A MYERS-TYPE THEOREM AND COMPACT RICCI SOLITONS

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Abstract. Let the Ricci curvature of a compact Riemannian manifold be greater, at every point, than the Lie derivative of the metric with respect to some fixed smooth vector field. It is shown that the fundamental group then has only finitely many conjugacy classes. This applies, in particular, to all compact shrinking Ricci solitons.

§0. Introduction

Myers's classical theorem [1, Theorem 6.51] implies that any compact Riemannian manifold with positive Ricci curvature has a finite fundamental group. Its standard proof uses a diameter estimate. In this note we observe that an easy upper bound (4) on the length spectrum leads to a seemingly weaker but similar assertion, valid in a more general case:

Theorem 1. *If* (M,g) *is a compact Riemannian manifold such that*

$$\mathcal{L}_w g \;+\; \mathrm{Ric} \;>\; 0 \tag{1}$$

for some C^∞ *vector field* w, *then* $\pi_1 M$ *has only finitely many conjugacy classes.*

Here $\mathcal{L}_w$ and Ric are the Lie derivative and the Ricci tensor, while positivity in (1) means positive-definiteness at every point. As $H_1(M,\mathbf{Z}) = \pi_1 M/[\pi_1 M, \pi_1 M]$, the following corollary is immediate:

Corollary 2. *For any compact Riemannian manifold* (M,g) *satisfying the assumptions of Theorem 1, the homology group* $H_1(M,\mathbf{Z})$ *is finite and* $b_1(M)=0$.

It is not known if the conclusion of Theorem 1 is actually weaker than finiteness of $\pi_1 M$. Whether *a finitely presented group with only finitely many conjugacy classes must itself be finite* is an open question, raised by Makowsky [9] in 1974. Another part of Myers's theorem, stating that any complete Riemannian manifold with $\mathrm{Ric} \ge cg > 0$ for a constant c is necessarily compact, fails when Ric is replaced by $\mathcal{L}_w g + \mathrm{Ric}$. Namely, Feldman, Ilmanen and Knopf [4, Theorem 1.5] provide examples of noncompact complete Kähler manifolds (M,g) with real holomorphic vector fields w satisfying the relation

$$\mathcal{L}_w g \;+\; \mathrm{Ric} \;=\; cg \quad \text{for a constant } c, \tag{2}$$

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and in those particular examples $c > 0$, so that $\mathcal{L}_w g + \mathrm{Ric} \ge cg > 0$.

A Riemannian manifold (M, g) admitting a C^∞ vector field w with (2) is called a *Ricci soliton* [4] – [6], [8], [10]; one then also refers to g as a *quasi-Einstein metric.* Such (M, g) is said to be a *shrinking soliton* if $c > 0$ in (2).

The assumptions (and hence conclusions) of Theorem 1 and Corollary 2 clearly hold when (M, g) is a compact shrinking Ricci soliton, or a small perturbation thereof. For compact shrinking Ricci solitons (M, g), some special instances of this fact are known: finiteness of $H_1(M, \mathbf{Z})$ is a trivial consequence of Theorem 1 of [7] (see §3); M is simply connected if g is also a Kähler metric (§3); and in Proposition 2.2.5 on p. 396 of [5] it is stated that $b_1(M) = 0$ under the additional assumption of a scalar curvature bound $\mathrm{Scal} \ge (\dim M - 2)c$, for c as in (2).

It is known that every non-Einstein compact Ricci soliton must be a shrinking soliton (cf. the end of §3). This leads to a further conclusion:

Corollary 3. *The Euler characteristic of every non-flat compact four-dimensional Ricci soliton is positive.*

In other words, Berger's inequality $\chi(M) > 0$, for the Euler characteristic of a non-flat compact four-dimensional Einstein manifold [1, Theorem 6.32], remains true for Ricci solitons (M, g). In fact, $\chi(M) = 2 + b_2(M)$ in the (orientable) non-Einstein case, since the soliton is shrinking and so, as stated above, $b_1(M) = 0$.

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§1. A simple estimate

Given $a, b \in \mathbf{R}$ with $a < b$, let a constant-speed geodesic $[a, b] \ni t \mapsto x(t)$ in an orientable Riemannian manifold (M, g) be smoothly closed, so that $x(b) = x(a)$ and $\dot{x}(b) = \dot{x}(a)$, and have the minimum length compared to all nearby smoothly closed C^∞ curves $[a, b] \to M$. Then

$$(3) \qquad (b - a) \int_a^b \mathrm{Ric}(\dot{x}, \dot{x})\, dt \quad \le \quad 2[\mathrm{dist}\,(I, \Psi)]^2 \quad \le \quad k\pi^2,$$

where k is the largest even integer with $k+1 \le \dim M$, while $\Psi \in \mathrm{SO}(m)$, for $m = \dim M - 1$, is the *holonomy matrix* of our smoothly closed geodesic, characterized by the matrix-product relation $le\,[e_1(a) \,\ldots\, e_m(a)] = [e_1(b) \,\ldots\, e_m(b)]\Psi$ for some (or any) system $[a, b] \ni t \mapsto e_j(t) \in T_{x(t)}M$ of m orthonormal vector fields parallel along the geodesic and normal to it. In addition, $I \in \mathrm{SO}(m)$ stands for the identity matrix, and dist denotes the geodesic distance function in $\mathrm{SO}(m)$ corresponding to its submanifold metric induced by the inner product Q in the ambient vector space $\mathfrak{gl}(m, \mathbf{R})$ of all $m \times m$ real matrices, given by $2Q(\Psi, \Phi) = \mathrm{tr}\, \Psi\Phi^*$.

We now verify (3). In any Riemannian manifold (M, g), if $t \mapsto w(t)$ is a C^∞ unit vector field normal to a fixed geodesic $[a, b] \ni t \mapsto x(t)$ having the properties listed in the lines preceding (3), and $w(b) = w(a)$, then $(R(\dot{x}, w)\dot{x}, w) \le (\dot{w}, \dot{w})$, where $(w, w') = \int_a^b \langle w, w' \rangle\, dt$ stands for the L^2 inner product of vector fields w, w' tangent to M along the geodesic, R is the curvature tensor, and $\dot{w} = \nabla_{\dot{x}} w$. This is a well-known consequence of the length-minimizing property of the geodesic; cf. [7, formula (3.1)]. Let us now select the m fields e_j as in the lines following (3). For any fixed C^∞ curve $[a,b] \ni t \mapsto \Phi(t) \in \mathrm{SO}(m)$ joining I to the holonomy matrix Ψ, we may apply the inequality $(R(\dot{x},w)\dot{x},w) \le (\dot{w},\dot{w})$ to each of the m fields $w = w_j$ given by the matrix-product formula $[w_1(t) \ldots w_m(t)] = [e_1(t) \ldots e_m(t)]\Phi(t)$. Summing the resulting inequalities over $j = 1,\ldots,m$, we get $\int_a^b \mathrm{Ric}(\dot{x},\dot{x})\,dt \le 2\int_a^b |\dot{\Phi}|^2\,dt$, with $|\;|$ corresponding to the inner product Q. Since the curve $t \mapsto \Phi(t)$ in $\mathrm{SO}(m)$ was arbitrary, we may choose it to be a constant-speed minimizing geodesic joining I to Ψ. The last inequality then yields the first relation in (3).

To obtain the remaining inequality in (3), it suffices to write any $\Psi \in \mathrm{SO}(m)$ as $\Phi(\theta_1,\ldots,\theta_p)$, which acts as the identity on $[\Pi_1 \oplus \ldots \oplus \Pi_p]^\perp$, and as a rotation by the angle $\theta_l \in [-\pi,\pi]$ on each Π_l, for some fixed set of mutually orthogonal planes Π_l in $\mathbf{R}^m$, $l = 1,\ldots,p$. The length of the curve $[0,1] \ni t \mapsto \Phi(t\theta_1,\ldots,t\theta_p)$, joining I to Ψ, then is $[\theta_1^2 + \cdots + \theta_p^2]^{1/2} \le \sqrt{k/2}\,\pi$.

Remark. The upper bounds in (3) reflect the following easily-verified facts, which we do not use: the metric in $\mathrm{SO}(m)$ corresponding to dist is bi-invariant, the diameter of $\mathrm{SO}(m)$ relative to dist equals $\sqrt{k/2}\,\pi$, for k as in (3), and the curve joining I to Ψ in $\mathrm{SO}(m)$, defined above, is a minimizing geodesic.

§2. Proof of Theorem 1

Give a compact, orientable Riemannian manifold (M,g) with a C^∞ vector field w satisfying (1), let us choose a constant $c > 0$ with $\mathcal{L}_w g + \mathrm{Ric} \ge cg$. We then have the following upper bound on the length spectrum:

$$L \le \sqrt{k/c}\,\pi \quad \text{for the largest even integer } k \le \dim M - 1, \tag{4}$$

L being the length of any smoothly closed constant-speed geodesic $[a,b] \ni t \mapsto x(t)$ in (M,g) which represents a local minimum of the length functional in its free homotopy class. Namely, (3) with $\mathcal{L}_w g + \mathrm{Ric} \ge cg$ gives $cL^2 = (b-a)c\int_a^b g(\dot{x},\dot{x})\,dt \le (b-a)\int_a^b \mathrm{Ric}(\dot{x},\dot{x})\,dt \le k\pi^2$, and (4) follows. (The Lie-derivative term in (2) does not contribute to the integral, as $(\mathcal{L}_w g)(\dot{x},\dot{x}) = 2d[g(w,\dot{x})]/dt$.)

The uniform bound (4) implies in turn that there are only finitely many free homotopy classes of closed curves in M. In fact, an infinite sequence of smoothly closed geodesics γ_j with uniformly bounded lengths L_j cannot represent infinitely many distinct free homotopy classes, as one sees choosing a point x_j on each γ_j with a unit vector u_j tangent to γ_j at x_j, and then selecting a convergent subsequence of the sequence (x_j,u_j,L_j).

Since the free homotopy classes are in a bijective correspondence with the conjugacy classes in the fundamental group of M, this completes the proof.

§3. Comments

In this section we elaborate on some comments made in the introduction.

Ricci solitons on a compact manifold M are precisely the fixed points of the Ricci flow $dg/dt = -2\,\mathrm{Ric}$ projected, from the space of metrics, onto its quotient under diffeomorphisms and scalings [6]. For shrinking solitons those scalings cause the metric to shrink to zero in finite time. Theorem 1 of [7] states that, under the Ricci flow with any initial compact Riemannian manifold (M,g), the lengths of curves representing a fixed element of infinite order in $H_1(M,\mathbf{Z})$ remain bounded away from zero. As a consequence, if (M,g) is a compact shrinking Ricci soliton, $H_1(M,\mathbf{Z})$ has no element of infinite order, and so it is finite.

Secondly, compact shrinking Ricci solitons (M,g) in which g is a Kähler metric are known to be simply connected. In fact, w in (2) then is holomorphic (cf. [3]), and so the real cohomology classes of the Ricci and Kähler forms of g are related by $[\rho] = c[\Omega]$, as (2) gives $\rho = c\Omega - d\xi$ for the 1-form $\xi = \imath_w\Omega$. Thus, $c_1(M) > 0$, and so $\pi_1 M = \{0\}$ by a result of Kobayashi [1, Theorem 11.26].

Finally, $c > 0$ for every non-Einstein compact Ricci soliton (M,g). In fact, by Ivey's Proposition 1 in [8], the scalar curvature Scal of g then must be positive, while taking the g-trace of (2) we see that c is the average value of Scal on M. (That $c > 0$ and $\mathrm{Scal} \geq 0$ is also stated in Propositions 2.2.2 – 2.2.4 on p. 396 of [5], while Scal is nonconstant, as shown by Bourguignon [2, Proposition 3.11].)

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