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Explicit Gross–Zagier and Waldspurger formulae

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We give an explicit Gross–Zagier formula which relates the height of an explicitly constructed Heegner point to the derivative central value of a Rankin L-series. An explicit form of the Waldspurger formula is also given.

1. Main results

1A. ***Introduction.*** The Gross–Zagier formula and the Waldspurger formula are probably the two most important analytic tools known at present for studying the still largely unproven conjecture of Birch and Swinnerton-Dyer. Much work has already been done on both formulae. In particular, the recent book by Yuan, Zhang and Zhang [Yuan et al. 2013] establishes what is probably the most general case

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of the Gross–Zagier formula. Nevertheless, when it comes to actual applications to the arithmetic of elliptic curves or abelian varieties, one very often needs a more explicit form of the Gross–Zagier formula than that given in [Yuan et al. 2013], and similarly a more explicit form of the Waldspurger formula than one finds in the existing literature. This is clearly illustrated, for example, by the papers [Bertolini and Darmon 1997; Tian 2014; Tian et al. 2013; Coates et al. 2014]. Our aim here is to establish what we believe are the most general explicit versions of both formulae, namely Theorems 1.5 and 1.6 for the Gross–Zagier formula, and Theorems 1.8 and 1.9 for the Waldspurger formula. Our methods have been directly inspired by [Yuan et al. 2013], and also the ideas of [Gross 1988] and [Gross and Prasad 1991].

In the remainder of this introduction, we would like to explain in detail our explicit formulae in the simplest and most important case of modular forms over $\mathbb{Q}$. Let ϕ be a newform of weight 2, level $\Gamma_0(N)$, with Fourier expansion $\phi=\sum_{n=1}^{\infty} a_n q^n$ normalized so that $a_1=1$. Let K be an imaginary quadratic field of discriminant D and χ a primitive ring class character over K of conductor c, i.e., a character of $\mathrm{Pic}(\mathbb{O}_c)$, where $\mathbb{O}_c$ is the order $\mathbb{Z}+c\mathbb{O}_K$ of K. Assume the Heegner conditions (first introduced by Birch in a special case):

(1) $(c, N)=1$, no prime divisor p of N is inert in K, and p must split in K if $p^2|N$.

(2) $\chi([\mathfrak{p}])\neq a_p$ for any prime $p|(N, D)$, where $\mathfrak{p}$ is the unique prime ideal of $\mathbb{O}_K$ above p and $[\mathfrak{p}]$ is its class in $\mathrm{Pic}(\mathbb{O}_c)$.

Let $L(s,\phi,\chi)$ be the Rankin L-series of ϕ and the theta series ϕ_χ associated to χ (without the local Euler factor at infinity). It follows from the Heegner conditions that the sign in the functional equation of $L(s,\phi,\chi)$ is -1. Let $(\phi,\phi)_{\Gamma_0(N)}$ denote the Petersson norm of ϕ:

$$(\phi,\phi)_{\Gamma_0(N)}=\iint_{\Gamma_0(N)\backslash\mathcal{H}} |\phi(z)|^2\,dx\,dy,\qquad z=x+iy.$$

Let $X_0(N)$ be the modular curve over $\mathbb{Q}$ whose $\mathbb{C}$-points parametrize isogenies $E_1\to E_2$ between elliptic curves over $\mathbb{C}$ whose kernels are cyclic of order N. By the Heegner conditions, there exists a proper ideal $\mathcal{N}$ of $\mathbb{O}_c$ such that $\mathbb{O}_c/\mathcal{N}\cong\mathbb{Z}/N\mathbb{Z}$. For any proper ideal $\mathfrak{a}$ of $\mathbb{O}_c$, let $P_\mathfrak{a}\in X_0(N)$ be the point representing the isogeny $\mathbb{C}/\mathfrak{a}\to\mathbb{C}/\mathfrak{a}\mathcal{N}^{-1}$, which is defined over the ring class field H_c over K of conductor c and only depends on the class of $\mathfrak{a}$ in $\mathrm{Pic}(\mathbb{O}_c)$. Let $J_0(N)$ be the Jacobian of $X_0(N)$. Writing ∞ for the cusp at infinity on $X_0(N)$, we have the morphism from $X_0(N)$ to $J_0(N)$ over $\mathbb{Q}$ given by $P\mapsto[P-\infty]$. Let P_χ be the point

$$P_\chi=\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathbb{O}_c)}[P_\mathfrak{a}-\infty]\otimes\chi([\mathfrak{a}])\in J_0(N)(H_c)\otimes_\mathbb{Z}\mathbb{C}$$

and write P_χ^ϕ for the ϕ-isotypical component of P_χ.

The following theorem was proved in the case $c = 1$ in the celebrated work by Gross and Zagier [1986], and follows immediately from the general explicit Gross–Zagier formula in Theorem 1.5 (see Special case 2, and the Example following).

Theorem 1.1. *Let ϕ, χ be as above satisfying the Heegner conditions* (1) *and* (2). *Then*

$$L'(1, \phi, \chi) = 2^{-\mu(N,D)} \cdot \frac{8\pi^2(\phi, \phi)_{\Gamma_0(N)}}{u^2\sqrt{|Dc^2|}} \cdot \hat{h}_K(P_\chi^\phi),$$

where $\mu(N, D)$ is the number of prime factors of the greatest common divisor of N and D, $u = [\mathbb{O}_c^\times : \mathbb{Z}^\times]$ is half of the number of roots of unity in $\mathbb{O}_c$, and $\hat{h}_K$ is the Néron–Tate height on $J_0(N)$ over K. In particular, if ϕ is associated to an elliptic curve E over $\mathbb{Q}$ via Eichler–Shimura theory and $f : X_0(N) \to E$ is a modular parametrization mapping the cusp ∞ to the identity $O \in E$, then the Heegner divisor $P_\chi^0(f) := \sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathbb{O}_c)} f(P_\mathfrak{a}) \otimes \chi([\mathfrak{a}]) \in E(H_c)_\mathbb{C}$ satisfies

$$L'(1, E, \chi) = 2^{-\mu(N,D)} \cdot \frac{8\pi^2(\phi, \phi)_{\Gamma_0(N)}}{u^2\sqrt{|Dc^2|}} \cdot \frac{\hat{h}_K(P_\chi^0(f))}{\deg f},$$

where $\hat{h}_K$ is the Néron–Tate height on E over K and $\deg f$ *is the degree of the morphism f.*

Comparing the above Gross–Zagier formula with the conjecture of Birch and Swinnerton-Dyer for $L(E/K, s)$, we immediately are led to:

Conjecture. *Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor N and let K be an imaginary quadratic field of discriminant D such that for any prime ℓ dividing N, either ℓ splits in K, or ℓ is ramified in K and E has nonsplit semistable reduction at ℓ. Let $f : X_0(N) \to E$ be a modular parametrization mapping ∞ to O. Let $\mathcal{N} \subset \mathbb{O}_K$ be any ideal with $\mathbb{O}_K/\mathcal{N} \cong \mathbb{Z}/N\mathbb{Z}$, let $P \in X_0(N)(H_K)$ be the point representing the isogeny $(\mathbb{C}/\mathbb{O}_K \to \mathbb{C}/\mathcal{N}^{-1})$, and write $P_K(f) := \mathrm{Tr}_{H_K/K}\, f(P) \in E(K)$. Assume $P_K(f)$ is not torsion. Then*

$$\sqrt{\#\text{Ш}(E/K)} = 2^{-\mu(N,D)} \cdot \frac{[E(K) : \mathbb{Z}P_K(f)]}{C \cdot [\mathbb{O}_K^\times : \mathbb{Z}^\times] \cdot \prod_{\ell | N/(N,D)} m_\ell},$$

where $m_\ell = [E(\mathbb{Q}_\ell) : E^0(\mathbb{Q}_\ell)]$ and C is the positive integer such that if ω_0 is a Néron differential on E then $f^*\omega_0 = \pm C \cdot 2\pi i \phi(z)\, dz$.

We next state our explicit Waldspurger formula over $\mathbb{Q}$. Let $\phi = \sum_{n=1}^\infty a_n q^n$ in $S_2(\Gamma_0(N))$ be a newform of weight 2 and level $\Gamma_0(N)$. Let K be an imaginary quadratic field and $\chi : \mathrm{Gal}(H_c/K) \to \mathbb{C}^\times$ a character of conductor c. Assume the conditions:

(i) $(c, N) = 1$ and, if $p | (N, D)$, then $p^2 \nmid N$.

(ii) Let S be the set of places $p|N\infty$ nonsplit in K such that, for a finite prime p, $\mathrm{ord}_p(N)$ is odd if p is inert in K, and $\chi([\mathfrak{p}]) = a_p$ if p is ramified in K. Then S has even cardinality.

It follows that the sign of the functional equation of the Rankin L-series $L(s, \phi, \chi)$ is $+1$. Let B be the quaternion algebra over $\mathbb{Q}$ ramified exactly at places in S. Note that condition (ii) implies that there exists an embedding of K into B, which we fix once and for all. Let $R \subset B$ be an order of discriminant N with $R \cap K = \mathcal{O}_c$. Such an order exists and is unique up to conjugation by $\widehat{K}^\times$. Here, for an abelian group M, we define $\widehat{M} = M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ with p running over all primes. By the reduction theory of definite quadratic forms, the coset $X := B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$ is finite, say of order n. Let $g_1, \dots, g_n$ in $\widehat{B}^\times$ represent the distinct classes $[g_1], \dots, [g_n]$. For each $i = 1, \dots, n$, let $\Gamma_i = (B^\times \cap g_i \widehat{R}^\times g_i^{-1})/\{\pm 1\}$. Then Γ_i is a finite group, and we denote its order by w_i. Let $\mathbb{Z}[X]$ denote the free $\mathbb{Z}$-module of formal sums $\sum_{i=1}^n a_i[g_i]$ with $a_i \in \mathbb{Z}$, and define a height pairing on $\mathbb{Z}[X]$ by

$$\left\langle \sum a_i[g_i], \sum b_i[g_i] \right\rangle = \sum_{i=1}^n a_i b_i w_i,$$

which is positive definite on $\mathbb{R}[X] := \mathbb{Z}[X] \otimes_{\mathbb{Z}} \mathbb{R}$ and has a natural Hermitian extension to $\mathbb{C}[X] := \mathbb{Z}[X] \otimes_{\mathbb{Z}} \mathbb{C}$. Define the degree of a vector $\sum a_i[g_i] \in \mathbb{Z}[X]$ to be $\sum a_i$ and let $\mathbb{Z}[X]^0$ denote the degree-0 submodule of $\mathbb{Z}[X]$. Then $\mathbb{Z}[X]$ and $\mathbb{Z}[X]^0$ are endowed with actions of Hecke operators T_p, S_p, $p \nmid N$, which are linear and defined as follows: For any prime $p \nmid N$, $B_p^\times / R_p^\times \cong \mathrm{GL}_2(\mathbb{Q}_p)/\mathrm{GL}_2(\mathbb{Z}_p)$ can be identified with the set of $\mathbb{Z}_p$-lattices in a 2-dimensional vector space over $\mathbb{Q}_p$. Then, for any $g = (g_v) \in \widehat{B}^\times$,

$$S_p([g]) = [g^{(p)} s_p(g_p)] \quad \text{and} \quad T_p([g]) = \sum_{h_p} [g^{(p)} h_p],$$

where $g^{(p)}$ is the p-off part of g, namely $g^{(p)} = (g_v^{(p)})$ with $g_v^{(p)} = g_v$ for all $v \neq p$ and $g_p^{(p)} = 1$; if g_p corresponds to lattice Λ, then $s_p(g_p)$ is the coset corresponding to the homothetic lattice $p\Lambda$; and h_p runs over $p+1$ lattices $\Lambda' \subset \Lambda$ with $[\Lambda : \Lambda'] = p$. There is a unique line $V_\phi \subset \mathbb{C}[X]^0$ where T_p acts as a_p and S_p acts trivially for all $p \nmid N$. Recall that the fixed embedding of K into B induces a map

$$\mathrm{Pic}(\mathcal{O}_c) = K^\times \backslash \widehat{K}^\times / \widehat{\mathcal{O}}_c^\times \longrightarrow X = B^\times \backslash \widehat{B}^\times / \widehat{R}^\times, \quad t \longmapsto x_t,$$

using which we define an element in $\mathbb{C}[X]$,

$$P_\chi := \sum \chi^{-1}(t) x_t,$$

and let P_χ^ϕ be its projection to the line V_ϕ. The following explicit height formula for P_χ^ϕ, which was proved by Gross [1987] in some cases, is a special case of the explicit Waldspurger formulas in Theorems 1.8 and 1.10 (with Proposition 3.8).

Theorem 1.2. *Let (ϕ, χ) be as above satisfying the conditions (i) and (ii). Then we have*

$$L(1, \phi, \chi) = 2^{-\mu(N,D)} \cdot \frac{8\pi^2(\phi, \phi)_{\Gamma_0(N)}}{u^2\sqrt{|Dc^2|}} \cdot \langle P_\chi^\phi, P_\chi^\phi \rangle,$$

where $\mu(N, D)$ and u are as in Theorem 1.1. Let $f = \sum_i f(g_i)w_i^{-1}[g_i]$ be any nonzero vector on the line V_ϕ, and let $P_\chi^0(f) = \sum_{t\in \mathrm{Pic}(\mathcal{O}_c)} f(t)\chi(t)$. Then the above formula can be rewritten as

$$L(1, \phi, \chi) = 2^{-\mu(N,D)} \cdot \frac{8\pi^2(\phi, \phi)_{\Gamma_0(N)}}{u^2\sqrt{|Dc^2|}} \cdot \frac{|P_\chi^0(f)|^2}{\langle f, f\rangle}.$$

Notation for first two sections. We denote by F the base number field of degree $d = [F : \mathbb{Q}]$ over $\mathbb{Q}$ and $\mathcal{O} = \mathcal{O}_F$ its ring of integers with different δ. Let $\mathbb{A} = F_\mathbb{A}$ be the adèle ring of F and $\mathbb{A}_f$ its finite part. For any $\mathbb{Z}$-module M, we let $\widehat{M} = M \otimes_\mathbb{Z} \widehat{\mathbb{Z}}$ and $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. For example, $\widehat{F} = \mathbb{A}_f$. Let $|\cdot|_\mathbb{A} : \mathbb{A}^\times \to \mathbb{R}_+^\times$ denote the standard adelic absolute value, so that $d(ab) = |a|_\mathbb{A}\, db$ for any Haar measure db on $\mathbb{A}$. Let $|\cdot|_v$ denote the absolute value on $F_v^\times$ for each place v of F, with $|x|_\mathbb{A} = \prod_v |x_v|_v$ for any $x = (x_v) \in \mathbb{A}^\times$. For any nonzero fractional ideal b of F, let $\|b\|$ denote the norm of b. For any $x \in \mathbb{A}_f^\times$, we also write $\|x\|$ for $\|b_x\|$, where b_x is the ideal corresponding to x, so that $\|x\| = |x|_\mathbb{A}^{-1}$; and for any nonzero fractional ideal b we also write $|b|_\mathbb{A}$ for $|x_b|_\mathbb{A}$ for any $x_b \in \mathbb{A}_f^\times$ whose corresponding ideal is b, so that $|b|_\mathbb{A} = \|b\|^{-1}$. For a finite place v, sometimes we also denote by v its corresponding prime ideal and write $q_v = \#\mathcal{O}/v$. For a fractional ideal b of F, we write $|b|_v = |x_b|_v$ for $x_b \in F_v$ with $x_b\mathcal{O}_v = b\mathcal{O}_v$, denote by $\mathrm{ord}_v(b)$ the additive valuation of b at v such that $\mathrm{ord}_v(v) = 1$, and write $v\|b$ if $\mathrm{ord}_v(b) = 1$. We denote by ∞ the set of infinite places of F. Denote by $L(s, 1_F)$ the complete L-series for the trivial Hecke character 1_F on $\mathbb{A}^\times$, so that $L(s, 1_F) = \Gamma_\mathbb{R}(s)^{r_1}\Gamma_\mathbb{C}(s)^{r_2}\zeta_F(s)$, where r_1 and r_2 are the number of real and complex places of F, $\zeta_F(s)$ is the usual Dedekind zeta function of F, $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$, and $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$. For each place v of F, let $L(s, 1_v)$ denote the local Euler factor of $L(s, 1_F)$ at v. Let D_F denote the absolute discriminant of F, and $\delta \subset \mathcal{O}$ the different of F, so that $\|\delta\| = |D_F|$.

In the first two sections, we let K be a quadratic extension over F, $D = D_{K/F} \subset \mathcal{O}$ be the relative discriminant of K over F, and D_K be the absolute discriminant of K. Let K^{ab} be the maximal abelian extension over K and $\sigma : K_\mathbb{A}^\times/K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ be the Artin reciprocity map in class field theory. For any nonzero ideal b of $\mathcal{O}$, let $\mathcal{O}_b = \mathcal{O} + b\mathcal{O}_K$ be the unique $\mathcal{O}$-order of K satisfying $[\mathcal{O}_K : \mathcal{O}_b] = \#\mathcal{O}/b$, and we call b its conductor. For any finite place v of F, $\mathcal{O}_{b,v} = \mathcal{O}_b \otimes_\mathcal{O} \mathcal{O}_v$ only depends on $\mathrm{ord}_v\, b$. Thus, for a fractional ideal b and a finite place v of F, $\mathcal{O}_{b,v}$ makes sense if

$\mathrm{ord}_v\, b \geq 0$. Let $\mathrm{Pic}_{K/F}(\mathbb{O}_b) = \widehat{K}^\times / K^\times \widehat{F}^\times \widehat{\mathbb{O}}_b^\times$. Then there is an exact sequence

$$\mathrm{Pic}(\mathbb{O}_F) \longrightarrow \mathrm{Pic}(\mathbb{O}_b) \longrightarrow \mathrm{Pic}_{K/F}(\mathbb{O}_b) \longrightarrow 0.$$

Let κ_b be the kernel of the first map, which has order 1 or 2 if F is totally real and K is a totally imaginary quadratic extension over F (see [Washington 1997, Theorem 10.3]).

For any algebraic group G over F, let $G_{\mathbb{A}} = G(\mathbb{A})$ be the group of adelic points on G. For a finite set S of places of F, let $G_S = \prod_{v \in S} G(F_v)$ (resp. $G_{\mathbb{A}}^{(S)} = G(\mathbb{A})^{(S)}$) be the S-part of $G_{\mathbb{A}}$ (resp. the S-off part of $G_{\mathbb{A}}$) viewed as a subgroup of $G_{\mathbb{A}}$ naturally so that the S-off components (resp. S-components) are constant 1. More generally, for a subgroup U of $G_{\mathbb{A}}$ of the form $U = U_T U^T$ for some set T of places disjoint with S, where $U_T \subset \prod_{v \in T} G(F_v)$ and $U^T = \prod_{v \notin T} U_v$ with U_v a subgroup of $G(F_v)$, we may define $U^{(S)}$, U_S, and view them as subgroups of U similarly. For any ideal b of $\mathbb{O}$, we also write $U^{(b)}$ for $U^{(S_b)}$ and U_b for U_{S_b}, where S_b is the set of places dividing b. Let $U_0(N)$ and $U_1(N)$ denote subgroups of $\mathrm{GL}_2(\widehat{\mathbb{O}})$ defined by

$$U_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{O}}) \,\middle|\, c \in N\widehat{\mathbb{O}} \right\},$$

$$U_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(N) \,\middle|\, d \equiv 1 \bmod N\widehat{\mathbb{O}} \right\}.$$

When F is a totally real field and σ is an automorphic cuspidal representation of level N such that σ_v is a discrete series for all $v|\infty$, for an automorphic form ϕ of level $U_1(N)$ we let $(\phi, \phi)_{U_0(N)}$ denote the Petersson norm defined using the invariant measure $dx\, dy/y^2$ on the upper half-plane.

1B. ***The explicit Gross–Zagier formula.*** Let F be a totally real number field of degree d, $\mathbb{A} = \mathbb{A}_F$ the adèle ring of F, and $\mathbb{A}_f$ its finite part. Let $\mathbb{B}$ be an incoherent quaternion algebra over $\mathbb{A}$, totally definite at infinity. For each open compact subgroup U of $\mathbb{B}_f^\times = (\mathbb{B} \otimes_{\mathbb{A}} \mathbb{A}_f)^\times$, let X_U be the Shimura curve over F associated to U and $\xi_U \in \mathrm{Pic}(X_U)_{\mathbb{Q}}$ the normalized Hodge class on X_U, that is, the unique line bundle which has degree one on each geometrically connected component and is parallel to

$$\omega_{X_U/F} + \sum_{x \in X_U(\overline{F})} (1 - e_x^{-1}) x.$$

Here $\omega_{X_U/F}$ is the canonical bundle of X_U and e_x is the ramification index of x in the complex uniformization of X_U, i.e., for a cusp x, $e_x = \infty$, so that $1 - e_x^{-1} = 1$; for a noncusp x, e_x is the ramification index of any preimage of x in the map $X_{U'} \to X_U$ for any sufficiently small open compact subgroup U' of U such that each geometrically connected component of $X_{U'}$ is a free quotient of $\mathcal{H}$ under the complex uniformization. For any two open compact subgroups $U_1 \subset U_2$ of $\mathbb{B}_f^\times$, there is a natural surjective morphism $X_{U_1} \to X_{U_2}$. Let X be the projective limit of the system $(X_U)_U$, which is endowed with the Hecke action of $\mathbb{B}^\times$ where $\mathbb{B}_\infty^\times$ acts trivially. Note that each X_U is the quotient of X by the action of U.

Let A be a simple abelian variety over F parametrized by X in the sense that there is a nonconstant morphism $X_U \to A$ over F for some U. Then, by Eichler–Shimura theory, A is of strict GL(2)-type in the sense that $M := \mathrm{End}^0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a field and $\mathrm{Lie}(A)$ is a free module of rank one over $M \otimes_{\mathbb{Q}} F$ by the induced action. Let

$$\pi_A = \mathrm{Hom}_\xi^0(X, A) := \varinjlim_U \mathrm{Hom}_{\xi_U}^0(X_U, A),$$

where $\mathrm{Hom}_{\xi_U}^0(X_U, A)$ denotes the morphisms in $\mathrm{Hom}(X_U, A) \otimes_{\mathbb{Z}} \mathbb{Q}$ using ξ_U as a base point: if ξ_U is represented by a divisor $\sum_i a_i x_i$ on $X_{U,\bar{F}}$, then for $f \in \mathrm{Hom}_F(X_U, A) \otimes_{\mathbb{Z}} \mathbb{Q}$,

$$f \in \pi_A \iff \sum_i a_i f(x_i) = 0 \text{ in } A(\bar{F})_{\mathbb{Q}} := A(\bar{F}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For each open compact subgroup U of $\mathbb{B}_f^\times$, let J_U denote the Jacobian of X_U. Then

$$\pi_A = \mathrm{Hom}^0(J, A) := \varinjlim_U \mathrm{Hom}^0(J_U, A),$$

where $\mathrm{Hom}^0(J_U, A) = \mathrm{Hom}_F(J_U, A) \otimes_{\mathbb{Z}} \mathbb{Q}$. The action of $\mathbb{B}^\times$ on X induces a natural $\mathbb{B}^\times$-module structure on π_A so that $\mathrm{End}_{\mathbb{B}^\times}(\pi_A) = M$ and there is a decomposition $\pi_A = \bigotimes_M \pi_{A,v}$, where $\pi_{A,v}$ are absolutely irreducible representations of $\mathbb{B}_v^\times$ over M. Using the Jacquet–Langlands correspondence, one can define the complete L-series of π_A,

$$L(s, \pi_A) = \prod_v L(s, \pi_{A,v}) \in M \otimes_{\mathbb{Q}} \mathbb{C},$$

as an entire function of $s \in \mathbb{C}$. Let $L(s, A, M)$ denote the L-series of the ℓ-adic Galois representation with coefficients in $M \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ associated to A (without local Euler factors at infinity); then $L_v(s, A, M) = L\left(s - \frac{1}{2}, \pi_v\right)$ for all finite places v of F. Let $A^\vee$ denote the dual abelian variety of A. There is a perfect $\mathbb{B}^\times$-invariant pairing

$$\pi_A \times \pi_{A^\vee} \longrightarrow M$$

given by

$$(f_1, f_2) = \mathrm{Vol}(X_U)^{-1}(f_{1,U} \circ f_{2,U}^\vee), \quad f_{1,U} \in \mathrm{Hom}(J_U, A),\ f_{2,U} \in \mathrm{Hom}(J_U, A^\vee),$$

where $f_{2,U}^\vee : A \to J_U$ is the dual of $f_{2,U}$ composed with the canonical isomorphism $J_U^\vee \simeq J_U$. Here $\mathrm{Vol}(X_U)$ is defined by a fixed invariant measure on the upper half-plane. It follows that $\pi_{A^\vee}$ is dual to π_A as representations of $\mathbb{B}^\times$ over M. For any fixed open compact subgroup U of $\mathbb{B}_f^\times$, define the U-pairing on $\pi_A\times\pi_{A^\vee}$ by

$$(f_1,f_2)_U=\mathrm{Vol}(X_U)(f_1,f_2),\qquad f_1\in\pi_A,\ f_2\in\pi_{A^\vee},$$

which is independent of the choice of measure defining $\mathrm{Vol}(X_U)$. If A is an elliptic curve and we identify $A^\vee$ with A canonically then, for any morphism $f:X_U\to A$, we have $(f,f)_U=\deg f$, the degree of the finite morphism f.

Let K be a totally imaginary quadratic extension over F with associated quadratic character η on $\mathbb{A}^\times$. Let L be a finite extension of M and $\chi:K^\times\backslash K_{\mathbb{A}}^\times\to L^\times$ an L-valued Hecke character of finite order. Let $L(s,A,\chi)$ be the L-series (without Euler factors at infinity) of the ℓ-adic Galois representations associated to A tensored with the induced representation of χ from $\mathrm{Gal}(\overline{K}/K)$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Assume that

$$\omega_A\cdot\chi|_{\mathbb{A}^\times}=1,$$

where ω_A is the central character of π_A on $\mathbb{A}_f^\times$ and that, for each finite place v of F,

$$\epsilon(\pi_{A,v},\chi_v)=\chi_v\eta_v(-1)\epsilon(\mathbb{B}_v),$$

where $\epsilon(\mathbb{B}_v)=1$ if $\mathbb{B}_v$ is split and is -1 otherwise, and $\epsilon(\pi_{A,v},\chi_v)=\epsilon\left(\frac{1}{2},\pi_{A,v},\chi_v\right)$ is the local root number of $L(s,\pi_A,\chi)$. It follows that the global root number of the L-series $L(s,\pi_A,\chi)$ is -1 and there is an embedding of $K_{\mathbb{A}}$ into $\mathbb{B}$ over $\mathbb{A}$. We fix such an embedding once for all and then view $K_{\mathbb{A}}^\times$ as a subgroup of $\mathbb{B}^\times$.

Let N be the conductor of π^{JL}, D the relative discriminant of K over F, and $c\subset\mathbb{O}$ the ideal that is maximal such that χ is trivial on $\prod_{v\nmid c}\mathbb{O}_{K_v}^\times\prod_{v|c}(1+c\mathbb{O}_{K,v})$. Define the set of places v of F dividing N,

$$\Sigma_1:=\{v|N \text{ nonsplit in } K \mid \mathrm{ord}_v(c)<\mathrm{ord}_v(N)\}.$$

Let $c_1=\prod_{\mathfrak{p}|c,\mathfrak{p}\notin\Sigma_1}\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}c}$ be the Σ_1-off part of c, N_1 the Σ_1-off part of N, and $N_2=N/N_1$.

Let v be a place of F and ϖ_v a uniformizer of F_v. Then there exists an $\mathbb{O}_v$-order R_v of $\mathbb{B}_v$ with discriminant $N\mathbb{O}_v$ such that $R_v\cap K_v=\mathbb{O}_{c_1,v}$. Such an order R_v is called admissible for $(\pi_v.\chi_v)$ if it also satisfies the conditions (1) and (2) that follow. Note that up to $K_v^\times$-conjugate there is a unique such order when $v\nmid(c_1,N)$, and that $\mathbb{B}$ must be split at places $v|(c_1,N)$ by Lemma 3.1.

(1) If $v|(c_1,N)$, then R_v is the intersection of two maximal orders R_v', R_v'' of $\mathbb{B}_v$ such that $R_v'\cap K_v=\mathbb{O}_{c,v}$ and

$$R_v''\cap K_v=\begin{cases}\mathbb{O}_{c/N,v} & \text{if } \mathrm{ord}_v(c/N)\geq 0,\\ \mathbb{O}_{K,v} & \text{otherwise.}\end{cases}$$

Note that, for $v|(c_1,N)$, there is a unique order up to $K_v^\times$-conjugate satisfying condition (1), unless $\mathrm{ord}_v(c_1)<\mathrm{ord}_v(N)$. In the case $0<\mathrm{ord}_v(c_1)<\mathrm{ord}_v(N)$,

v must split in K by the definition of Σ_1 and there are exactly two $K_v^\times$-conjugacy classes of orders satisfying condition (1), which are conjugate to each other by a normalizer of $K_v^\times$ in $\mathbb{B}_v^\times$. Fix an F_v-algebra isomorphism $K_v \cong F_v^2$ and identify $\mathbb{B}_v$ with $\mathrm{End}_{F_v}(K_v)$. Then the two classes contain, respectively, orders $R_{i,v} = R'_{i,v} \cap R''_{i,v}$, $i = 1, 2$ as in (1) such that $R'_{i,v} = \mathrm{End}_{\mathbb{O}}(\mathbb{O}_c)$, $i = 1, 2$, $R''_{1,v} = \mathrm{End}_{\mathbb{O}_v}((\varpi_v^{n-c}, 1)\mathbb{O}_{K_v})$ and $R''_{2,v} = \mathrm{End}_{\mathbb{O}_v}((1, \varpi_v^{n-c})\mathbb{O}_{K_v})$.

(2) If $0 < \mathrm{ord}_v(c_1) < \mathrm{ord}_v(N)$, then R_v is $K_v^\times$-conjugate to some $R_{i,v}$ such that χ_i has conductor $\mathrm{ord}_v(c)$, where χ_i, $i = 1, 2$, is defined by $\chi_1(a) = \chi_v(a, 1)$ and $\chi_2(b) = \chi_v(1, b)$.

Definition 1.3. An $\widehat{\mathbb{O}}$-order $\mathcal{R}$ of $\mathbb{B}_f$ is called admissible for (π, χ) if, for every finite place v of F, $\mathcal{R}_v := \mathcal{R} \otimes_{\widehat{\mathbb{O}}} \mathbb{O}_v$ is admissible for (π_v, χ_v). Note that an admissible order $\mathcal{R}$ for (π, χ) is of discriminant $N\widehat{\mathbb{O}}$ such that $\mathcal{R} \cap \widehat{K} = \widehat{\mathbb{O}}_{c_1}$.

Let $\mathcal{R}$ be an $\widehat{\mathbb{O}}$-order of $\mathbb{B}_f$ with discriminant N such that $\mathcal{R} \cap K_{\mathbb{A}_f} = \widehat{\mathbb{O}}_{c_1}$ and that $\mathcal{R}_v := \mathcal{R} \otimes_{\widehat{\mathbb{O}}} \mathbb{O}_v$ is admissible for (π_v, χ_v) at all places v. Note that $\mathcal{R}_v$ is unique up to $K_v^\times$-conjugate for any $v \nmid (c_1, N)$.

Let $U = \mathcal{R}^\times$ and $U^{(N_2)} := \mathcal{R}^\times \cap \mathbb{B}_f^{\times (N_2)}$. For any finite place $v | N_1$, $\mathbb{B}_v$ must be split (by Lemma 3.1(5)). Let $Z \cong \mathbb{A}_f^\times$ denote the center of $\mathbb{B}_f^\times$. The group $U^{(N_2)}$ has a decomposition $U^{(N_2)} = U' \cdot (Z \cap U^{(N_2)})$, where $U' = \prod_{v \nmid N_2 \infty} U'_v$ is so that, for any finite place $v \nmid N_2$, $U'_v = U_v$ if $v \nmid N$ and $U'_v \cong U_1(N)_v$ otherwise. View ω as a character on Z. We may define a character on $U^{(N_2)}$ that is ω on $Z \cap U^{(N_2)}$ and trivial on U'. This character is also denoted by ω.

Definition 1.4. Let $V(\pi, \chi)$ denote the space of forms $f \in \pi_A \otimes_M L$ which are ω-eigenforms under $U^{(N_2)}$ and χ_v^{-1}-eigenforms under $K_v^\times$ for all places $v \in \Sigma_1$. The space $V(\pi, \chi)$ is actually a one-dimensional L-space (see Proposition 3.7).

Consider the Hecke action of $K_{\mathbb{A}}^\times \subset \mathbb{B}^\times$ on X. Let $X^{K^\times}$ be the F-subscheme of X of fixed points of X under $K^\times$. The theory of complex multiplication asserts that every point in $X^{K^\times}(\bar{F})$ is defined over K^{ab} and that the Galois action is given by the Hecke action under the reciprocity law. Fix a point $P \in X^{K^\times}$ and let $f \in V(\pi, \chi)$ be a nonzero vector. Define a Heegner cycle associated to (π, χ) by

$$P_\chi^0(f) := \sum_{t \in \mathrm{Pic}_{K/F}(\mathbb{O}_{c_1})} f(P)^{\sigma_t} \chi(t) \in A(K^{\mathrm{ab}})_{\mathbb{Q}} \otimes_M L,$$

where $\mathrm{Pic}_{K/F}(\mathbb{O}_{c_1}) = \widehat{K}^\times / K^\times \widehat{F}^\times \widehat{\mathbb{O}}_{c_1}^\times$ and $t \mapsto \sigma_t$ is the reciprocity law map in class field theory. The Néron–Tate height pairing over K gives a $\mathbb{Q}$-linear map $\langle\, , \rangle_K : A(\bar{K})_{\mathbb{Q}} \otimes_M A^\vee(\bar{K})_{\mathbb{Q}} \to \mathbb{R}$. Let $\langle\, , \rangle_{K,M} : A(\bar{K})_{\mathbb{Q}} \otimes_M A^\vee(\bar{K})_{\mathbb{Q}} \to M \otimes_{\mathbb{Q}} \mathbb{R}$ be the unique M-bilinear pairing such that $\langle\, , \rangle_K = \mathrm{tr}_{M \otimes \mathbb{R}/\mathbb{R}} \langle\, , \rangle_{K,M}$. The pairing $\langle\, , \rangle_{K,M}$ induces an L-linear Néron–Tate pairing over K,

$$\langle\, , \rangle_{K,L} : (A(\bar{K})_{\mathbb{Q}} \otimes_M L) \otimes_L (A^\vee(\bar{K})_{\mathbb{Q}} \otimes_M L) \longrightarrow L \otimes_{\mathbb{Q}} \mathbb{R}.$$

The $\mathbb{B}^\times$-invariant M-linear pairing $(\ ,\ )_U : \pi_A \times \pi_{A^\vee} \to M$ induces a $\mathbb{B}^\times$-invariant L-linear pairing

$$(\ ,\ )_U : (\pi_A \otimes_M L) \times (\pi_{A^\vee} \otimes_M L) \longrightarrow L.$$

The Hilbert newform ϕ in the Jacquet–Langlands correspondence σ of π_A on $\mathrm{GL}_2(\mathbb{A})$ is the form satisfying these conditions:

- ϕ is of level $U_1(N)$.
- For each $v|\infty$, the action of $\mathrm{SO}_2(\mathbb{R}) \subset \mathrm{GL}_2(F_v)$ on ϕ is given by $\sigma(k_\theta)\phi = e^{4\pi i\theta}\phi$, where $k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in \mathrm{SO}_2(\mathbb{R})$.
- Let $d^\times a$ be the Tamagawa measure so that $\mathrm{Res}_{s=1} \int_{|a|\le 1, a\in F^\times\backslash\mathbb{A}^\times} |a|^{s-1}\, d^\times a = \mathrm{Res}_{s=1} L(s, 1_F)$; then

  $$L(s,\pi) = 2^d \cdot |\delta|_{\mathbb{A}}^{s-1/2} \cdot Z(s,\phi) \quad \text{with } Z(s,\phi) = \int_{F^\times\backslash\mathbb{A}^\times} \phi\begin{pmatrix} a & \\ & 1\end{pmatrix} |a|_{\mathbb{A}}^{s-1/2}\, d^\times a,$$

  where δ is the different of F.

Note that $\phi(g)\bar{\phi}(g)$ is a function on

$$\mathrm{GL}_2(F)_+\backslash \mathrm{GL}_2(F_\infty)_+ \times \mathrm{GL}_2(\mathbb{A}_f)/Z(\mathbb{A})\cdot(U_{1,\infty}\times U_0(N)) \cong \mathrm{GL}_2(F)_+\backslash \mathcal{H}^d \times \mathrm{GL}_2(\mathbb{A}_f)/U_0(N)\mathbb{A}_f^\times.$$

We define the Petersson norm $(\phi,\phi)_{U_0(N)}$ by the integration of $\phi\bar{\phi}$ with measure $dx\,dy/y^2$ on each upper half-plane. One main result of this paper is the following:

Theorem 1.5 (explicit Gross–Zagier formula). *Let F be a totally real field of degree d. Let A be an abelian variety over F parametrized by a Shimura curve X over F and ϕ the Hilbert holomorphic newform of parallel weight* 2 *on* $\mathrm{GL}_2(\mathbb{A})$ *associated to A. Let K be a totally imaginary quadratic extension over F with relative discriminant D and discriminant D_K. Let $\chi : K_{\mathbb{A}}^\times/K^\times \to L^\times$ be a finite Hecke character of conductor c over some finite extension L of $M :=$* $\mathrm{End}^0(A)$. *Assume that*:

(1) $\omega_A \cdot \chi|_{\mathbb{A}^\times} = 1$, *where ω_A is the central character of π_A*;

(2) *for any place v of F, $\epsilon(\pi_{A,v},\chi_v) = \chi_v\eta_v(-1)\epsilon(\mathbb{B}_v)$.*

For any nonzero forms $f_1 \in V(\pi_A,\chi)$ and $f_2 \in V(\pi_{A^\vee},\chi^{-1})$, we have an equality in $L\otimes_{\mathbb{Q}}\mathbb{C}$,

$$L'^{(\Sigma)}(1,A,\chi) = 2^{-\#\Sigma_D} \cdot \frac{(8\pi^2)^d \cdot (\phi,\phi)_{U_0(N)}}{u_1^2\sqrt{|D_K|\|c_1^2\|}} \cdot \frac{\langle P_\chi^0(f_1), P_{\chi^{-1}}^0(f_2)\rangle_{K,L}}{(f_1,f_2)_{\mathcal{R}^\times}},$$

where

$$\Sigma := \{v|(N, Dc) \mid \text{if } v\|N \text{ then } \mathrm{ord}_v(c/N) \geq 0\},$$
$$\Sigma_D := \{v|(N, D) \mid \mathrm{ord}_v(c) < \mathrm{ord}_v(N)\},$$

the ideal $c_1|c$ *is the* Σ_1*-off part of* c *as before,* $u_1 = \#\kappa_{c_1} \cdot [\mathbb{O}_{c_1}^\times : \mathbb{O}^\times]$ *and* κ_{c_1} *is the kernel of the morphism from* $\mathrm{Pic}(\mathbb{O})$ *to* $\mathrm{Pic}(\mathbb{O}_{c_1})$*, which has order* 1 *or* 2*, and* $(\phi, \phi)_{U_0(N)}$ *is the Petersson norm with respect to the measure* $dx\,dy/y^2$ *on the upper half-plane.*

Remark. The assumption $\omega_A|_{\mathbb{A}^\times} \cdot \chi = 1$ implies $L(s, A, \chi) = L(s, A^\vee, \chi^{-1})$. Let $\phi^\vee$ be the Hilbert newform associated to $A^\vee$. Then $(\phi^\vee, \phi^\vee)_{U_0(N)} = (\phi, \phi)_{U_0(N)}$.

We may state the above theorem in simpler way under some assumptions. First assume that ω_A is unramified and, if $v \in \Sigma_1$, then $v \nmid c$.

Given this, $c_1 = c$. Fix an infinite place τ of F and let B be the nearby quaternion algebra whose ramification set is obtained from that of $\mathbb{B}$ by removing τ. Then there is an F-embedding of K into B which we fix once and for all and view $K^\times$ as an F-subtorus of $B^\times$. Let R be an admissible $\mathbb{O}$-order of B for (π, χ), by which we mean that $\widehat{R}$ is an admissible $\widehat{\mathbb{O}}$-order of $\mathbb{B}_f = \widehat{B}$ for (π, χ). Note that R is of discriminant N and that $R \cap K = \mathbb{O}_c$. Let $U = \widehat{R}^\times \subset \widehat{B}^\times$ and let X_U be the Shimura curve of level U, so that it has complex uniformization

$$X_{U,\tau}(\mathbb{C}) = B_+^\times \backslash \mathcal{H} \times \widehat{B}^\times / U \cup \{\text{cusps}\},$$

where $B_+^\times$ is the subgroup of elements $x \in B^\times$ with totally positive norms. Let $u = \#\kappa_c \cdot [\mathbb{O}_c^\times : \mathbb{O}^\times]$. By Proposition 3.8, we have that $V(\pi_A, \chi) \subset (\pi_A \otimes_M L)^{\widehat{R}^\times}$.

Special case 1. Further assume that $(N, Dc) = 1$. Then there is a nonconstant morphism $f : X_U \to A$ mapping a Hodge class on X_U to the torsion of A and, for any two such morphisms $f_1, f_2 : X_U \to A$, $n_1 f_1 = n_2 f_2$ for some nonzero integers n_1, n_2. Let h_0 be the unique fixed point of $K^\times$ and let $P = [h_0, 1] \in X_U$. Replace χ by χ^{-1}; there is a nonconstant morphism $X_U \to A^\vee$ with similar uniqueness. For any such $f_1 : X_U \to A$ and $f_2 : X_U \to A^\vee$, let $(f_1, f_2) = f_1 \circ f_2^\vee$. Then we have an equality in $L \otimes_{\mathbb{Q}} \mathbb{C}$,

$$L'(1, A, \chi) = \frac{(8\pi^2)^d (\phi, \phi)_{U_0(N)}}{u^2 \cdot \sqrt{|D_K| \|c^2\|}} \cdot \frac{\langle P_\chi^0(f_1), P_{\chi^{-1}}^0(f_2)\rangle_{K,L}}{(f_1, f_2)_U}.$$

Special case 2. Further assume that ω_A is trivial — or, more generally, that $\omega_A(\varpi_v)$ is in $\mathrm{Aut}(A)^2 \subset M^{\times 2}$ for all places v dividing (N, D) but not c, where ϖ_v is a uniformizer of F_v. For each place v that divides (N, D) but not c, $K_v^\times$ normalizes $R_v^\times$ (see Lemma 3.4) and a uniformizer ϖ_{K_v} of K_v induces an automorphism $T_{\varpi_{K_v}} : X_U \to X_U$ over F. Note that $\chi_v(\varpi_{K_v}) \in \mathrm{Aut}(A) \subset M^\times$. There exists a nonconstant morphism $f : X_U \to A$ mapping a Hodge class to the torsion point such that $T_{\varpi_{K_v}} f = \chi^{-1}(\varpi_{K_v}) f$ for each place v dividing (N, D) but not c. Such an f has the same uniqueness property as in special case 1. Then, for any such $f_1 : X_U \to A$ and $f_2 : X_U \to A^\vee$, we have an equality in $L \otimes_{\mathbb{Q}} \mathbb{C}$,

$$L'^{(\Sigma)}(1, A, \chi) = 2^{-\#\Sigma_D} \cdot \frac{(8\pi^2)^d (\phi, \phi)_{U_0(N)}}{u^2 \cdot \sqrt{|D_K| \|c^2\|}} \cdot \frac{\langle P_\chi^0(f_1), P_{\chi^{-1}}^0(f_2)\rangle_{K,L}}{(f_1, f_2)_U},$$

where Σ is now the set of places $v|(cD, N)$ of F such that, if $v\|N$, then $v \nmid D$.

Example. Let $\phi \in S_2(\Gamma_0(N))$ be a newform. Let K be an imaginary quadratic field of discriminant D and χ a primitive character of $\mathrm{Pic}(\mathcal{O}_c)$. Assume that (ϕ, χ) satisfies the Heegner conditions (1)–(2) in Theorem 1.1; then, by Lemma 3.1(1) and (3), $\epsilon(\phi, \chi) = -1$ and $B = M_2(\mathbb{Q})$. The Heegner conditions also imply that there exist $a, b \in \mathbb{Z}$ with $(N, a, b) = 1$ such that $a^2 - 4Nb = Dc^2$. Fix an embedding of K into B by

$$(Dc^2 + \sqrt{Dc^2})/2 \longmapsto \begin{pmatrix} (Dc^2 + a)/2 & -1 \\ Nb & (Dc^2 - a)/2 \end{pmatrix}.$$

Then $R := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \;\middle|\; N|c \right\}$ is an order of B such that $\widehat{R} \cap K = \mathcal{O}_c$. Let A be an abelian variety associated to ϕ via Eichler–Shimura theory and $f : X_0(N) \to A$ any nonconstant morphism mapping cusp ∞ to $O \in A$. Then $f \in V(\pi_A, \chi)$. Let $z \in \mathcal{H}$ be the point fixed by $K^\times$; then $Nbz^2 - az + 1 = 0$, $\mathcal{O}_c = \mathbb{Z} + \mathbb{Z}z^{-1}$, and $\mathfrak{n}^{-1} = \mathbb{Z} + \mathbb{Z}N^{-1}z^{-1}$, so that $\mathcal{O}_c/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$. The point on $X_0(N)$ corresponding to z via complex uniformization represents the isogeny $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}z) \to \mathbb{C}/(N^{-1}\mathbb{Z} + \mathbb{Z}z)$, or $\mathbb{C}/\mathcal{O}_c \to \mathbb{C}/\mathfrak{n}^{-1}$. Thus Theorem 1.1 now follows from Theorem 1.5.

For various arithmetic applications, we may need explicit formulas for different test vectors, which we now give. Let v be a finite place of F, fix $\langle\, , \rangle_v$ a $\mathbb{B}_v^\times$-invariant pairing on $\pi_{A,v} \times \pi_{A^\vee,v}$ and a Haar measure dt_v on $F_v^\times \backslash K_v^\times$. For any $f'_{1,v} \in \pi_{A,v}$, $f'_{2,v} \in \pi_{A^\vee,v}$ with $\langle f'_{1,v}, f'_{2,v}\rangle_v \neq 0$, let

$$\beta^0(f'_{1,v}, f'_{2,v}) = \beta^0(f'_{1,v}, f'_{2,v}, dt_v) = \int_{F_v^\times \backslash K_v^\times} \frac{\langle \pi_{A,v}(t_v) f'_{1,v}, f'_{2,v}\rangle_v}{\langle f'_{1,v}, f'_{2,v}\rangle_v} \chi_v(t_v)\, dt_v.$$

For any two nonzero pure tensor forms $f' = \bigotimes_v f'_v$, $f'' = \bigotimes_v f''_v \in \pi$, we say that f' and f'' differ at a place v if f'_v and f''_v are not parallel, and that they coincide otherwise. This is independent of the decompositions. In particular, if two nonzero pure tensor forms coincide locally everywhere then they are the same up to a scalar.

Theorem 1.6 (variation of the Gross–Zagier formula). *Let (A, χ), $f_1 \in V(\pi_A, \chi)$ and $f_2 \in V(\pi_{A^\vee}, \chi^{-1})$ be as in Theorem 1.5. Let S be a finite set of finite places of F, $f'_1 \in \pi_A$, $f'_2 \in \pi_{A^\vee}$ be vectors such that f'_i and f_i coincide for any $v \notin S$,*

$i = 1, 2$, and $\langle f'_{1,v}, f'_{2,v}\rangle_v \neq 0$ and $\beta^0(f'_{1,v}, f'_{2,v}) \neq 0$ for any $v \in S$. Define

$$P^0_\chi(f'_1) = \frac{\#\operatorname{Pic}(\mathcal{O}_{c_1})}{\operatorname{Vol}(K^\times \widehat{F}^\times \backslash \widehat{K}^\times, dt)} \cdot \int_{K^\times \widehat{F}^\times \backslash \widehat{K}^\times} f'_1(P)^{\sigma_t} \chi(t)\, dt,$$

and define $P^0_{\chi^{-1}}(f'_2)$ similarly. Then, with notations as in Theorem 1.5, we have

$$L'^{(\Sigma)}(1, A, \chi) = 2^{-\#\Sigma_D} \cdot \frac{(8\pi^2)^d \cdot (\phi, \phi)_{U_0(N)}}{u_1^2 \sqrt{|D_K| \|c_1^2\|}} \cdot \frac{\langle P^0_\chi(f'_1), P^0_{\chi^{-1}}(f'_2)\rangle_{K,L}}{(f'_1, f'_2)^\times_{\mathcal{R}}} \cdot \prod_{v \in S} \frac{\beta^0(f_{1,v}, f_{2,v})}{\beta^0(f'_{1,v}, f'_{2,v})},$$

which is independent of the choice of Haar measure dt_v for $v \in S$.

Example. Let A be the elliptic curve $X_0(36)$ with the cusp ∞ as the identity point and let $K = \mathbb{Q}(\sqrt{-3})$. Let $p \equiv 2 \bmod 9$ be a prime; then the field $L' = K(\sqrt[3]{p})$ is contained in H_{3p}. Let $\chi : \operatorname{Gal}(L'/K) \to K^\times$ be the character mapping σ to $(\sqrt[3]{p})^{\sigma - 1}$. Fix the embedding $K \to M_2(\mathbb{Q})$ mapping $w := (-1 + \sqrt{-3})/2$ to $\left(\begin{smallmatrix} -1 & -p/6 \\ 6/p & 0 \end{smallmatrix}\right)$.

For $f' = \mathrm{id} : X_0(36) \to A$, let $P \in X_0(36)$ be the point corresponding to $-pw/6 \in \mathcal{H}$. The Heegner divisor $P^0_\chi(f')$ is

$$P^0_\chi(f') = \frac{1}{9} \sum_{t \in \operatorname{Pic}(\mathcal{O}_{6p})} f'(P)^{\sigma_t} \chi(t).$$

One can show that $P^0_\chi(f')$ is nontrivial (see [Satgé 1987; Dasgupta and Voight 2009; Cai et al. 2014]) and then it follows that the prime p is the sum of two rational cubes. By the variation formula, one can easily obtain the height formula of $P^0_\chi(f')$: let $\phi \in S_2(\Gamma_0(36))$ be the newform associated to A, and note that $\#\Sigma_D = 1$, $u_1 = 1$ and $c_1 = p$ in the variation

$$L'(1, A, \chi) = 9 \cdot \frac{8\pi^2 \cdot (\phi, \phi)_{\Gamma_0(36)}}{\sqrt{3p^2}} \cdot \langle P^0_\chi(f'), P^0_{\chi^{-1}}(f')\rangle_{K,K}.$$

In fact, $U = \mathcal{R}^\times$ in Theorem 1.5 is given by

$$\mathcal{R} = \left\{ \begin{pmatrix} a & b/6 \\ 6c & d \end{pmatrix} \in M_2(\widehat{\mathbb{Q}}) \;\middle|\; a, b, c, d \in \widehat{\mathbb{Z}},\ p^{-1}b + pc,\ a + pc - d \in 6\widehat{\mathbb{Z}} \right\}$$

and $f \in V(\pi_A, \chi)$ is a χ_v^{-1}-eigenform for $v = 2, 3$. Then

$$(f', f') = \frac{\operatorname{Vol}(X_U)}{\operatorname{Vol}(X_0(36))} = \frac{2}{9}.$$

The ratio $\beta^0(f_v, f_v)/\beta^0(f'_v, f'_v)$ equals 1 at $v = 2$, and 4 at $v = 3$.

1C. ***The explicit Waldspurger formula.*** Let F be a general base number field. Let B be a quaternion algebra over F and π a cuspidal automorphic representation of $B_{\mathbb{A}}^\times$ with central character ω. Let K be a quadratic field extension of F and η the quadratic Hecke character on $F^\times\backslash\mathbb{A}^\times$ associated to the quadratic extension. Let χ be a Hecke character on $K_{\mathbb{A}}^\times$. Write $L(s,\pi,\chi)$ for the Rankin L-series $L(s,\pi^{\mathrm{JL}}\times\pi_\chi)$, where π^{JL} is the Jacquet–Langlands correspondence of π on $\mathrm{GL}_2(\mathbb{A})$ and π_χ is the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ corresponding to the theta series of χ, so that $L(s,\pi_\chi)=L(s,\chi)$. Assume that

$$\omega\cdot\chi|_{\mathbb{A}^\times}=1.$$

Then, for any place v of F, the local root number $\epsilon\big(\frac{1}{2},\pi_v,\chi_v\big)$ of the Rankin L-series is independent of the choice of additive character. We also assume that, for all places v of F,

$$\epsilon\big(\tfrac{1}{2},\pi_v,\chi_v\big)=\chi_v\eta_v(-1)\epsilon(B_v),$$

where $\epsilon(B_v)=-1$ if B_v is division and $+1$ otherwise. It follows that the global root number $\epsilon\big(\frac{1}{2},\pi,\chi\big)$ equals $+1$ and there exists an F-embedding of K into B. We fix such an embedding once and for all and view $K^\times$ as an F-subtorus of $B^\times$.

Let N be the conductor of π^{JL}, D the relative discriminant of K over F, $c\subset\mathcal{O}$ the ideal maximal such that χ is trivial on $\prod_{v\nmid c}\mathcal{O}_{K_v}^\times\prod_{v|c}(1+c\mathcal{O}_{K,v})$. Define the following set of places v of F dividing N:

$$\Sigma_1:=\{v|N \text{ nonsplit in } K\mid \mathrm{ord}_v(c)<\mathrm{ord}_v(N)\},$$

Let $c_1=\prod_{\mathfrak{p}|c,\mathfrak{p}\notin\Sigma_1}\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}c}$ be the Σ_1-off part of c, N_1 the Σ_1-off part of N, and $N_2=N/N_1$ the Σ_1-part of N.

Let R be an admissible $\mathcal{O}$-order of B for (π,χ) in the sense that R_v is admissible for (π_v,χ_v) for every finite place v of F. It follows that R is an $\mathcal{O}$-order with discriminant N such that $R\cap K=\mathcal{O}_{c_1}$.

Let $U=\prod_v U_v\subset B_{\mathbb{A}}^\times$ be a compact subgroup such that, for any finite place v, $U_v=R_v^\times$, and that, for any infinite place v of F, U_v is a maximal compact subgroup of $B_v^\times$ such that $U_v\cap K_v^\times$ is the maximal compact subgroup of $K_v^\times$. For any finite place $v|N_1$, B_v must be split. Let $Z\cong\mathbb{A}_f^\times$ denote the center of $\widehat{B}^\times$. The group $U^{(N_2\infty)}$ has a decomposition $U^{(N_2\infty)}=U'\cdot(Z\cap U^{(N_2\infty)})$, where $U'=\prod_{v\nmid N_2\infty}U_v'$ is such that, for any finite place $v\nmid N_2$, $U_v'=U_v$ if $v\nmid N$ and $U_v'\cong U_1(N)_v$ otherwise. View ω as a character on Z and we may define a character on $U^{(c_2\infty)}$ that is ω on $Z\cap U^{(c_2\infty)}$ and trivial on U'; we also denote this character by ω.

Definition 1.7. Let $V(\pi,\chi)$ denote the space of forms $f=\bigotimes_v f_v\in\pi$ such that f is an ω-eigenform under $U^{(N_2\infty)}$; for all places $v\in\Sigma_1$, f is a χ_v^{-1}-eigenform under $K_v^\times$; and, for any infinite place v, f is a χ_v^{-1}-eigenform under $U_v\cap K_v^\times$ with weight minimal. The space $V(\pi,\chi)$ is actually one-dimensional (see Proposition 3.7).

Let r, s, t be integers such that $B \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}^r \times M_2(\mathbb{R})^s \times M_2(\mathbb{C})^t$, and let X_U denote the U-level real manifold

$$X_U = B_+^\times \backslash (\mathcal{H}_2^s \times \mathcal{H}_3^t) \times \widehat{B}^\times / U,$$

which has finitely many connected components, where $\mathcal{H}_2$, $\mathcal{H}_3$ are the usual hyperbolic spaces of dimension two and three, respectively. Define the volume of X_U, denoted by $\mathrm{Vol}(X_U)$, as follows:

- If $s+t>0$, then X_U is the disjoint union of manifolds of dimension $2s+3t$,

$$X_U = B_+^\times \backslash (\mathcal{H}_2^s \times \mathcal{H}_3^t) \times \widehat{B}^\times / U = \bigsqcup_i \Gamma_i \backslash (\mathcal{H}_2^s \times \mathcal{H}_3^t),$$

  for some discrete subgroup $\Gamma_i \subset B_+^\times \cap \prod_{v\mid\infty,\, B_v \text{ not division}} (B_v)^\times$, then define the volume of X_U with the measure $dx\,dy/(4\pi y^2)$ on $\mathcal{H}_2$ and the measure $dx\,dy\,dv/\pi^2 v^3$ on $\mathcal{H}_3$. Here the notation $\mathcal{H}_3$ is the same as in [Vignéras 1980].
- If $s+t=0$, then F is totally real and B is totally definite. For any open compact subgroup U of $\widehat{B}^\times$, the double coset $B^\times \backslash \widehat{B}^\times / U$ is finite; let $g_1, \dots, g_n \in \widehat{B}^\times$ be a complete set of representatives for the coset. Let $\mu_Z = \widehat{F}^\times \cap U$; then, for any $g \in \widehat{B}^\times$, $B^\times \cap gUg^{-1}/\mu_Z$ is a finite set. Define the volume of X_U to be the mass of U:

$$\mathrm{Vol}(X_U) = \mathrm{Mass}(U) = \sum_{i=1}^n \frac{1}{\#(B^\times \cap g_i U g_i^{-1})/\mu_Z}.$$

For any automorphic forms $f_1 \in \pi$ and $f_2 \in \widetilde{\pi}$, $\langle f_1, f_2 \rangle_{\mathrm{Pet}}$ is the Petersson pairing of f_1, f_2, defined by

$$\langle f_1, f_2 \rangle_{\mathrm{Pet}} = \int_{B^\times \mathbb{A}^\times \backslash B_{\mathbb{A}}^\times} f_1(g) f_2(g)\, dg,$$

where dg is the Tamagawa measure on $F^\times \backslash B^\times$, so that $B^\times \mathbb{A}^\times \backslash B_{\mathbb{A}}^\times$ has total volume 2. For any $f_1 \in V(\pi, \chi)$ and $f_2 \in V(\widetilde{\pi}, \chi^{-1})$, one may define the U-level pairing as

$$\langle f_1, f_2 \rangle_U = \tfrac{1}{2} \langle f_1, f_2 \rangle_{\mathrm{Pet}} \cdot \mathrm{Vol}(X_U).$$

For any $f \in V(\pi, \chi)$, define the c_1-level period of $f \in V(\pi, \chi)$ as follows: let $\overline{K_\infty^\times / F_\infty^\times}$ be the closure of $K_\infty^\times / F_\infty^\times$ in the compact group $K_{\mathbb{A}}^\times / \mathbb{A}^\times K^\times$ and endow $\overline{K_\infty^\times / F_\infty^\times}$ with the Haar measure dh of total volume one; then, let

$$P_\chi^0(f) = \sum_{t \in \mathrm{Pic}_{K/F}(\mathcal{O}_{c_1})} f^0(t)\chi(t), \qquad f^0(t) = \int_{\overline{K_\infty^\times / F_\infty^\times}} f(th)\chi(h)\, dh.$$

The function $f^0(t)\chi(t)$ on $K_{\mathbb{A}}^\times$ is constant on $K_{\Sigma_1}^\times$, so can be viewed as a function on $\mathrm{Pic}_{K/F}(\mathcal{O}_{c_1}) = \widehat{K}^\times / K^\times \widehat{F}^\times \widehat{\mathcal{O}}_{c_1}^\times$. Note that, when F is totally real and all infinite places v of F are inert in K, $f^0 = f$.

Notations. Let b be an integral ideal of F; we define the relative regulator R_b to be the quotient of the regulator of $\mathcal{O}_b^\times$ by the regulator of $\mathcal{O}^\times$ and $w_b = \#\mathcal{O}_{b,\mathrm{tor}}^\times / \#\mathcal{O}_{\mathrm{tor}}^\times$. Denote by κ_b the kernel of the natural homomorphism from $\mathrm{Pic}(\mathcal{O})$ to $\mathrm{Pic}(\mathcal{O}_b)$. Define $\nu_b = 2^{-r_{K/F}} R_b^{-1} \cdot \#\kappa_b \cdot w_b$, where $r_{K/F} = \operatorname{rank} \mathcal{O}_K^\times - \operatorname{rank} \mathcal{O}^\times$. For example, if F is a totally real field of degree d and K is a totally imaginary quadratic field extension over F, then $\nu_b = 2^{1-d} \cdot \#\kappa_b \cdot [\mathcal{O}_b^\times : \mathcal{O}^\times]$, where $\kappa_b \subset \kappa_1$ and $\#\kappa_1 = 1$ or 2 [Washington 1997, Theorem 10.3].

For an infinite place v of F, let U_v denote the maximal compact subgroup of $\mathrm{GL}_2(F_v)$, which is O_2 if v is real and U_2 if v is complex, and let $U_{1,v} \subset U_v$ denote its subgroup of diagonal matrices $\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$ for $a \in F_v^\times$ with $|a|_v = 1$. For a generic $(\mathfrak{g}_v, U_v)$-module σ_v and a nontrivial additive character ψ_v of F_v, let $\mathcal{W}(\sigma_v, \psi_v)$ be the ψ_v-Whittaker model of σ_v. There is an invariant bilinear pairing on $\mathcal{W}(\sigma_v, \psi_v) \times \mathcal{W}(\widetilde{\sigma}_v, \psi^{-1})$,

$$\langle W_1, W_2 \rangle_v := \int_{F_v^\times} W_1\left[\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right] W_2\left[\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right] d^\times a,$$

with the measure $d^\times a = L(1, 1_v)\, da/|a|_v$, where da equals $[F_v : \mathbb{R}]$ times the usual Lebesgue measure on F_v. Let $W_0 \in \mathcal{W}(\sigma_v, \psi_v)$ be the vector invariant under $U_{1,v}$ with minimal weight such that

$$L(s, \pi_v) = Z(s, W_0), \quad \text{where } Z(s, W_0) := \int_{F_v^\times} W_{\sigma_v}\left[\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right] |a|_v^{s-1/2}\, d^\times a$$

with $d^\times a$ the Tamagawa measure. Similarly, define $\widetilde{W}_0$ for $\widetilde{\sigma}_v$. Then $\Omega_{\sigma_v} := \langle W_0, \widetilde{W}_0 \rangle_v$ is an invariant of σ_v which is independent of the choice of ψ_v (see an explicit formula for Ω_{σ_v} before Lemma 3.14). We associate to (σ_v, χ_v) a constant by

$$C(\sigma_v, \chi_v) := \begin{cases} 2^{-1}\pi \cdot \Omega_{\sigma_v}^{-1} & \text{if } K_v \text{ is nonsplit,} \\ \Omega_{\sigma_v \otimes \chi_{1,v}} \cdot \Omega_{\sigma_v}^{-1} & \text{if } K_v \text{ is split,} \end{cases} \tag{1-1}$$

where for split $K_v \cong F_v^2$, embedded into $M_2(F_v)$ diagonally, the character χ_1 is given by $\chi_{1,v}(a) := \chi_v\left[\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)\right]$. If v is a real place of F and σ_v is a discrete series of weight k, then $C(\sigma_v, \chi_v) = 4^{k-1}\pi^{k+1}\Gamma(k)^{-1}$ when $K_v \cong \mathbb{C}$, and $C(\sigma_v, \chi_v) = 1$ when $K_v \cong \mathbb{R}^2$.

Let σ be the Jacquet–Langlands correspondence of π to $\mathrm{GL}_2(\mathbb{A})$; the normalized new vector $\phi^0 = \bigotimes_v \phi_v \in \sigma$ is the one fixed by $U_1(N)$ and ϕ_v is fixed by $U_{1,v}$ with weight minimal for all $v|\infty$ such that

$$L(s, \sigma) = |\delta|_{\mathbb{A}}^{s-1/2} Z(s, \phi^0), \quad \text{where } Z(s, \phi^0) := \int_{F^\times \backslash \mathbb{A}^\times} \phi^0\begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|_{\mathbb{A}}^{s-1/2}\, d^\times a$$

with the Tamagawa measure on $\mathbb{A}^\times$, so that

$$\mathrm{Res}_{s=1} \int_{|a| \le 1, a \in F^\times \backslash \mathbb{A}^\times} |a|^{s-1}\, d^\times a = \mathrm{Res}_{s=1} L(s, 1_F).$$

When F is a totally real field and σ a cuspidal automorphic representation such that σ_v is a discrete series for any infinite place v, the normalized new vector ϕ^0 is not parallel to the Hilbert newform ϕ: they are different at infinity. If σ is unitary and ϕ^0 is the normalized new vector of σ, then $\bar{\sigma} \cong \widetilde{\sigma}$ and $\overline{\phi^0}$ is the normalized new vector of $\bar{\sigma}$. We will see that $(\phi, \phi)_{U_0(N)} = (2\pi)^d \langle \phi_0, \bar{\phi}_0 \rangle_{U_0(N)}$.

Theorem 1.8 (explicit Waldspurger formula). *Let F be a number field. Let B be a quaternion algebra over F and π an irreducible cuspidal automorphic representation of $B_{\mathbb{A}}^{\times}$ with central character ω. Let K be a quadratic field extension of F and χ a Hecke character of $K_{\mathbb{A}}^{\times}$. Assume that*:

(1) $\omega \cdot \chi|_{\mathbb{A}^\times} = 1$;

(2) $\epsilon\big(\frac{1}{2}, \pi_v, \chi_v\big) = \chi_v \eta_v(-1)\epsilon(B_v)$ *for all places v of F.*

Then, for any nonzero forms $f_1 \in V(\pi, \chi)$ and $f_2 \in V(\widetilde{\pi}, \chi^{-1})$, we have

$$L^{(\Sigma)}\big(\tfrac{1}{2}, \pi, \chi\big) = 2^{-\#\Sigma_D+2} \cdot C_\infty \cdot \frac{\langle \phi_1^0, \phi_2^0 \rangle_{U_0(N)}}{\nu_{c_1}^2 \sqrt{|D_K| \|c_1\|^2}} \cdot \frac{P_\chi^0(f_1) P_{\chi^{-1}}^0(f_2)}{\langle f_1, f_2 \rangle_{\widehat{R}^\times}},$$

where $\phi_1^0 \in \pi^{\mathrm{JL}}$ and $\phi_2^0 \in \widetilde{\pi}^{\mathrm{JL}}$ are normalized new vectors, Σ is the set of places $v|(cD, N)\infty$ of F such that if $v\|N$ then $\mathrm{ord}_v(c/N) \geq 0$, and if $v|\infty$ then $K_v \cong \mathbb{C}$. The constant $C_\infty = \prod_{v|\infty} C_v$, $c_1|c$ and Σ_D are the same as in Theorem 1.5, and $C_v = C(\pi_v^{\mathrm{JL}}, \chi_v)$ is given in (1-1).

For many applications, we need an explicit form of the Waldspurger formula for different test vectors. The following variation of the formula is useful. For each place v of F, fix a $B_v^\times$-invariant pairing $\langle\,,\rangle_v$ on $\pi_v \times \widetilde{\pi}_v$. Here, if $v|\infty$, we mean it is the restriction of a $B_v^\times$-invariant pairing on the corresponding smooth representations. For any $f'_{1,v} \in \pi_v$, $f'_{2,v} \in \widetilde{\pi}_v$ with $\langle f'_{1,v}, f'_{2,v} \rangle_v \neq 0$, define $\beta^0(f'_{1,v}, f'_{2,v})$ as in Theorem 1.6.

Theorem 1.9 (variation of the Waldspurger formula). *Let (π, χ) and $f_1 \in V(\pi, \chi)$, $f_2 \in V(\widetilde{\pi}, \chi^{-1})$ be as in Theorem 1.8. Let S be a finite set of places of F, $f'_1 \in \pi$, $f'_2 \in \widetilde{\pi}$ be pure vectors which coincide with f_1, f_2 respectively outside S such that $\langle f'_{1,v}, f'_{2,v} \rangle_v \neq 0$ and $\beta^0(f'_{1,v}, f'_{2,v}) \neq 0$ for all $v \in S$. Here β^0 is similarly defined as in Theorem 1.6. Define*

$$P_\chi^0(f'_1) = \frac{\#\mathrm{Pic}_{K/F}(\mathcal{O}_{c_1})}{\mathrm{Vol}(K^\times \mathbb{A}^\times \backslash K_{\mathbb{A}}^\times, dt)} \cdot \int_{K^\times \mathbb{A}^\times \backslash K_{\mathbb{A}}^\times} f'(t)\chi(t)\,dt,$$

and define $P_{\chi^{-1}}^0(f'_2)$ similarly. Then, in the notation of Theorem 1.8, we have

$$L^{(\Sigma)}\big(\tfrac{1}{2}, \pi, \chi\big) = 2^{-\#\Sigma_D+2} \cdot C_\infty \cdot \frac{\langle \phi_1^0, \phi_2^0 \rangle_{U_0(N)}}{\nu_{c_1}^2 \sqrt{|D_K| \|c_1^2\|}} \cdot \frac{P_\chi^0(f'_1) P_{\chi^{-1}}^0(f'_2)}{\langle f'_1, f'_2 \rangle_{\widehat{R}^\times}} \cdot \prod_{v \in S} \frac{\beta^0(f_{1,v}, f_{2,v})}{\beta^0(f'_{1,v}, f'_{2,v})},$$

Example. Let $\phi = \sum a_n q^n \in S_2(\Gamma_0(N))$ be a newform of weight 2 and p a good ordinary prime of ϕ, K an imaginary quadratic field of discriminant D and χ a character of $\mathrm{Gal}(H_c/K)$ of conductor c that is prime to p. Assume that the conditions (i)–(ii) in Theorem 1.2 are satisfied. Let B be the quaternion algebra, π the cuspidal automorphic representation on $B_{\mathbb{A}}^\times$, and identify $\widetilde{\pi}$ with $\bar{\pi}$. Let $f \in \pi^{\widehat{R}^\times} = V(\pi, \chi)$ be a nonzero test vector as in Theorem 1.8. Define the p-stabilization of f by

$$f^\dagger = f - \alpha^{-1}\pi \begin{pmatrix} 1 & \\ & p \end{pmatrix} f,$$

where α is the unit root of $X^2 - a_p X + p$ and $\beta = p/\alpha$ is another root. By the variation of the Waldspurger formula and Theorem 1.2, one may easily obtain a formula for $P_\chi^0(f^\dagger)$, which is used to give the interpolation property of anticyclotomic p-adic L-functions:

$$L(1, \phi, \chi) = 2^{-\mu(N,D)} \cdot \frac{8\pi^2(\phi, \phi)_{\Gamma_0(N)}}{[\mathcal{O}_c^\times : \mathbb{Z}^\times]^2\sqrt{|Dc^2|}} \cdot \frac{|P_\chi^0(f^\dagger)|^2}{(f^\dagger, f^\dagger)_{\widehat{R}^\times}} \cdot e_p,$$

where

$$e_p = \frac{\beta^0(W, \overline{W})}{\beta^0(W^\dagger, \overline{W^\dagger})} = \frac{L(2, 1_p)}{L(1, \pi_p, \mathrm{ad})} \cdot (1 - \alpha^{-1}\chi_1(p))^{-1}(1 - \beta^{-1}\chi_1^{-1}(p))^{-1}.$$

Here W is a new vector of the Whittaker model $\mathcal{W}(\pi_p, \psi_p)$ with $\psi_p(x) = e^{-2\pi i \iota(x)}$, where $\iota : \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}$ is the natural embedding and $W^\dagger := W - \alpha^{-1}\pi_p\left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right)W$ is its stabilization, where $K_p^\times \cong \mathbb{Q}_p^{\times 2}$ is embedded into $\mathrm{GL}_2(\mathbb{Q}_p)$ as a diagonal subgroup and $\chi_1(a) = \chi\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$.

Now we consider the situation that:

(1) F is a totally real field and K is a totally imaginary quadratic extension over F,

(2) for any place $v|\infty$ of F, π_v^{JL} is a unitary discrete series of weight 2,

(3) $(c, N) = 1$.

Let ϕ be the Hilbert newform as in Theorem 1.5 (which is different from the one we chose in Theorem 1.8). We are going to give an explicit form of the Waldspurger formula following [Gross 1988], which is quoted in many references. Let $X = B^\times \backslash \widehat{B}^\times / \widehat{R}^\times$ and let $g_1, \ldots, g_n \in \widehat{B}^\times$ be a complete set of representatives of X. Write $[g] \in X$ for the class of an element $g \in \widehat{B}^\times$. For each g_i, let $\Gamma_i = (B^\times \cap g_i \widehat{R}^\times g_i^{-1})/\mathcal{O}^\times$, which is finite, and denote by w_i its order. Let $\mathbb{Z}[X]$ be the free $\mathbb{Z}$-module (of rank $\#X$) of formal sums $\sum_i a_i[g_i]$. There is a height pairing on $\mathbb{Z}[X] \times \mathbb{Z}[X]$ defined by

$$\left\langle \sum a_i[g_i], \sum b_i[g_i] \right\rangle = \sum_i a_i b_i w_i.$$

By Eichler's norm theorem, the norm map

$$\mathrm{N} : X \longrightarrow C_+, \quad \text{where } X := B^\times \backslash \widehat{B}^\times / \widehat{R}^\times,\ C_+ := F_+^\times \backslash \widehat{F}^\times / \widehat{\mathcal{O}}^\times,$$

is surjective. For each $c \in C_+$, let $X_c \subset X$ be the preimage of c and $\mathbb{Z}[X_c]$ be the submodule of $\mathbb{Z}[X]$ supported on X_c. Then $\mathbb{Z}[X] = \bigoplus_{c \in C_+} \mathbb{Z}[X_c]$. Let $\mathbb{Z}[X_c]^0$ be the submodule of classes $\sum a_i[g_i] \in \mathbb{Z}[X_c]$ with degree $\sum_i a_i = 0$, and let $\mathbb{Z}[X]^0 = \bigoplus_{c \in C_+} \mathbb{Z}[X_c]^0$ and $\mathbb{C}[X]^0 = \mathbb{Z}[X]^0 \otimes_{\mathbb{Z}} \mathbb{C}$. Note that $V(\pi, \chi) \subset \pi^{\widehat{R}^\times}$ by Proposition 3.8, and then there is an injection

$$V(\pi, \chi) \longrightarrow \mathbb{C}[X]^0, \quad f \mapsto \sum f([g_i]) w_i^{-1} [g_i],$$

so we can view $V(\pi, \chi)$ as a line on $\mathbb{C}[X]^0$. It follows that $\langle f, f \rangle = \langle f, f \rangle_{\widehat{R}^\times}$. The fixed embedding $K \to B$ induces a map

$$\mathrm{Pic}(\mathbb{O}_c) \longrightarrow X, \quad t \longmapsto x_t,$$

using which we define an element in $\mathbb{C}[X]$,

$$P_\chi := \sum_{t \in \mathrm{Pic}(\mathbb{O}_c)} \chi^{-1}(t) x_t,$$

and let P_χ^π be its projection to the line $V(\pi, \chi)$. Then the explicit formula in Theorem 1.8 implies:

Theorem 1.10. *Let* (π, χ) *be as above with conditions* (1)–(3). *The height of* P_χ^π *is given by the formula*

$$L^{(\Sigma)}\big(\tfrac{1}{2}, \pi, \chi\big) = 2^{-\#\Sigma_D} \cdot \frac{(8\pi^2)^d \cdot (\phi, \phi)_{U_0(N)}}{u^2 \sqrt{|D_K| \|c\|^2}} \cdot \langle P_\chi^\pi, P_\chi^\pi \rangle,$$

where

$$\Sigma := \{v | (N, D)\infty \,|\, \text{if } v \| N \text{ then } v \nmid D\}, \quad \Sigma_D := \{v | (N, D)\},$$

$u = \#\kappa_c \cdot [\mathbb{O}_c^\times : \mathbb{O}^\times]$, *and* $\phi \in \pi^{\mathrm{JL}}$ *is the Hilbert newform as in Theorem 1.5. For any nonzero vector* $f \in V(\pi, \chi)$, *let* $P_\chi^0(f) = \sum_{t \in \mathrm{Pic}(\mathbb{O}_c)} f(t) \chi(t)$; *then we have*

$$L^{(\Sigma)}\big(\tfrac{1}{2}, \pi, \chi\big) = 2^{-\#\Sigma_D} \cdot \frac{(8\pi^2)^d \cdot (\phi, \phi)_{U_0(N)}}{u^2 \sqrt{|D_K| \|c\|^2}} \cdot \frac{|P_\chi^0(f)|^2}{\langle f, f \rangle}.$$

Remark. When c and N have a common factor, one can still formulate an explicit formula in the spirit of Gross by defining a system of height pairings $\langle \, , \rangle_U$ in the same way as Theorem 1.8.

As a byproduct, we obtain the following result about the relation between the Petersson norm of a newform and a special value of the adjoint L-function:

Proposition 1.11. *Let* F *be a totally real field and* σ *a cuspidal unitary automorphic representation of* $\mathrm{GL}_2(\mathbb{A})$ *of conductor* N *such that, for any* $v | \infty$, σ_v *is a discrete* *series of weight k_v. Let ϕ be the Hilbert newform in σ as in Theorem 1.5. Then*

$$\frac{L^{(S)}(1,\sigma,\mathrm{ad})}{(\phi,\phi)_{U_0(N)}}=2^{d-1+\sum_{v|\infty}k_v}\cdot\|N\delta^{-2}\|^{-1}\cdot h_F^{-1},$$

where S is the set of finite places v of F with $\mathrm{ord}_v(N)\geq 2$ and $\mathrm{ord}_v(N)>\mathrm{ord}_v(C)$, C is the conductor of the central character of σ, h_F is the ideal class number of F, and

$$(\phi,\phi)_{U_0(N)}=\iint_{X_{U_0(N)}}|\phi|^2\Big(\bigwedge_{v|\infty}y_v^{k_v-2}\,dx_v\,dy_v\Big),\quad z_v=x_v+y_vi.$$

Or, equivalently,

$$\frac{L^{(S\infty)}(1,\sigma,\mathrm{ad})}{(\phi,\phi)_{U_0(N)}}=\tfrac{1}{2}\cdot\|N\delta^{-2}\|^{-1}\cdot h_F^{-1}\cdot\prod_{v|\infty}\frac{4^{k_v}\pi^{k_v+1}}{\Gamma(k_v)}.$$

Proof. This follows from Proposition 2.1, Lemma 2.2, and Proposition 3.11. Here [Tunnell 1978, Proposition 3.4] is also used. □

Example. Assume that $F=\mathbb{Q}$ and σ is the cuspidal automorphic representation associated to a cuspidal newform $\phi\in S_k(\mathrm{SL}_2(\mathbb{Z}))$. Then we have that

$$L(1,\sigma,\mathrm{ad})=2^k\cdot(\phi,\phi)_{\mathrm{SL}_2(\mathbb{Z})},\quad L^{(\infty)}(1,\sigma,\mathrm{ad})=\frac{2^{2k-1}\pi^{k+1}}{\Gamma(k)}\cdot(\phi,\phi)_{\mathrm{SL}_2(\mathbb{Z})}.$$

2. Reduction to local theory

We now explain how to obtain the explicit formulas in Theorems 1.5 and 1.8 from the original Waldspurger formula and the general Gross–Zagier formula proved in [Yuan et al. 2013]. We first consider the Waldspurger formula. Let B be a quaternion algebra over a number field F and π a cuspidal automorphic representation on $B_{\mathbb{A}}^\times$ with central character ω. Let K be a quadratic field extension over F and χ be a Hecke character on $K_{\mathbb{A}}^\times$. Assume that: (1) $\omega\cdot\chi|_{\mathbb{A}^\times}=1$; and (2) for any place v of F, $\epsilon\big(\frac{1}{2},\pi_v,\chi_v\big)=\chi_v\eta_v(-1)\epsilon(B_v)$. Define the Petersson pairing on $\pi\otimes\widetilde{\pi}$ by

$$\langle f_1,f_2\rangle_{\mathrm{Pet}}=\int_{B^\times\mathbb{A}^\times\backslash B_{\mathbb{A}}^\times}f_1(g)f_2(g)\,dg$$

with the Tamagawa measure, so that the volume of $B^\times\mathbb{A}^\times\backslash B_{\mathbb{A}}^\times$ is 2. Let P_χ denote the period functional on π

$$P_\chi(f)=\int_{K^\times\mathbb{A}^\times\backslash K_{\mathbb{A}}^\times}f(t)\chi(t)\,dt\quad\text{for all } f\in\pi.$$

Then Waldspurger's period formula [Waldspurger 1985; Yuan et al. 2013, Theorem 1.4] says that, for any pure tensors $f_1\in\pi$, $f_2\in\widetilde{\pi}$ with $\langle f_1,f_2\rangle_{\mathrm{Pet}}\neq 0$,

$$\frac{P_\chi(f_1)P_{\chi^{-1}}(f_2)}{\langle f_1, f_2\rangle_{\mathrm{Pet}}} = \frac{L\left(\frac{1}{2}, \pi, \chi\right)}{2L(1, \pi, \mathrm{ad})L(2, 1_F)^{-1}} \cdot \prod_v \beta(f_{1,v}, f_{2,v}), \tag{2-1}$$

where $L(1, \pi, \mathrm{ad})$ is defined using the Jacquet–Langlands lifting of π. Here, for any place v of F, let $\langle\, , \rangle_v : \pi_v \times \widetilde{\pi}_v \to \mathbb{C}$ be a nontrivial invariant pairing; then

$$\beta(f_{1,v}, f_{2,v}) = \frac{L(1, \eta_v)L(1, \pi_v, \mathrm{ad})}{L\left(\frac{1}{2}, \pi_v, \chi_v\right)L(2, 1_{F_v})} \int_{K_v^\times/F_v^\times} \frac{\langle \pi(t_v) f_{1,v}, f_{2,v}\rangle_v}{\langle f_{1,v}, f_{2,v}\rangle_v}\, \chi(t_v)\, dt_v,$$

where local Haar measures dt_v are chosen so that $\bigotimes_v dt_v = dt$ is the Haar measure on $K_{\mathbb{A}}^\times/\mathbb{A}^\times$ in the definitions of P_χ and $P_{\chi^{-1}}$, and the volume of $K^\times\backslash K_{\mathbb{A}}^\times/\mathbb{A}^\times$ with respect to dt is $2L(1, \eta)$. Note that the Haar measure dt is different from the one used in [Yuan et al. 2013, Theorem 1.4]. To obtain the explicit formula, we first relate $P_\chi(f)$, $L(1, \pi, \mathrm{ad})$, and $\langle f_1, f_2\rangle_{\mathrm{Pet}}$ to the corresponding objects with levels in Theorem 1.8, and reduce to local computation.

For our purpose, it is more convenient to normalize local additive characters and local Haar measures as follows. Take the additive character $\psi = \bigotimes_v \psi_v$ on $\mathbb{A}$ given by

$$\psi_v(a) = \begin{cases} e^{2\pi i a} & \text{if } F_v = \mathbb{R}, \\ e^{4\pi i \mathrm{Re}(a)} & \text{if } F_v = \mathbb{C}, \\ \psi_p(\mathrm{tr}_{F/\mathbb{Q}_p}(a)) & \text{if } F_v \text{ is a finite extension of } \mathbb{Q}_p \text{ for some prime } p, \end{cases}$$

where $\psi_p(b) = e^{-2\pi i \iota(b)}$ and $\iota : \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}$ is the natural embedding. It turns out that ψ is a character on $F\backslash\mathbb{A}$. For any place v of F, let da_v denote the Haar measure on F_v self-dual to ψ_v and let $d^\times a_v$ denote the Haar measure on $F_v^\times$ defined by $d^\times a_v = L(1, 1_v)\, da_v/|a_v|_v$. Let L be a separable quadratic extension of F_v or a quaternion algebra over F_v, and q the reduced norm on L; then (L, q) is a quadratic space over F_v. Fix the Haar measure dx on L to be the one self-dual with respect to ψ_v and q, in the sense that $\widehat{\widehat{\Phi}}(x) = \Phi(-x)$ for any $\Phi \in S(L)$, where $\widehat{\Phi}(y) := \int_L \Phi(x)\psi_v(\langle x, y\rangle)\, dx$ is the Fourier transform of Φ and $\langle x, y\rangle = q(x+y) - q(x) - q(y)$ is the bilinear form on L associated to q. Fix the Haar measure $d^\times x$ on $L^\times$ to be the one defined by

$$d^\times x = \begin{cases} L(1, 1_v)^2 \dfrac{dx}{|q(x)|_v} & \text{if } L = F_v^2, \\ L(1, 1_L) \dfrac{dx}{|q(x)|_v} & \text{if } L \text{ is a quadratic field extension over } F_v, \\ L(1, 1_v) \dfrac{dx}{|q(x)|_v^2} & \text{if } L \text{ is a quaternion algebra.} \end{cases}$$

Endow $L^\times/F_v^\times$ with the quotient Haar measure. Let K be a quadratic field extension of F and B a quaternion algebra over F. For local Haar measures on $K_v^\times/F_v^\times$ and

$B_v^\times/F_v^\times$, their product Haar measures on $K_{\mathbb{A}}^\times/\mathbb{A}^\times$ and $B_{\mathbb{A}}^\times/\mathbb{A}^\times$ satisfy

$$\mathrm{Vol}(K^\times\backslash K_{\mathbb{A}}^\times/\mathbb{A}^\times)=2L(1,\eta)\quad\text{and}\quad\mathrm{Vol}(B^\times\backslash B_{\mathbb{A}}^\times/\mathbb{A}^\times)=2.$$

Thus, these measures can be taken as the ones used in the above statement of Waldspurger's formula. From now on, we always use these measures and the additive character ψ on $\mathbb{A}$.

2A. ***Petersson pairing formula.*** Let σ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ and $\widetilde{\sigma}$ its contragredient; let N be the unipotent subgroup $N=\left\{\left(\begin{smallmatrix}1&x\\&1\end{smallmatrix}\right)\middle|\,x\in F\right\}$ of GL_2. View ψ as a character on $N(F)\backslash N(\mathbb{A})$ and the Haar measure da on $\mathbb{A}$ as the one on $N(\mathbb{A})$. For any $\phi\in\sigma$, let $W_\phi\in\mathcal{W}(\sigma,\psi)$ be the Whittaker function associated to ϕ,

$$W_\phi(g):=\int_{N(F)\backslash N(\mathbb{A})}\phi(ng)\overline{\psi(n)}\,dn.$$

Recall there is a $\mathrm{GL}_2(F_v)$-pairing on $\mathcal{W}_{\sigma_v,\psi_v}\times\mathcal{W}_{\widetilde{\sigma}_v,\psi_v^{-1}}$: for any local Whittaker functions $W_{1,v}\in\mathcal{W}(\sigma_v,\psi_v)$, $W_{2,v}\in\mathcal{W}(\widetilde{\sigma}_v,\psi_v^{-1})$,

$$\langle W_{1,v},W_{2,v}\rangle_v=\int_{F_v^\times}W_{1,v}\begin{pmatrix}a&\\&1\end{pmatrix}W_{2,v}\begin{pmatrix}a&\\&1\end{pmatrix}d^\times a.$$

Define the Petersson pairing on $\sigma\times\widetilde{\sigma}$ by

$$\langle\phi_1,\phi_2\rangle_{\mathrm{Pet}}:=\int_{Z(\mathbb{A})\,\mathrm{GL}_2(F)\backslash\,\mathrm{GL}_2(\mathbb{A})}\phi_1(g)\phi_2(g)\,dg,\qquad\phi_1\in\sigma,\ \phi_2\in\widetilde{\sigma},$$

where $Z\cong F^\times$ is the center of GL_2.

Proposition 2.1. *For any pure tensors $\phi_1\in\sigma$, $\phi_2\in\widetilde{\sigma}$, with $W_{\phi_i}=\bigotimes_v W_{i,v}$, $i=1,2$,*

$$\langle\phi_1,\phi_2\rangle_{\mathrm{Pet}}=2L(1,\sigma,\mathrm{ad})L(2,1_F)^{-1}\prod_v\alpha(W_{1,v},W_{2,v}),\tag{2-2}$$

where, for any place v of F,

$$\alpha(W_{1,v},W_{2,v})=\frac{1}{L(1,\sigma_v,\mathrm{ad})L(1,1_v)L(2,1_v)^{-1}}\cdot\langle W_{1,v},W_{2,v}\rangle.$$

Proof. We may assume that the cuspidal automorphic representation σ is also unitary and identify $\widetilde{\sigma}$ with $\overline{\sigma}$. Let $G=\mathrm{GL}_2$ over F, P the parabolic subgroup of upper triangular matrices in G, and let $U=\prod_v U_v$ be a maximal compact subgroup of $G(\mathbb{A})$. For any place v of F, with respect to the Iwasawa decomposition of $G(F_v)$,

$$g=a\begin{pmatrix}1&x\\&1\end{pmatrix}\begin{pmatrix}1&\\&b\end{pmatrix}k\in G(F_v),\qquad a,b\in F_v^\times,\ x\in F_v,\ k\in U_v.$$

Choose the measure dk on U_v such that $dg=|b|\,dx\,d^\times a\,d^\times b\,dk$ is the fixed local Haar measure on $G(F_v)$. For v nonarchimedean, U_v has volume $L(2,1_v)^{-1}|\delta_v|^{1/2}$

with respect to dk and has volume $L(2, 1_v)^{-1}|\delta_v|^2$ with respect to the fixed measure on $G(F_v)$; for v archimedean, U_v has volume $L(2, 1_v)^{-1}$ with respect to dk.

By [Jacquet and Chen 2001, Lemma 2.3], for any Bruhat–Schwartz function $\Phi_v \in \mathcal{S}(F_v^2)$ we have

$$\int_{F_v^\times \times U_v} \Phi([0, b]k)|b|^2\, d^\times b\, dk = \widehat{\Phi}_v(0),$$

where $\widehat{\Phi}_v$ is the Fourier transformation of Φ_v and $\widehat{\Phi}_v(0)$ is independent of the choice of the additive character ψ_v. For any $\Phi \in \mathcal{S}(\mathbb{A}^2)$, let

$$F(s, g, \Phi) = |\det g|^s \int_{\mathbb{A}^\times} \Phi([0, b]g)|b|^{2s}\, d^\times b,$$

and define the Eisenstein series

$$E(s, g, \Phi) := \sum_{\gamma \in P(F)\backslash G(F)} F(s, \gamma g, \Phi), \quad \operatorname{Re}(s) \gg 0.$$

By the Poisson summation formula,

$$\begin{aligned} E(s, g, \Phi) &= |\det g|^s \int_{F^\times\backslash\mathbb{A}^\times} \Bigl(\sum_{\xi \in F^2\backslash\{0\}} \Phi(a\xi g) \Bigr) |a|^{2s}\, d^\times a \\ &= |\det g|^s \int_{|a|\geq 1} \Bigl(\sum_{\xi \in F^2\backslash\{0\}} \Phi(a\xi g) \Bigr) |a|^{2s}\, d^\times a \\ &\quad + |\det g|^{s-1} \int_{|a|\geq 1} \Bigl(\sum_{\xi \in F^2\backslash\{0\}} \widehat{\Phi}(g^{-1}\xi^t a) \Bigr) |a|^{2-2s}\, d^\times a \\ &\quad + |\det g|^{s-1}\widehat{\Phi}(0) \int_{|a|\leq 1} |a|^{2s-2}\, d^\times a - |\det g|^s \Phi(0) \int_{|a|\leq 1} |a|^{2s}\, d^\times a. \end{aligned}$$

It follows that $E(s, g, \Phi)$ has meromorphic continuation to the whole s-plane, has possible poles only at $s = 0$ and 1, and its residue at $s = 1$ is equal to

$$\operatorname{Res}_{s=1} E(s, g, \Phi) = \widehat{\Phi}(0) \lim_{s\to 1}(s-1) \int_{|a|\leq 1} |a|^{2s-2}\, d^\times a = \tfrac{1}{2}\widehat{\Phi}(0) \operatorname{Res}_{s=1} L(s, 1_F),$$

which is independent of g. By unfolding the Eisenstein series and Fourier expansions of ϕ_i,

$$\begin{aligned} Z(s, \phi_1, \phi_2, \Phi) &:= \int_{[Z\backslash G]} \phi_1(g)\phi_2(g)E(s, g, \Phi)\, dg \\ &= \int_{N(\mathbb{A})\backslash G(\mathbb{A})} |\det g|^s W_{\phi_1}(g) W_{\phi_2}(g) \Phi([0, 1]g)\, dg \end{aligned}$$

has an Euler product if $\Phi \in S(\mathbb{A}^2)$ is a pure tensor. For each place v of F and $\Phi_v \in S(F_v^2)$, denote

$$Z(s, W_{1,v}, W_{2,v}, \Phi_v) = \int_{N(F_v)\backslash G(F_v)} |\det g|^s W_{1,v}(g) W_{2,v}(g) \Phi_v([0, 1]g)\, dg,$$

which has meromorphic continuation to the whole s-plane; and moreover, for $v \nmid \infty$, the fractional ideal of $\mathbb{C}[q_v^s, q_v^{-s}]$ of all $Z(s, W_{1,v}, W_{2,v}, \Phi_v)$ with $W_{1,v} \in \mathcal{W}(\sigma_v, \psi_v)$, $W_{2,v} \in \mathcal{W}(\widetilde{\sigma}_v, \psi_v^{-1})$ and $\Phi_v \in \mathcal{S}(F_v^2)$ is generated by $L(s, \sigma_v \times \widetilde{\sigma}_v)$. It is also known ([Jacquet and Chen 2001, p. 51]) that, for each v,

$$\begin{aligned} &Z(1, W_{1,v}, W_{2,v}, \Phi_v) \\ &\qquad = \int_{F_v^\times} W_{1,v}\begin{pmatrix} a & \\ & 1 \end{pmatrix} W_{2,v}\begin{pmatrix} a & \\ & 1 \end{pmatrix} d^\times a \cdot \iint_{F_v^\times \times U_v} \Phi_v([0, b]k)|b|^2\, d^\times b\, dk, \end{aligned}$$

with the Haar measures chosen above. Let $\Phi = \bigotimes_v \Phi_v \in \mathcal{S}(\mathbb{A}^2)$ be a pure tensor such that $\widehat{\Phi}(0) \neq 0$ and take residue at $s = 1$ on the two sides of

$$Z(s, \phi_1, \phi_2, \Phi) = \prod_v Z(s, W_{1,v}, W_{2,v}, \Phi_v).$$

We have

$$\langle \phi_1, \phi_2 \rangle_{\mathrm{Pet}} \operatorname{Res}_{s=1} E(s, g, \Phi) = \operatorname{Res}_{s=1} L(s, \sigma \times \widetilde{\sigma}) \widehat{\Phi}(0) \prod_v \frac{\langle W_{1,v}, W_{2,v} \rangle_v}{L(1, \sigma_v \times \widetilde{\sigma}_v)},$$

or

$$\frac{L(1, \sigma, \mathrm{ad})}{\langle \phi_1, \phi_2 \rangle_{\mathrm{Pet}}} = \frac{1}{2} \prod_v \frac{L(1, \sigma_v, \mathrm{ad}) L(1, 1_{F_v})}{\langle W_{1,v}, W_{2,v} \rangle_v}.$$

The formula in the proposition follows. □

2B. *U-level pairing.*

Lemma 2.2. *Let B be a quaternion algebra over a number field F and denote by r, s, t integers such that $B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}^r \times M_2(\mathbb{R})^s \times M_2(\mathbb{C})^t$. For $U \subset \widehat{B}^\times$ an open compact subgroup, the volume of X_U, defined after Definition 1.7, is given by*

$$\mathrm{Vol}(X_U) = 2(4\pi^2)^{-d} \#(\mathbb{A}_f^\times / F^\times U_Z) \cdot \frac{\mathrm{Vol}(U_Z)}{\mathrm{Vol}\, U},$$

where $U_Z = U \cap \widehat{F}^\times$ and the volumes $\mathrm{Vol}(U_Z)$ and $\mathrm{Vol}\, U$ are with respect to Tamagawa measure, so that

$$\mathrm{Vol}(\mathrm{GL}_2(\mathcal{O}_v)) = L(2, 1_v)^{-1} \mathrm{Vol}(\mathcal{O}_v)^4,$$
$$\mathrm{Vol}(B_v^\times) = L(2, 1_v)^{-1} \mathrm{Vol}(\mathcal{O}_v)^4 (q_v - 1)^{-1} \quad \textit{for } B_v \textit{ division.}$$

In particular, if U contains $\widehat{\mathcal{O}}^\times$ then — where h_F is the class number of F —

$$\mathrm{Vol}(X_U) = 2(4\pi^2)^{-d} |D_F|^{-1/2} \cdot h_F \cdot \mathrm{Vol}(U)^{-1}.$$

Proof (see also [Yuan et al. 2013] *for the case $s=1$ and $t=0$).* Let q be the reduced norm on B, and $B^1 := \{b \in B^\times \mid q(b)=1\}$. For each place v of F, we have the exact sequence

$$1 \longrightarrow B_v^1 \longrightarrow B_v^\times \longrightarrow q(B_v^\times) \longrightarrow 1,$$

and define the Haar measure dh_v on B_v^1 so that the Haar measure on $q(B_v^\times)$ — obtained by the restriction of the Haar measure on $F_v^\times$ — equals the quotient of the Haar measure on $B_v^\times$ by dh_v. The product of these local measures give the Tamagawa measure on $B_{\mathbb{A}}^1$, so that $\mathrm{Vol}(B^1\backslash B_{\mathbb{A}}^1)=1$. This follows from the fact that the Tamagawa numbers of B^1 and $B^\times$ are 1 and 2, respectively. Assume that $B \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}^r \times M_2(\mathbb{R})^s \times M_2(\mathbb{C})^t$. We assume that $s+t>0$ first and let $\Sigma \subset \infty$ be the subset of infinite places of F where B splits. By the strong approximation theorem, $B_{\mathbb{A}}^1 = B^1 B_\infty^1 U^1$, where $U^1 = U \cap B_{\mathbb{A}_f}^1$ is an open compact subgroup of $B_{\mathbb{A}_f}^1$. It follows that

$$B^1\backslash B_{\mathbb{A}}^1 = B^1\backslash B^1 B_\infty^1 U^1 = (\Gamma\backslash B_\Sigma^1) B_\infty^{1,\Sigma} U^1,$$

where $\Gamma = B^1 \cap U^1$, and we identify $\Gamma\backslash B_\Sigma^1$ with the fundamental domain of this quotient.

For a real place v of F, $B_v^1 \cong \mathrm{SL}_2(\mathbb{R})$. By the Iwasawa decomposition, any element is uniquely of the form

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad x \in \mathbb{R},\ y \in \mathbb{R}_+,\ \theta \in [0, 2\pi).$$

The measure on B_v^1 is $dx\,dy\,d\theta/2y^2$ with $dx\,dy$ the usual Lebesgue measure, and θ has volume 2π. For a complex place v of F, $B_v^1 \cong \mathrm{SL}_2(\mathbb{C})$. By the Iwasawa decomposition, any element in $\mathrm{SL}_2(\mathbb{C})$ is uniquely of form

$$\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & \\ & v^{-1/2} \end{pmatrix} u, \quad z \in \mathbb{C},\ v \in \mathbb{R}_+,\ u \in \mathrm{SU}_2,$$

The measure on B_v^1 is $dx\,dy\,dv\,du/v^3$ with $z = x + yi$, dx, dy, dv the usual Lebesgue measure, and du has volume $8\pi^2$ (see [Vignéras 1980]). It follows that

$$\mathrm{Vol}(\Gamma\backslash B_\Sigma^1) = 2^{-t}(4\pi^2)^{s+2t} w_U^{-1} \cdot \mathrm{Vol}\left(\Gamma\backslash(\mathcal{H}_2^s \times \mathcal{H}_3^t),\ \frac{dx\,dy}{4\pi y^2} \wedge \frac{dx\,dy\,dv}{\pi^2 v^3}\right),$$

where $w_U = \#\{\pm 1\} \cap U$. But also, for any infinite place $v \notin \Sigma$, $\mathrm{Vol}(B_v^1) = 4\pi^2$. Thus,

$$w_U^{-1} \cdot 2^{-t}(4\pi^2)^d \cdot \mathrm{Vol}\left(\Gamma\backslash(\mathcal{H}_2^s \times \mathcal{H}_3^t),\ \frac{dx\,dy}{4\pi y^2} \wedge \frac{dx\,dy\,dv}{\pi^2 v^3}\right) \cdot \mathrm{Vol}(U^1) = 1,$$

where $d = [F : \mathbb{Q}]$. Let $B_+^\times \subset B^\times$ be the subgroup of elements whose norms are positive at all real places. Now consider the natural map

$$(B^1 \cap U^1)\backslash(\mathcal{H}_2^s \times \mathcal{H}_3^t) \longrightarrow (B_+^\times \cap U)\backslash(\mathcal{H}_2^s \times \mathcal{H}_3^t),$$

whose degree is just

$$[(B_+^\times \cap U) : (B^1 \cap u^1)\mu_U] = [\det(B_+^\times \cap U) : \mu_U^2] = [\mu_U' : \mu_U^2].$$

Here $\mu_U = F^\times \cap U$ and $\mu_U' = F_+^\times \cap \det U$, subgroups of $\mathcal{O}_F^\times$ with finite index. It follows that

$$\begin{aligned}\operatorname{Vol}(X_U) &= \operatorname{Vol}((B_+^\times \cap U)\backslash(\mathcal{H}_2^s \times \mathcal{H}_3^t)) \cdot \#(F_+^\times \backslash \widehat{F}^\times / \det U)\\ &= \frac{2^t w_U}{(4\pi^2)^d \cdot \operatorname{Vol}(U^1) \cdot [\mu_U' : \mu_U^2]} \cdot \#(F_+^\times \backslash \widehat{F}^\times / \det U).\end{aligned}$$

Note that

$$\frac{\#(\widehat{F}^\times / F_+^\times \det U)}{\#(\widehat{F}^\times \backslash F^\times U_Z)} = [F^\times U_Z : F_+^\times \det U] = [F^\times : F_+^\times] \frac{\operatorname{Vol}(U_Z)}{\operatorname{Vol}(\det U)} [\mu_U' : \mu_U].$$

Since $[F^\times : F_+^\times] = 2^{r+s}$, $[\mu_U : \mu_U^2] = 2^{r+s+t-1} w_U$, and $\operatorname{Vol} U = \operatorname{Vol}(U^1)\operatorname{Vol}(\det U)$, we have

$$\operatorname{Vol}(X_U) = 2(4\pi^2)^{-d} \#(\widehat{F}^\times / F^\times U_Z) \cdot \frac{\operatorname{Vol}(U_Z)}{\operatorname{Vol}(U)}.$$

Now assume $s = t = 0$. The Tamagawa number of $B^\times$ is 2, $\operatorname{Vol}(B_v^\times / F_v^\times) = 4\pi^2$ for any $v|\infty$, and the decomposition

$$B^\times \mathbb{A}^\times \backslash B_{\mathbb{A}}^\times = F_\infty^\times \backslash B_\infty^\times \times B^\times \widehat{F}^\times \backslash \widehat{B}^\times.$$

It follows that $\operatorname{Vol}(B^\times \widehat{F}^\times \backslash \widehat{B}^\times) = 2(4\pi^2)^{-d}$. Let $\gamma_1, \ldots, \gamma_h$ be a complete set of representatives in $\widehat{B}^\times$ of the coset $B^\times \backslash \widehat{B}^\times / U$. Consider the natural map

$$B^\times \backslash B^\times \gamma_i U \longrightarrow B^\times \widehat{F}^\times \backslash B^\times \widehat{F}^\times \gamma_i U,$$

whose degree is $\#\widehat{F}^\times / F^\times U_Z$. Now

$$\operatorname{Vol}(B^\times \widehat{F}^\times \backslash B^\times \widehat{F}^\times \gamma_i U) = \operatorname{Vol}\left(\frac{\gamma_i (U/U_Z)\gamma_i^{-1}}{(B^\times \cap \gamma_i U \gamma_i^{-1})/\mu_Z}\right) = \frac{\operatorname{Vol}(U)/\operatorname{Vol}(U_Z)}{\#(B^\times \cap \gamma_i U \gamma_i^{-1})/\mu_Z}.$$

Thus,

$$\begin{aligned}2(4\pi^2)^{-d} &= \operatorname{Vol}(B^\times \widehat{F}^\times \backslash \widehat{B}^\times)\\ &= (\#\widehat{F}^\times / F^\times U_Z)^{-1} \cdot \frac{\operatorname{Vol}(U)}{\operatorname{Vol}(U_Z)} \cdot \sum_{i=1}^h \frac{1}{\#(B^\times \cap \gamma_i U \gamma_i^{-1})/\mu_Z}. \qquad \square\end{aligned}$$

2C. c_1-level periods. Now take $f_1 \in V(\pi, \chi)$, $f_2 \in V(\widetilde{\pi}, \chi^{-1})$ to be nonzero test vectors as defined before. Let $\sigma = \pi^{\mathrm{JL}}$ and take $\phi_1 \in \sigma$ and $\phi_2 \in \widetilde{\sigma}$ to be normalized new vectors. The c_1-level periods $P^0_\chi(f_1)$, $P^0_{\chi^{-1}}(f_2)$ are related to the periods in Waldspurger's formula by the following lemma:

Lemma 2.3. *Let $b \subset \mathbb{O}$ be a nonzero ideal of F and denote by $\mathrm{Pic}_{K/F}(\mathbb{O}_b)$ the group $\widehat{K}^\times / K^\times \widehat{F}^\times \widehat{\mathbb{O}}_b^\times$. Then there is a relative class number formula,*

$$L^{(b)}(1, \eta) \cdot \|D_{K/F} b^2 \delta\|^{1/2} \cdot 2^{-r_{K/F}} = \frac{\#\,\mathrm{Pic}_{K/F}(\mathbb{O}_b) \cdot R_b}{\#\kappa_b \cdot w_b},$$

where $r_{K/F} = \mathrm{rank}\, \mathbb{O}_K^\times - \mathrm{rank}\, \mathbb{O}^\times$, $w_b = [\mathbb{O}^\times_{b,\mathrm{tor}} : \mathbb{O}^\times_{\mathrm{tor}}]$, R_b is the quotient of the regulator of $\mathbb{O}_b^\times$ by that of $\mathbb{O}^\times$, and κ_b is the kernel of the natural morphism from $\mathrm{Pic}(\mathbb{O})$ *to* $\mathrm{Pic}(\mathbb{O}_b)$. *Define a constant $\nu_b := 2^{-r_{K/F}} R_b^{-1} \cdot \#\kappa_b w_b$. Then*

$$P_\chi(f) = 2L_{c_1}(1, \eta) \|D c_1^2 \delta\|^{-1/2} \nu_{c_1}^{-1} \cdot P^0_\chi(f).$$

Proof. There are exact sequences

$$1 \longrightarrow \kappa_b \longrightarrow \widehat{F}^\times / F^\times \widehat{\mathbb{O}}_F^\times \longrightarrow \widehat{K}^\times / K^\times \widehat{\mathbb{O}}_b^\times \longrightarrow \widehat{K}^\times / K^\times \widehat{F}^\times \widehat{\mathbb{O}}_b^\times \longrightarrow 1$$

and

$$1 \longrightarrow \mathbb{O}_K^\times / \mathbb{O}_b^\times \longrightarrow \widehat{\mathbb{O}}_K^\times / \widehat{\mathbb{O}}_b^\times \longrightarrow \widehat{K}^\times / K^\times \widehat{\mathbb{O}}_b^\times \longrightarrow \widehat{K}^\times / K^\times \widehat{\mathbb{O}}_K^\times \longrightarrow 1.$$

It follows that

$$\#\,\mathrm{Pic}_{K/F}(\mathbb{O}_b) = \#\widehat{K}^\times / K^\times \widehat{F}^\times \widehat{\mathbb{O}}_b^\times = \frac{h_K}{h_F} \cdot [\widehat{\mathbb{O}}_K^\times : \widehat{\mathbb{O}}_b^\times] \cdot [\mathbb{O}_K^\times : \mathbb{O}_b^\times]^{-1} \cdot \#\kappa_b,$$

where $h_K = \#\widehat{K}^\times / K^\times \widehat{\mathbb{O}}_K^\times$ is the ideal class number of K and similarly for h_F. By the class number formula for F and K,

$$\mathrm{Res}_{s=1} L(s, 1_F) = 2^{r_F+1} \frac{R_F h_F}{w_F \sqrt{|D_F|}}, \quad \mathrm{Res}_{s=1} L(s, 1_K) = 2^{r_K+1} \frac{R_K h_K}{w_K \sqrt{|D_K|}},$$

where $r_F = \mathrm{rank}\, \mathbb{O}_F^\times$, D_F is the discriminant of F, R_F is the regulator of $\mathbb{O}^\times$, h_F the ideal class number of F, $w_F = \#\mathbb{O}^\times_{\mathrm{tor}}$, and similar for r_K, D_K, R_K, h_F and w_K. Noting that $|D_K|/|D_F| = |D_{K/F}\delta|_{\mathbb{A}}^{-1}$ and $[\widehat{\mathbb{O}}_K^\times : \widehat{\mathbb{O}}_b^\times]^{-1} = L_b(1, \eta)|b|$, we have that

$$\begin{aligned} L(1, \eta) &= \frac{h_K}{h_F} \cdot 2^{r_{K/F}} \frac{R_K w_K^{-1}}{R_F w_F^{-1}} \cdot \|D_{K/F}\delta\|^{-1/2} \\ &= \#\,\mathrm{Pic}_{K/F}(\mathbb{O}_b) \cdot L_b(1, \eta) \cdot 2^{r_{K/F}} \cdot [\mathbb{O}_K^\times : \mathbb{O}_b^\times] \frac{R_K w_K^{-1}}{R_F w_F^{-1}} (\#\kappa_b)^{-1} \cdot \|D_{K/F} b^2 \delta\|^{-1/2}. \end{aligned}$$

The relative class number formula then follows. $\square$

Let N be the conductor of $\sigma = \pi^{\mathrm{JL}}$, let $U \subset \widehat{B}^\times$ be an open compact subgroup, and recall

$$\langle f_1, f_2\rangle_U = \tfrac{1}{2}\langle f_1, f_2\rangle_{\mathrm{Pet}} \operatorname{Vol}(X_U), \quad \langle \phi_1, \phi_2\rangle_{U_0(N)} = \tfrac{1}{2}\langle \phi_1, \phi_2\rangle_{\mathrm{Pet}} \operatorname{Vol}(X_{U_0(N)}).$$

Applying Proposition 2.1, Lemma 2.2, and Lemma 2.3, Waldspurger's formula (2-1) implies the following:

Proposition 2.4. *Let $U = \prod_v U_v \subset \widehat{B}^\times$ be an open compact subgroup with $\widehat{\mathcal{O}}^\times \subset U$. Let $\gamma_v = \operatorname{Vol}(U_0(N)_v)^{-1} \operatorname{Vol}(U_v)$ for all finite places v and $\gamma_v = 1$ for $v|\infty$. Let $\phi_1 \in \pi^{\mathrm{JL}}$, $\phi_2 \in \widetilde{\pi}^{\mathrm{JL}}$ be any forms with $\langle \phi_1, \phi_2\rangle_{U_0(N)} \neq 0$ and let $\alpha(W_{1,v}, W_{2,v})$ be the corresponding local constants defined in Proposition 2.1. Let $f_1 \in \pi$, $f_2 \in \widetilde{\pi}$ be any pure tensors with $(f_1, f_2)_{\mathrm{Pet}} \neq 0$ and $\beta(f_{1,v}, f_{2,v})$ the corresponding constants defined in (2-1). Then we have*

$$(2L_{c_1}(1, \eta)|Dc_1^2\delta|_{\mathbb{A}}^{1/2} v_{c_1}^{-1})^2 \cdot \frac{P_\chi^0(f_1) P_{\chi^{-1}}^0(f_2)}{\langle f_1, f_2\rangle_U} = \frac{L\left(\frac{1}{2}, \pi, \chi\right)}{\langle \phi_1, \phi_2\rangle_{U_0(N)}} \cdot \prod_v \alpha(W_{1,v}, W_{2,v})\beta(f_{1,v}, f_{2,v})\gamma_v, \tag{2-3}$$

where v_{c_1} is defined as in Lemma 2.3.

It is now clear that the explicit Waldspurger formula will follow from the computation of these local factors. In the next section, we will choose ϕ_1, ϕ_2 to be normalized new vectors in π^{JL} and $\widetilde{\pi}^{\mathrm{JL}}$, respectively, choose nonzero $f_1 \in V(\pi, \chi)$, $f_2 \in V(\widetilde{\pi}, \chi)$, and compute the related local factors in (2-3).

We obtain the explicit Gross–Zagier formula from the Yuan–Zhang–Zhang formula in a similar way. Let F be a totally real field and X a Shimura curve over F associated to an incoherent quaternion algebra $\mathbb{B}$. Let A be an abelian variety over F parametrized by X and let $\pi_A = \operatorname{Hom}_\xi^0(X, A)$ be the associated automorphic representation of $\mathbb{B}^\times$ over the field $M := \operatorname{End}^0(A)$ and ω its central character. Let K be a totally imaginary quadratic extension over F and $\chi : K_{\mathbb{A}}^\times \to L^\times$ a finite-order Hecke character over a finite extension L of M such that $\omega \cdot \chi|_{\mathbb{A}^\times} = 1$ and, for all places v of F, $\epsilon\left(\frac{1}{2}, \pi_A, \chi\right) = \chi_v \eta_v(-1)\epsilon(\mathbb{B}_v)$. Fix an embedding $K_{\mathbb{A}} \to \mathbb{B}$ with $K_{\mathbb{A}}^\times \to \mathbb{B}^\times$, let $P \in X^{K^\times}(K^{\mathrm{ab}})$, and define

$$P_\chi(f) = \int_{K_{\mathbb{A}}^\times / K^\times \mathbb{A}^\times} f(P)^{\sigma_t} \otimes_M \chi(t)\, dt \in A(K^{\mathrm{ab}})_{\mathbb{Q}} \otimes_M L,$$

where we use the Haar measure so that the total volume of $K_{\mathbb{A}}^\times / K^\times \mathbb{A}^\times$ is $2L(1, \eta)$, and η is the quadratic Hecke character on $\mathbb{A}^\times$ associated to the extension K/F. We further assume for all nonarchimedean places v that the compact subgroup $\mathcal{O}_{K_v}^\times / \mathcal{O}_v^\times$ has a volume in $\mathbb{Q}^\times$, and fix a local invariant pairing $(\,,)_v$ on $\pi_{A,v} \times \pi_{A^\vee,v}$ with values in M. Define $\beta(f_{1,v}, f_{2,v}) \in L$ for $(f_{1,v}, f_{2,v})_v \neq 0$ by

$$\beta(f_{1,v}, f_{2,v}) = \frac{L(1,\eta_v)L(1,\pi_v,\mathrm{ad})}{L\big(\frac{1}{2},\pi_v,\chi_v\big)L(2,1_{F_v})}\int_{K_v^\times/F_v^\times}\frac{(\pi(t_v)f_{1,v}, f_{2,v})_v}{(f_{1,v}, f_{2,v})_v}\chi(t_v)\,dt_v \in L,$$

where we take an embedding of L into $\mathbb{C}$, and the above integral lies in L and does not depend on the embedding.

Then, for any pure tensors $f_1 \in \pi_A$, $f_2 \in \pi_{A^\vee}$ with $(f_1, f_2) \neq 0$, Yuan et al. [2013] obtained the following celebrated formula as an identity in $L \otimes_{\mathbb{Q}} \mathbb{C}$:

$$\frac{\langle P_\chi(f_1), P_{\chi^{-1}}(f_2)\rangle_{K,L}}{\mathrm{Vol}(X_U)^{-1}(f_1, f_2)_U} = \frac{L'\big(\frac{1}{2},\pi_A,\chi\big)}{L(1,\pi_A,\mathrm{ad})L(2,1_F)^{-1}}\prod_v \beta(f_{1,v}, f_{2,v}). \tag{2-4}$$

Note that we use height over K whereas that used in [Yuan et al. 2013] is over F, the Haar measure to define $P_\chi(f)$ is different from theirs by $2L(1,\eta)$, and the measure to define $\mathrm{Vol}(X_U)$ is different from theirs by 2. Similar to Proposition 2.4, we have:

Proposition 2.5. *Let $U = \prod_v U_v \subset \widehat{B}^\times$ be a pure product open compact subgroup such that $\widehat{\mathbb{O}}^\times \subset U$. Let $\gamma_v = \mathrm{Vol}(U_0(N)_v)\,\mathrm{Vol}(U_v)^{-1}$ for all finite places v and $\gamma_v = 1$ for $v|\infty$. Let $\phi \in \pi_A^{\mathrm{JL}}$ be any nonzero form and let $\alpha(W_v, \overline{W_v})$ be the corresponding local constants defined in Proposition 2.1. Let $f_1 \in \pi_A$, $f_2 \in \pi_{A^\vee}$ be any pure tensors with $(f_1, f_2) \neq 0$ and $\beta(f_{1,v}, f_{2,v})$ the corresponding constants defined in (2-4). Then we have*

$$(2L_{c_1}(1,\eta)|Dc_1^2\delta|_{\mathbb{A}}^{1/2}v_{c_1}^{-1})^2\cdot\frac{\langle P_\chi^0(f_1), P_{\chi^{-1}}^0(f_2)\rangle_{K,L}}{(f_1, f_2)_U}$$
$$= \frac{L'\big(\frac{1}{2},\pi_A,\chi\big)}{\langle\phi,\phi\rangle_{U_0(N)}}\prod_v \alpha_v(W_{1,v}, W_{2,v})\beta_v(f_{1,v}, f_{2,v})\gamma_v. \tag{2-5}$$

We will study the local factors appearing in formulas in Propositions 2.4 and 2.5 in the next section.

2D. ***Proofs of main results.*** In this subsection, we prove Theorems 1.5, 1.6, 1.8, 1.9 and 1.10, assuming local results proved in Section 3.

Proof of Theorem 1.8. We first give a proof of the explicit Waldspurger formula. In (2-3), take nonzero $f_1 \in V(\pi,\chi)$, $f_2 \in V(\widetilde{\pi},\chi^{-1})$, and ϕ_1^0 (resp. ϕ_2^0) the normalized new vector of π^{JL} (resp. $\widetilde{\pi}^{\mathrm{JL}}$). Let $W_{\phi_i^0} := W_i = \bigotimes_v W_{i,v}$ be the corresponding Whittaker functions of ϕ_i^0, $i = 1, 2$. Let $R \subset B$ be the order, as defined in Theorem 1.8, and $U = \widehat{R}^\times$. Denote

$$\alpha := \alpha(W_{1,v}, W_{2,v})\cdot|\delta|_v^{1/2}, \quad \beta := \beta(f_{1,v}, f_{2,v})\cdot|D\delta|_v^{-1/2}.$$

Then (2-3) becomes

$$4|Dc_1^2\delta^2|_{\mathbb{A}}^{1/2}\nu_{c_1}^{-2}\frac{P_\chi^0(f_1)P_{\chi^{-1}}^0(f_2)}{\langle f_1, f_2\rangle_U} = \frac{L^{(\Sigma)}\big(\frac{1}{2},\pi,\chi\big)}{\langle\phi_1^0,\phi_2^0\rangle_{U_0(N)}}L_\Sigma\big(\tfrac{1}{2},\pi,\chi\big)L_{c_1}(1,\eta)^{-2}|c_1|_{\mathbb{A}}^{-1}\prod_v\alpha_v\beta_v\gamma_v\,.$$

Let Σ be the set in Theorem 1.8, $\Sigma_\infty = \Sigma\cap\infty$ and $\Sigma_f = \Sigma\setminus\Sigma_\infty$. Comparing with the formula (2-3), the proof of the explicit formula in Theorem 1.8 is reduced to showing that

$$L_{\Sigma_f}\big(\tfrac{1}{2},\pi,\chi\big)L_{c_1}(1,\eta)^{-2}|c_1|_{\mathbb{A}}^{-1}\prod_{v\nmid\infty}\alpha_v\beta_v\gamma_v = 2^{\#\Sigma_D}$$

and

$$L_{\Sigma_\infty}\big(\tfrac{1}{2},\pi,\chi\big)\prod_{v\mid\infty}\alpha_v\beta_v\gamma_v = C_\infty^{-1},$$

which are given by Lemma 3.13 and Lemma 3.14. □

Proof of Theorem 1.10. Given the hypotheses of Theorem 1.10, identify $\widetilde{\pi}$ with $\bar{\pi}$; by Theorem 1.8,

$$L^{(\Sigma)}\big(\tfrac{1}{2},\pi,\chi\big) = 2^{-\#\Sigma_D+2}(4\pi^3)^d\frac{\langle\phi^0,\overline{\phi^0}\rangle_{U_0(N)}}{\nu_{c_1}^2\sqrt{|D_K|\,\|c_1^2\|}}\frac{|P_\chi^0(f)|^2}{\langle f_1, f_2\rangle_U}\,.$$

The formula in Theorem 1.10 follows by noting these facts:

(i) $\nu_{c_1} = 2^{1-d}u_1$.

(ii) $\langle\phi^0,\overline{\phi^0}\rangle_{U_0(N)} = (2\pi)^{-d}(\phi,\phi)_{U_0(N)}$, where ϕ is the Hilbert newform of π_A^{JL}. This is obtained by applying the formula in Proposition 2.1 to ϕ and ϕ^0, and the comparison of local Whittaker pairings at infinity; see the discussion before Proposition 3.12.

(iii) Let $g_1,\dots,g_n\in\widehat{B}^\times$ be a complete set of representatives of $X = B^\times\backslash\widehat{B}^\times/\widehat{R}^\times$ and let $w_i = \#(B^\times\cap g_i\widehat{R}^\times g_i^{-1}/\mathcal{O}^\times)$; then, as in the proof of Lemma 2.2, for $U=\widehat{R}^\times$,

$$\begin{aligned}\langle f,\bar{f}\rangle_U = 2^{-1}\operatorname{Vol}(X_U)\langle f,\bar{f}\rangle_{\mathrm{Pet}} &= \sum_{i=1}^n|f(g_i)|^2w_i^{-1}\\ &= \Big\langle\sum f(g_i)w_i^{-1}[g_i],\sum f(g_i)w_i^{-1}[g_i]\Big\rangle\\ &= \langle f, f\rangle,\end{aligned}$$

where we identify f with its image under the map $V(\pi,\chi)\to\mathbb{C}[X]$ and $\langle\,,\rangle$ is the height pairing on $\mathbb{C}[X]$. □

Proof of Theorem 1.5. To show the explicit Gross–Zagier formula in Theorem 1.5, similarly to above, we apply the formula (2-5) in Proposition 2.5 to nonzero forms $f_1 \in V(\pi_A, \chi)$, $f_2 \in V(\pi_{A^\vee}, \chi^{-1})$, ϕ^0 the normalized new vector of π_A^{JL}, and $U = \mathcal{R}^\times$ as in Theorem 1.5. By Lemma 3.13 and Lemma 3.14, we have

$$L'^{(\Sigma)}\big(\tfrac{1}{2}, \pi, \chi\big) = 2^{-\#\Sigma_D+2}(4\pi^3)^d \frac{\langle \phi^0, \overline{\phi^0}\rangle_{U_0(N)}}{\nu_{c_1}^2\sqrt{|D_K|\,\|c_1^2\|}} \frac{\langle P_\chi^0(f_1), P_{\chi^{-1}}^0(f_2)\rangle_{K,L}}{(f_1, f_2)_U}.$$

Then the explicit Gross–Zagier formula follows again by noting facts (i) and (ii) above. □

Proof of Theorems 1.9 and 1.6. We now show that the variations of the explicit Waldspurger formula in Theorem 1.9 follow from the Waldspurger formula (2-1) and its explicit form in Theorem 1.8, and similarly for the variation of the explicit Gross–Zagier formula in Theorem 1.6.

Let $f_1' = \bigotimes_v f_{1,v}' \in \pi$, $f_2' = \bigotimes_v f_{2,v}' \in \widetilde{\pi}$ be forms different from the test vectors $f_1 = \bigotimes_v f_{1,v} \in V(\pi, \chi)$, $f_2 = \bigotimes_v f_{2,v} \in V(\widetilde{\pi}, \chi^{-1})$ at a finite set S of places of F, respectively, such that $\langle f_{1,v}', f_{2,v}'\rangle_v \neq 0$ and $\beta(f_{1,v}', f_{2,v}') \neq 0$ for any $v \in S$. By the Waldspurger formula (2-1), we have the formulas

$$\frac{P_\chi^0(f_1)\cdot P_{\chi^{-1}}^0(f_2)}{\langle f_1, f_2\rangle_U} = \mathcal{L}(\pi, \chi)\prod_v \beta(f_{1,v}, f_{2,v}),$$
$$\frac{P_\chi^0(f_1')\cdot P_{\chi^{-1}}^0(f_2')}{\langle f_1', f_2'\rangle_U} = \mathcal{L}(\pi, \chi)\prod_v \beta(f_{1,v}', f_{2,v}'),$$

where

$$\mathcal{L}(\pi, \chi) = \left(\frac{\#\operatorname{Pic}_{K/F}(\mathcal{O}_{c_1})}{2L(1,\eta)}\right)^2 \cdot \frac{2}{\operatorname{Vol}(X_U)} \cdot \frac{L\big(\tfrac{1}{2}, \pi, \chi\big)}{2L(1, \pi, \mathrm{ad})L(2, 1_F)^{-1}}.$$

It follows that

$$\frac{P_\chi^0(f_1)\cdot P_{\chi^{-1}}^0(f_2)}{\langle f_1, f_2\rangle_U} = \frac{P_\chi^0(f_1')\cdot P_{\chi^{-1}}^0(f_2')}{\langle f_1', f_2'\rangle_U} \cdot \prod_{v\in S} \frac{\beta(f_{1,v}, f_{2,v})}{\beta(f_{1,v}', f_{2,v}')}.$$

The variation formula follows immediately. □

3. Local theory

Notations. In this section, we denote by F a local field of characteristic zero, i.e., a finite field extension of $\mathbb{Q}_v$ for some place v of $\mathbb{Q}$. Denote by $|\cdot|$ the absolute value of F such that $d(ax) = |a|\,dx$ for a Haar measure dx on F. Take an element $\delta \in F^\times$ such that $\delta\mathcal{O}$ is the different of F over $\mathbb{Q}_v$ for v finite and $\delta = 1$ for v infinite. For F nonarchimedean, denote by $\mathcal{O}$ the ring of integers in F, ϖ a uniformizer, $\mathfrak{p}$ its maximal ideal, and q the cardinality of its residue field. Let $v : F \to \mathbb{Z} \cup \{\infty\}$ be the additive valuation on F such that $v(\varpi) = 1$. For μ a (continuous) character on $F^\times$, denote by $n(\mu)$ the conductor of μ, that is, the minimal nonnegative integer n such that μ is trivial on $(1 + \varpi^n \mathcal{O}) \cap \mathcal{O}^\times$. We will always use the additive character ψ on F and the Haar measure da on F as in Section 2, so that da is self-dual to ψ.

Denote by K a separable quadratic extension of F and, for any $t \in K$, write $t \mapsto \bar{t}$ for the nontrivial automorphism of K over F. We use similar notations as those for F with a subscript K. If F is nonarchimedean and K is nonsplit, denote by e the ramification index of K/F. Denote by $\mathrm{tr}_{K/F}$ and $\mathrm{N}_{K/F}$ the trace and norm maps from K to F, and let $D \in \mathcal{O}$ be an element such that $D\mathcal{O}$ is the relative discriminant of K over F. For an integer $c \geq 0$, denote by $\mathcal{O}_c$ the order $\mathcal{O} + \varpi^c \mathcal{O}_K$ in K. Let $\eta : F^\times \to \{\pm 1\}$ be the character associated to the extension K over F. Let B be a quaternion algebra over F. Let $\epsilon(B) = +1$ and $\delta(B) = 0$ if $B \cong M_2(F)$ is split, and $\epsilon(B) = -1$ and $\delta(B) = 1$ if B is division. Denote by G the algebraic group $B^\times$ over F, and we also write G for $G(F)$. We take the Haar measure on $F^\times$, $K^\times$ and $K^\times/F^\times$ as in Section 2. In particular, $\mathrm{Vol}(\mathcal{O}^\times, d^\times a) = \mathrm{Vol}(\mathcal{O}, da) = |\delta|^{1/2}$ and

$$\mathrm{Vol}(K^\times/F^\times) = \begin{cases} 2 & \text{if } F = \mathbb{R} \text{ and } K = \mathbb{C}, \\ |\delta|^{1/2} & \text{if } K \text{ is the unramified extension field of } F, \\ 2|D\delta|^{1/2} & \text{if } K/F \text{ is ramified.} \end{cases}$$

For F nonarchimedean and n a nonnegative integer, define the following subgroups of $\mathrm{GL}_2(\mathcal{O})$:

$$U_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}) \,\middle|\, c \in \mathfrak{p}^n \right\}, \quad U_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(n) \,\middle|\, d \in 1 + \varpi^n \mathcal{O} \right\}.$$

Let π be an irreducible admissible representation of G, which is always assumed to be generic if $G \cong \mathrm{GL}_2$. Denote by ω the central character of π and by $\sigma = \pi^{\mathrm{JL}}$ the Jacquet–Langlands correspondence of π to $\mathrm{GL}_2(F)$. Let χ be a character on $K^\times$ such that

$$\chi|_{F^\times} \cdot \omega = 1.$$

For F nonarchimedean, let n be the conductor of σ, i.e., the minimal nonnegative integer such that the invariant subspace $\sigma^{U_1(n)}$ is nonzero, and let c be the minimal nonnegative integer such that χ is trivial on $(1 + \varpi^c \mathcal{O}_K) \cap \mathcal{O}_K^\times$.

Denote by

$$L(s, \pi, \chi) := L(s, \sigma \times \pi_\chi) \quad \text{and} \quad \epsilon(s, \pi, \chi) := \epsilon(s, \sigma \times \pi_\chi, \psi)$$

the Rankin–Selberg L-factor and ϵ-factor of $\sigma \times \pi_\chi$, where π_χ is the representation on $\mathrm{GL}_2(F)$ constructed from χ via Weil representation. Denote by π_K the base change lifting of σ to $\mathrm{GL}_2(K)$; then we have

$$L(s,\pi,\chi)=L(s,\pi_K\otimes\chi),\quad \epsilon(s,\pi,\chi)=\eta(-1)\epsilon(s,\pi_K\otimes\chi,\psi_K)$$

Note that $\epsilon(\pi,\chi):=\epsilon\left(\frac{1}{2},\pi,\chi\right)$ equals ±1 and is independent of the choice of ψ. In the following, we denote by $L(s,\pi,\mathrm{ad}):=L(s,\sigma,\mathrm{ad})$ the adjoint L-factor of σ.

3A. ***Local toric integrals.*** Let $\mathcal{P}(\pi,\chi)$ denote the functional space

$$\mathcal{P}(\pi,\chi):=\mathrm{Hom}_{K^\times}(\pi,\chi^{-1}).$$

By a theorem of Tunnell [1983] and Saito [1993], the space $\mathcal{P}(\pi,\chi)$ has dimension at most one, and equals one if and only if

$$\epsilon(\pi,\chi)=\chi\eta(-1)\epsilon(B).$$

Lemma 3.1. *Let the pair (π,χ) be as above with $\epsilon(\pi,\chi)=\chi\eta(-1)\epsilon(B)$.*

(1) *If K is split or π is a principal series, then B is split.*

(2) *If $K/F=\mathbb{C}/\mathbb{R}$, σ is the discrete series of weight k, and $\chi(z)=|z|_{\mathbb{C}}^s(z/\sqrt{|z|_{\mathbb{C}}})^m$ with $s\in\mathbb{C}$ and $m\equiv k \pmod 2$, then B is split if and only if $m\geq k$.*

Furthermore, assume F is nonarchimedean. Then:

(3) *If K/F is nonsplit and σ is the special representation $\mathrm{sp}(2)\otimes\mu$ with μ a character of $F^\times$, then B is division if and only if $\mu_K\chi=1$ with $\mu_K:=\mu\circ N_{K/F}$.*

(4) *If K/F is inert and $c=0$, then B is split if and only if n is even.*

(5) *If K is nonsplit with $c\geq n$, then B is split.*

Proof. See [Tunnell 1983, Propositions 1.6, 1.7] for (1), (3), and [Gross 1988, Propositions 6.5, 6.3(2)] for (2), (4). We now give a proof of (5). If π is a principal series then, by (1), B is split. If σ is a supercuspidal representation then, by [Tunnell 1983, Lemma 3.1], B is split if $n(\chi)\geq ne/2+(2-e)$. It is then easy to check that, if $c\geq n$, this condition always holds. Finally, assume $\sigma=\mathrm{sp}(2)\otimes\mu$ with μ a character of $F^\times$. By (2), B is division if and only if $\mu_K\chi=1$. If μ is unramified, then $n=1$ and χ is ramified, which implies that B must be split. Assume μ is ramified; then $n=2n(\mu)$ and, by [Tunnell 1983, Lemma 1.8], $fn(\mu_K)=n(\mu)+n(\mu\eta)-n(\eta)$, where f is the residue degree of K/F. If K/F is unramified and $\mu_K\chi=1$, then $c=n(\mu_K)=n(\mu)=n/2$, a contradiction. If K/F is ramified and $\mu_K\chi=1$, then $2c-1\leq n(\mu_K)<2n(\mu)=n$, a contradiction again. Hence, if $c\geq n$, B is always split. $\square$

Assume that the pair (π,χ) is *essentially unitary*, in the sense that there exists a character $\mu=|\cdot|^s$ on $F^\times$ with $s\in\mathbb{C}$ such that both $\pi\otimes\mu$ and $\chi\otimes\mu_K^{-1}$ are unitary. In particular, if π is a local component of some global cuspidal representation, then (π,χ) is essentially unitary. Under such an assumption, we study the space

$\mathcal{P}(\pi, \chi)$ via the toric integral

$$\int_{F^\times\backslash K^\times} \langle \pi(t) f_1, f_2\rangle \chi(t)\, dt,$$

where $f_1 \in \pi$, $f_2 \in \widetilde{\pi}$, and $\langle \cdot, \cdot \rangle$ is any invariant pairing on $\pi \times \widetilde{\pi}$. The following basic properties for this toric integral are established in [Waldspurger 1985]:

- It is absolutely convergent for any $f_1 \in \pi$ and $f_2 \in \widetilde{\pi}$.
- $\mathcal{P}(\pi, \chi) \neq 0$ if and only if $\mathcal{P}(\pi, \chi) \otimes \mathcal{P}(\widetilde{\pi}, \chi^{-1}) \neq 0$, and in this case the above integral defines a generator of $\mathcal{P}(\pi, \chi) \otimes \mathcal{P}(\widetilde{\pi}, \chi^{-1})$.
- For $f_1 \in \pi$, $f_2 \in \widetilde{\pi}$ such that $\langle f_1, f_2\rangle \neq 0$, define the toric integral

  $$\beta(f_1, f_2) := \frac{L(1, \eta) L(1, \pi, \mathrm{ad})}{L(2, 1_F) L\big(\frac{1}{2}, \pi, \chi\big)} \int_{F^\times\backslash K^\times} \frac{\langle \pi(t) f_1, f_2\rangle}{\langle f_1, f_2\rangle} \chi(t)\, dt.$$

  Then $\beta(f_1, f_2) = 1$ in the case that $B = M_2(F)$, K is an unramified extension of F, both π and χ are unramified, dt is normalized such that $\mathrm{Vol}(\mathbb{O}_K^\times/\mathbb{O}^\times) = 1$, and f_1, f_2 are spherical.

For any pair (π, χ), β is invariant if we replace (π, χ) by $(\pi \otimes \mu, \chi \otimes \mu_K^{-1})$ for any character μ of $F^\times$. Therefore, we may assume π and χ are both unitary from now on and identify $(\widetilde{\pi}, \chi^{-1})$ with $(\bar{\pi}, \bar{\chi})$. Let $(\,,\,) : \pi \times \pi \to \mathbb{C}$ be the Hermitian pairing defined by $(f_1, f_2) = \langle f_1, \overline{f_2}\rangle$.

Let $\beta(f) := \beta(f, \bar{f})$. Then the functional space $\mathcal{P}(\pi, \chi)$ is nontrivial if and only if β is nontrivial. Assume $\mathcal{P}(\pi, \chi)$ is nonzero in the following. A nonzero vector f of π is called a *test vector* for $\mathcal{P}(\pi, \chi)$ if $\ell(f) \neq 0$ for some (thus any) nonzero $\ell \in \mathcal{P}(\pi, \chi)$ or, equivalently, if $\beta(f)$ is nonvanishing.

The notion of new vectors in an irreducible smooth admissible representation of $\mathrm{GL}_2(F)$ (see [Casselman 1973a] for F nonarchimedean and [Popa 2008] for F archimedean) can be viewed as a special case of test vectors. Let π be an irreducible admissible representation of $\mathrm{GL}_2(F)$. Recall the definition of *new vector line* in π, as follows. Denote by $T = K^\times$ the diagonal torus in $\mathrm{GL}_2(F)$. Write $T = ZT_1$ with $T_1 = \left\{\left(\begin{smallmatrix} * & \\ & 1\end{smallmatrix}\right)\right\}$.

- If F is nonarchimedean, then the new vector line is the invariant subspace $\pi^{U_1(n)}$.
- If F is archimedean, take U to be $O_2(\mathbb{R})$ if $F = \mathbb{R}$ and U_2 if $F = \mathbb{C}$. The new vector line consists of vectors $f \in \pi$ which are invariant under $T_1 \cap U$ with weight minimal.

It is known that new vectors satisfy the following properties:

(1) For any $s \in \mathbb{C}$, denote by ω_s the character on T such that $\omega_s|_Z = \omega$ and $\omega_s|_{T_1} = |\cdot|^{s-1/2}$. Then any nonzero f in the new vector line is a test vector for $\mathcal{P}(\pi, \omega_s^{-1})$.

(2) If $\mathcal{W}(\pi, \psi)$ is the Whittaker model of π with respect to ψ, then there is a vector W_0 in the new vector line, called the *normalized new vector* of π, such that the local zeta integral $|\delta|^{s-1/2}Z(s, W_0)$ equals $L(s, \pi)$.

3B. ***Local orders of quaternions.*** Assume F is nonarchimedean in this subsection.

First, in the case that the quaternion algebra B is split, given nonnegative integers m and k we want to classify all the $K^\times$ conjugacy classes of Eichler orders R in B with discriminant m such that $R \cap K = \mathcal{O}_k$. For this, identify B with the F-algebra $\mathrm{End}_F(K)$ which contains K as an F-subalgebra by multiplication. Recall that an Eichler order in B is the intersection of two maximal orders in B. Then, any Eichler order must be of the form $R(L_1, L_2) := R(L_1) \cap R(L_2)$, where L_i, $i = 1, 2$, are two $\mathcal{O}$-lattices in K and $R(L_i) := \mathrm{End}_{\mathcal{O}}(L_i)$. Denote by $d(L_1, L_2)$ the discriminant of $R(L_1, L_2)$. For any maximal order $R(L)$, there exists a unique integer $j \geq 0$ such that $L = t\mathcal{O}_j$ for some $t \in K^\times$. In fact, $\mathcal{O}_j = \{x \in K \mid xL \subset L\}$. Thus, any $K^\times$-conjugacy class of Eichler order contains an order of the form $R(\mathcal{O}_j, t\mathcal{O}_{j'})$ with $0 \leq j' \leq j$ and $t \in K^\times$ and the conjugacy class is exactly determined by the integers $j' \leq j$ and the class of $t \in K^\times$ modulo $F^\times \mathcal{O}_{j'}^\times$. The question is reduced to solving the equation with variables k' and $[t]$,

$$d(\mathcal{O}_k, t\mathcal{O}_{k'}) = m, \quad 0 \leq k' \leq k, \quad [t] \in K^\times / F^\times \mathcal{O}_{k'}^\times.$$

If $(k', [t])$ is a solution, then so is $\left(k', \left[\bar{t}\right]\right)$. A complete representative system (k', t) with $t \in K^\times$ of solutions to the above equation corresponds to a complete system $R(\mathcal{O}_k, t\mathcal{O}_{k'})$ for $K^\times$-conjugacy classes of Eichler orders R with discriminant m and $R \cap K = \mathcal{O}_k$.

Lemma 3.2. *Let m, k be nonnegative integers. Let $\tau \in K^\times$ be such that $\mathcal{O}_K = \mathcal{O}[\tau]$, if K is split then $\tau^2 - \tau = 0$, and if K is nonsplit then $v(\tau) = (e-1)/2$. Let $d := k + k' - m$. Then a complete representative system of (k', t) is the following*:

- *For $0 \leq m \leq 2k$, $k' \in [|m-k|, k]$ with d even, so $d \in 2 \cdot [0, k']$, and*

$$t = 1 + \varpi^{d/2}\tau u, \quad u \in (\mathcal{O}/\varpi^{k'-d/2}\mathcal{O})^\times.$$

  In the case $k' = k - m \geq 0$, the unique class of t is also represented by 1.

- *For split $K \cong F^2$ and $k + 1 \leq m$, $k' \in [0, \min(m-k-1, k)]$, so $d \in [k-m, 0)$, and*

$$t = (\varpi^{\pm d}u, 1), \qquad u \in (\mathcal{O}/\varpi^{k'}\mathcal{O})^\times.$$

- *For nonsplit K and $k + 1 \leq m \leq 2k + e - 1$, $k' = m - k - e + 1$, i.e., $d = 1 - e$, and*

$$t = \varpi x + \tau, \quad x \in \mathcal{O}/\varpi^{k'+e-2}\mathcal{O}.$$

Proof. The discriminant $d(L_1, L_2)$ of the Eichler order $R(L_1, L_2)$ can be computed as follows. Let e_i, e_i' be an $\mathbb{O}$-basis of L_i, $i = 1, 2$, and let $A = (a_{ij}) \in \mathrm{GL}_2(F)$ so that $A\binom{e_1}{e_1'} = \binom{e_2}{e_2'}$. Let $v : F \to \mathbb{Z} \cup \{\infty\}$ be the additive valuation on F such that $v(\varpi) = 1$. Let $\alpha = \min_{i,j} v(a_{ij})$ and $\beta = v(\det A)$. Then $d(L_1, L_2) = |2\alpha - \beta|$. Now solve the equation

$$d(\mathbb{O}_k, t\mathbb{O}_{k'}) = m, \quad k' \in [0, k], \ \ t \in K^\times/F^\times\mathbb{O}_{k'}^\times. \qquad \square$$

Define

$$c_1 = \begin{cases} 0 & \text{if } K \text{ is nonsplit and } c < n, \\ c & \text{otherwise.} \end{cases}$$

Lemma 3.3. *There exists an order R of discriminant n and $R \cap K = \mathbb{O}_{c_1}$ satisfying the condition that, if $nc_1 \neq 0$, then R is the intersection of two maximal orders R' and R'' of B such that $R' \cap K = \mathbb{O}_{c_1}$, $R'' \cap K = \mathbb{O}_{\max\{0, c_1 - n\}}$. Such an order is unique up to $K^\times$-conjugacy unless $0 < c_1 < n$. In the case $0 < c_1 < n$, there are exactly two $K^\times$-conjugacy classes which are conjugate to each other by a normalizer of $K^\times$.*

Proof. If $nc_1 = 0$, this is proved in [Gross 1988, Propositions 3.2, 3.4]. Now assume that $nc_1 \neq 0$; then B is split and one can apply Lemma 3.2. $\square$

Let R be an $\mathbb{O}$-order of B of discriminant n such that $R \cap K = \mathbb{O}_{c_1}$. Such an order R is called admissible for (π, χ) if the following conditions are satisfied:

(1) If $nc_1 \neq 0$ (thus B is split), then R is the intersection of two maximal orders R' and R'' of B such that $R' \cap K = \mathbb{O}_{c_1}$ and $R'' \cap K = \mathbb{O}_{\max\{0, c_1 - n\}}$.

(2) If $0 < c_1 < n$, fix an F-algebra isomorphism $K \cong F^2$ and identify B with $\mathrm{End}_F(K)$. The two $K^\times$-conjugacy classes of $\mathbb{O}$-orders in B satisfying the above condition (1) contain, respectively, the orders $R_i = R_i' \cap R_i''$, $i = 1, 2$ with $R_1' = R_2' = \mathrm{End}_{\mathbb{O}}(\mathbb{O}_c)$, $R_1'' = \mathrm{End}_{\mathbb{O}}((\varpi^{n-c}, 1)\mathbb{O}_K)$ and $R_2'' = \mathrm{End}_{\mathbb{O}}((1, \varpi^{n-c})\mathbb{O}_K)$. Let $\chi_1(a) = \chi(a, 1)$ and $\chi_2(b) = \chi(1, b)$. Then R is $K^\times$-conjugate to some R_i such that the conductor of χ_i is c_1.

Lemma 3.4. *If K is nonsplit, $n > 0$ and $c = 0$, then there is a unique admissible order R for (π, χ).*

Proof. Let $\mathbb{O}_B$ be a maximal order containing $\mathbb{O}_K$; then, by [Gross 1988, (3.3)], any admissible order for (π, χ) is $K^\times$-conjugate to $R := \mathbb{O}_K + I\mathbb{O}_B$, where I is a nonzero ideal of $\mathbb{O}_K$ such that $n = \delta(B) + \mathrm{length}_{\mathbb{O}}(\mathbb{O}_K/I)$. If B is nonsplit, then $\mathbb{O}_B$ is invariant under $B^\times$-conjugations and R is unique. Assume B is split. As $\mathbb{O}_K^\times \subset \mathbb{O}_B^\times$, $\mathbb{O}_B$ is invariant under $F^\times\mathbb{O}_K^\times$-conjugations. In particular, if K is unramified, then $K^\times = F^\times\mathbb{O}_K^\times$ and R is unique. Consider the case that K is ramified. Then $K^\times = F^\times\mathbb{O}_K^\times \cup \varpi_K F^\times\mathbb{O}_K^\times$ and it suffices to show that ϖ_K normalizes R. For this, embed K into $B = M_2(F)$ by $\varpi_K \mapsto \begin{pmatrix} \mathrm{tr}\,\varpi_K & 1 \\ -N\varpi_K & 0 \end{pmatrix}$ and take $\mathbb{O}_B = M_2(\mathbb{O})$. Then $R = \mathbb{O}_K + \varpi_K^n M_2(\mathbb{O})$. Note that $R_0(1) = \mathbb{O}_K + \varpi_K M_2(\mathbb{O})$ with $R_0(1) = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathfrak{p} & \mathbb{O} \end{pmatrix}$ the

Iwahori order in $M_2(F)$. Denote by m the maximal integer such that $2m \le n$. Then $R = \mathcal{O}_K + \varpi^{m-1}\varpi_K R_0(1)$ if n is even, and $R = \mathcal{O}_K + \varpi^m R_0(1)$ if n is odd. As ϖ_K normalizes $R_0(1)$, it also normalizes R and R is unique. $\square$

In the following, take an admissible $\mathcal{O}$-order R of B. Let $U = R^\times$ and define

$$\gamma := \frac{\mathrm{Vol}(U)}{\mathrm{Vol}(U_0(n))},$$

where the Haar measure is taken, so that $\mathrm{Vol}(\mathrm{GL}_2(\mathcal{O})) = L(2, 1_F)^{-1}|\delta|^2$ and $\mathrm{Vol}(\mathcal{O}_B^\times) = L(2, 1_F)^{-1}(q-1)^{-1}|\delta|^2$ if B is division.

Lemma 3.5. *If either R is not maximal or B is nonsplit, then*

$$\gamma = L(1, 1_F)(1 - e(R)q^{-1})$$

where $e(R)$ is the Eichler symbol of R, defined as follows: Let $\kappa(R) = R/\mathrm{rad}(R)$ with $\mathrm{rad}(R)$ the Jacobson radical of R and let κ be the residue field of F. Then

$$e(R) = \begin{cases} 1 & \text{if } \kappa(R) = \kappa^2, \\ -1 & \text{if } \kappa(R) \text{ is a quadratic field extension of } \kappa, \\ 0 & \text{if } \kappa(R) = \kappa. \end{cases}$$

Proof. Let R_0 be a maximal order of B containing R. Then we have the formula (for example, see [Yu 2013])

$$\frac{[R_0^\times : R^\times]}{[R_0 : R]} = \frac{|\kappa(R_0)^\times|/|\kappa(R^\times)|}{|\kappa(R_0)|/\kappa(R)|}.$$

If B is split and R is not maximal, then

$$[R_0 : R] = q^n, \quad \frac{|\kappa(R_0)^\times|}{|\kappa(R_0)|} = (1-q^{-2})(1-q^{-1}),$$
$$\frac{|\kappa(R)|}{|\kappa(R)^\times|} = (1-q^{-1})^{-1}(1-e(R)q)^{-1},$$

while, if B is division, then

$$[R_0 : R] = q^{n-1}, \quad \frac{|\kappa(R_0)^\times|}{|\kappa(R_0)|} = 1 - q^{-2}, \quad \frac{|\kappa(R)|}{|\kappa(R)^\times|} = (1-q^{-1})^{-1}(1-e(R)q)^{-1}.$$

Summing up,

$$[R_0^\times : R^\times] = (q-1)^{-\delta(B)} q^n (1-q^{-2})(1-e(R)q^{-1})^{-1},$$

where $\delta(B)$ equals 0 if B is split and 1 if B is ramified. Thus

$$\begin{aligned}\gamma^{-1}&=\frac{\mathrm{Vol}(U_0(n))}{\mathrm{Vol}(U)}=\frac{\mathrm{Vol}(\mathrm{GL}_2(\mathbb{O}))}{\mathrm{Vol}(R_0^\times)}\frac{[R_0^\times:U]}{[\mathrm{GL}_2(\mathbb{O}):U_0(n)]}\\&=\frac{L(2,1)^{-1}}{(q-1)^{-\delta(B)}L(2,1)^{-1}}\frac{(q-1)^{-\delta(B)}q^n(1-q^{-2})(1-e(R)q^{-1})^{-1}}{q^n(1-q^{-2})(1-q^{-1})^{-1}}\\&=L(1,1_F)^{-1}(1-e(R)q^{-1})^{-1}.\end{aligned}$$ □

3C. *Test vector spaces.*

Definition 3.6. Define $V(\pi,\chi)\subset\pi$ to be the subspace of vectors f satisfying the following conditions:

- For nonarchimedean F, K split or $c\ge n$, let $U\subset G$ be the compact subgroup defined before Lemma 3.5, then f is an ω-eigenform under U. Here, write $U=(U\cap Z)U'$ so that $U'=U$ if $cn=0$ and $U'\cong U_1(n)$ otherwise, and view ω as a character on $U\cap Z$ that extends to U by making it trivial on U'.
- For nonarchimedean F, K nonsplit and $c<n$, f is a χ^{-1}-eigenform under the action of $K^\times$.
- For archimedean F, let U be a maximal compact subgroup of G such that $U\cap K^\times$ is the maximal compact subgroup of $K^\times$; then f is a χ^{-1}-eigenform under $U\cap K^\times$ with weight minimal.

Proposition 3.7. *The dimension of $V(\pi,\chi)$ is one, and any nonzero vector in $V(\pi,\chi)$ is a test vector for $\mathcal{P}(\pi,\chi)$.*

Proof. If F is nonarchimedean, the claim that $\dim V(\pi,\chi)=1$ follows from local newform theory [Casselman 1973a]. Assume F is archimedean. If K is nonsplit, then $V(\pi,\chi)$ is the χ^{-1}-eigenline of $K^\times$. If K is split, then without loss of generality embed $K^\times$ into $G\cong\mathrm{GL}_2(F)$ as the diagonal matrices and decompose $K^\times=F^\times K^1$ so that the image of K^1 in G is $\begin{pmatrix}*&\\&1\end{pmatrix}$. Then $V(\pi,\chi)$ is the new vector line for $\pi\otimes\chi_1$ with $\chi_1:=\chi|_{K^1}$.

We shall prove any nonzero vector in $V(\pi,\chi)$ is a test vector in the next subsection by computing the toric integral β. □

Proposition 3.8. *Assume K/F is a quadratic extension of nonarchimedean fields with $n>0$ and $c=0$. Then $V(\pi,\chi)\subseteq\pi^{R^\times}$ and $\dim\pi^{R^\times}=\dim\pi^{\mathbb{O}_K^\times}\le 2$. The dimension of $\pi^{R^\times}$ is one precisely when K/F is inert or K/F is ramified and $\epsilon(\pi,\chi_1)\ne\epsilon(\pi,\chi_2)$, where χ_i, $i=1,2$, are unramified characters of $K^\times$ with $\chi_i|_{F^\times}\cdot\omega=1$.*

The proof of this proposition is in [Gross 1988; Gross and Prasad 1991] except for the case that π is a supercuspidal representation on $G=\mathrm{GL}_2(F)$. For this case, the proof in [Gross 1988, §7] is based on a character formula for odd residue characteristic. We next prove this case with arbitrary residue characteristic.

Let $R_0 = M_2(\mathcal{O})$ if $e = 1$ and the Iwahori order $\left(\begin{smallmatrix}\mathcal{O} & \mathcal{O}\\ \mathfrak{p} & \mathcal{O}\end{smallmatrix}\right)$ if $e = 2$. Fix an embedding of K into $M_2(F)$ such that $R_0 \cap K = \mathcal{O}_K$. Consider the filtration of open compact subgroups of G and $K^\times$

$$\mathcal{K}(r) := (1 + \varpi^r R_0) \cap \mathrm{GL}_2(\mathcal{O}), \quad \mathcal{E}(r) := \mathcal{K}(r) \cap K^\times, \quad r \geq 0.$$

Denote by m the minimal integer such that $2m + 1 \geq n$. The proof is based on:

Proposition 3.9. *For any integer $r \geq m$, $\pi^{\mathcal{K}(r)} = \pi^{\mathcal{E}(r)}$.*

Proof. Firstly, note that it is enough to prove Proposition 3.9 for the case π is minimal, that is, π has minimal conductor among its twists. In fact, assume that π is not minimal. Denote by n_0 the minimal conductor of π. Take a character μ so that $\pi_0 := \pi \otimes \mu$ has conductor n_0. Then, by [Tunnell 1978, Proposition 3.4], $n_0 \leq \max(n, 2n(\mu))$ with equality if π is minimal or $n \neq 2n(\mu)$. In particular, $n = 2m$ with $n(\mu) = m$. Hence, for any $r \geq m$, $\pi^{\mathcal{K}(r)} = \pi_0^{\mathcal{K}(r)}$ and $\pi^{\mathcal{E}(r)} = \pi_0^{\mathcal{E}(r)}$. Since $r \geq n_0/2$, one can apply the proposition to the minimal representation π_0.

Assume π is minimal in the following. Since $\mathcal{K}(r) \supset \mathcal{E}(r)$, $\pi^{\mathcal{K}(r)} \subset \pi^{\mathcal{E}(r)}$. It remains to prove that $\pi^{\mathcal{K}(r)}$ and $\pi^{\mathcal{E}(r)}$ have the same dimension. Denote by π_D the representation on $D^\times$, where D is the division quaternion algebra over F, so that the Jacquet–Langlands lifting of π_D to G is π. Then π_D has conductor n, that is, $\pi_D^{1+\varpi_D^{n-1}\mathcal{O}_D} = \pi_D$ and $\pi_D^{1+\varpi_D^{n-2}\mathcal{O}_D} = 0$, where ϖ_D is a uniformizer of D. Moreover, by [Carayol 1984, Proposition 6.5],

$$\dim \pi_D = \begin{cases} 2q^{m-1} & \text{if } n \text{ is even,} \\ q^m + q^{m-1} & \text{if } n \text{ is odd.} \end{cases}$$

For any $r \geq m$, $\mathcal{E}(r) \subset (1 + \varpi_D^{n-1}\mathcal{O}_D) \cap \mathcal{O}_K^\times$. Therefore, by the Tunnell–Saito theorem, if we denote by $\mathcal{X}(r)$ the set of all the characters μ on $K^\times$ such that $\mu|_{F^\times}\omega = 1$ and $\mu|_{\mathcal{E}(r)} = 1$, then

$$\dim \pi^{\mathcal{E}(r)} + \dim \pi_D = \sum_{\mu \in \mathcal{X}(r)} \dim \pi^\mu + \sum_\mu \dim \pi_D^\mu = \sum_{\mu \in \mathcal{X}(r)} (\dim \pi^\mu + \dim \pi_D^\mu) = \#\mathcal{X}(r)$$

and, on the other hand, the lemma below implies that

$$\dim \pi^{\mathcal{K}(r)} + \dim \pi_D = \#\mathcal{X}(r),$$

and then the equality $\dim \pi^{\mathcal{E}(r)} = \dim \pi^{\mathcal{K}(r)}$ holds. $\square$

Lemma 3.10. *Let π be minimal. For any integer $r \geq m$, we have the dimension formula*

$$\dim \pi^{\mathcal{K}(r)} = \begin{cases} q^r + q^{r-1} - 2q^{m-1} & \text{if } n \text{ is even and } e = 1, \\ q^r + q^{r-1} - (q^{m-1} + q^{m-2}) & \text{if } n \text{ is odd and } e = 1, \\ 2q^r - (q^m + q^{m-1}) & \text{if } n \text{ is odd and } e = 2, \\ 2q^r - 2q^{m-1} & \text{if } n \text{ is even and } e = 2. \end{cases}$$

Proof. For $r = m$ and $e = 1$, this formula occurs in [Casselman 1973b, Theorem 3]. We now use the method in [Casselman 1973b] to prove the dimension formula for the case n is even and $e = 1$, while the other cases are similar. Firstly, recall some basics about the Kirillov model. Let ψ be an unramified additive character of F. Associated to ψ, we can realize π on the space $C_c^\infty(F^\times)$ of Schwartz functions on the multiplicative group. For any $f \in C_c^\infty(F^\times)$ and any character μ of $\mathbb{O}^\times$, define

$$f_k(\mu) = \int_{\mathbb{O}^\times} f(u\varpi^k)\mu(u)\,du,$$

where we choose the Haar measure on $\mathbb{O}^\times$ so that the total measure is 1. Define further the formal power series

$$\hat{f}(\mu, t) = \sum_{k\in\mathbb{Z}} f_k(\mu)t^k,$$

which is actually a Laurent polynomial in t as f has compact support on $F^\times$. Because f is locally constant, this vanishes identically for all but a finite number of μ. By Fourier duality for $F^\times$, knowing $f(\mu, t)$ for all μ is equivalent to knowing f. For each μ, there is a formal power series $C(\mu, t)$ such that, for all $f \in C_c^\infty(F^\times)$,

$$(\pi(w)f)\hat{}(\mu, t) = C(\mu, t)\hat{f}(\mu^{-1}\omega_0^{-1}, t^{-1}z_0^{-1}),$$
$$C(\mu, t) = C_0(\mu)t^{n_\mu}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\omega_0 = \omega|_{\mathbb{O}^\times}$, $z_0 = \omega(\varpi)$ and n_μ is an integer, $n_\mu \le -2$. Moreover, if $\mu = 1$, then $-n_1 = n$. For any character μ of $\mathbb{O}^\times$,

$$-n_\mu = \begin{cases} n & \text{if } n(\mu) \le m, \\ 2n(\mu) & \text{if } n(\mu) > m. \end{cases}$$

In fact, if we take any character Ω on $F^\times$ such that $\Omega|_{\mathbb{O}^\times} = \mu$, denote $\pi' = \pi \otimes \Omega$ and $C'(\cdot\,,\cdot)$ the monomial that occurs in the above functional equation, then for any character ν on $\mathbb{O}^\times$, $C'(\nu, t) = C(\nu\mu, \Omega(\varpi)t)$. Therefore, $-n_\mu = n(\pi') = \max(n, 2n(\mu))$.

On the other hand, by [Casselman 1973b, Corollary to Lemma 2], for any $r \ge m$, the subspace $\pi^{\mathcal{K}(r)}$ is isomorphic to the space of all functions $\hat{f}(\mu, t)$ such that

(1) $\hat{f}(\mu, t) = 0$ unless $n(\mu) \le r$;

(2) for each μ, $f_k(\mu) = 0$ unless $-r \le k \le n_\mu + r$.

Summing up, for a given μ with conductor $n(\mu) \le r$, the dimension of the space consisting of those $\hat{f}(\mu, t)$ with $f \in \pi^{\mathcal{K}(r)}$ is

$$\begin{cases} 2(r-m)+1 & \text{if } n(\mu) \le m, \\ 2(r-n(\mu))+1 & \text{if } n(\mu) > m. \end{cases}$$

Therefore,

$$\begin{aligned}\dim \pi^{\mathcal{K}(r)} &= (q^m - q^{m-1})(2(r-m)+1) + \sum_{m<k\le r} (q^k - 2q^{k-1} + q^{k-2})(2(r-k)+1)\\ &= q^r + q^{r-1} - 2q^{m-1}.\end{aligned}$$

□

Proof of Proposition 3.8. Note that $R^\times = \mathbb{O}_K^\times \mathcal{K}(m)$ unless K is ramified with n even and, once this equation holds, Proposition 3.8 follows directly from Proposition 3.9. So consider the case K is ramified with n even. Here, $R^\times = \mathbb{O}_K^\times \mathcal{K}'(m)$ with $\mathcal{K}'(m) = 1 + \varpi_K^{2m-1} R_0$. We want to show $\pi^{\mathcal{K}'(m)} = \pi^{\mathscr{E}'(m)}$ with $\mathscr{E}'(m) = \mathcal{K}'(m) \cap K^\times$, and Proposition 3.8 then holds. By [Tunnell 1983, Proposition 3.5], π is not minimal. Take a character μ such that $\pi_0 = \pi \otimes \mu$ has minimal conductor n_0. Then $n(\mu) = m$. Apply Proposition 3.9:

$$\pi^{\mathcal{K}'(m)} = \pi_0^{\mathcal{K}'(m)} \supset \pi_0^{\mathcal{K}(m-1)} = \pi_0^{\mathscr{E}(m-1)}.$$

We claim that $\pi_0^{\mathscr{E}(m-1)} = \pi_0^{\mathscr{E}'(m)}$. If so, $\pi_0^{\mathscr{E}(m-1)} = \pi^{\mathscr{E}'(m)}$ and then $\pi^{\mathcal{K}'(m)} = \pi^{\mathscr{E}'(m)}$. To prove this, note that $\mathscr{E}'(m) \subset \mathscr{E}(m-1) \subset 1 + \varpi_D^{n_0-1} \mathbb{O}_D$. Using the Tunnell–Saito theorem,

$$\dim \pi_0^{\mathscr{E}(m-1)} + \dim \pi_{0,D} = \#\mathscr{X}(m-1), \qquad \dim \pi_0^{\mathscr{E}'(m)} + \dim \pi_{0,D} = \#\mathscr{X}'(m),$$

where the set $\mathscr{X}(m-1)$ consists of characters Ω of $K^\times$ such that $\Omega|_{F^\times} \cdot \omega_{\pi_0} = 1$ with $\Omega|_{\mathscr{E}(m-1)} = 1$, and the set $\mathscr{X}'(m)$ is defined similarly. As they are nonempty,

$$\#\mathscr{X}(m-1) = \#K^\times / F^\times \mathscr{E}(m-1) = \#K^\times / F^\times \mathscr{E}'(m) = \#\mathscr{X}'(m).$$

Thus, $\pi_0^{\mathscr{E}(m-1)} = \pi_0^{\mathscr{E}'(m)}$ and the proof is complete. □

3D. ***Local computations.*** Let $\mathcal{W}(\sigma, \psi)$ be the Whittaker model of σ with respect to ψ and recall that we have an invariant Hermitian form on $\mathcal{W}(\sigma, \psi)$ defined by

$$(W_1, W_2) := \int_{F^\times} W_1\left[\begin{pmatrix} a & \\ & 1\end{pmatrix}\right] \overline{W_2\left[\begin{pmatrix} a & \\ & 1\end{pmatrix}\right]}\, d^\times a.$$

For any $W \in \sigma$, denote

$$\alpha(W) = \frac{(W, W)}{L(1, \sigma, \mathrm{ad}) L(1, 1_F) L(2, 1_F)^{-1}}.$$

Proposition 3.11. *Denote by W_0 the normalized new vector of σ. If F is nonarchimedean, then*

$$\alpha(W_0)|\delta|^{1/2} = \begin{cases} 1 & \textit{if } \sigma \textit{ is unramified,}\\ L(2, 1_F) L(1, 1_F)^{-1} L(1, \sigma, \mathrm{ad})^{-\delta_\sigma} & \textit{otherwise,}\end{cases}$$

where $\delta_\sigma \in \{0, 1\}$ *and equals* 0 *precisely when* σ *is a subrepresentation of the induced representation* $\mathrm{Ind}(\mu_1, \mu_2)$ *with at least one* μ_i *unramified. If* $F = \mathbb{R}$ *and* σ *is the discrete series* $\mathcal{D}_\mu(k)$, *then* $\alpha(W_0) = 2^{-k}$.

The proposition follows from the explicit form of W_0. If F is nonarchimedean, W_0 is the one in the new vector line such that

$$W_0\left[\begin{pmatrix} \delta^{-1} & \\ & 1 \end{pmatrix}\right] = |\delta|^{-1/2}$$

and we have the following list (see [Schmidt 2002, p. 23]):

(1) If $\sigma = \pi(\mu_1, \mu_2)$ is a principal series, then

$$W_0\left[\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right] = \begin{cases} |y|^{1/2} \displaystyle\sum_{\substack{k+l=v(y\delta) \\ k,l\geq 0}} \mu_1(\varpi)^k \mu_2(\varpi)^l 1_{\mathcal{O}}(\delta y) & \text{if } n(\mu_1) = n(\mu_2) = 0, \\ |y|^{1/2}\mu_1(\delta y) 1_{\mathcal{O}}(\delta y) & \text{if } n(\mu_1) = 0,\, n(\mu_2) > 0, \\ |\delta|^{-1/2} 1_{\mathcal{O}^\times}(\delta y) & \text{if } n(\mu_1) > 0,\, n(\mu_2) > 0. \end{cases}$$

(2) If $\sigma = \mathrm{sp}(2) \otimes \mu$ is a special representation, then

$$W_0\left[\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right] = \begin{cases} |\delta|^{-1/2}\mu(\delta y)|\delta y| 1_{\mathcal{O}}(\delta y) & \text{if } n(\mu) = 0, \\ |\delta|^{-1/2} 1_{\mathcal{O}^\times}(\delta y) & \text{if } n(\mu) > 0. \end{cases}$$

(3) If σ is supercuspidal, then

$$W_0\left[\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right] = |\delta|^{-1/2} 1_{\mathcal{O}^\times}(\delta y).$$

If $F = \mathbb{R}$ and σ is the discrete series $\mathcal{D}_\mu(k)$, then

$$W_0\left[\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right] = |y|^{k/2} e^{-2\pi|y|}$$

and, in general, for archimedean cases it is expressed by the Bessel function [Popa 2008]. For $F = \mathbb{R}$ and σ a unitary discrete series of weight k, let $W \in \mathcal{W}(\sigma, \psi)$ be the vector satisfying

$$W\left[\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right] = |y|^{k/2} e^{-2\pi|y|} 1_{\mathbb{R}_+^\times}(y).$$

Then W can be realized as a local component of a Hilbert newform and

$$(W_0, W_0) = 2(W, W), \qquad Z(s, W) = \tfrac{1}{2} L(s, \sigma).$$

Proposition 3.12. *If F is nonarchimedean, let f be a nonzero vector in the one-dimensional space $V(\pi, \chi)$; then $\beta(f)|D\delta|^{-1/2}$ equals*:

$$\begin{cases} 1 & \text{if } n = c = 0, \\ L(1,\eta)^2|\varpi^c| & \text{if } n = 0 \text{ and } c > 0, \\ \dfrac{L(1,1_F)}{L(2,1_F)} L(1,\pi,\mathrm{ad})^{\delta_\pi} & \text{if } n > 0,\ c = 0 \text{ and } K \text{ is split}, \\ \dfrac{L(1,1_F)}{L(2,1_F)} L(1,\eta)^2|\varpi^c| \dfrac{L(1,\pi,\mathrm{ad})^{\delta_\pi}}{L\big(\frac{1}{2},\pi,\chi\big)} & \text{if } nc > 0, \text{ either } K \text{ is split or } c \geq n, \\ e(1-q^{-e}) \dfrac{L(1,\pi,\mathrm{ad})}{L\big(\frac{1}{2},\pi,\chi\big)} & \text{if } n > c \text{ and } K \text{ is nonsplit}, \end{cases}$$

which is independent of the choice of $f \in V(\pi, \chi)$.

The proof of Proposition 3.12 is reduced to computing the integral

$$\beta^0 = \int_{F^\times \backslash K^\times} \frac{(\pi(t)f, f)}{(f, f)} \chi(t)\, dt,$$

where f is any nonzero vector in $V(\pi, \chi)$.

In the case that $n > c$ and K is nonsplit, f is a χ^{-1}-eigenform and it is easy to see that $\beta^0 = \mathrm{Vol}(F^\times \backslash K^\times)$.

From now on assume $n \leq c$ or K is split. Then $B = M_2(F)$ by Lemma 3.1(5). Recall that the space $V(\pi, \chi)$ depends on a choice of an admissible order R for (π, χ). Let f be a test vector in $V(\pi, \chi)$ defined by R. For any $t \in K^\times$, $f' := \pi(t)f$ is a test vector defined by the admissible order $R' = tRt^{-1}$. It is easy to check that $\beta(f') = \beta(f)$. Thus, for a $K^\times$-conjugacy class of admissible orders, we can pick a particular order to compute β^0. There is a unique $K^\times$-conjugacy class of admissible orders except for the exceptional case $0 < c_1 < n$ and $n(\chi_1) = n(\chi_2) = c$. In this case, there are exactly two $K^\times$-conjugacy classes of admissible orders, which are conjugate to each other by a normalizer of $K^\times$ in $B^\times$.

Any admissible order (in the case $n \leq c$ or K is split) is an Eichler order of discriminant n, i.e., conjugate to $R_0(n) := \left(\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^n & \mathcal{O} \end{smallmatrix}\right)$. Choose an embedding of K into $M_2(F)$ as follows, so that $R_0(n)$ is an admissible order for (π, χ):

(1) If K is split, fix an F-algebra isomorphism $K \cong F^2$. If $c \geq n$ or $n(\chi_1) = c$, embed K into $M_2(F)$ by

$$\iota_1 : (a, b) \longmapsto \gamma_c^{-1} \begin{pmatrix} a & \\ & b \end{pmatrix} \gamma_c, \qquad \gamma_c = \begin{pmatrix} 1 & \varpi^{-c} \\ & 1 \end{pmatrix}.$$

If $n(\chi_1) < c < n$, embed K into $M_2(F)$ by

$$\iota_2 : (a, b) \longmapsto \gamma_c^{-1} \begin{pmatrix} b & \\ & a \end{pmatrix} \gamma_c.$$

Note that, for any $t \in K^\times$, $\iota_1(t) = j\iota_2(t)j^{-1}$ with $j = \gamma_c^{-1} w \gamma_c$ and $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

(2) If K is a field, take $\tau \in \mathcal{O}_K$ such that $\mathcal{O}_K = \mathcal{O}[\tau]$ and, if K/F is ramified, then τ is a uniformizer. Embed K into $M_2(F)$ by

$$a + b\tau \longmapsto \gamma_c^{-1} \begin{pmatrix} a + b\operatorname{tr}\tau & b\mathrm{N}\tau \\ -b & a \end{pmatrix} \gamma_c, \quad \text{where } \gamma_c = \begin{pmatrix} \varpi^c N\tau & \\ & 1 \end{pmatrix}.$$

Assume $K \cong F^2$. If $n(\chi_1) < c < n$,

$$\beta^0 = \int_{F^\times \backslash K^\times} \frac{(\pi(\iota_2(t))W_0, W_0)}{(W_0, W_0)} \chi(t)\, dt = \int_{F^\times \backslash K^\times} \frac{(\pi(\iota_1(t))W_0, W_0)}{(W_0, W_0)} \bar{\chi}(t)\, dt,$$

where $\bar{\chi}_1 = \chi_2$, $\bar{\chi}_2 = \chi_1$ and $n(\bar{\chi}_1) = n(\chi_2) = c$. We reduce to the case $c \geq n$ or $n(\chi_1) = c$. For the exceptional case, if we take $\pi(j)W_0$ as a test vector, then

$$\beta^0 = \int_{F^\times \backslash K^\times} \frac{(\pi(\iota_1(t)j)W_0, \pi(j)W_0)}{(W_0, W_0)} \chi(t)\, dt = \int_{F^\times \backslash K^\times} \frac{(\pi(\iota_1(t))W_0, W_0)}{(W_0, W_0)} \chi(\bar{t})\, dt$$

with $n(\bar{\chi}_1) = n(\chi_2) = c$. Thus, even for the exceptional case, we only need to consider W_0 as a test vector. Thus,

$$\begin{aligned} \beta^0 &= (W_0, W_0)^{-1} \iint_{(F^\times)^2} \pi(\gamma_c)W_0\left[\begin{pmatrix} ab & \\ & 1 \end{pmatrix}\right] \overline{\pi(\gamma_c)W_0\left[\begin{pmatrix} b & \\ & 1 \end{pmatrix}\right]} \chi_1(a)\, d^\times b\, d^\times a \\ &= (W_0, W_0)^{-1} \left| Z\left(\tfrac{1}{2}, \pi(\gamma_c)W_0, \chi_1\right)\right|^2. \end{aligned}$$

If $c = 0$, $Z\left(\frac{1}{2}, W_0, \chi_1\right) = \chi_1(\delta)^{-1} L\left(\frac{1}{2}, \pi \otimes \chi_1\right)$ and so $\beta^0 = (W_0, W_0)^{-1} L\left(\frac{1}{2}, \pi, \chi\right)$. If $c > 0$, then

$$\begin{aligned} Z\left(\tfrac{1}{2}, \pi(\gamma_c)W_0, \chi_1\right) &= \int_{F^\times} W_0\left[\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right] \psi(a\varpi^{-c}) \chi_1(a)\, d^\times a \\ &= \sum_{k \in \mathbb{Z}} W_0\left[\begin{pmatrix} \varpi^k & \\ & 1 \end{pmatrix}\right] \int_{\varpi^k \mathcal{O}^\times} \psi(a\varpi^{-c}) \chi_1(a)\, d^\times a. \end{aligned}$$

Assume $n(\chi_1) = c$; then the integral $\int_{\varpi^k \mathcal{O}^\times} \psi(a\varpi^{-c})\chi_1(a)\, d^\times a$ vanishes unless $k = -v(\delta)$, while

$$\left| \int_{\delta^{-1}\mathcal{O}^\times} \psi(a\varpi^{-c}) \chi_1(a)\, d^\times a \right| = L(1, 1_F)|\delta|^{1/2} q^{-c/2}.$$

Thus,

$$\beta^0 = (W_0, W_0)^{-1} L(1, 1_F)^2 q^{-c}.$$

Assume $c \geq n$ and $n(\chi_1) < c$. Let j be a normalizer of $K^\times$ with $jt = \bar{t}j$ for any $t \in K^\times$. As $c \geq n$, there exists some $t_0 \in K^\times$ such that $t_0 U_0(n) t_0^{-1} = jU_0(n)j^{-1}$

and $\pi(t_0)W_0, \pi(j)W_0$ are in the same line. Thus,

$$\beta^0 = \int_{F^\times\backslash K^\times} \frac{(\pi(t)W_0, W_0)}{(W_0, W_0)}\,\overline{\chi}(t)\,dt = (W_0, W_0)^{-1}L(1, 1_F)^2 q^{-c}$$

as $n(\overline{\chi}_1) = n(\chi_2) = c$.

Remark. Assume $n(\chi_1) < c < n$ and R is the intersection of two maximal orders R' and R'' with $R' \cap K = \mathcal{O}_c$ and $R'' \cap K = \mathcal{O}_K$. If R is not admissible, then the toric integral for ω-eigenforms f under $R^\times$ must vanish if $c > 1$. In the case $c = 1$, so that $n(\chi_1) = 0$,

$$\int_{F^\times\backslash K^\times} \frac{(\pi(\iota_1(t))W_0, W_0)}{(W_0, W_0)}\,\chi(t)\,dt = (W_0, W_0)^{-1}L(1, 1_F)^2 q^{-2}.$$

It remains to consider the case K is a field and $c \geq n$. Let $\Psi(g)$ denote the matrix coefficient:

$$\Psi(g) := \frac{(\pi(g)W_0, W_0)}{(W_0, W_0)}, \quad g \in \mathrm{GL}_2(F).$$

Then

$$\beta^0 = \frac{\mathrm{Vol}(K^\times/F^\times)}{\#K^\times/F^\times\mathcal{O}_c^\times} \sum_{t\in K^\times/F^\times\mathcal{O}_c^\times} \Psi(t)\chi(t).$$

In the case $c = 0$, π is unramified. Furthermore, if K/F is unramified, then $\beta^0 = \mathrm{Vol}(K^\times/F^\times) = |\delta|^{1/2}$ and, if K/F is ramified, $\beta^0 = |D\delta|^{1/2}(1 + \Psi(\tau)\chi(\tau))$, where $\Psi(\tau)$ is expressed by the MacDonald polynomial and one has $\beta(f) = |D\delta|^{1/2}$. It remains to consider the case $c > 0$. Denote

$$S_i = \{1 + b\tau \mid b \in \mathcal{O}/\mathfrak{p}^c,\ v(b) = i\}, \quad 0 \leq i \leq c-1,$$

and

$$S' = \begin{cases} \{a + \tau \mid a \in \mathfrak{p}/\mathfrak{p}^c\} & \text{if } e = 1, \\ \{a\varpi + \tau \mid a \in \mathcal{O}/\mathfrak{p}^c\} & \text{if } e = 2. \end{cases}$$

Then a complete representatives of $K^\times/F^\times\mathcal{O}_c^\times$ can be taken as

$$\{1\} \sqcup \bigsqcup_i S_i \sqcup S'.$$

Note that Ψ is a function on $U_1(n)\backslash G/U_1(n)$. The following observation is key to our computation: the images of S_i, $0 \leq i \leq c-1$, and S' under the natural map

$$\mathrm{pr} : K^\times \to U_1(n)\backslash G/U_1(n)$$

are constant. Precisely,

$$\mathrm{pr}(S_i) = \left[\begin{pmatrix} 1 & \varpi^{i-c} \\ & 1 \end{pmatrix}\right], \quad \mathrm{pr}(S') = \left[\begin{pmatrix} & \varpi^{-c} \\ -\varpi^{c+e-1} & \end{pmatrix}\right].$$

From this, it follows that

$$\sum_{t\in K^\times/F^\times\mathbb{O}_c^\times}\Psi(t)\chi(t)=1+\sum_{i=0}^{c-1}\Psi_i\sum_{t\in S_i}\chi(t)+\Psi'\sum_{t\in S'}\chi(t),$$

where Ψ_i (resp. Ψ') are the valuations of $\Psi(t)$ on S_i (resp. S').

Assume the central character ω is unramified; then we may take $\omega=1$. If $e=c=1$, we have

$$\sum_{t\in S_0}\chi(t)=-\chi(\tau)-1 \quad\text{and}\quad \sum_{t\in S'}\chi(t)=\chi(\tau).$$

Otherwise,

$$\sum_{t\in S_i}\chi(t)=\begin{cases}0 & \text{if } c>1 \text{ and } 0\le i\le c-2,\\ -1 & \text{if } i=c-1,\end{cases}\qquad\text{and}\qquad \sum_{t\in S'}\chi(t)=0.$$

Therefore,

$$\sum_{t\in K^\times/F^\times\mathbb{O}_c^\times}\Psi(t)\chi(t)=\begin{cases}1+(-\chi(\tau)-1)\Psi_0+\chi(\tau)\Psi' & \text{if } e=c=1,\\ 1-\Psi_{c-1} & \text{otherwise.}\end{cases}$$

Note that, if $e=1$, then $\begin{pmatrix} & \varpi^{-c}\\ -\varpi^c & \end{pmatrix}$ equals $\begin{pmatrix}1 & \varpi^{-c}\\ & 1\end{pmatrix}$ in $ZU_1(n)\backslash G/U_1(n)$ and, since $\omega=1$, $\Psi'=\Psi_0$. We obtain

$$\sum_{t\in K^\times/F^\times\mathbb{O}_c^\times}\Psi(t)\chi(t)=1-\Psi_{c-1}$$

and reduce to the evaluation of Ψ_{c-1}. If $n=0$, the matrix coefficient Ψ_{c-1} is expressed by the MacDonald polynomial. In particular, if the Satake parameter of π is (α,α^{-1}), then

$$1-\Psi_{c-1}=\frac{(1-\alpha^2q^{-1})(1-\alpha^{-2}q^{-1})}{1+q^{-1}}.$$

If $n=1$, then $\pi=\mathrm{sp}(2)\otimes\mu$ with μ an unramified quadratic character on $F^\times$. By definition,

$$\begin{aligned}\Psi_{c-1}&=|\delta|^{1/2}L(1,\pi,\mathrm{ad})^{-1}\int_{F^\times}W_0\left[\begin{pmatrix}a & \\ & 1\end{pmatrix}\begin{pmatrix}1 & \varpi^{-1}\\ & 1\end{pmatrix}\right]\overline{W_0\left[\begin{pmatrix}a & \\ & 1\end{pmatrix}\right]}\,d^\times a\\ &=|\delta|^{3/2}L(1,\pi,\mathrm{ad})^{-1}\int_{\varpi^{-n(\psi)}\mathbb{O}}\psi(a\varpi^{-1})|a|^2\,d^\times a\\ &=|\delta|^{3/2}L(1,\pi,\mathrm{ad})^{-1}(-q^{-1})L(1,\pi,\mathrm{ad})|\delta|^{-3/2}=-q^{-1}.\end{aligned}$$

If $n\ge 2$, then

$$\Psi_{c-1}=|\delta|^{-1/2}\int_{\varpi^{-1-n(\psi)}\mathbb{O}^\times}\psi(x)\,d^\times x=-q^{-1}L(1,1_F).$$

With these results, we obtain

$$\beta^0 = \frac{\mathrm{Vol}(K^\times/F^\times)}{\# K^\times/F^\times \mathbb{O}_c^\times} \times \begin{cases} \dfrac{L(1, 1_F)}{L(1, \pi, \mathrm{ad})(1+q^{-1})} & \text{if } n = 0, \\ 1+q^{-1} & \text{if } n = 1, \\ L(1, 1_F) & \text{if } n \geq 2. \end{cases}$$

Finally, we deal with the case that ω is ramified. As above, it is routine to check that Ψ_i for $i < c-1$ and Ψ' are vanishing. Moreover, $\Psi_{c-1} = 0$ if and only if $\delta_\pi = 0$ and, for $\delta_\pi = 1$,

$$\Psi_{c-1} = -q^{-1} L(1, 1_F).$$

By the definition of δ_π, if $\delta_\pi = 1$ then $c \geq 2$ and $n(\omega) < n \leq c$. Thus, for $\delta_\pi = 1$,

$$\begin{aligned} 0 &= \sum_{t \in 1+\varpi^{c-1}\mathbb{O}_K/1+\varpi^c \mathbb{O}_K} \chi(t) \\ &= \sum_{t \in 1+\varpi^{c-1}\mathbb{O}_K/(1+\varpi^{c-1}\mathbb{O})(1+\varpi^c\mathbb{O}_K)} \chi(t) \sum_{a \in 1+\varpi^{c-1}\mathbb{O}/1+\varpi^c\mathbb{O}} \omega^{-1}(a) \\ &= q \sum_{b \in \mathfrak{p}^{c-1}/\mathfrak{p}^c} \chi(1+b\tau). \end{aligned}$$

Therefore, if $\delta_\pi = 1$, then $\sum_{t \in S_{c-1}} \chi(t) = -1$ and

$$\beta^0 = \frac{\mathrm{Vol}(K^\times/F^\times)}{\# K^\times/F^\times \mathbb{O}_c^\times} \times \begin{cases} 1 & \text{if } \delta_\pi = 0, \\ L(1, 1_F) & \text{if } \delta_\pi = 1. \end{cases}$$

The proof of Proposition 3.12 is now complete. □

We finish our discussions of $\alpha(W_0)$, $\beta(f)$ and γ with Lemmas 3.13 and 3.14.

Lemma 3.13. *Let F be nonarchimedean and f a nonzero element in $V(\pi, \chi)$; then*

$$\alpha(W_0)\beta(f)\gamma|D|^{-1/2} = 2^{\delta(\Sigma_D)} L\big(\tfrac{1}{2}, \pi, \chi\big)^{-\delta(\Sigma)} L(1, \eta)^{2\delta(c_1)} q^{-c_1},$$

where these $\delta \in \{0, 1\}$ are given by:

- $\delta(\Sigma_D) = 1$ *if and only if K is ramified, $n > 0$ and $c < n$*;
- $\delta(\Sigma) = 1$ *if and only if $n > 0$, K is either ramified or $c > 0$ and, if $n = 1$, then $c \geq n$*;
- $\delta(c_1) = 1$ *if and only if $c_1 \neq 0$.*

Proof. We have computed $\alpha(W_0)$ in Proposition 3.11 and $\beta(f)$ in Proposition 3.12. When $n > 0$, by Lemma 3.5, $\gamma = L(1, 1_F)(1 - e(R)q^{-1})$ and it suffices to compute $e(R)$:

(i) $e(R) = 1$ and $\gamma = 1$ if K is split, or if K is ramified, $n = 1$ and B is split, or if K is nonsplit and $c \geq n$;

(ii) $e(R) = -1$ and $\gamma = L(1, 1_F)(1+q^{-1})$ if K is inert and $c < n$, or if K is ramified, $n = 1$, B is division and $c = 0$;

(iii) $e(R) = 0$ and $\gamma = L(1, 1_F)$ if K is ramified, $n \geq 2$ and $c < n$. $\square$

For archimedean places, using Barnes' lemma we have the following list for (W_0, W_0) (see [Tadić 2009] for the classification of unitary dual of $\mathrm{GL}_2(F)$):

(1) Assume $F = \mathbb{R}$, σ is the infinite-dimensional subquotient of the induced representation $\mathrm{Ind}(\mu_1, \mu_2)$, where $\mu_i(a) = |a|^{s_i}\,\mathrm{sgn}(a)^{m_i}$ with $s_i \in \mathbb{C}$ and $m_i \in \{0, 1\}$. Let $k = s_1 - s_2 + 1$, $\mu = s_1 + s_2$.

(a) If $\sigma = \mathcal{D}_\mu(k)$ is the discrete series with $k \geq 2$, then (W_0, W_0) equals

$$2(4\pi)^{-k}\Gamma(k).$$

(b) If $\sigma = \pi(\mu_1, \mu_2)$ is a principal series, then (W_0, W_0) equals

$$\pi^{-1-m_1-m_2}\Gamma\big(\tfrac{1}{2}(1+2m_1)\big)\Gamma\big(\tfrac{1}{2}(1+2m_2)\big)B\big(\tfrac{1}{2}(k+m_1+m_2), \tfrac{1}{2}(2-k+m_1+m_2)\big),$$

where $B(x, y) := \Gamma(x)\Gamma(y)\Gamma(x+y)^{-1}$ is the beta function.

(2) Assume $F = \mathbb{C}$, $\sigma = \pi(\mu_1, \mu_2)$ is a principal series with $\mu_i(z) = |z|^{s_i}\Big(\dfrac{z}{\sqrt{|z|_{\mathbb{C}}}}\Big)^{m_i}$ and $s_i \in \mathbb{C}$ and $m_i \in \mathbb{Z}$; then (W_0, W_0) equals

$$8(2\pi)^{-1-|m_1|-|m_2|}\Gamma(1+|m_1|)\Gamma(1+|m_2|) \\ \times B\big(1+s_1-s_2+\tfrac{1}{2}(|m_1|+|m_2|), 1-s_1+s_2+\tfrac{1}{2}(|m_1|+|m_2|)\big).$$

For a pair (π, χ), define

$$C(\pi, \chi) = \begin{cases} 2^{-1}\pi(W_0, W_0)^{-1} & \text{if } K/F = \mathbb{C}/\mathbb{R}, \\ (W_0', W_0')(W_0, W_0)^{-1} & \text{if } K = F^2. \end{cases}$$

In the split case, W_0' is the new vector of $\pi \otimes \chi_1$, where K is embedded into $M_2(F)$ diagonally and $\chi_1(a) = \chi\big(\big(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\big)\big)$.

Lemma 3.14. *For F archimedean, let f be a nonzero vector in $V(\pi, \chi)$; then*

$$\alpha(W_0)\beta(f) = C(\pi, \chi)^{-1}\begin{cases} L\big(\tfrac{1}{2}, \pi, \chi\big)^{-1} & \text{if } K/F = \mathbb{C}/\mathbb{R}, \\ 1 & \text{if } K = F^2. \end{cases}$$

In particular, if $\sigma = \mathcal{D}_\mu(k)$ is a discrete series with weight k, then

$$C(\pi, \chi) = \begin{cases} 4^{k-1}\pi^{k+1}\Gamma(k)^{-1} & \text{if } K = \mathbb{C}, \\ 1 & \text{if } K = \mathbb{R}^2. \end{cases}$$

Proof. By definition,

$$\alpha(W_0)\beta(f) = \frac{L(1, \eta)}{L(1, 1_F)}\, L\big(\tfrac{1}{2}, \pi, \chi\big)^{-1}(W_0, W_0)\beta^0$$

with

$$\beta^0 = \int_{F^\times\backslash K^\times} \frac{(\pi(t)f, f)}{(f, f)}\,\chi(t)\,dt, \qquad f \in V(\pi,\chi).$$

If $K/F = \mathbb{C}/\mathbb{R}$, then $\beta^0 = \mathrm{Vol}(K^\times/F^\times) = 2$. If K is split, taking $f = W_0'$, then $\beta^0 = L\big(\frac{1}{2}, \pi, \chi\big)(W_0', W_0')^{-1}$. If $\sigma = \mathscr{D}_\mu(k)$, the value for (W_0, W_0) is given in (1a) in the above list and we note that, if $K = \mathbb{R}^2$, then $(W_0', W_0') = (W_0, W_0)$ as, for any χ_1, $\pi \otimes \chi_1$ and π have the same weight. □

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