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A note on q-analogues of Dirichlet L-functions

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In this note, we consider the special values of q-analogues of Dirichlet L-functions, namely, the values of the functions

$$L_q(s,\chi) = \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi\left(\frac{n}{d}\right) d^{s-1} \right) q^n$$

at positive integers s, where χ is a primitive Dirichlet character and $q = e^{2\pi i\tau}$ is a complex number such that $|q| < 1$. We prove that if $\chi(-1) = (-1)^k$ and q is algebraic, then $L_q(k,\chi)$ is transcendental. We also prove that if $\chi(-1) = (-1)^k$ and $j(\tau)$ is algebraic, then there exists a transcendental number ω_τ which depends only on τ and is $\overline{\mathbb{Q}}$-linearly independent with π such that $(\pi/\omega_\tau)^k(L(1-k,\chi) + 2L_q(k,\chi))$ is algebraic. These results can be viewed as an analogue of the classical result of Hecke on the arithmetic nature of the special values $L(k,\chi)$ for $\chi(-1) = (-1)^k$.

Keywords: q-Analogues; modular functions; transcendental values of modular forms; Nesterenko's theorem.

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1. Introduction and Statement of the Main Result

Let N be a positive integer, and let χ be a primitive Dirichlet character modulo N. We define the q-analogue of a Dirichlet L-function as

$$L_q(s,\chi) = \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi\left(\frac{n}{d}\right) d^{s-1} \right) q^n.$$

It would seem more natural to define the q-analogue of a Dirichlet L-function $L(s,\chi)$ by

$$Z_q(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{(1-q^n)^s}$$

since one immediately sees that $\lim_{q\to 1}(1-q)^s Z_q(s,\chi) = L(s,\chi)$ for $\Re(s) > 1$. In what follows, we justify that the values $L_q(s,\chi)$ for integers $s > 1$, as normalized above, can also be considered as q-analogues of the values of the classical Dirichlet L-function $L(s,\chi)$. To this end, we need to recall the definition of the Stirling numbers of the second kind, $S(s,j)$, where s and j are positive integers such that $j \le s$:

$$m^s = \sum_{j=1}^{s} S(s,j)m(m-1)\cdots(m-(j-1)).$$

We have

$$\begin{aligned}
L_q(s,\chi) &= \sum_{n=1}^{\infty}\chi(n)\sum_{m=1}^{\infty} m^{s-1}q^{nm} \\
&= \sum_{n=1}^{\infty}\chi(n)\sum_{m=0}^{\infty}(m+1)^{s-1}q^{n(m+1)} \\
&= \sum_{j=1}^{s-1}(-1)^{s-1-j}S(s-1,j)j!\sum_{n=1}^{\infty}\chi(n)\sum_{m=0}^{\infty}\binom{m+j}{j}q^{n(m+1)} \\
&= \sum_{j=1}^{s-1}(-1)^{s-1-j}S(s-1,j)j!\sum_{n=1}^{\infty}\chi(n)\frac{q^n}{(1-q^n)^{j+1}} \\
&= \sum_{j=1}^{s-1}(-1)^{s-1-j}S(s-1,j)j!Z_q(j+1,\chi),
\end{aligned}$$

which clearly implies that $\lim_{q\to 1}(1-q)^s L_q(s,\chi) = (s-1)!L(s,\chi)$. We mention here that this line of argument is used in [6, 10] where q-analogues of the values of the Riemann zeta function are considered.

We are interested in the special values $L_q(k,\chi)$ when k is a positive integer. It turns out that much like the case with the classical Dirichlet L-functions, the study of these special values depends on whether $\chi(-1) = (-1)^k$ or not. In this note, we consider the case when χ and k have the same parity, and we obtain the following results.

Theorem 1.1. *Let N and k be positive integers, and let χ be a primitive Dirichlet character of conductor N such that $\chi(-1) = (-1)^k$. If q is an algebraic number with $|q| < 1$, then $L_q(k,\chi)$ is transcendental.*

Theorem 1.2. *Let N and k be positive integers. If $k = 2$, we assume that $N \neq 1$. Let χ be a primitive Dirichlet character of conductor N such that $\chi(-1) = (-1)^k$. If $q = e^{2\pi i\tau}$ and $j(\tau) \in \overline{\mathbb{Q}}$, then there exists a transcendental number ω_τ which depends only on τ and is $\overline{\mathbb{Q}}$-linearly independent with π such that $(\pi/\omega_\tau)^k(L(1-k,\chi) + 2L_q(k,\chi))$ is algebraic.*

The point of departure in this work is the fact that $L_q(k,\chi)$ is essentially modular of level N and weight k whenever χ and k share the same parity. In fact, the arguments used to prove Theorems 1.1 and 1.2 could be easily modified to obtain similar results for all modular forms of level N and weight k (see Theorem 3.1, Corollary 3.3 and Theorem 4.3).

Much like the classical case, the problem becomes more challenging when χ and k are not of the same parity in which case a more analytic approach is needed. This topic is the subject of an ongoing project of the authors.

Throughout the paper, $\mathbb{H}$ denotes the upper half plane and $\overline{\mathbb{Q}}$ is a fixed algebraic closure of $\mathbb{Q}$ embedded into $\mathbb{C}$. An element $\tau \in \mathbb{H}$ is a CM point if $K_\tau := \mathbb{Q}(\tau)$ is an imaginary quadratic field.

2. Notation and Preliminaries

In this section, we recall some definitions and collect some standard results that are fundamental to the proof of Theorems 1.1 and 1.2.

For an even integer $k > 2$, the Eisenstein series of weight k is defined as

$$G_k(\tau) := \sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)}} \frac{1}{(c\tau+d)^k}, \quad \tau \in \mathbb{H}.$$

It is well known that G_k is a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Upon dividing $G_k(\tau)$ by $2\zeta(k)$, the leading coefficient in its Fourier expansion, one obtains the (normalized) Eisenstein series

$$E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $q = e^{2\pi i\tau}$, $\sigma_{k-1}(n) = \sum_{\substack{d\mid n\\ d>0}} d^{k-1}$ and B_k is the kth Bernoulli number. We note that

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n$$

is not a modular form; rather it is a quasi-modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ (see [5]). Nonetheless, we refer to E_2 as the Eisenstein series of weight 2.

Let us now recall Nesterenko's theorem on the algebraic independence of values of the Eisenstein series,

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty} \sigma_5(n) q^n.$$

The reader is referred to [9, Chap. 3] for a proof of this result.

Theorem 2.1. *For any $\tau \in \mathbb{H}$, the transcendence degree of the field*

$$\mathbb{Q}(e^{2\pi i\tau}, E_2(\tau), E_4(\tau), E_6(\tau))$$

is at least 3. *In particular, if $q = e^{2\pi i\tau}$ is algebraic, then $E_2(\tau)$, $E_4(\tau)$ and $E_6(\tau)$ are algebraically independent.*

Essential to our work are the Ramanujan cusp form $\Delta(\tau)$ and the modular invariant $j(\tau)$ given by

$$\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728}$$

and

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}.$$

It is very well known that Δ is a cusp form of weight 12 for the full modular group and j is a weight zero modular function for the full modular group as well.

An immediate consequence of Theorem 2.1 is the following result which was originally conjectured by Mahler in 1969 and first proved by Barré-Sirieix *et al.* [1].

Theorem 2.2. *For any $\tau \in \mathbb{H}$, at least one of the two numbers $e^{2\pi i\tau}$ and $j(\tau)$ is transcendental.*

Next, we present the formulae for the values of E_2, E_4 and E_6 at the points $\tau \in \mathbb{H}$ with $j(\tau) \in \overline{\mathbb{Q}}$. To this end, we adapt the discussion in [2, Sec. 2.1] leading to Theorem 2.1 therein.

The Eisenstein series generalize to functions of a variable lattice $L \subset \mathbb{C}$,

$$G_k(L) = \sum_{\substack{\omega \in L \\ \omega \neq 0}} \frac{1}{\omega^k}, \quad k > 2 \text{ even},$$

so that $G_k(L_\tau) = G_k(\tau)$ for $L_\tau := \mathbb{Z} \oplus \tau\mathbb{Z}$ and $\tau \in \mathbb{H}$. The elliptic curve corresponding to a lattice $L \subset \mathbb{C}$ is given by the Weierstrass equation

$$E_L : y^2 = 4x^3 - g_2(L)x - g_3(L),$$

where $g_2(L) = 60G_4(L)$ and $g_3(L) = 140G_6(L)$.

Given $\tau \in \mathbb{H}$ such that $j(\tau)$ is algebraic, there exists a unique (up to an algebraic multiple) non-zero complex number ω_τ for which $g_2(\omega_\tau L_\tau)$ and $g_3(\omega_\tau L_\tau)$ are algebraic numbers. We have the formulae

$$E_2(\tau) = 3\frac{\omega_\tau}{\pi}\frac{\eta(\omega_\tau)}{\pi}, \tag{2.1}$$

$$E_4(\tau) = \frac{3}{4}\left(\frac{\omega_\tau}{\pi}\right)^4 g_2(\omega_\tau L_\tau), \quad E_6(\tau) = \frac{27}{8}\left(\frac{\omega_\tau}{\pi}\right)^6 g_3(\omega_\tau L_\tau). \tag{2.2}$$

Notice that ω_τ is a non-zero period of the elliptic curve $E_\tau := E_{\omega_\tau L_\tau}$ which is defined over the algebraic numbers with j-invariant $j(\tau)$. If η denotes the quasi-period function associated to E_τ, it follows by a fundamental theorem of Schneider [12] (see also [11, Corollary 12.4]) that ω_τ and $\eta(\omega_\tau)$ are transcendental. Moreover, a theorem of Chudnovsky [3] asserts that the numbers $\frac{\omega_\tau}{\pi}$ and $\frac{\eta(\omega_\tau)}{\pi}$ are algebraically independent.

In what follows, we state two standard results pertaining to the structure of the field of modular functions at level N. For proofs and an elaborate discussion on this topic, the reader is referred to [4, Chap. 7; 13 Chap. 6].

Proposition 2.3. (a) *The field of all modular functions of level N with Fourier coefficients (with respect to $e^{\frac{2\pi i\tau}{N}}$) in $\mathbb{Q}(e^{\frac{2\pi i}{N}})$ is a finite Galois extension of $\mathbb{Q}(j)$.*
(b) *The field of all modular functions of level N with algebraic Fourier coefficients (with respect to $e^{\frac{2\pi i\tau}{N}}$) is a finite Galois extension of $\overline{\mathbb{Q}}(j)$.*

Proposition 2.4. *The field of all modular functions of level N is $\mathbb{C}(j, f_0, f_1)$, where*

$$f_0(\tau) = \frac{9E_4(\tau)}{2\pi^2 E_6(\tau)}\wp_\tau\left(\frac{\tau}{N}\right), \quad f_1(\tau) = \frac{9E_4(\tau)}{2\pi^2 E_6(\tau)}\wp_\tau\left(\frac{1}{N}\right).$$

The function $\wp_\tau(z)$ that appears above is the Weierstrass $\wp$-function attached to the lattice L_τ. More precisely,

$$\wp_\tau(z) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(z+m\tau+n)^2} - \frac{1}{(m\tau+n)^2}\right), \quad z \in \mathbb{C}, z \notin L_\tau.$$

Finally, we recall some Eisenstein series of level N, weight k and character χ. If $k = 2$, we assume that $N > 1$. We denote the trivial character modulo 1 by $\mathbf{1}$. For any two primitive Dirichlet characters ψ modulo u and ϕ modulo v such that $uv = N$ and $(\psi\phi)(-1) = (-1)^k$, we have the Eisenstein series

$$E_k^{\psi,\phi}(\tau) = \delta(\psi)L(1-k,\phi) + 2\sum_{n=1}^{\infty} \sigma_{k-1}^{\psi,\phi}(n)e^{2\pi i n\tau} \in M_k(N, \psi\phi),$$

where $\delta(\psi)$ is 1 if $\psi = \mathbf{1}$ and 0 otherwise, and $\sigma_{k-1}^{\psi,\phi}(n) = \sum_{m|n}\phi(\frac{n}{m})\psi(m)m^{k-1}$ (see [4, Theorems 4.5.1 and 4.5.2]). Here, we denote by $M_k(N,\psi\phi)$ the space of weight k modular forms of level N and nebentypus $\psi\phi$. Let χ be a primitive Dirichlet character modulo N such that $\chi(-1) = (-1)^k$. Upon substituting $\psi = \mathbf{1}$ and $\phi = \chi$, we get

$$E_k^{\mathbf{1},\chi}(\tau) = L(1-k,\chi) + 2L_q(k,\chi),$$

where $q = e^{2\pi i\tau}$. For ease of notation, we shall henceforth write $E_{k,\chi}$ instead of $E_k^{\mathbf{1},\chi}$. It is known that $L(1-k,\chi)$ is algebraic for all $k \geq 1$. In fact, we have

$$L(1-k,\chi) = -\frac{B_{k,\chi}}{k} = -\frac{N^{k-1}}{k}\sum_{c=0}^{N-1}\chi(c)B_k(c/N),$$

where $B_k(x)$ is the kth Bernoulli polynomial (see [4, §4.7]).

3. Proof of Theorem 1.1

Let $\tau \in \mathbb{H}$ be such that $q = e^{2\pi i\tau}$ is algebraic. First, notice that

$$L_q(2, \mathbf{1}) = \frac{1 - E_2(\tau)}{24},$$

and so it follows immediately from Theorem 2.1 that $L_q(2, \mathbf{1})$ is transcendental. We may henceforth assume that $N > 1$ if $k = 2$. In view of the above discussion, we see that $L_q(k, \chi)$ is transcendental if and only if $E_{k,\chi}(\tau)$ is transcendental. We proceed to show that if $\tau \in \mathbb{H}$ is such that $q = e^{2\pi i\tau}$ is algebraic, then $E_{k,\chi}(\tau)$ is transcendental.

Consider the function $\frac{E_{k,\chi}^{12}}{\Delta^k}$; this is a weight zero modular function of level N with Fourier coefficients in $\mathbb{Q}(e^{\frac{2\pi i}{N}})$. It follows from Proposition 2.3(a) that there exists a non-zero polynomial

$$P(x) = \sum_{r=0}^{m} a_r(j)x^r \in \mathbb{Q}[j][x]$$

such that $P(\frac{E_{k,\chi}(\tau)^{12}}{\Delta(\tau)^k}) = 0$ for any $\tau \in \mathbb{H}$. Here, we express the polynomial $a_r(y) \in \mathbb{Q}[y]$ as

$$a_r(y) = \sum_{s=0}^{d_r} a_{r,s}y^s.$$

For any positive integer l and any $\tau \in \mathbb{H}$, we have

$$\begin{aligned} 0 &= \Delta(\tau)^l P\left(\frac{E_{k,\chi}(\tau)^{12}}{\Delta(\tau)^k}\right) \\ &= \sum_{r=0}^{m} a_r(j)E_{k,\chi}(\tau)^{12r}\Delta(\tau)^{l-kr} \\ &= \sum_{r=0}^{m}\sum_{s=0}^{d_r} a_{r,s}j(\tau)^s E_{k,\chi}(\tau)^{12r}\Delta(\tau)^{l-kr} \\ &= \sum_{r=0}^{m}\sum_{s=0}^{d_r} a_{r,s}E_4(\tau)^{3s}E_{k,\chi}(\tau)^{12r}\Delta(\tau)^{l-kr-s}. \end{aligned}$$

If we choose $l \geq km + \max(d_0, d_1, \ldots, d_m)$ so that only positive powers of $\Delta(\tau)$ appear in the above equations, then we get the following relation:

$$0 = \sum_{r=0}^{m}\sum_{s=0}^{d_r}\sum_{t=0}^{l-kr-s} \frac{(-1)^t a_{r,s}}{1728^{l-kr-s}}\binom{l-kr-s}{t}E_{k,\chi}(\tau)^{12r}E_4(\tau)^{3(l-kr-t)}E_6(\tau)^{2t}. \tag{3.1}$$

Now let $\tau \in \mathbb{H}$ such that $q = e^{2\pi i\tau}$ is algebraic. Notice that $E_{k,\chi}(\tau) \neq 0$, for otherwise $a_0(j(\tau)) = 0$ which would imply that $j(\tau)$ is algebraic contradicting Corollary 2.2. If $E_{k,\chi}(\tau)$ is algebraic, then Eq. (3.1) is an algebraic dependence relation between $E_4(\tau)$ and $E_6(\tau)$. This yields a contradiction to Theorem 2.1.

Using Proposition 2.3(b), the above proof is easily modified to obtain the following more general result.

Theorem 3.1. *Let f be a non-zero modular form of level N and weight k with algebraic Fourier coefficients. If $\tau \in \mathbb{H}$ is such that $e^{2\pi i\tau}$ is algebraic, then $f(\tau)$ is transcendental.*

A complementary result to Theorem 3.1 is shown by Chang [2]. More precisely, this result [2, Corollary 2.3], in parts, states the following.

Theorem 3.2. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and k be a positive integer. Suppose that f is holomorphic on $\mathbb{H}$ so that f^2 becomes a modular form of weight k for Γ with algebraic Fourier coefficients. Then for any $\tau \in \mathbb{H}$ such that $j(\tau)$ is algebraic, $f(\tau)$ is transcendental unless $f(\tau) = 0$.*

Combining Theorems 3.2 and 2.2 with Theorem 3.1, one deduces the following result.

Corollary 3.3. *Let f be a non-zero modular form of level N and weight k with algebraic Fourier coefficients. For any $\tau \in \mathbb{H}$ such that $f(\tau) \neq 0$, at least two of*

$$e^{2\pi i\tau}, \quad j(\tau), \quad f(\tau)$$

are transcendental.

4. Proof of Theorem 1.2

Let $\tau \in \mathbb{H}$ be such that $j(\tau)$ is algebraic. Applying Eq. (3.1) and formula (2.1), we get

$$0 = \sum_{r,s,t} b_{r,s,t} g_2^{l-kr-t} g_3^{2t} ((\pi/\omega_\tau)^k E_{k,\chi}(\tau))^{12r},$$

for some rational coefficients $b_{r,s,t}$. Hence, $(\pi/\omega_\tau)^k E_{k,\chi}(\tau)$ is an algebraic number since it satisfies a non-trivial polynomial equation with algebraic coefficients. This concludes the proof of Theorem 1.2.

Remark 4.1. If $k = 2$ and $N = 1$, then $L_q(2, \mathbf{1}) = \frac{1-E_2(\tau)}{24}$. In view of formula (2.1), $\frac{\pi^2}{\omega_\tau \eta(\omega_\tau)}(\zeta(-1) + L_q(2, \mathbf{1}))$ is algebraic.

The proof of Theorem 1.2 is easily modified to prove a similar result for all modular forms of level N (Theorem 4.3(a)). An additional result (Theorem 4.3(b)) is obtained upon specializing to the case of complex multiplication.

Lemma 4.2. *Let $\tau \in \mathbb{H}$ be a CM point. Then, $f_0(\tau)$ and $f_1(\tau)$ belong to the ray class field of K_τ of conductor N.*

Proof. Recall that $f_0(\tau) = \frac{9E_4(\tau)}{2\pi^2 E_6(\tau)} \wp_\tau(\frac{\tau}{N})$ and $f_1(\tau) = \frac{9E_4(\tau)}{2\pi^2 E_6(\tau)} \wp_\tau(\frac{1}{N})$. The first Weber function $w_\tau(z)$ is obtained by normalizing the Weierstrass $\wp$-function $\wp_\tau(z)$

to be homogeneous of degree zero with respect to L_τ. More precisely, we have

$$w_\tau(z) = -\frac{3}{\pi^2}\frac{E_4(\tau)E_6(\tau)}{\Delta(\tau)}\wp_\tau(z).$$

It is a classical result that for a CM point $\tau \in \mathbb{H}$, the value of w_τ at a non-trivial N-division point of L_τ is algebraic, and together with $j(\tau)$ generate the ray class field of K_τ of conductor N (see [7, Secs. 1 and 2, Chap. 10]). Using formula (2.2), an easy calculation shows that $f_0(\tau) = \delta w_\tau(\frac{\tau}{N})$ and $f_1(\tau) = \delta w_\tau(\frac{1}{N})$ for some algebraic number δ in $\mathbb{Q}(j(\tau))$. Hence, $f_0(\tau)$ and $f_1(\tau)$ belong to the ray class field of K_τ of conductor N as desired. □

Theorem 4.3. *Let f be a non-zero modular form of level N and weight k with algebraic Fourier coefficients. Let $\tau \in \mathbb{H}$ be such that $j(\tau)$ is algebraic.*

(a) *There exists a transcendental number ω_τ which depends only on τ and is $\overline{\mathbb{Q}}$-linearly independent with π such that $(\pi/\omega_\tau)^k f(\tau)$ is algebraic.*
(b) *If τ is a CM point, let H_τ^N be the ray class field of K_τ of conductor N. Then for any automorphism $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/H_\tau^N)$, we have $((\pi/\omega_\tau)^k f(\tau))^\sigma = (\pi/\omega_\tau)^k f^\sigma(\tau)$.*

Proof. Consider the function $g = \frac{f^{12}}{\Delta^k}$ which is clearly a modular function of level N. To prove part (b), let $\tau \in \mathbb{H}$ be a CM point. By Proposition 2.4, we have

$$g(\tau) = P(j(\tau), f_0(\tau), f_1(\tau))$$

for some polynomial $P(x, y, z) = \sum_{l,m,n} a_{m,n,l}x^l y^m z^l$ with algebraic coefficients. It follows that

$$(\pi/\omega_\tau)^{12k} f^{12}(\tau) = \beta \sum_{l,m,n} a_{l,m,n} j^l(\tau) f_0^m(\tau) f_1^n(\tau),$$

for some algebraic number β in $\mathbb{Q}(j(\tau))$. Using Lemma 4.2, we see that for a Galois automorphism $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/H_\tau^N)$, we have

$$((\pi/\omega_\tau)^{12k} f^{12}(\tau))^\sigma = \beta \sum_{l,m,n} (a_{l,m,n})^\sigma j^l(\tau) f_0^m(\tau) f_1^n(\tau).$$

Therefore, we get $((\pi/\omega_\tau)^k f(\tau))^\sigma = (\pi/\omega_\tau)^k f^\sigma(\tau)$. □

As a side remark, we note that if τ is a CM point, then ω_τ and π are algebraically independent (see e.g., [11, Corollary 17.11]).

Corollary 4.4. *Let N and k be positive integers. If $k = 2$, we assume that $N \neq 1$. Let χ be a primitive Dirichlet character of conductor N such that $\chi(-1) = (-1)^k$. If $q = e^{2\pi i\tau}$ and τ is a CM point, then for any automorphism $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/H_\tau^N)$, we have $((\pi/\omega_\tau)^k(L(1-k,\chi) + 2L_q(k,\chi)))^\sigma = (\pi/\omega_\tau)^k(L(1-k,\chi^\sigma) + 2L_q(k,\chi^\sigma))$.*

Finally, we note that Theorem 4.3 is a generalization to higher levels of the main theorem in [8]. Similar results for full level modular forms have been discussed in [14] as well.

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