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A Table of Integrals of the Error Functions*

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This is a compendium of indefinite and definite integrals of products of the Error function with elementary or transcendental functions. A substantial portion of the results are new.

Key Words: Astrophysics; atomic physics; Error functions; indefinite integrals; special functions; statistical analysis.

1. Introduction

Integrals of the error function occur in a great variety of applications, usually in problems involving multiple integration where the integrand contains exponentials of the squares of the arguments. Examples of applications can be cited from atomic physics [16],[1] astrophysics [13], and statistical analysis [15]. This paper is an attempt to give an up-to-date exhaustive tabulation of such integrals.

All formulas for indefinite integrals in sections 4.1, 4.2, 4.5, and 4.6 below were derived from integration by parts and checked by differentiation of the resulting expressions. Section 4.3 and the second half of 4.5 cover all formulas given in [7], with omission of trivial duplications and with a number of additions; section 4.4 covers essentially formulas given in [4], Vol. **I,** pp. 233–235. All these formulas have been re-derived and checked, either from the integral representation or from the hypergeometric series of the error function. Sections 4.7, 4.8 and 4.9 originated in a more varied way. Some formulas were derived from multiple integrals involving elementary functions, others from existing formulas for integrals of confluent hypergeometric functions, and still others, a small portion, were compiled directly from existing literature. In connection with the last three sections, the reader should refer to [3] and [4], Vol. **II,** pp. 402, 409–411.

Throughout this paper, we have adhered to the notations used in the NBS Handbook [9] and we have also assumed the reader's familiarity with the properties of the error functions, for which he is referred to [5]. In addition, the reader should also attend to the following conventions:

(i) $z=x+iy=r\exp(i\theta)$ is a complex variable,

$$\mathscr{R}(z)=x,\ \mathscr{I}(z)=y,\ |z|=r,\ \arg z=\theta;$$

(ii) the parameters a, b, and c are real and positive except where otherwise stated;

(iii) unless otherwise specified, the parameters n and k represent the integers 0, 1, 2 . . ., whereas the parameters p, q, and ν may be nonintegral;

(iv) the integration constants have been omitted for the indefinite integrals;

(v) when x is used (instead of z) as the integration variable, it means that the formula has been established only for real x, though it may still be valid for certain complex values;

(vi) the integration symbol $⨍$ denotes a Cauchy principal value.

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[1] Figures in brackets indicate the literature references at the end of this paper.

2. Glossary of Functions and Notation

$A(x)$	Gaussian Probability Function	$\frac{1}{\sqrt{2\pi}}\int_{-x}^{x} e^{-t^2/2}dt$
$C(z)$	Fresnel Integral	$\int_0^z \cos\left(\frac{\pi}{2}t^2\right)dt$
$Ci(z)$	Cosine Integral	$-\int_z^\infty \frac{\cos t}{t}dt$
$D_\nu(z)$	Parabolic Cylinder Function	
$e_n(z)$	Truncated Exponential	$\sum_{k=0}^{n}\frac{z^k}{k!}$
$-Ei(-z)\equiv E_1(z)$	Exponential Integral	$\int_z^\infty \frac{e^{-t}}{t}dt$
$Ei(x)$	Exponential Integral	$-\fint_{-x}^{\infty}\frac{e^{-t}}{t}dt$
${}_1F_1(a;\, b;\, z)\equiv M(a,\, b,\, z)$	Confluent Hypergeometric Function	$\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}$
${}_kF_l(a_1\ .\ .\ .\ a_k;\, b_1\ .\ .\ .\ b_l;\, z)$	Generalized Hypergeometric Function	$\sum_{n=0}^{\infty}\frac{(a_1)_n\ .\ .\ .\ (a_k)_n}{(b_1)_n\ .\ .\ .\ (b_l)_n}\frac{z^n}{n!}$
$H_n(x)$	Hermite Polynomial	
$\mathbf{H}_\nu(x)$	Struve Function	
$I_\nu(z)$	Modified Bessel Function	
$J_\nu(z)$	Bessel Function	
$K_\nu(z)$	Modified Bessel Function	
L_n^α	Generalized Laguerre Polynomial	
$M_{p,\,q}(z)$	Whittaker Function	
$Y_\nu(z)$	Neumann Function (Bessel Function of Second Kind)	
$P(x)$	Probability Function	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$
$(p)_n$	Pochhammer's Symbol	$\Gamma(p+n)/\Gamma(p)$
$P_n(x)$	Legendre Polynomial	
$P_\nu^\mu(z)$	Associated Legendre Function of the First Kind	
$S(z)$	Fresnel Integral	$\int_0^z \sin\left(\frac{\pi}{2}t^2\right)dt$
$si(z)$	Sine Integral	$-\int_z^\infty \sin t\,\frac{dt}{t}$
$U(a,\, b,\, z)\equiv\Psi(a,\, b,\, z)$	Confluent Hypergeometric Function	
$W_{p,\,q}(z)$	Whittaker Function	

γ	Euler's Constant	0.5772156649 . . .
$\Gamma(p)$	Gamma Function	
$\gamma(p, z)$	Incomplete Gamma Function	$\int_0^z e^{-t}t^{p-1}dt$
$\Gamma(p, z)$	Incomplete Gamma Function	$\int_z^\infty e^{-t}t^{p-1}dt$
$\zeta(z)$	Riemann's Zeta Function	$\sum_{k=0}^{\infty} k^{-z}$
$\psi(z)$	Psi Function	$\frac{d}{dz}[\ln \Gamma(z)]$

3. Definition and Integral Representations

3.1. Definitions and Other Notations

1. $\text{erf}\,(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt,$

2. $\text{erfc}\,(z) \equiv \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt = 1 - \text{erf}\,(z),$

3. $\text{erfi}\,(z) \equiv -i\ \text{erf}\,(iz) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt.$

Some authors use the above notations without the factor $\frac{2}{\sqrt{\pi}}$, and some use $\Phi(z)$ for erf (z).

4. $w(z) \equiv e^{-z^2}\ \text{erfc}\,(-iz).$

For real x, Dawson's integral is defined as

5. $F(x) \equiv \frac{\sqrt{\pi}}{2} e^{-x^2}\ \text{erfi}\,(x) = \frac{\sqrt{\pi}}{2} \mathscr{I}[w(x)].$

The error function is also closely related to the Gaussian probability functions:

6. $\text{erf}\,(x) = 2P(x\sqrt{2}) - 1 = A(x\sqrt{2}).$

3.2. Integral Representations

1. $\text{erf}\,(az) = \frac{2az}{\sqrt{\pi}} \int_0^1 e^{-a^2z^2t^2}\, dt$

2. $\text{erfc}\,(az) = \frac{2}{\sqrt{\pi}} e^{-a^2z^2} \int_0^\infty e^{-(t^2+2azt)} dt$

3. $\text{erf}\left(az + \frac{w}{a}\right) = \frac{2a}{\sqrt{\pi}} \exp\left(c - \frac{w^2}{a^2}\right) \int e^{-(a^2z^2+2wz+c)} dz.$

4. $\operatorname{erfc}\left(\frac{z}{a}\right)=\frac{2a}{\sqrt{\pi}}\exp\left(c-\frac{z^2}{a^2}\right)\int_0^\infty e^{-(a^2t^2+2zt+c)}dt$

5. $e^{2ab}\operatorname{erf}\left(ax+\frac{b}{x}\right)+e^{-2ab}\operatorname{erf}\left(ax-\frac{b}{x}\right)=\frac{4a}{\sqrt{\pi}}\int e^{-a^2x^2-b^2/x^2}dx$

6. $\operatorname{erfc}(az)=\frac{2a}{\sqrt{\pi}}e^{-a^2z^2}\int_0^\infty\frac{e^{-a^2t^2}tdt}{(z^2+t^2)^{1/2}},\qquad \mathscr{R}(a)>0,\ \mathscr{R}(z)>0.$

7. $\operatorname{erfc}(az)=\frac{2z}{\pi}e^{-a^2z^2}\int_0^\infty\frac{e^{-a^2t^2}dt}{(t^2+z^2)},\qquad \mathscr{R}(a)>0,\ |\arg z|<\pi,\ z\neq 0.$

8. $1-[\operatorname{erf}(x)]^2=\frac{4}{\pi}e^{-x^2}\int_0^1\frac{e^{-x^2t^2}dt}{(t^2+1)},\qquad x>0$

9. $\operatorname{erf}\left(x+\frac{iy}{2}\right)+\operatorname{erf}\left(x-\frac{iy}{2}\right)=\frac{4}{\sqrt{\pi}}e^{y^2/4}\int e^{-x^2}\cos xy\,dx$

10. $\operatorname{erf}\left(x+\frac{iy}{2}\right)-\operatorname{erf}\left(x-\frac{iy}{2}\right)=\frac{4}{i\sqrt{\pi}}e^{y^2/4}\int e^{-x^2}\sin xy\,dx$

11. $\operatorname{erf}(x)=\frac{x}{\sqrt{\pi}}\int_0^\pi e^{x^2\cos\theta}\cos(x^2\sin\theta+\theta/2)d\theta,\qquad x\neq 0$

12. $\operatorname{erf}(x)=\frac{1}{\pi}\int_0^\infty e^{-t}\sin(2x\sqrt{t})\frac{dt}{t}.$

4. Integrals of Product of Error Functions With Other Functions

4.1. Combination of Error Function With Powers

1. $\int\operatorname{erf}(az)dz=z\operatorname{erf}(az)+\frac{1}{a\sqrt{\pi}}\exp(-a^2z^2)$

2. $\int\operatorname{erfc}(az)dz=z\operatorname{erfc}(az)-\frac{1}{a\sqrt{\pi}}\exp(-a^2z^2)$

3. $\int_0^\infty\operatorname{erfc}(ax)dx=\frac{1}{a\sqrt{\pi}},\qquad |\arg a|<\frac{\pi}{4}$

4. $\int\operatorname{erf}(az)z\,dz=\frac{1}{2}z^2\operatorname{erf}(az)-\frac{1}{4a^2}\operatorname{erf}(az)+\frac{z}{2a\sqrt{\pi}}\exp(-a^2z^2)$

5. $\int\operatorname{erfc}(az)z\,dz=\frac{1}{2}z^2\operatorname{erfc}(az)+\frac{1}{4a^2}\operatorname{erf}(az)-\frac{z}{2a\sqrt{\pi}}\exp(-a^2z^2)$

6. $\int_0^\infty\operatorname{erfc}(ax)x\,dx=\frac{1}{4a^2},\qquad |\arg a|<\frac{\pi}{4}$

7. $$\int \operatorname{erf}(az) z^n dz = \frac{z^{n+1}}{n+1} \operatorname{erf}(az) + \frac{e^{-a^2z^2}}{a\sqrt{\pi}(n+1)} \sum_{k=0}^{l-1} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}-k+1\right)} \frac{z^{n-2k}}{a^{2k}}$$

$$-\frac{1-j}{n+1} \frac{\Gamma\left(l+\frac{1}{2}\right)}{a^{n+1}\sqrt{\pi}} \operatorname{erf}(az), \qquad j=0 \text{ or } 1,\ 2l-j=n+1$$

8. $$\int \operatorname{erf}(az) z^{n+2} dz = \frac{(n+2)(n+1)}{2(n+3)a^2} \int \operatorname{erf}(az) z^n dz + \left(z^2 - \frac{n+2}{2a^2}\right) \frac{z^{n+1}}{(n+3)} \operatorname{erf}(az)$$

$$+\frac{1}{a\sqrt{\pi}(n+3)} z^{n+2} e^{-a^2z^2}$$

9. $$\int \operatorname{erfc}(az) z^n dz = \frac{z^{n+1}}{(n+1)} \operatorname{erfc}(az) - \frac{e^{-a^2z^2}}{a\sqrt{\pi}(n+1)} \sum_{k=0}^{l-1} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}-k+1\right)} \frac{z^{n-2k}}{a^{2k}}$$

$$+\frac{1-j}{n+1} \frac{\Gamma\left(l+\frac{1}{2}\right)}{a^{n+1}\sqrt{\pi}} \operatorname{erf}(az), \qquad j=0 \text{ or } 1,\ 2l-j=n+1$$

10. $$\int \operatorname{erfc}(az) z^{n+2} dz = \frac{(n+2)(n+1)}{2(n+3)a^2} \int \operatorname{erfc}(az) z^n dz + \left(z^2 - \frac{n+2}{2a^2}\right) \frac{z^{n+1}}{(n+3)} \operatorname{erfc}(az)$$

$$-\frac{1}{a\sqrt{\pi}(n+3)} z^{n+2} e^{-a^2z^2}$$

11. $$\int_0^\infty \operatorname{erfc}(ax) x^n dx = \frac{\Gamma\left(\frac{n}{2}+1\right)}{(n+1)\sqrt{\pi}a^{n+1}}, \qquad |\arg a| < \frac{\pi}{4}$$

12.[2] $$\int \operatorname{erf}(az) z^{-1} dz = \ln z \operatorname{erf}(az) - \frac{2a}{\sqrt{\pi}} \int \ln z e^{-a^2z^2} dz$$

13. $$\int \operatorname{erfc}(az) z^{-1} dz = \ln z \operatorname{erfc}(az) + \frac{2a}{\sqrt{\pi}} \int \ln z e^{-a^2z^2} dz$$

14. $$\int \operatorname{erf}(az) z^{-n} dz = -\frac{\operatorname{erf}(az)}{(n-1)z^{n-1}} + \frac{2a}{(n-1)\sqrt{\pi}} \int \frac{1}{z^{n-1}} e^{-a^2z^2} dz, \qquad n \geqslant 2$$

15. $$\int \operatorname{erfc}(az) z^{-n} dz = -\frac{\operatorname{erfc}(az)}{(n-1)z^{n-1}} - \frac{2a}{(n-1)\sqrt{\pi}} \int \frac{1}{z^{n-1}} e^{-a^2z^2} dz, \qquad n \geqslant 2$$

16. $$\int \operatorname{erf}(az) z^p dz = \frac{z^{p+1}}{p+1} \operatorname{erf}(az) - \frac{1}{(p+1)a^{p+1}\sqrt{\pi}} \gamma\left(\frac{p}{2}+1,\ a^2z^2\right), \qquad p > -2,\ p \neq -1$$

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[2] See appendix for integrals on the right-hand sides of eqs (12 to 15).

17. $\int \operatorname{erfc}(az)z^p dz = \frac{z^{p+1}}{p+1} \operatorname{erfc}(az) + \frac{1}{(p+1)a^{p+1}\sqrt{\pi}} \gamma\left(\frac{p}{2}+1, a^2z^2\right), \qquad p > -1$

18. $\int_0^\infty \operatorname{erfc}(ax)x^p dx = \frac{1}{(p+1)a^{p+1}\sqrt{\pi}} \Gamma\left(\frac{p}{2}+1\right), \qquad |\arg a| < \frac{\pi}{4}, \; p > -1$

19. $\int_0^\infty \operatorname{erf}(ax)x^{p-2} dx = \frac{a^{1-p}}{\sqrt{\pi}(1-p)} \Gamma\left(\frac{p}{2}\right), \qquad |\arg a| < \frac{\pi}{4}, \; 0 < p < 1.$

4.2. Combination of Error Functions With Exponentials and Powers [3]

1. $\int \operatorname{erf}(az)e^{bz} dz = \frac{1}{b} e^{bz} \operatorname{erf}(az) - \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erf}\left(az - \frac{b}{2a}\right)$

2. $\int \operatorname{erfc}(az)e^{bz} dz = \frac{1}{b} e^{bz} \operatorname{erfc}(az) + \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erf}\left(az - \frac{b}{2a}\right)$

3. $\int_0^\infty \operatorname{erf}(ax)e^{-bx} dx = \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erfc}\left(\frac{b}{2a}\right), \qquad \mathscr{R}(b) > 0, \; |\arg a| < \pi/4$

4. $\int_0^\infty \operatorname{erfc}(ax)e^{bx} dx = \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left[1 + \operatorname{erf}\left(\frac{b}{2a}\right)\right] - \frac{1}{b}, \; |\arg(b-a)| < \frac{\pi}{4}$

5. $\int \operatorname{erf}(az)e^{bz} z\, dz = \frac{1}{b} \operatorname{erf}(az)e^{bz}\left(z - \frac{1}{b}\right) - \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left\{ \left(\frac{b}{2a^2} - \frac{1}{b}\right) \operatorname{erf}(t) - \frac{1}{a\sqrt{\pi}} e^{-t^2} \right\}, \; t = az - \frac{b}{2a}$

6. $\int \operatorname{erfc}(az)e^{bz} z\, dz = \frac{1}{b} \operatorname{erfc}(az)e^{bz}\left(z - \frac{1}{b}\right)$

$$+ \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left\{ \left(\frac{b}{2a^2} - \frac{1}{b}\right) \operatorname{erf}(t) - \frac{1}{a\sqrt{\pi}} e^{-t^2} \right\}, \; t = az - \frac{b}{2a}$$

7. $\int_0^\infty \operatorname{erf}(ax)e^{-bx} x\, dx = \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left[\frac{1}{b} - \frac{b}{2a^2}\right] \operatorname{erfc}\left(\frac{b}{2a}\right) + \frac{1}{ab\sqrt{\pi}}, \; \mathscr{R}(b) > 0, \; |\arg a| < \frac{\pi}{4}$

8. $\int_0^\infty \operatorname{erfc}(ax)e^{bx} x\, dx = \frac{1}{b} \exp\left(\frac{b^2}{4a^2}\right) \left[\frac{b}{2a^2} - \frac{1}{b}\right] \left[1 + \operatorname{erf}\left(\frac{b}{2a}\right)\right] + \frac{1}{b^2} + \frac{1}{ab\sqrt{\pi}}, \; |\arg(b-a)| < \frac{\pi}{4}$

9.[4] $\int \operatorname{erf}(az)e^{bz} z^n dz = (-1)^n \frac{n!}{b^{n+1}} e^{bz} \operatorname{erf}(az) e_n(-bz)$

$$- \frac{2a}{\sqrt{\pi}} (-1)^n \frac{n!}{b^{n+1}} \sum_{k=0}^{n} \frac{(-b)^k}{k!} \int z^k \exp(-a^2z^2 + bz) dz$$

[3] In this section a and b can take any value on the complex plane other than the origin, except where otherwise stated.

[4] See appendix for the integrals on the right-hand sides of eqs (9 to 12).

10. $$b\int \operatorname{erf}(az)e^{bz}z^n dz + n\int \operatorname{erf}(az)e^{bz}z^{n-1}dz$$

$$= e^{bz}z^{n-1}\left[z \operatorname{erf}(az) + \frac{1}{a\sqrt{\pi}}e^{-a^2z^2}\right]$$

$$-\frac{1}{a\sqrt{\pi}}\int (bz^n + nz^{n-1}) \exp(-a^2z^2 + bz)dz$$

11. $$\int \operatorname{erfc}(az)e^{bz}z^n dz = (-1)^n \frac{n!}{b^{n+1}} e^{bz} \operatorname{erfc}(az)e_n(-bz)$$

$$+\frac{2a}{\sqrt{\pi}}\frac{(-1)^n n!}{b^{n+1}}\sum_{k=0}^{n}\frac{(-b)^k}{k!}\int z^k \exp(-a^2z^2 + bz)dz$$

12. $$b\int \operatorname{erfc}(az)e^{bz}z^n dz + n\int \operatorname{erfc}(az)e^{bz}z^{n-1}dz$$

$$= e^{bz}z^{n-1}\left[z \operatorname{erfc}(az) - \frac{1}{a\sqrt{\pi}}e^{-a^2z^2}\right]$$

$$+\int (bz^n + nz^{n-1}) \exp(-a^2z^2 + bz)dz$$

13. $$\int_0^\infty \operatorname{erf}(ax)e^{-bx}x^n dx = \left(\frac{2}{\pi}\right)^{1/2}\sum_{k=0}^{n}\frac{n!}{b^{k+1}}(2a)^{\frac{k-n}{2}}\exp\left(\frac{b^2}{8a^2}\right)D_{-n+k-1}\left(\frac{b}{a\sqrt{2}}\right)$$

$$= \sum_{k=0}^{n}\frac{(-1)^{n-k}}{b^{k+1}}\frac{n!(a)^{k-n}}{(n-k)!}\frac{d^{n-k}}{dq^{n-k}}\left[\exp\left(\frac{q^2}{4}\right)\operatorname{erfc}\left(\frac{q}{2}\right)\right],$$

$$q = \frac{b}{a}, \qquad \mathscr{R}(b) > 0, \qquad |\arg a| < \frac{\pi}{4}$$

14. $$\int_0^\infty \operatorname{erfc}(ax)e^{bx}x^n dx = (-1)^{n+1}\frac{n!}{b^{n+1}}$$

$$+\left(\frac{2}{\pi}\right)^{1/2}\sum_{k=0}^{n}(-1)^k\frac{n!}{b^{k+1}}(2a^2)^{\frac{k-n}{2}}\exp\left(\frac{b^2}{8a^2}\right)D_{-n+k-1}\left(-\frac{b}{a\sqrt{2}}\right)$$

$$|\arg(b-a)| < \frac{\pi}{4}.$$

4.3. Combination of Error Function With Exponentials of More Complicated Arguments

1. $$\int_0^a e^{-x^2}\operatorname{erf}(x)dx = \frac{\sqrt{\pi}}{4}(\operatorname{erf} a)^2$$

2. $$\int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2}dx = \frac{\sqrt{\pi}}{2b} - \frac{1}{b\sqrt{\pi}}\tan^{-1}\frac{b}{a}$$

3. $$\int_0^\infty \operatorname{erfc}(ax)e^{b^2x^2}dx = \frac{1}{2\sqrt{\pi}\,b}\ln\left[\frac{a+b}{a-b}\right], \qquad b \text{ may be complex}, \qquad |\arg a| < \frac{\pi}{4}$$

4. $\int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2}x\,dx=\frac{a}{2b^2}(a^2+b^2)^{-1/2}, \qquad \mathscr{R}(b^2)>\mathscr{R}(a^2), \qquad \mathscr{R}(b^2)>0$

5. $\int_0^\infty \operatorname{erfc}(ax)e^{b^2x^2}x\,dx=\frac{1}{2b^2}\left[\frac{a}{(a^2-b^2)^{1/2}}-1\right], \qquad \mathscr{R}(a^2)>\mathscr{R}(b^2)$

6. $\int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2}x^2\,dx=\frac{\sqrt{\pi}}{4b^3}-\frac{1}{2\sqrt{\pi}}\left[\frac{1}{b^3}\tan^{-1}\frac{b}{a}-\frac{a}{b^2(a^2+b^2)}\right], \qquad |\arg a|<\frac{\pi}{4}$

7. $\int_0^\infty \operatorname{erfc}(ax)e^{-b^2x^2}x^2\,dx=\frac{1}{2\sqrt{\pi}}\left[\frac{1}{b^3}\tan^{-1}\frac{b}{a}-\frac{a}{b^2(a^2+b^2)}\right], \qquad |\arg a|<\frac{\pi}{4}$

8. $\int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2}x^p\,dx=\frac{a}{\sqrt{\pi}}\,b^{-p-2}\Gamma\left(\frac{p}{2}+1\right)\,{}_2F_1\left(\frac{1}{2},\frac{p}{2}+1;\frac{3}{2};-\frac{a^2}{b^2}\right),$

$\mathscr{R}(b^2)>0, \qquad \mathscr{R}(p)>-2$

9. $\int_0^\infty \operatorname{erfc}(ax)e^{b^2x^2}x^p\,dx=\frac{\Gamma(\frac{1}{2}p+1)}{\sqrt{\pi}\,(p+1)a^{p+1}}\,{}_2F_1\left(\frac{p+1}{2},\frac{p+2}{2};\frac{p+3}{2};\frac{b^2}{a^2}\right),$

$\mathscr{R}(b^2)<\mathscr{R}(a^2), \qquad \mathscr{R}(p)>-1$

10. $\int_0^\infty \operatorname{erfc}(x)e^{x^2}x^{p-1}\,dx=\frac{1}{2}\sec\left(\frac{p\pi}{2}\right)\Gamma\left(\frac{p}{2}\right), \qquad 0<p<1$

11. $\int_0^\infty \operatorname{erf}(ax)e^{-b^2x^2}\frac{dx}{x}=\frac{1}{2}\ln\frac{(a^2+b^2)^{1/2}+a}{(a^2+b^2)^{1/2}-a}$

$=\ln\frac{a+(a^2+b^2)^{1/2}}{b}, \qquad \mathscr{R}(b^2)>0$

12. $\int_0^\infty \operatorname{erf}(iax)e^{-a^2x^2-bx}\,dx=\frac{1}{2ai\sqrt{\pi}}\exp\left(\frac{b^2}{4a^2}\right)Ei\left(-\frac{b^2}{4a^2}\right), \qquad \mathscr{R}(b)>0, \qquad |\arg a|<\frac{\pi}{4}$

13. $\int_{-\infty}^\infty \operatorname{erf}(x)e^{-(ax+b)^2}\,dx=-\frac{\sqrt{\pi}}{a}\operatorname{erf}\left(\frac{b}{\sqrt{a^2+1}}\right), \qquad \mathscr{R}(a^2)>0$

14. $\int_0^\infty \operatorname{erf}(ix)e^{-(x^2+ax)}x\,dx=\frac{i}{\sqrt{\pi}}\left[\frac{1}{a}+\frac{a}{4}Ei\left(-\frac{a^2}{4}\right)\exp\left(\frac{a^2}{4}\right)\right], \qquad \mathscr{R}(a)>0$

15. $\int_0^\infty \operatorname{erf}(ax)e^{-bx^4}x^3\,dx=\frac{a^2}{8\sqrt{\pi}\,b^{3/2}}\exp\left(\frac{a^4}{8b}\right)K_{1/4}\left(\frac{a^4}{8b}\right)$

16. $\int_1^\infty \operatorname{erfc}(ax)e^{a^2x}x^{-3}\,dx=\frac{1}{2}(1-2a^2)e^{a^2}\operatorname{erfc}(a)+\frac{a}{\sqrt{\pi}}$

17. $\int_1^\infty \operatorname{erfc}(ax)e^{a^2x}\frac{(x-1)^{p/2-1}}{x^{p+1}}\,dx=\frac{1}{\sqrt{\pi}}\,e^{a^2/2}2^{5p/4-2}a^{p/2-1}\Gamma\left(\frac{p}{2}\right)D_{-1-p}(a\sqrt{2}), \qquad p>0$

18. $\int_0^1 \operatorname{erfc}\left(\frac{ax}{\sqrt{2}}\right)e^{\frac{a^2x^2}{2}}x^{p-1}(1-x^2)^{-(p+1)/2}\,dx=\frac{1}{\sqrt{\pi}}\Gamma(p)\Gamma\left(\frac{1-p}{2}\right)2^{-p/2}e^{a^2/4}D_{-p}(a), \qquad 0<p<1$

19. $\int_0^a \operatorname{erf}(x) e^{x^2}(a^2-x^2)^{-1/2} x dx = \frac{\sqrt{\pi}}{2}(e^{a^2}-1), \qquad a>0$

20. $\int_0^a \operatorname{erfc}(x) e^{x^2}(a^2-x^2)^{-1/2} x dx = \frac{\sqrt{\pi}}{2}[1-e^{a^2}\operatorname{erfc}(a)], \qquad a>0$

21. $\int_0^a \operatorname{erf}(x) e^{x^2}(a^2-x^2)^{p-1} x dx = \frac{1}{2} e^{a^2} \frac{\Gamma(p)}{\Gamma(p+\frac{1}{2})} \gamma(p+\tfrac{1}{2}, a^2), \qquad p>0$

22. $\int_0^\infty \operatorname{erf}\left(\frac{a}{x}\right) e^{-b^2x^2} x dx = \frac{1}{2b^2}(1-e^{-2ab}), \qquad \mathscr{R}(a)>0, \qquad \mathscr{R}(b^2)>0$

23. $\int_0^\infty \operatorname{erfc}\left(\frac{a}{x}\right) e^{-b^2x^2} x dx = \frac{1}{2b^2} e^{-2ab}, \qquad \mathscr{R}(a)>0, \qquad \mathscr{R}(b^2)>0$

24. $\int_0^\infty \operatorname{erfc}\left(\frac{a}{x}\right) e^{-b^2x^2} \frac{dx}{x} = -Ei(-2ab), \qquad \mathscr{R}(a)>0, \qquad \mathscr{R}(b^2)>0$

25. $\int_0^\infty \operatorname{erfc}(ax) e^{-\frac{b^2}{4x^2}} x dx = \frac{1}{4a^2} e^{-ab}(1+ab) - \frac{b^2}{4}[-Ei(-ab)]$

26. $\int_0^\infty \operatorname{erf}(ax)[1-e^{-\frac{b^2}{4x^2}}] \frac{dx}{x} = \gamma + \ln(ab) + [-Ei(-ab)]$

27. $\int_0^\infty \operatorname{erfc}(ax) e^{-\frac{b^2}{x^2}} \frac{dx}{x} = -Ei(-2ab), \qquad \mathscr{R}(a)>0, \qquad \mathscr{R}(b^2)>0$

28. $\int_0^\infty \operatorname{erf}(ax) e^{-\frac{b^2}{4x^2}} \frac{dx}{x^3} = \frac{2}{b^2}(1-e^{-ab})$

29. $\int_0^\infty \operatorname{erfc}(ax) e^{-\frac{b^2}{4x^2}} \frac{dx}{x^3} = \frac{2}{b^2} e^{-ab}$

30. $\int_0^\infty \operatorname{erfc}\left(\frac{1}{x}\right) \exp\left(\frac{1}{x^2}-b^2x^2\right) dx = \frac{1}{b\sqrt{\pi}}[\sin 2bCi(2b) - \cos 2bsi(2b)]$

31. $\int_0^\infty \operatorname{erfc}\left(\frac{1}{x}\right) \exp\left(\frac{1}{x^2}-b^2x^2\right) x dx = \frac{\pi}{2b}[\mathbf{H}_1(2b) - Y_1(2b)] - \frac{1}{b}, \qquad |\arg b| < \frac{\pi}{4}$

32. $\int_0^\infty \operatorname{erfc}\left(\frac{1}{x}\right) \exp\left(\frac{1}{x^2}-b^2x^2\right) \frac{dx}{x} = \frac{\pi}{2}[\mathbf{H}_0(2b) - Y_0(2b)], \ |\arg b| < \frac{\pi}{4}$

33. $\int_0^\infty \left[\operatorname{erfc}\left(\frac{a}{x}\right)(x^2+2a^2) - \frac{2}{\sqrt{\pi}} a x e^{-a^2/x^2}\right] e^{-b^2x^2} x dx = \frac{1}{2b^4} e^{-2ab}, \qquad |\arg b| < \frac{\pi}{4}, \ \mathscr{R}(a)>0$

34. $\int_0^\infty \operatorname{erfc}\left(ax+\frac{b}{x}\right) e^{-c^2x^2} x dx$

$= \frac{1}{2}(a^2+c^2)^{-1/2}[a+(a^2+c^2)^{1/2}]^{-1} \exp[-2b(a+\sqrt{a^2+c^2})], \qquad \mathscr{R}(b)>0, \ \mathscr{R}(a^2+c^2)>0$

35. $\int_0^\infty \left\{2\cosh ab - e^{-ab}\operatorname{erf}\left(\frac{b-2ax^2}{2x}\right) - e^{ab}\operatorname{erf}\left(\frac{b+2ax^2}{2x}\right)\right\} e^{-(c^2-a^2)x^2} x dx = \frac{1}{c^2-a^2} e^{-bc},$

$a>0, \ b>0, \ \mathscr{R}(c^2)>0$

36. $\int_0^\infty \cosh(2bx) \exp[(a \cosh x)^2] \operatorname{erfc}(a \cosh x) dx$

$$= \tfrac{1}{2} \sec(b\pi) e^{a^2/2} K_b(a^2), \qquad \mathscr{R}(a) > 0, -\tfrac{1}{2} < \mathscr{R}(b) < \tfrac{1}{2}$$

37. $\int_0^\infty \{\exp[-(x-a)^2] - \exp[-(x+a)^2]\} \operatorname{erf}(x) dx = \sqrt{\pi} \operatorname{erf}(a/\sqrt{2})$

38. $\int_0^\infty \{\exp[-(x-a)^2] + \exp[-(x+a)^2]\} \operatorname{erf}(x) dx = \frac{\sqrt{\pi}}{2} \{1 + [\operatorname{erf}(a/\sqrt{2})]^2\}\cdot$

4.4 Definite Integrals From Laplace Transforms Involving Erf ($\sqrt{ax}$)

1. $\int_0^\infty e^{-bx}[e^{ax} \operatorname{erf}(\sqrt{ax})] dx = (b-a)^{-1}(a/b)^{1/2}, \qquad b > a$

2. $\int_0^\infty e^{-bx}[e^{ax} \operatorname{erfc}(\sqrt{ax}) dx] = b^{-1/2}(\sqrt{b} + \sqrt{a})^{-1}$

3. $\int_0^\infty e^{-bx}[e^{ax} \operatorname{erf}(\sqrt{ax}) - 2(ax/\pi)^{1/2}] dx = (b-a)^{-1}(a/b)^{3/2}, \qquad b > a$

4. $\int_0^\infty e^{-bx}[(\pi x)^{-1/2} - a^{1/2} e^{ax} \operatorname{erfc}(\sqrt{ax})] dx = (\sqrt{b} + \sqrt{a})^{-1}$

5. $\int_0^\infty e^{-bx}[1 - e^{ax} \operatorname{erfc}(\sqrt{ax})] dx = \sqrt{a} b^{-1}(\sqrt{b} + \sqrt{a})^{-1}$

6. $\int_0^\infty e^{-bx}[e^{ax} \operatorname{erfc}(\sqrt{ax}) + 2(ax/\pi)^{1/2} - 1] dx = a(\sqrt{b} + \sqrt{a})^{-1} b^{-3/2}$

7. $\int_0^\infty e^{-bx}[1 - 2(ax/\pi)^{1/2} + (2ax - 1) e^{ax} \operatorname{erfc}(\sqrt{ax})] dx = ab^{-1}(\sqrt{b} + \sqrt{a})^{-2}$

8. $\int_0^\infty e^{-bx}[(x/\pi)^{1/2} - a^{1/2} x e^{ax} \operatorname{erfc}(\sqrt{ax})] dx = (4b)^{-1/2}(\sqrt{b} + \sqrt{a})^{-2}$

9. $\int_0^\infty e^{-bx}[8axe^{ax} \operatorname{erfc}(\sqrt{ax}) - 8(ax/\pi)^{1/2} + 1] dx = b^{-1} \left(\frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}}\right)^2$

10. $\int_0^\infty e^{-bx}[e^{ax} \sqrt{ax}\,(2ax + 3) \operatorname{erfc}(\sqrt{ax}) - 2(x/\pi)^{1/2}(ax + 1)] = -(\sqrt{b} + \sqrt{a})^{-3}$

11. $\int_0^\infty e^{-bx}[(2a^2x^2 + 5ax + 1) e^{ax} \operatorname{erfc}(\sqrt{ax}) - 2(ax + 2)(ax/\pi)^{1/2}] = \sqrt{b}(\sqrt{b} + \sqrt{a})^{-3}$

12. $\int_0^\infty e^{-bx}[2(8a^2x^2 + 8ax + 1) e^{ax} \operatorname{erf}(\sqrt{ax}) - 8(ax/\pi)^{1/2}(2ax + 1) - 1] dx$

$$= b^{-1}(\sqrt{b} - \sqrt{a})^3(\sqrt{b} + \sqrt{a})^{-3}, \; b > a$$

13. $\int_0^\infty e^{-bx}[(2a^{1/2}x^{1/2} + 1) x e^{ax} \operatorname{erf}(\sqrt{ax}) - 2(ax^3/\pi)^{1/2}] dx = b^{-1/2}(\sqrt{b} + \sqrt{a})^{-3}, \qquad b > a$

14. $\int_0^\infty e^{-bx}[(4a^2x^2+12ax+3)xe^{ax}\operatorname{erfc}(\sqrt{ax})-2(a^3x^5/\pi)^{1/2}(2ax+5)]dx=3(\sqrt{b}+\sqrt{a})^{-4}$

15. $\int_0^\infty e^{-bx}[ae^{ax}\operatorname{erfc}(\sqrt{ax})+\sqrt{ac}e^{cx}\operatorname{erfc}(\sqrt{cx})-ce^{cx}]dx=(a-c)(b-c)^{-1}\sqrt{b}(\sqrt{b}+\sqrt{a})^{-1}$

16. $\int_0^\infty e^{-bx}[a^{1/2}e^{cx}\operatorname{erf}(\sqrt{bx})+c^{1/2}e^{ax}\operatorname{erfc}(\sqrt{ax})-c^{1/2}e^{cx}]dx$

$$=(a-c)\sqrt{c}(b-c)^{-1}b^{-1/2}(\sqrt{b}+\sqrt{a})^{-1},\ b>c$$

17. $\int_0^\infty e^{-bx}\left[2\left(\frac{x}{\pi}\right)^{1/2}e^{-a^2/(4x)}-a\operatorname{erfc}\left(\frac{a}{2\sqrt{x}}\right)\right]dx=\frac{e^{-a\sqrt{b}}}{b^{3/2}}.$

18. $\int_0^\infty e^{-bx}\left[a\left(\frac{x}{\pi}\right)^{1/2}e^{-a^2/(4x)}+\left(x+\frac{a^2}{2}\right)\operatorname{erf}\left(\frac{a}{2\sqrt{x}}\right)-\frac{a^2}{2}\right]dx=\frac{1}{b^2}(1-e^{-a\sqrt{b}}).$

4.5. Combination of Error Function With Trigonometric Functions

1. $\int \operatorname{erf}(az)\sin bz\,dz=-\frac{1}{b}\cos bz\operatorname{erf}(az)+\frac{1}{2b}\exp\left(-\frac{b^2}{4a^2}\right)\left\{\operatorname{erf}\left(az-i\frac{b}{2a}\right)+\operatorname{erf}\left(az+\frac{ib}{2a}\right)\right\}$

2. $\int \operatorname{erf}(az)\cos bz\,dz=\frac{1}{b}\sin bz\operatorname{erf}(az)+\frac{i}{2b}\exp\left(-\frac{b^2}{4a^2}\right)\left\{\operatorname{erf}\left(az-\frac{ib}{2a}\right)-\operatorname{erf}\left(az+\frac{ib}{2a}\right)\right\}$

3. $\int \operatorname{erfc}(az)\sin bz\,dz=-\frac{1}{b}\cos bz\operatorname{erfc}(az)-\frac{1}{2b}\exp\left(-\frac{b^2}{4a^2}\right)\left\{\operatorname{erf}\left(az-\frac{ib}{2a}\right)+\operatorname{erf}\left(az+\frac{ib}{2a}\right)\right\}$

4. $\int \operatorname{erfc}(az)\cos bz\,dz=\frac{1}{b}\sin bz\operatorname{erfc}(az)-\frac{i}{2b}\exp\left(-\frac{b^2}{4a^2}\right)\left\{\operatorname{erf}\left(az-\frac{ib}{2a}\right)-\operatorname{erf}\left(az+\frac{ib}{2a}\right)\right\}$

5. $\int_0^\infty \operatorname{erfc}(ax)\sin bx\,dx=\frac{1}{b}\left[1-\exp\left(-\frac{b^2}{4a^2}\right)\right],\qquad |\arg a|<\frac{\pi}{4}$

6. $\int_0^\infty \operatorname{erfc}(ax)\cos bx\,dx=-\frac{i}{b}\exp\left(-\frac{b^2}{4a^2}\right)\operatorname{erf}\left(\frac{ib}{2a}\right),\qquad |\arg a|<\frac{\pi}{4}$

7. $\int_0^\infty \operatorname{erfc}(ax)\cos(bx)x\,dx=\frac{1}{2a^2}\exp\left(-\frac{b^2}{4a^2}\right)-\frac{1}{b^2}\left[1-\exp\left(-\frac{b^2}{4a^2}\right)\right]$

8. $\int_0^\infty \operatorname{erfc}(\sqrt{ax})\sin bx\,dx=\frac{1}{b}-\left(\frac{a/2}{a^2+b^2}\right)^{1/2}[(a^2+b^2)^{1/2}-a]^{-1/2},\qquad \mathscr{R}(a)>|\mathscr{I}(b)|$

9. $\int_0^\infty \operatorname{erfc}(\sqrt{ax})\cos bx\,dx=\left(\frac{a/2}{a^2+b^2}\right)^{1/2}[(a^2+b^2)^{1/2}+a]^{-1/2},\qquad \mathscr{R}(a)>|\mathscr{I}(b)|$

10. $\int_0^\infty \operatorname{erf}(ax)\sin b^2x^2\,dx=\frac{1}{4b\sqrt{2\pi}}\left(\ln\frac{a^2+b^2+ab\sqrt{2}}{a^2+b^2-ab\sqrt{2}}+2\tan^{-1}\frac{ab\sqrt{2}}{b^2-a^2}\right),\qquad a>0$

11. $\int_0^\infty \operatorname{erfc}(ax) \sin bx x^p dx = \frac{\Gamma\left(\frac{p+3}{2}\right) b}{a^{p+2}\sqrt{\pi}(p+2)} {}_2F_2\left(\frac{p+2}{2}, \frac{p+3}{2}; \frac{3}{2}, \frac{p+4}{2}; -\frac{b^2}{4a^2}\right),$

$\mathscr{R}(a) > 0, \mathscr{R}(p) > 0$

12. $\int_0^\infty \operatorname{erfc}(ax) \cos bx x^p dx = \frac{\Gamma\left(\frac{p}{2}+1\right)}{a^{p+1}\sqrt{\pi}(p+1)} {}_2F_2\left(\frac{p+1}{2}, \frac{p+2}{2}; \frac{1}{2}, \frac{p+3}{2}; -\frac{b^2}{4a^2}\right),$

$\mathscr{R}(a) > 0, \mathscr{R}(p) > 0$

13. $\int_0^\infty \operatorname{erf}(ax) \frac{\sin bx}{x^2} dx = \frac{b}{2}\left[-Ei\left(-\frac{b^2}{4a^2}\right)\right] + \sqrt{\pi} \operatorname{erf}\left(\frac{b^2}{4a^2}\right)$

14. $\int_0^\infty \operatorname{erf}(ax) \frac{\cos bx}{x} dx = \frac{1}{2}\left[-Ei\left(-\frac{b^2}{4a^2}\right)\right]$

15. $\int_0^\infty [\operatorname{erfc}(ax) - \operatorname{erfc}(bx)] \frac{\cos px}{x} dx = \frac{1}{2}\left\{Ei\left(-\frac{p^2}{4a^2}\right) - Ei\left(-\frac{p^2}{4b^2}\right)\right\}$

16. $\int_0^\infty \operatorname{erfc}\left(\sqrt{\frac{a}{x}}\right) \sin bx dx = \frac{1}{b} \exp[-(2ab)^{1/2}] \cos[(2ab)^{1/2}],$ $\mathscr{R}(a) > 0, \mathscr{R}(b) > 0$

17. $\int_0^\infty \operatorname{erf}\left(\sqrt{\frac{a}{x}}\right) \cos bx dx = -\frac{1}{b} \exp[-(2ab)^{1/2}] \sin[(2ab)^{1/2}],$ $\mathscr{R}(a) > 0, \mathscr{R}(b) > 0$

18. $⨍_0^\infty \operatorname{erfc}(ax) \tan x dx = \sum_{k=1}^\infty \frac{(-1)^k}{k} \exp\left(-\frac{k^2}{a^2}\right) + \ln 2$

19. $\int_0^\infty \operatorname{erfc}(ax) \sin^2 bx \frac{dx}{x} = \frac{1}{4}\left\{\gamma + 2\ln\left(\frac{b}{a}\right) + \left[-Ei\left(-\frac{b^2}{a^2}\right)\right]\right\}$

20. $\int_0^\infty \operatorname{erf}(ax) \sin bx \sin cx \frac{dx}{x} = \frac{1}{4}\left\{\left[-Ei\left(-\frac{(c-b)^2}{4a^2}\right)\right] - \left[-Ei\left(-\frac{(c+b)^2}{4a^2}\right)\right]\right\},$ $c \neq b$

21. $\int_0^\infty \operatorname{erfc}(ax) \sin bx \sin cx \frac{dx}{x} = \frac{1}{2} \ln\left|\frac{c+b}{c-b}\right| - \int_0^\infty \operatorname{erf}(ax) \sin bx \sin cx \frac{dx}{x},$ $c \neq b$

22. $\int_0^\infty \operatorname{erf}(ax) \cos bx \cos cx \frac{dx}{x} = \frac{1}{4}\left\{\left[-Ei\left(-\frac{(c-b)^2}{4a^2}\right)\right] + \left[-Ei\left(-\frac{(c+b)^2}{4a^2}\right)\right]\right\},$ $c \neq b$

23. $\int_0^\infty \operatorname{erfc}(ax) \sin bx e^{a^2x^2} dx = \frac{\sqrt{\pi}}{2a} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erfc}\left(\frac{b}{2a}\right)$

24. $\int_0^\infty \operatorname{erf}(iax) \sin bx e^{-a^2x^2} dx = \frac{i\sqrt{\pi}}{2a} \exp\left(-\frac{b^2}{4a^2}\right)$

25. $\int_0^\infty \operatorname{erfc}(ax) \cos bx e^{a^2x^2} dx = \frac{1}{2a\sqrt{\pi}} \exp\left(\frac{b^2}{4a^2}\right)\left[-Ei\left(-\frac{b^2}{4a^2}\right)\right]$

26. $\int_0^\infty \operatorname{erfc}(ax) \sin x \cosh x dx = \frac{1}{2}\left[\sin\left(\frac{1}{2a^2}\right) - \cos\left(\frac{1}{2a^2}\right) + 1\right]$

27. $\int_0^\infty \operatorname{erfc}(ax)\cos x \sinh x dx = \frac{1}{2}\left[\sin\left(\frac{1}{2a^2}\right) + \cos\left(\frac{1}{2a^2}\right) - 1\right]$

28. $\int_0^\infty \operatorname{erfc}(ax)\, x \cos x \cosh x dx = \left[\frac{1}{2a^2}\cos\left(\frac{1}{2a^2}\right) - \frac{1}{2}\sin\left(\frac{1}{2a^2}\right)\right]$

29. $\int_0^\infty \operatorname{erfc}(ax)\, x \sin x \sinh x dx = \left[\frac{3}{2}\cos\left(\frac{1}{2a^2}\right) + \frac{1}{a^2}\sin\left(\frac{1}{2a^2}\right) - \frac{3}{2}\right]$

30. $\int_0^\infty \left[e^{-bx}\operatorname{erfc}\left(ab - \frac{x}{2a}\right) - e^{bx}\operatorname{erfc}\left(ab + \frac{x}{2a}\right)\right]\sin px dx = \frac{2p}{(p^2+b^2)}\exp\left[-a^2(b^2+p^2)\right].$

4.6. Combination of Error Function With Logarithms and Powers

1. $\int \operatorname{erf}(az)\ln z dz = (\ln z - 1)\left[z \operatorname{erf}(az) + \frac{1}{a\sqrt{\pi}}e^{-a^2z^2}\right] - \frac{1}{2a\sqrt{\pi}}Ei(-a^2z^2)$

2. $\int \operatorname{erfc}(az)\ln z dz = (\ln z - 1)\left[z \operatorname{erfc}(az) - \frac{1}{a\sqrt{\pi}}e^{-a^2z^2}\right] + \frac{1}{2a\sqrt{\pi}}Ei(-a^2z^2)$

3. $\int_0^\infty \operatorname{erfc}(ax)\ln x dx = -\frac{1}{a\sqrt{\pi}}\left[1 + \frac{\gamma}{2} + \ln a\right]$

4.[5] $\int \operatorname{erf}(az) z \ln z dz = \frac{1}{2}z^2 \operatorname{erf}(az)\ln z + \frac{1}{2a\sqrt{\pi}} z \ln z e^{-a^2z^2}$

$$- \frac{1}{2}\int z \operatorname{erf}(az) dz - \frac{1}{4a^2}\operatorname{erf}(az) - \frac{1}{2a\sqrt{\pi}}\int \ln z e^{-a^2z^2} dz$$

5. $\int \operatorname{erfc}(az) z \ln z dz = \frac{1}{2}z^2 \operatorname{erf}(az)\ln z - \frac{z\ln z}{2a\sqrt{\pi}} e^{-a^2z^2}$

$$- \frac{1}{2}\int z \operatorname{erfc}(az) dz + \frac{1}{4a^2}\operatorname{erf}(az) + \frac{1}{2a\sqrt{\pi}}\int \ln z e^{-a^2z^2} dz$$

6. $\int_0^\infty \operatorname{erfc}(ax) x \ln x dx = \frac{1}{8a^2} + \frac{1}{2a\sqrt{\pi}}\int_0^\infty \ln x e^{-a^2x^2} dx$

7. $(k+1)\int \operatorname{erf}(az) z^k \ln z dz = z^{k+1}\operatorname{erf}(az)\ln z + \frac{1}{a\sqrt{\pi}} z^k \ln z e^{-a^2z^2}$

$$- \int z^k \operatorname{erf}(az) dz - \frac{1}{a\sqrt{\pi}}\int z^{k-1} e^{-a^2z^2} dz$$

$$- \frac{k}{a\sqrt{\pi}}\int z^{k-1}\ln z e^{-a^2z^2} dz$$

8. $(k+1)\int \operatorname{erfc}(az) z^k \ln z dz = z^{k+1}\operatorname{erfc}(az)\ln z - \frac{1}{a\sqrt{\pi}} z^k \ln z e^{-a^2z^2}$

$$- \int z^k \operatorname{erfc}(az) dz + \frac{1}{a\sqrt{\pi}}\int z^{k-1} e^{-a^2z^2} dz$$

$$+ \frac{k}{a\sqrt{\pi}}\int z^{k-1} e^{-a^2z^2}\ln z dz$$

[5] For the elementary integrals in eqs. (4 to 8), see appendix.

9. $(k+1)\int_0^\infty \operatorname{erfc}(ax)x^k \ln x\,dx = \frac{\Gamma(k/2)}{2\sqrt{\pi}a^{k+1}}\left[\frac{1}{k+1}+\frac{k}{2}\,\psi\left(\frac{k}{2}\right)-k\ln a\right].$

4.7. Combination of Two Error Functions

1. $\int_0^\infty \operatorname{erf}(bx)\operatorname{erfc}(ax)\,dx = \frac{1}{b\sqrt{\pi}}\left(\frac{\sqrt{a^2+b^2}}{a}-1\right)$

2. $\int_0^\infty \operatorname{erfc}(bx)\operatorname{erfc}(ax)\,dx = \frac{1}{ab\sqrt{\pi}}(a+b-\sqrt{a^2+b^2})$

3. $\int_0^\infty \operatorname{erf}\left(\frac{b}{x}\right)\operatorname{erfc}(ax)\,dx$

$$= \int_0^\infty \operatorname{erf}(bx)\operatorname{erfc}\left(\frac{a}{x}\right)\frac{dx}{x^2} = \frac{1}{a\sqrt{\pi}}(1-e^{-2ab}) - \frac{2b}{\sqrt{\pi}}[-Ei(-2ab)]$$

4. $\int_0^\infty \operatorname{erfc}\left(\frac{b}{x}\right)\operatorname{erfc}(ax)\,dx$

$$= \int_0^\infty \operatorname{erfc}(bx)\operatorname{erfc}\left(\frac{a}{x}\right)\frac{dx}{x^2} = \frac{1}{a\sqrt{\pi}}e^{-2ab} + \frac{2b}{\sqrt{\pi}}[-Ei(-2ab)]$$

5. $\int_0^\infty \operatorname{erfc}(ax)\operatorname{erf}(bx)e^{b^2x^2}\,dx = \frac{-1}{2b\sqrt{\pi}}\ln\left(1-\frac{b^2}{a^2}\right), \qquad a^2 > b^2$

6. $\int_0^\infty \operatorname{erfc}(ax)\operatorname{erfc}(bx)e^{b^2x^2}\,dx = \frac{1}{b\sqrt{\pi}}\ln\left(1+\frac{b}{a}\right), \qquad a+b>0$

7. $\int_0^1 \operatorname{erf}(x)\operatorname{erf}(\sqrt{1-x^2})\,x\,dx = \frac{1}{4}\left(\frac{3}{e}-1\right)$

8. $\int_0^\infty \operatorname{erfc}(bx)\left[(x^2+a^2)\operatorname{erfc}\left(\frac{a}{x\sqrt{2}}\right) - \frac{2}{\sqrt{\pi}}axe^{-\frac{a^2}{2x^2}}\right]dx$

$$= \frac{1}{3b^3\sqrt{\pi}}\{e^{-2ab}(2a^2b^2-ab+1)-[-Ei(-2ab)]\}$$

9. $\int_0^\infty \operatorname{erfc}(cx)e^{a^2x^2}\left[2\cosh ab - e^{-ab}\operatorname{erf}\left(\frac{b-2ax^2}{2x}\right) - e^{ab}\operatorname{erf}\left(\frac{b+2ax^2}{2x}\right)\right]dx$

$$= \frac{1}{a\sqrt{\pi}}\{e^{-ab}[-Ei(-bc+ba)] - e^{ab}[-Ei(-bc-ba)]\}.$$

4.8. Combination of Error Function With Bessel Functions

1. $\int_0^\infty \operatorname{erf}(ax)J_0(bx)\,dx = \frac{1}{b}\operatorname{erfc}\left(\frac{b}{2a}\right)$

2. $\int_0^\infty \operatorname{erfc}(ax)J_0(bx)\,dx = \frac{1}{b}\operatorname{erf}\left(\frac{b}{2a}\right)$

3. $$\int_0^\infty \left[2\ \text{erfc}\ (x)-1\right] J_0(bx)dx=-\frac{1}{b}\left[2\ \text{erfc}\left(\frac{b}{2}\right)-1\right]$$

4. $$\int_0^\infty \text{erf}\ (x)J_1(bx)dx=\frac{1}{b}\ e^{-b^2/8}I_0\left(\frac{b^2}{8}\right)$$

5. $$\int_0^\infty \text{erf}\left(\frac{x}{\sqrt{2}}\right) J_{3/2}(bx)x^{-1/2}dx=b^{-3/2}\ \text{erf}\left(\frac{b}{\sqrt{2}}\right)$$

6. $$\int_0^\infty \text{erf}\ (ax)J_p(bx)x^p dx=\frac{1}{\sqrt{\pi}}\left(\frac{2}{b}\right)^p\frac{1}{b}\ \Gamma\left(p+\frac{1}{2},\frac{b^2}{4a^2}\right),\qquad -1<p<\frac{1}{2}$$

7. $$\int_0^\infty \text{erfc}\ (ax)J_p(bx)x^p dx=\frac{1}{\sqrt{\pi}}\left(\frac{2}{b}\right)^p\frac{1}{b}\ \gamma\left(p+\frac{1}{2},\frac{b^2}{4a^2}\right),\qquad p>-1$$

8. $$\int_0^\infty \text{erfc}\ (ax)J_p(bx)x^{p+1}dx=\frac{1}{2\sqrt{\pi}}\left(\frac{b}{2}\right)^p\frac{1}{a^{2p+2}}\frac{\Gamma\left(p+\frac{3}{2}\right)}{\Gamma(p+2)}\ {}_1F_1\left(p+\frac{3}{2};\ p+2;\ -\frac{b^2}{4a^2}\right),\qquad p>-1$$

9. $$\int_0^\infty \text{erf}\ (ax)J_p(bx)x^{1-p}dx=\frac{1}{b}\left(\frac{b}{2}\right)^{p-1}\frac{1}{\Gamma(p)}\ {}_1F_1\left(\frac{1}{2};\ p;\ -\frac{b^2}{4a^2}\right),\qquad p>\frac{1}{2}$$

10. $$\int_0^\infty \text{erfc}\ (ax)J_\nu(bx)x^p dx=\frac{2^{-(p+2\nu+1)}b^\nu\Gamma(p+\nu+1)}{a^{p+\nu+1}\Gamma(\nu+1)\Gamma\left(\frac{p+\nu+3}{2}\right)}$$
$$\times\ {}_2F_2\left(\frac{p+\nu+1}{2},\frac{p+\nu+2}{2};\ \nu+1,\frac{p+\nu+3}{2};-\frac{b^2}{4a^2}\right),\qquad p+\nu>-1$$

11. $$\int_0^\infty \text{erfc}\ (ax)J_0(bx)e^{a^2x^2}x\,dx=\frac{1}{ab\sqrt{\pi}}\left[1-\sqrt{\pi}\left(\frac{b}{2a}\right)e^{b^2/4a^2}\ \text{erfc}\left(\frac{b}{2a}\right)\right]$$

12. $$\int_0^\infty \text{erfc}\ (ax)J_p(bx)e^{a^2x^2}x^{p+1}dx=\frac{1}{2\pi}\left(\frac{b}{2}\right)^p\frac{1}{a^{2p+2}}\ \Gamma\left(p+\frac{3}{2}\right)e^{b^2/4a^2}\Gamma\left(-p-1,\frac{b^2}{4a^2}\right),$$
$$-1<p<\frac{1}{2}$$

13. $$\int_0^\infty \text{erfc}\ (ax)J_p(bx)e^{a^2x^2}x^p dx=\frac{b^{p+1/2}\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}a^{3p/2+1}2^{p+1}}\ U\left(p+\frac{1}{2},\ p+1,\frac{b^2}{4a^2}\right)$$
$$=\frac{1}{\sqrt{\pi}}\frac{1}{ba^p}\ \Gamma\left(p+\frac{1}{2}\right)e^{b^2/8a^2}W_{-p/2,\,p/2}\left(\frac{b^2}{8a^2}\right),\qquad -1<p<\frac{3}{2}$$

14. $$\int_0^\infty \text{erfc}\ (ax)J_p(bx)e^{a^2x^2}x^{p+1}dx=\frac{1}{2\pi}\left(\frac{b}{2}\right)^p\frac{1}{a^{2p+2}}\ \Gamma\left(p+\frac{3}{2}\right)e^{b^2/4a^2}\Gamma\left(-p-\frac{1}{2},\frac{b^2}{4a^2}\right),$$
$$-1<p<\frac{1}{2}$$

15. $$\int_0^\infty \text{erfc}\ (x)Y_p(bx)e^{x^2}x^{p+1}dx=\frac{1}{2\pi}\left(\frac{b}{2}\right)^p\Gamma(p+1)e^{b^2/4}\Gamma\left(-p,\frac{b^2}{4}\right),\qquad -1<p<\frac{1}{2}$$

16. $$\int_0^\infty \text{erfc}\ (x)Y_p(bx)e^{x^2}x^{p+3}dx=-\frac{1}{b\pi}\ \Gamma(p+2)e^{b^2/8}W_{-(p+3)/2,\,p/2}\left(\frac{b^2}{4}\right),\qquad -2<p<-\frac{3}{2}$$

17. $$\int_0^\infty \operatorname{erfc}(x) I_p\left(\frac{1}{2}x^2\right) e^{x^2/2} x^{2p+1} dx = \frac{\Gamma\left(2p+\frac{3}{2}\right)\Gamma(-p)}{2\pi^{3/2}\left(p+\frac{1}{2}\right)}, \qquad -\frac{1}{2} < p < 0$$

18. $$\int_0^\infty \operatorname{erfc}(x) I_p\left(\frac{1}{2}x^2\right) e^{x^2/2} x^{2p} dx = \frac{\Gamma\left(2p+\frac{1}{2}\right)}{2\Gamma(p+1)\cos p\pi}, \qquad -\frac{1}{4} < p < \frac{1}{2}$$

19. $$\int_0^\infty \operatorname{erfc}(ax) I_p(x^2) e^{-(1-a^2)x^2} x^{2p+1} dx$$

$$= \frac{\Gamma\left(2p+\frac{3}{2}\right)\Gamma(-p)}{\pi\left(p+\frac{1}{2}\right)} 2^{p-2} a^{-4p-2} {}_2F_1\left(p+\frac{1}{2}, 2p+\frac{3}{2}; p+\frac{3}{2}; 1-\frac{2}{a^2}\right),$$

$$p \neq -\frac{1}{2}, \qquad \mathscr{R}(a^2) > 1, \qquad -1 < \mathscr{R}(p) < 0$$

20. $$\int_0^\infty \operatorname{erfc}\left(\frac{a}{\sqrt{2x}}\right) K_p(x) e^{a^2/2x - x} \frac{dx}{x} = \frac{\pi^{5/2}}{4} \sec p\pi \{[J_p(a)]^2 + [Y_p(a)]^2\}, \qquad -\frac{1}{2} < p < \frac{1}{2}$$

21. $$\int_0^\infty \operatorname{erfc}(x) J_{\lambda+\nu}(ax) J_{\lambda-\nu}(ax) x^p dx$$

$$= \frac{a^{2\lambda}\Gamma\left(\lambda+\frac{1}{2}p+1\right) {}_4F_4\left(1+\lambda, \frac{1}{2}+\lambda, 1+\lambda+\frac{p}{2}, \frac{1}{2}+\lambda+\frac{p}{2}; 1+\lambda+\nu, 1+\lambda-\nu, 1+2\lambda, \frac{3}{2}+\lambda+\frac{p}{2}; -a^2\right)}{\sqrt{\pi}\, 2^{2\lambda+1}\Gamma(\lambda+\nu+1)\Gamma(\lambda-\nu+1)\left(\lambda+\frac{1}{2}p+\frac{1}{2}\right)},$$

$$\lambda + \frac{1}{2}p > 0, \qquad 1+2\lambda \neq -n.$$

4.9. Combination of Error Function With Other Special Functions

1. $$\int_0^\infty \operatorname{erfc}(ax)\left[-Ei\left(-\frac{1}{4}x^2\right)\right]\frac{dx}{x} = (\gamma + \ln a)^2 + \zeta(2) + 2\sum_{k=0}^{\infty} \frac{(-a)^{k+1}}{k!(k+1)^3}$$

2. $$\int_0^\infty \operatorname{erfc}(ax)\left[-Ei\left(-\frac{b^2}{x^2}\right)\right]\frac{dx}{x^3} = \frac{1}{2b^2}(1-2ab)e^{-2ab} + 2a^2[-Ei(-2ab)]$$

3. $$\int_0^\infty \operatorname{erfc}(x) si(2px) dx = (e^{-1/4a^2} - 1) - \frac{\sqrt{\pi}}{2a}\operatorname{erfc}\left(\frac{1}{2a}\right)$$

4. $$\int_0^\infty \operatorname{erf}(ax) Ci(x)\frac{dx}{x} = -\frac{1}{8}[\zeta(2) + (\gamma - \ln 4a^2)^2] - \frac{1}{4}\sum_{k=0}^{\infty}\left(-\frac{1}{4a^2}\right)^{k+1}\frac{1}{k!(k+1)^3}$$

5. $$\int_1^\infty \operatorname{erfc}(ax) P_\nu^\mu(x) e^{a^2x^2}(x^2-1)^{-\mu/2} dx$$

$$= \frac{2^{\mu-1}}{\pi} a^{\mu-3/2} e^{a^2/4}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right) W_{(1-2\mu)/4,\,(1+2\mu)/4}(a^2),$$

$$\mu < 1, \qquad \mu < \nu; \qquad \mu+\nu > -1, \qquad \mu - \nu \neq -2n$$

6. $\int_0^\infty \operatorname{erfc}(x) L_\nu^{(p)}(x^2) x^{2p+1} dx = \frac{\Gamma\left(p+\frac{3}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{2\pi\nu!(p+\nu+1)}, \qquad p>-1$

7. $\int_0^\infty \operatorname{erfc}(x) L_\nu^{(p)}(ax^2) e^{-(a-1)x^2} x^{2p+1} dx$

$$= \frac{\Gamma\left(p+\frac{3}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{2\pi(p+\nu+1)\Gamma(\nu+1)}\, {}_2F_1\left(p+\nu+1,\, p+\frac{3}{2};\, p+\nu+2;\, 1-a\right), \qquad |1-a|<1, \qquad p>-1$$

8. $\int_0^\infty \operatorname{erfc}(x)\, {}_1F_1\left(b-a+\frac{1}{2};\, 2b+1;\, x^2\right) x^{4b} dx = \frac{\Gamma(4b+1)\Gamma\left(a-b+\frac{1}{2}\right)}{2^{4b+1}\Gamma(a+b+1)}$

$$\mathscr{R}(b) > -\frac{1}{4},\ \mathscr{R}(b-a) < \frac{1}{2}.$$

9. $\int_0^\infty \operatorname{erf}(px)\, {}_1F_1(a;\, b;\, -p^2x^2) x^{2(b-1)} dx = \frac{\Gamma(b)}{\sqrt{\pi} p^{2b-1}(2b-1)}\, {}_2F_1\left(a,\, b-\frac{1}{2};\, b+\frac{1}{2};\, -1\right), \qquad b>\frac{1}{2},\ a<\frac{1}{2}$

10. $\int_0^\infty \operatorname{erf}(px)\, {}_1F_1(a;\, b;\, -q^2x^2) x^{2(b-1)} dx = \frac{\Gamma(b)}{\sqrt{\pi} p^{2b-1}(2b-1)}\, {}_2F_1\left(a,\, b-\frac{1}{2};\, b+\frac{1}{2};\, \frac{-q^2}{p^2}\right), \qquad b>\frac{1}{2},\ p2>q2$

11. $\int_0^\infty \operatorname{erf}(px)\, {}_1F_1(a;\, b;\, -q^2x^2) x^\nu dx$

$$= \frac{1}{p^{\nu+1}\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2}\nu+1\right)}{(\nu+1)}\, {}_3F_2\left(a,\, \frac{\nu+2}{2},\, \frac{\nu+1}{2};\, b,\, \frac{\nu+3}{2};\, \frac{-q^2}{p^2}\right), \qquad p\neq 0,\ \nu>-1, \qquad p^2>q^2$$

12. $\int_0^\infty \operatorname{erf}(x)\Psi\left(a-\frac{1}{2};\, b-\frac{1}{2};\, px^2\right) x^{2b-2} dx$

$$= \frac{1}{2p^b}\frac{\Gamma(b)}{(a-b)\Gamma(a)}\, {}_2F_1\left(\frac{1}{2},\, b;\, a;\, 1-\frac{1}{p}\right), \qquad \mathscr{R}(p) \geqslant \frac{1}{2}, a\neq b$$

13. $\int_0^\infty \operatorname{erf}(x)\, {}_1F_1\left(a; \frac{3}{2}\, qx^2\right) e^{-px^2} x dx$

$$= \frac{(p+1)^{a-1/2}}{2p(p+1-q)^a}\, {}_2F_1\left[1,\, a;\, \frac{3}{2};\, \frac{q}{p(p+1-q)}\right], \qquad p\neq 0,\ p+1\neq q,\ \mathscr{R}(p) > \mathscr{R}(q)$$

14. $\int_0^\infty \operatorname{erfc}(x)\, {}_1F_1(a;\, b;\, -px^2) e^{x^2} x^{2b-1} dx = \frac{\Gamma(2b)\Gamma\left(a-b+\frac{1}{2}\right)}{\sqrt{\pi}\, 2^{2b}\Gamma(a+1)}\, {}_2F_1\left(a;\, b+\frac{1}{2};\, a+1;\, 1-p\right),$

$$\mathscr{R}(b)>0, \qquad \mathscr{R}(b-a)<\frac{1}{2}, \qquad |1-p|<1$$

15. $\int_0^1 \operatorname{erf}(x)\, {}_1F_1(a;\, b;\, 1-x^2) e^{x^2}(1-x^2)^{b-1} x dx = \frac{\Gamma(b)}{2\Gamma\left(b+\frac{3}{2}\right)}\, {}_1F_1\left(a+1;\, b+\frac{3}{2};\, 1\right),$

$$\mathscr{R}(b)>0$$

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16. $\int_0^q \operatorname{erf}(ax)\,{}_1F_1\left[p+\frac{1}{2};\, p;\, b(q^2-x^2)\right] e^{a^2x^2}(q^2-x^2)^{p-1}x\,dx$

$$=\frac{a}{2}\, q^{2p+1} e^{a^2q^2}\, \frac{\Gamma(p)}{\Gamma\left(p+\frac{3}{2}\right)}\, {}_1F_1[1;\, p+\tfrac{3}{2};\, q^2(b-a^2)], \qquad p \geqslant 1$$

17. $\int_0^\infty \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) D_\nu(\pm x) e^{x^2/4} dx = \frac{2^{(\nu+1)/2}}{\nu+1}\left[\frac{1}{\Gamma\left(\frac{1-\nu}{2}\right)} \mp \frac{\sqrt{\pi}}{\Gamma\left(-\frac{\nu}{2}\right)}\right],$

$$\mathscr{R}(\nu) \neq -1; \qquad \mathscr{R}(\nu) > -1 \quad \text{for lower sign}$$

18. $\int_0^\infty \operatorname{erfc}\left(\frac{a}{\sqrt{2}}\, x\right) [D_\nu(x) - D_\nu(-x)]\, e^{(2a-1)x^2/4} x\,dx$

$$=\frac{2^{\frac{\nu}{2}+2} a^{-3/2}}{(\nu\pi)}\, {}_2F_1\left(\frac{1}{2}\nu+1,\, 2;\, \frac{1}{2}\nu+2;\, 1-\frac{1}{a}\right),$$

$$\mathscr{R}(a) > \frac{1}{2}, \qquad \mathscr{R}(\nu) > 0$$

19. $\int_0^\infty \operatorname{erfc}(x) M_{\mu,\nu}(ax^2)\, e^{(2-a)x^2/2} x^{2\nu-1} dx$

$$=\frac{\Gamma(4\nu+1)\Gamma(\mu-\nu+\frac{1}{2})\; a^{\nu+1/2}}{\Gamma(\mu+\nu+1)2^{4\nu+1}}\, {}_2F_1\left(\mu+\nu+\frac{1}{2},\, 2\nu+\frac{1}{2};\, \mu+\nu+1;\, 1-a\right),$$

$$|a-1|<1, \qquad \mathscr{R}(\nu) > -\frac{1}{4}, \qquad \mathscr{R}(\nu-\mu) < \frac{1}{2}.$$

20. $\int_0^\infty \operatorname{erfc}(x) M_{\mu,\nu}(ax^2) e^{-1/2(a-2)x^2} x^{2\nu} dx$

$$=\frac{\Gamma(2\nu+1)\Gamma\left(2\nu+\frac{3}{2}\right)\Gamma(\mu-\nu)p^{\nu+1/2}}{2\pi\Gamma\left(\mu+\nu+\frac{3}{2}\right)}\, {}_2F_1\left(\mu+\nu+\frac{1}{2},\, \frac{3}{2}+2\nu;\, \mu+\nu+\frac{3}{2};\, 1-a\right),$$

$$|a-1|<1, \qquad \mathscr{R}(\mu) > \qquad \mathscr{R}(\nu) > -\frac{1}{2}$$

21. $\int_0^\infty \operatorname{erfc}(x)\, M_{\lambda,\mu}(ax^2) x^p \exp(ax^2/2)\, dx$

$$=\frac{\Gamma\left(\mu+\frac{1}{2}p+\frac{3}{2}\right) a^{\mu+1/2}}{2\sqrt{\pi}\left(\mu+\frac{1}{2}p+1\right)}\, {}_3F_2\left(\lambda+\mu+\frac{1}{2},\, \mu+\frac{p}{2}+\frac{3}{2},\, \mu+\frac{p}{2}+1;\, 2\mu+1,\, \mu+\frac{p}{2}+2,\, -a\right),$$

$$\mu+\frac{1}{2}p+1>0.$$

5. Appendix. Some Relevant Integrals Involving Elementary Functions

(A1) $\int z^n e^{az} dz = e^{az} \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} \frac{z^{n-k}}{a^{k+1}}$

(A2) $\int e^{-a^2z^2+bz} dz = \frac{\sqrt{\pi}}{2a} \exp\left(\frac{b^2}{4a^2}\right) \operatorname{erf}\left(az-\frac{b}{2a}\right)$

(A3) $$\int_0^\infty e^{-a^2x^2+bx}dx=\frac{\sqrt{\pi}}{2a}\exp\left(\frac{b^2}{4a^2}\right)\left[1+\operatorname{erf}\left(\frac{b}{2a}\right)\right]$$

(A4) $$\int ze^{-a^2z^2+bz}dz=\frac{1}{2a^2}\exp\left(\frac{b^2}{4a^2}\right)\left[\frac{b\sqrt{\pi}}{2a}\operatorname{erf}\left(az-\frac{b}{2a}\right)-e^{-(az-b/2a)^2}\right]$$

(A5) $$\int_0^\infty xe^{-a^2x^2+bx}dx=\frac{1}{2a^2}\left[\frac{\sqrt{\pi}\,b}{2a}e^{b^2/4a^2}\operatorname{erfc}\left(-\frac{b}{2a}\right)+1\right]$$

(A6) $$\int_0^\infty x^ne^{-a^2x^2+bx}dx=(2a^2)^{-(n+1)/2}n!\exp\left(\frac{b^2}{8a^2}\right)D_{-(n+1)}\left(-\frac{b}{a\sqrt{2}}\right)$$

(A7) $$\int z^ne^{-a^2z^2+bz}dz=a^{-n-1}\exp\left(\frac{b^2}{4a^2}\right)\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{b}{2a}\right)^{n-k}\int u^ke^{-u^2}du,\qquad u=az-\frac{b}{2a}$$

(A8) $$\int u^ke^{-u^2}du=-\frac{e^{-u^2}}{2}\sum_{j=0}^{r-1}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}-j\right)}u^{k-2j-1}+\frac{1}{2}(1-s)\Gamma\left(r+\frac{1}{2}\right)\operatorname{erf}(u),$$

$$k=2r-s,\qquad s=0\text{ or }1$$

(A9) $$(n-1)\int z^{-n}e^{-a^2z^2}dz=-z^{-n+1}e^{-a^2z^2}-2a^2\int z^{-n+2}e^{-a^2z^2}dz$$

(A10) $$\int z^{-n}e^{-a^2z^2}dz=\frac{e^{-a^2z^2}}{2a^2\Gamma\left(\frac{n+1}{2}\right)}\sum_{k=1}^{n/2}(-1)^k\Gamma\left(\frac{n+1}{2}-k\right)a^{2k}z^{2k-n-1}$$

$$+\frac{(-1)^{n/2}\pi a^{n-1}}{2\Gamma\left(\frac{n+1}{2}\right)}\operatorname{erf}(az),\qquad n\text{ even positive integer}$$

$$=\frac{e^{-a^2z^2}}{2a^2\Gamma\left(\frac{n+1}{2}\right)}\sum_{k=1}^{(n-1)/2}(-1)^k\Gamma\left(\frac{n+1}{2}-k\right)a^{2k}z^{2k-n-1}$$

$$+\frac{(-1)^{(n-1)/2}}{2\left(\frac{n-1}{2}\right)!}Ei(-a^2z^2),\qquad n>1\text{ odd positive integer}$$

(A11) $$\int f(z)e^{-(a^2z^2+2bz)}dz=\frac{1}{a}e^{b^2/a^2}\int f\left(\frac{au-b}{a^2}\right)e^{-u^2}du,\qquad\text{where } u=a\left(z+\frac{b}{a^2}\right)$$

(A12) $$\int(z-a)^ne^{-c^2(z-b)^2}dz=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{(b-a)^{n-k}}{c^{k+1}}\int u^ke^{-u^2}du,\qquad\text{where } u=c(z-b)$$

(A13) $$B_1(p,\alpha,u)\equiv\int u^pe^{-\alpha u}\ln u\,du=-u^{p+1}\left[\sum_{j=0}^{\infty}\frac{(-\alpha u)^j}{j!(p+j+1)^2}-\ln u\sum_{j=0}^{\infty}\frac{(-\alpha u)^j}{j!(p+j+1)}\right],\qquad p>-1$$

(A14) $$\alpha B_1(p,\alpha,u)=pB_1(p-1,\alpha,u)+\frac{1}{\alpha^p}\gamma(p,\alpha u)$$

(A15) $$\int_u^\infty\ln x\,e^{-ax}\frac{dx}{x}=\frac{1}{2}\{(\gamma+\ln au)^2+\zeta(2)+2\ln u\,E_1(au)\}+\sum_{k=0}^{\infty}\frac{(-au)^{k+1}}{k!(k+1)^3}$$

(A16) $$\int_0^\infty x^pe^{-ax}\ln x\,dx=\frac{\Gamma(p+1)}{a^{p+1}}[\psi(p+1)-\ln a]$$

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