- ability to compute the sol. over a long span of indep. variables.



* Disadvantages of MOC
  - difficulties of keeping track of the locations of the char. lines and the values of variables in 3D.
  - difficulties in handling mixed-type PDE (e.g. hyperbolic in one area and elliptic in other area)



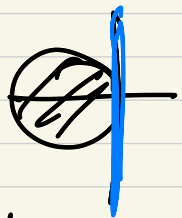
⓪ Explicit methods for hyperbolic eqs.

Convection eq. $\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$ $\quad x - ct = \text{const}.$

wave eq. $\underbrace{\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}}$ $\rightarrow$ $\begin{matrix} x - ct = \text{const} \\ x + ct = \text{const} \end{matrix} \left(\frac{dx}{dt} = \pm c \right)$

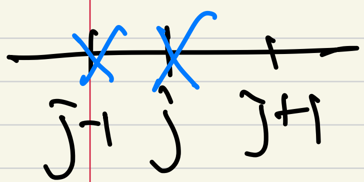
$\hookrightarrow \begin{matrix} u = \phi \\ w = u' \end{matrix} \rightarrow \frac{\partial u}{\partial t} = c\frac{\partial w}{\partial x}$ & $\frac{\partial w}{\partial t} = c\frac{\partial u}{\partial x}$

For $\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$, $(c>0)$

EE + CD2 :  unstable

leapfrog : stable, no amp. error, spurious root.
(CFL<1)

$c>0$ $\rightarrow$

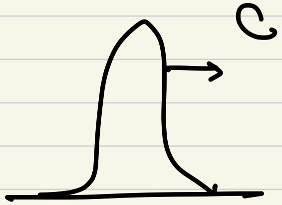


EE + upwind scheme : $\frac{\phi_j^{n+1} - \phi_j^{n}}{\Delta t} + c\,\frac{\phi_j^{n} - \phi_{j-1}^{n}}{\Delta x} = 0$

↳ complex eigenvalue

Von Neumann stability analysis $\rightarrow$ $\frac{c\Delta t}{\Delta x} < 1$ (CFL<1) for stability

→ too dissipative



@ t

$c<0$, $\frac{\partial \phi}{\partial x} = \frac{\phi_{j+1} - \phi_j}{\Delta x}$

For wave eq., two char. lines $x - ct = \text{const}$ $\xrightarrow{c}$

$x + ct = \text{const}$ $\leftarrow$

$\Rightarrow$ no upwind scheme!

CD scheme + EE $\Rightarrow$ unstable

+ leapfrog $\Rightarrow$ stable for CFL < 1
spurious root

+ RK4 $\rightarrow$ conditionally stable.

* Lax - Wendroff scheme

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$$

$$\phi(x, t+\Delta t) = \phi(x,t) + \Delta t \underbrace{\frac{\partial \phi}{\partial t}(x,t)}_{= -c\frac{\partial \phi}{\partial x}} + \frac{1}{2}\Delta t^2 \frac{\partial^2 \phi}{\partial t^2}(x,t) + \frac{1}{6}\Delta t^3 \frac{\partial^3 \phi}{\partial t^3} + \cdots$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t}\right) = \frac{\partial}{\partial t}\left(-c\frac{\partial \phi}{\partial x}\right)$$

$$= -c\frac{\partial}{\partial x}\frac{\partial \phi}{\partial t} = c^2\frac{\partial^2 \phi}{\partial x^2}$$

CD2 : $\boxed{\phi_j^{n+1} = \phi_j^n - c\Delta t \dfrac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x} + \dfrac{1}{2} c^2 \Delta t^2 \dfrac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta x^2}} + O(\Delta t^3)$

$\Delta t^3 \dfrac{\partial^3 \phi}{\partial t^3}$ ↙ dispersive error

Lax-Wendroff scheme

explicit

stable for CFL < 1

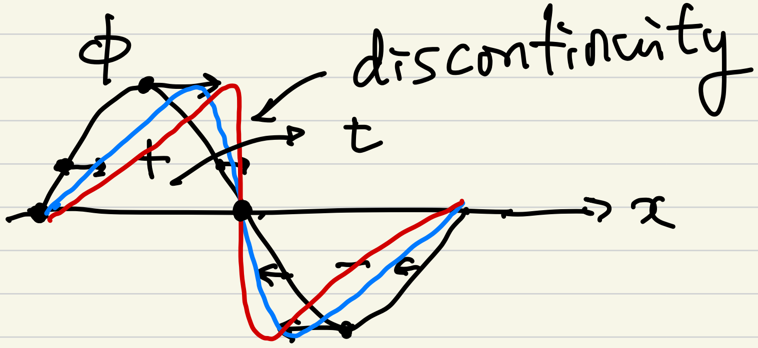
2nd-order accurate in space & time.

disadvantage : c is not constant f(t)
nonlinear, $c = c(\phi)$
higher dimension
→ difficulty

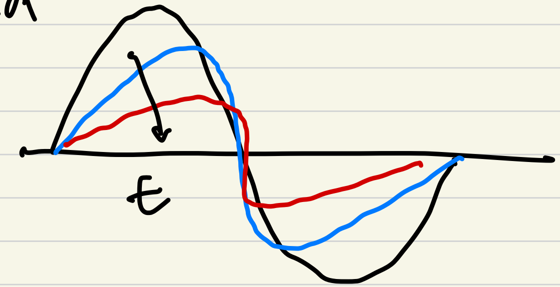
this method can be applied to

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} \rightarrow \left\{ \begin{array}{l} \dfrac{\partial u}{\partial t} = c \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial t} = c \dfrac{\partial u}{\partial x} \end{array} \right.$$

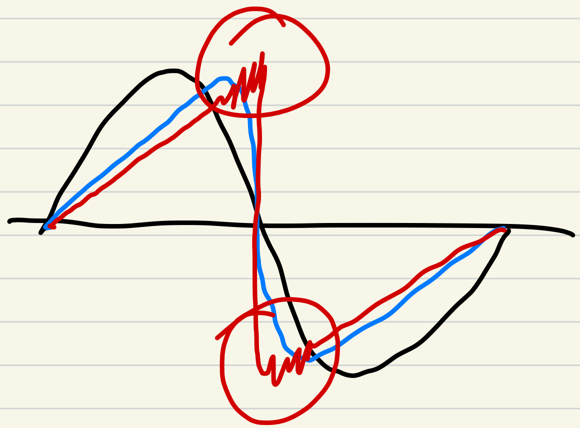
• $\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} = 0$ $(c=\phi)$ nonlinear



upwind



CD2



TVD
ENO
schemes

◉ Implicit method for hyperbolic eqs.

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$$

CD2 + CN : $$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + c \cdot \frac{1}{2}\left[\frac{\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1}}{2\Delta x} + \frac{\phi_{j+1}^{n} - \phi_{j-1}^{n}}{2\Delta x} \right] = 0$$

no amplitude error (no dissipation)

dispersive error

stable

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2}$$

① splitting into two eqs, u & v and applying CN to two eqs.

② applying CN directly to wave eq.

$$\frac{\phi_j^{n+1} - 2\phi_j^n + \phi_j^{n-1}}{\Delta t^2} = \frac{1}{2}c^2\left[\frac{\phi_{j+1}^{n+1} - 2\phi_j^{n+1} + \phi_{j-1}^{n+1}}{\Delta x^2} + \frac{\phi_{j+1}^{n} - 2\phi_j^{n} + \phi_{j-1}^{n}}{\Delta x^2}\right]$$

stable

a little more accurate than the method applied to two splitted eqs.

spurious root

- higher dimension (2D/3D) $\rightarrow$ ADI
- nonlinear eq. $\frac{\partial\phi}{\partial t} + \phi\frac{\partial\phi}{\partial x} = 0$ $\rightarrow$ iterative method for implicit scheme.

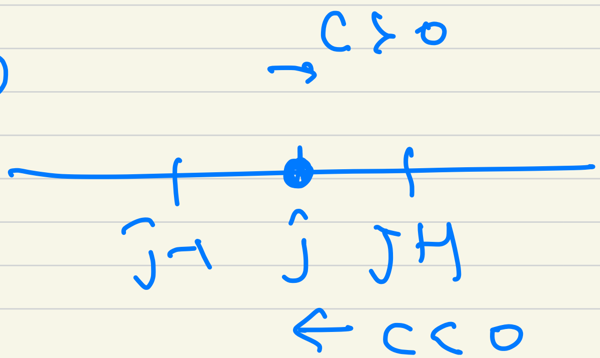
- MacCormack scheme – very popular $\nearrow$ EE + down wind

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \qquad \left(\begin{array}{ll} \phi_j^* = \phi_j^n - \frac{c\Delta t}{\Delta x}\left(\phi_{j+1}^n - \phi_j^n\right) & \text{predictor} \\ \phi_j^{n+1} = \frac{1}{2}\left(\phi_j^n + \phi_j^*\right) - \frac{c\Delta t}{\Delta x}\left(\phi_j^* - \phi_{j-1}^*\right) & \text{corrector} \end{array}\right.$$

$\downarrow$ upwind

- explicit
- equivalent to Lax-Wendroff scheme for linear prob.
  - $\hookrightarrow O(\Delta x^2)$ & $O(\Delta t^2)$
- stable for CFL < 1
- readily extended to 2D & 3D
- accurate and easier to program
- desirable nonlinear properties.

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$$



$$\left.\frac{\partial \phi}{\partial x}\right|_j = \frac{\phi_j - \phi_{j-1}}{\Delta x} \quad (c > 0) \text{ upwind}$$

$$= \frac{\phi_{j+1} - \phi_j}{\Delta x} \quad (c < 0) \text{ down wind}$$

Ch. 6. Discrete Transform methods : spectral method

6.1 Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$$

k : wavenumber

$\hat{f}_k$ ↳ Fourier coeff.

$f(x)$ ↑ continuous ft

$$f'(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k \, ik \, e^{ikx} = \sum_{k=-\infty}^{\infty} ik\hat{f}_k e^{ikx}$$

$ik\hat{f}_k$ ↳ Fourier coeff. of f'

$$f(x) \xrightarrow{FT} \hat{f}_k \longrightarrow ik\hat{f}_k \xrightarrow{IFT} f'(x)$$

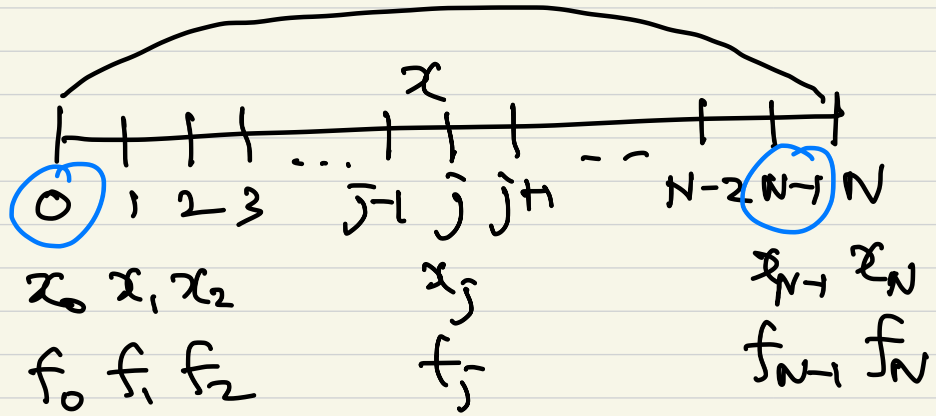
6.1.1 Discrete Fourier series

$\boxed{f : \text{periodic ft.}}$

$f_0 = f_N$

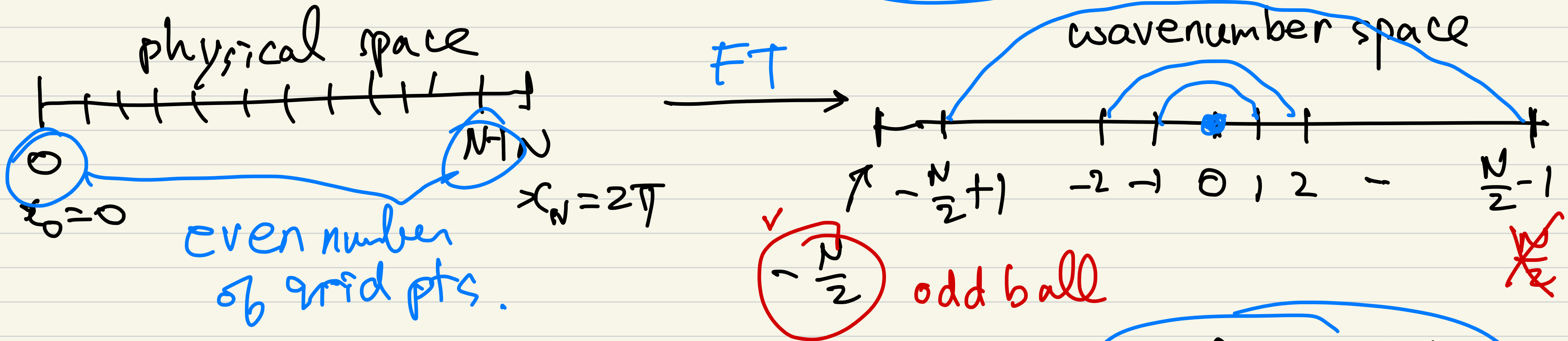
N grid pts.

N : even number



Let us take N to be even number and period of f to be 2π.

$$e^{ikx} : \Delta k \cdot 2\pi = 2\pi \Rightarrow \Delta k = 1$$



$$x_j = jh, \quad h = \frac{2\pi}{N} \text{ (grid spacing)}$$

uniform grids

Discrete Fourier transformation (DFT)

$$\star \quad \boxed{f_j = \sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{ikx_j}}, \quad j = 0, 1, 2, \cdots, N-1$$

$O(N^2)$ operations

discrete Fourier coeff. of f_j

(If a period of f is L, $\Delta k \cdot L = 2\pi \rightarrow \Delta k = 2\pi / L$)

$$f_j \xrightarrow{\text{DFT}} \hat{f}_k \Rightarrow A \hat{f}_k = f_j$$

full matrix

to get $\hat{f}_k$, $O(N^3)$ operations. too expensive!

so, use orthogonality.

$$I = \sum_{j=0}^{N-1} e^{ikx_j} e^{-ik'x_j} = \sum_{j=0}^{N-1} e^{i(k-k')x_j} = \begin{cases} N & \text{if } k = k' + mN \\ 0 & \text{otherwise} \end{cases}$$

$m = 0, \pm 1, \pm 2, \cdots$

$$f_j = \sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{ikx_j}$$

$$\sum_{j=0}^{N-1} f_j e^{-ik'x_j} = \sum_{j=0}^{N-1} \sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{ikx_j} e^{-ik'x_j}$$

$$= \sum_{k=-N/2}^{N/2-1} \sum_{j=0}^{N-1} \hat{f}_k e^{i(k-k')x_j} = N \hat{f}_{k'}$$

$$\rightarrow \boxed{\hat{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-ikx_j}}$$

$k = -\frac{N}{2}, -\frac{N}{2}+1, \cdots, 0, \cdots, \frac{N}{2}-1$

$O(N^2)$ operations.

FFT $\longrightarrow$ $O(N\log_2 N)$ operations
(fast FT) see 'Numerical Recipes' for this routine.

$$\text{DFT}: \begin{cases} f_j = \displaystyle\sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{ikx_j}, & j = 0, 1, 2, \cdots, N-1 \quad N \\ \hat{f}_k = \dfrac{1}{N}\displaystyle\sum_{j=0}^{N-1} f_j e^{-ikx_j}, & k = -\frac{N}{2}, \cdots, 0, \cdots, \frac{N}{2}-1 \quad N \end{cases}$$

N

~~N~~

2N

↳ complex number

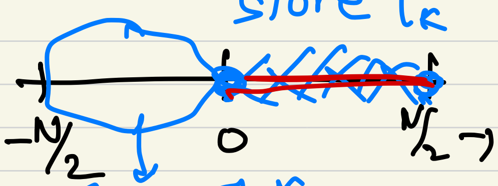
$a+ib$

• FT of a real ft.

$$f: \text{real ft} \xrightarrow{\text{FT}} \hat{f}_k : \text{complex}$$

$$\left(\hat{f}_k = \frac{1}{N}\sum f_j e^{-ikx_j}\right)^* \rightarrow \hat{f}_k^* = \frac{1}{N}\sum f_j e^{ikx_j} = \hat{f}_{-k}$$

$$\Rightarrow \boxed{\hat{f}_{-k} = \hat{f}_k^*} \qquad \underset{0 \qquad\qquad 2\pi}{\overset{f_j}{\longmapsto}} \quad \xrightarrow{\text{FT}}$$



No need to store $\hat{f}_{-k}$ when f is real. $\hat{f}_{-k} = \hat{f}_k^*$

· Higher dimensions

$f(x,y)$: doubly periodic in x & y directions

$$\boxed{f(x_m, y_\ell) = \sum_{k_1=-\frac{N_1}{2}}^{\frac{N_1}{2}-1} \sum_{k_2=-\frac{N_2}{2}}^{\frac{N_2}{2}-1} \hat{f}_{k_1,k_2}\, e^{ik_1 x_m}\, e^{ik_2 y_\ell}}$$

$m = 0, 1, \cdots, N_1-1$

$\ell = 0, 1, \cdots, N_2-1$

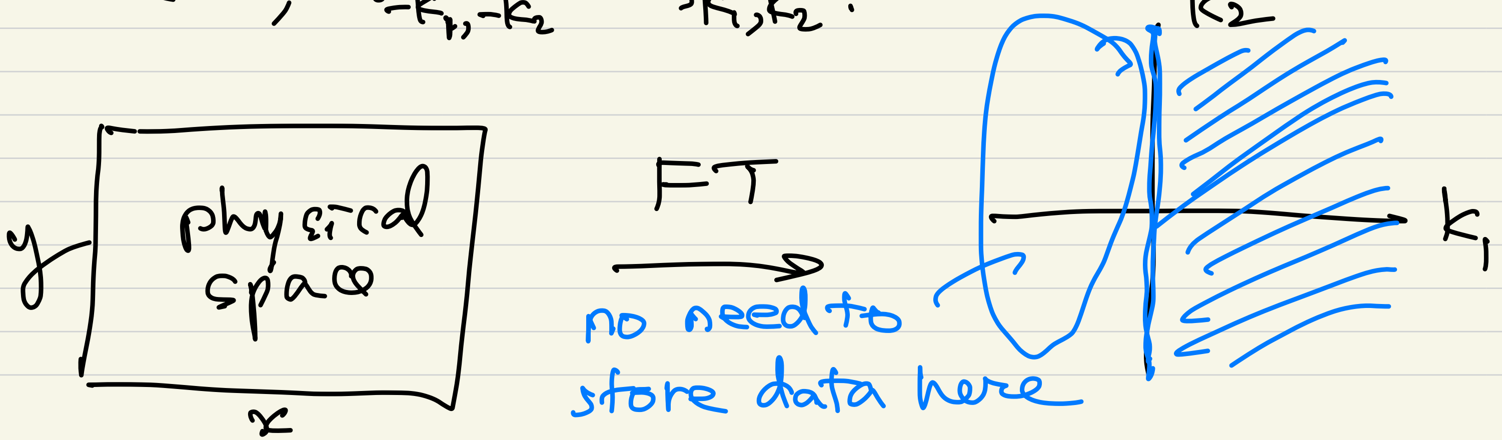
use orthogonality

$$\boxed{\hat{f}_{k_1,k_2} = \frac{1}{N_1}\frac{1}{N_2}\sum_{m=0}^{N_1-1}\sum_{\ell=0}^{N_2-1} f_{m,\ell}\, e^{-ik_1 x_m}\, e^{-ik_2 y_\ell}}$$

$k_1 = -\frac{N_1}{2}, \cdots, 0, \cdots, \frac{N_1}{2}-1$

$k_2 = -\frac{N_2}{2}, \cdots, 0, \cdots, \frac{N_2}{2}-1$

If f is real, $\hat{f}_{-k_1,-k_2} = \hat{f}^*_{k_1,k_2}$.



$$H(x) = f(x)\, g(x) \qquad \hat{H} \doteq ?$$

[aliasing error
" control]