FUNCTIONS

EXAM QUESTIONS

Question 1 (**)

The function f is given by

$$f : x \mapsto \frac{x}{x+3}, \; x \in \mathbb{R}, \; x \neq -3.$$

a) Find an expression for $f^{-1}(x)$.

The function g is defined as

$$g : x \mapsto \frac{2}{x}, \; x \in \mathbb{R}, \; x \neq 0.$$

b) Evaluate $fg\left(\frac{2}{3}\right)$.

$\boxed{}$, $\boxed{f^{-1} : x \mapsto \frac{3x}{1-x}}$, $\boxed{fg\left(\frac{2}{3}\right) = \frac{1}{2}}$

(a) $f(x) = \frac{x}{x+3}$

$\Rightarrow y = \frac{x}{x+3}$

$\Rightarrow yx + 3y = x$

$\Rightarrow 3y = x - yx$

$\Rightarrow 3y = x(1-y)$

$\Rightarrow x = \frac{3y}{1-y}$

$\therefore f^{-1}(x) = \frac{3x}{1-x}$

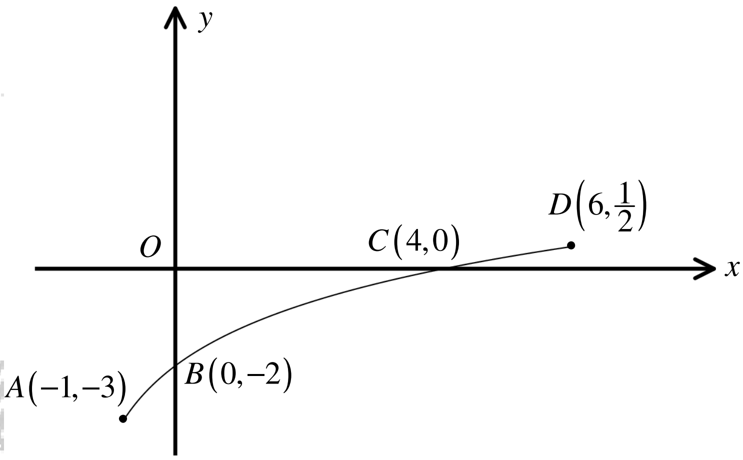
(b) $f(g(\frac{2}{3})) = f\left(\frac{2}{2/3}\right)$

$= f(3)$

$= \frac{3}{3+3}$

$= \frac{1}{2}$

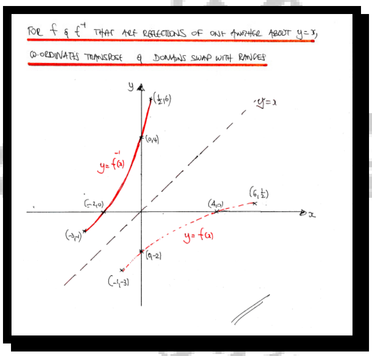
Question 2 (**)



The figure above shows the graph of the function $f(x)$, defined for $-1 \le x \le 6$.

Sketch the graph of $f^{-1}(x)$, marking clearly the end points of the graph and any points where it crosses the coordinate axes.

graph



Question 3 (**)

The function f is given by

$$f(x) = \ln(4x-2), \quad x \in \mathbb{R}, \quad x > \frac{1}{2}.$$

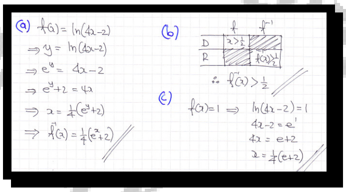
a) Find an expression for $f^{-1}(x)$, in its simplest form.

b) State the range of $f^{-1}(x)$.

c) Solve the equation

$$f(x) = 1.$$

$\boxed{f^{-1}(x) = \frac{1}{4}\left(e^x + 2\right)}$, $\boxed{f^{-1}(x) > \frac{1}{2}}$, $\boxed{x = \frac{1}{4}(e+2)}$



Question 4 **(**)**

The function f is given by

$$f(x) = 3 - \ln x, \ x \in \mathbb{R}, \ x > 0.$$

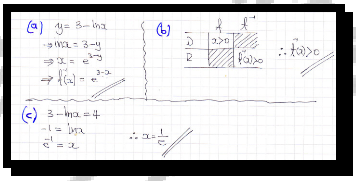
a) Find an expression for $f^{-1}(x)$.

b) State the range of $f^{-1}(x)$.

c) Solve the equation

$$f(x) = 4.$$

$$\boxed{f^{-1}(x) = \mathrm{e}^{3-x}}, \ \boxed{f^{-1}(x) > 0}, \ \boxed{x = \frac{1}{\mathrm{e}}}$$



Question 5 **(**+)**

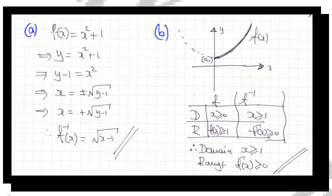
A function f is defined by

$$f(x) = x^2 + 1, \ x \in \mathbb{R}, \ x \geq 0.$$

a) Find an expression for $f^{-1}(x)$.

b) State the domain and range of $f^{-1}(x)$.

$$\boxed{f^{-1}(x) = \sqrt{x-1}}, \ \boxed{x \geq 1}, \ \boxed{f^{-1}(x) \geq 0}$$



Question 6 **(**+)**

The functions f and g are given by

$$f(x) = x^2, \ x \in \mathbb{R}.$$

$$g(x) = \frac{1}{x+2}, \ x \in \mathbb{R}, \ x \neq -2.$$

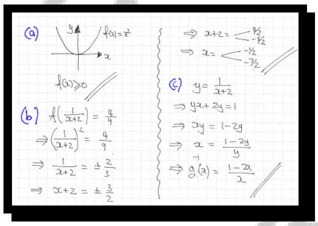
a) State the range of $f(x)$.

b) Solve the equation

$$fg(x) = \tfrac{4}{9}.$$

c) Find, in its simplest form, an expression for $g^{-1}(x)$.

$\boxed{f(x) \geq 0}$, $\boxed{x = -\frac{1}{2}, -\frac{7}{2}}$, $\boxed{g^{-1}(x) = \frac{1}{x} - 2 = \frac{1-2x}{x}}$



Question 7 (**+)

The functions f and g satisfy

$$f(x) = \ln(4-2x), \ x \in \mathbb{R}, \ x < 2.$$

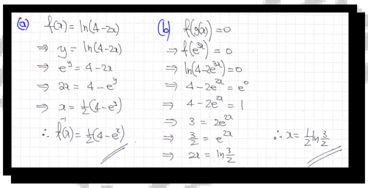
$$g(x) = e^{3x}, \ x \in \mathbb{R}.$$

a) Find an expression for $f^{-1}(x)$.

b) Solve the equation

$$fg(x) = 0.$$

$$\boxed{f^{-1}(x) = 2 - \tfrac{1}{2}e^{x}}, \ \boxed{x = \tfrac{1}{3}\ln\left(\tfrac{3}{2}\right)}$$


(a) $f(x) = \ln(4-2x)$
$\Rightarrow y = \ln(4-2x)$
$\Rightarrow e^{y} = 4-2x$
$\Rightarrow 2x = 4-e^{y}$
$\Rightarrow x = \frac{1}{2}(4-e^{y})$
$\therefore f^{-1}(x) = \frac{1}{2}(4-e^{x})$

(b) $f(g(x)) = 0$
$\Rightarrow f(e^{3x}) = 0$
$\Rightarrow \ln(4-2e^{3x}) = 0$
$\Rightarrow 4-2e^{2x} = e^{0}$
$\Rightarrow 4-2e^{2x} = 1$
$\Rightarrow 3 = 2e^{2x}$
$\Rightarrow \frac{3}{2} = e^{2x}$
$\Rightarrow 2x = \ln\frac{3}{2}$
$\therefore x = \frac{1}{2}\ln\frac{3}{2}$

Question 8 (+)**

The functions f and g satisfy

$$f(x) = 1 + \tfrac{1}{2}\ln(x+3), \ \ x \in \mathbb{R}, \ \ x > 3.$$

$$g(x) = e^{2(x-1)} - 3, \ \ x \in \mathbb{R}.$$

a) Find, in its **simplest** form, an expression for $fg(x)$.

b) Hence, or otherwise, write down an expression for $f^{-1}(x)$.

$\boxed{fg(x) = x}$, $\boxed{f^{-1}(x) = e^{2(x-1)} - 3}$

(a) $f(g(x)) = f\left(e^{2(x-1)} - 3\right) = 1 + \frac{1}{2}\ln\left[\left(e^{2(x-1)} - 3\right) + 3\right]$

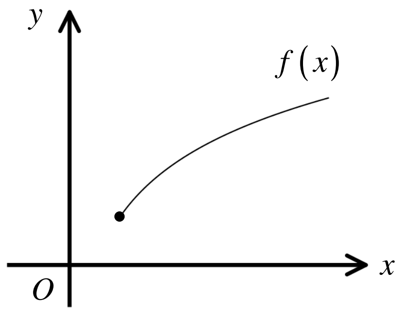
$= 1 + \frac{1}{2}\ln\left(e^{2(x-1)}\right) = 1 + \frac{1}{\cancel{2}} \times \cancel{2}(x-1)$

$= \cancel{1} + x \cancel{-1} = x$

(b) SINCE $f(g(x)) = x$ $\quad g(x) = f^{-1}(x)$

$\therefore f^{-1}(x) = e^{2(x-1)} - 3$

Question 9 (**+)



The diagram above shows the graph of the function f, defined as

$$f(x) \equiv \frac{1}{1-x} + 4, \quad x \in \mathbb{R}, \; x \geq 2.$$

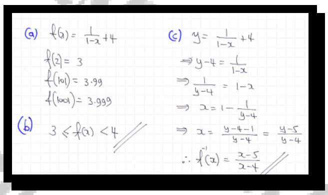
a) Evaluate $f(2)$, $f(101)$, $f(1001)$.

b) State the range of $f(x)$.

The inverse function is denoted by $f^{-1}(x)$.

c) Determine an expression for $f^{-1}(x)$, as a simplified fraction.

$$\boxed{f(2) = 3, \; f(101) = 3.99, \; f(1001) = 3.999}, \quad \boxed{3 \leq f(x) < 4}, \quad \boxed{f^{-1}(x) = \frac{x-5}{x-4}}$$



Question 10 (**+)

The function f is given by

$$f(x) = 4 - \ln(2x-1), \; x \in \mathbb{R}, \; x > \frac{1}{2}.$$

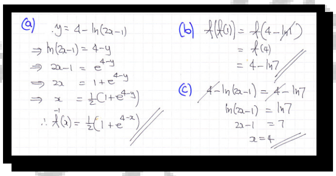
a) Find an expression for $f^{-1}(x)$, in its simplest form.

b) Determine the exact value of $ff(1)$.

c) Hence, or otherwise, solve the equation

$$f(x) = ff(1).$$

$\boxed{\quad}$, $\boxed{f^{-1}(x) = \frac{1}{2}\left(1 + e^{4-x}\right)}$, $\boxed{ff(1) = 4 - \ln 7}$, $\boxed{x = 4}$



Question 11 (**+)

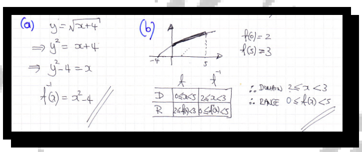
A function f is defined by

$$f(x) = \sqrt{x+4}, \; x \in \mathbb{R}, \; 0 \le x < 5.$$

a) Find an expression for $f^{-1}(x)$.

b) Determine the domain and the range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = x^2 - 4}$, $\boxed{2 \le x < 3}$, $\boxed{0 \le f^{-1}(x) < 5}$



Question 12 (**+)

The function f is defined

$$f : x \mapsto \frac{2x-3}{x-2}, \quad x \in \mathbb{R}, \quad x \neq 2.$$

a) Find an expression for $f^{-1}(x)$ in its simplest form.

b) Hence, or otherwise, find in its simplest form $ff(k+2)$.

$\boxed{}$, $\boxed{f^{-1} : x \mapsto \dfrac{2x-3}{x-2}}$, $\boxed{k+2}$

a) FOLLOWING THE USUAL METHODOLOGY

$\Rightarrow f(x) = \dfrac{2x-3}{x-2}$

$\Rightarrow y = \dfrac{2x-3}{x-2}$

$\Rightarrow y(x-2) = 2x-3$

$\Rightarrow yx - 2y = 2x - 3$

$\Rightarrow yx - 2x = 2y - 3$

$\Rightarrow x(y-2) = 2y-3$

$\Rightarrow x = \dfrac{2y-3}{y-2}$

$\therefore f^{-1}(x) = \dfrac{2x-3}{x-2}$

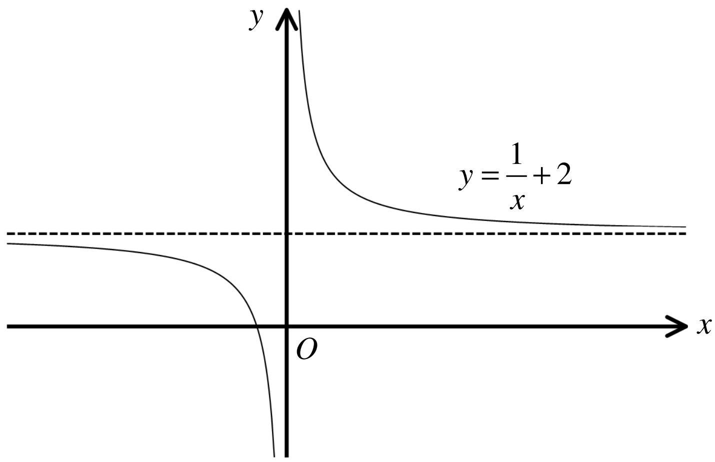
b) AS $f(x)$ IS SELF INVERSE, IE $f(x) = f^{-1}(x)$ THEN WE HAVE

$\Rightarrow f(f^{-1}(x)) = x$

$\Rightarrow f(f(x)) = x$

$\Rightarrow f(f(k+2)) = k+2$

Question 13 (***)



The figure above shows the graph of

$$y = \frac{1}{x} + 2, \ x \neq 0.$$

a) State the equation of the horizontal asymptote to the curve, marked as a dotted line in the figure.

The function f is defined

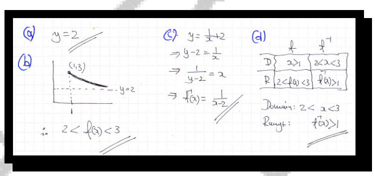
$$f(x) = \frac{1}{x} + 2, \ x \in \mathbb{R}, \ x > 1.$$

b) State the range of $f(x)$.

c) Obtain an expression for $f^{-1}(x)$.

d) State the domain and range of $f^{-1}(x)$.

$\boxed{}$, $\boxed{y = 2}$, $\boxed{2 < f(x) < 3}$, $\boxed{f^{-1}(x) = \frac{1}{x-2}}$, $\boxed{2 < x < 3}$, $\boxed{f^{-1}(x) > 1}$



Question 14 (***)

The functions f and g are given by

$$f(x) = \frac{2x+3}{2x-3}, \quad x \in \mathbb{R},\ x \neq \frac{3}{2}.$$

$$g(x) = x^2 + 2, \quad x \in \mathbb{R}.$$

a) State the range of $g(x)$.

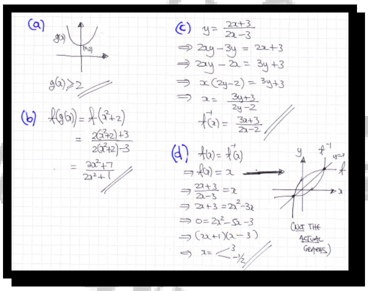
b) Find an expression, as a simplified algebraic fraction, for $fg(x)$.

c) Determine an expression, as a simplified algebraic fraction, for $f^{-1}(x)$.

d) Solve the equation

$$f^{-1}(x) = f(x).$$

$\boxed{}$, $\boxed{g(x) \geq 2}$, $\boxed{fg(x) = \frac{2x^2+7}{2x^2+1}}$, $\boxed{f^{-1}(x) = \frac{3x+3}{2x-2}}$, $\boxed{x = -\frac{1}{2}, 3}$



Question 15 (***)

$$f(x) = e^{2x} - 4, \quad x \in \mathbb{R}.$$

$$g(x) = \frac{1}{x-11}, \quad x \in \mathbb{R}, \ x \neq 11.$$

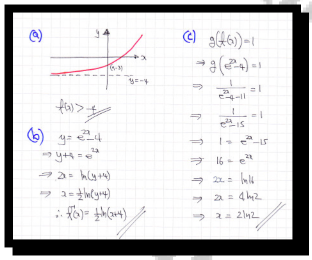
a) Determine the range of $f(x)$.

b) Find an expression for the inverse function $f^{-1}(x)$.

c) Solve the equation

$$gf(x) = 1.$$

$\boxed{f(x) > -4}$, $\boxed{f^{-1}(x) = \frac{1}{2}\ln(x+4)}$, $\boxed{x = 2\ln 2}$



Question 16 (***)

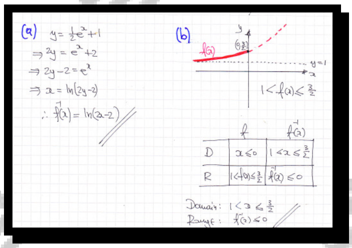
A function f is defined by

$$f(x) = \frac{1}{2}e^{x} + 1, \quad x \in \mathbb{R}, \ x \le 0.$$

a) Find an expression for $f^{-1}(x)$.

b) State the domain and range of $f^{-1}(x)$.

$$\boxed{f^{-1}(x) = \ln(2x-2)}, \ \boxed{1 < x \le \frac{3}{2}}, \ \boxed{f^{-1}(x) \le 0}$$



Question 17 (***)

The function f is given by

$$f(x) = (x-3)^2 + 1, \; x \in \mathbb{R}, \; x \geq 4 .$$

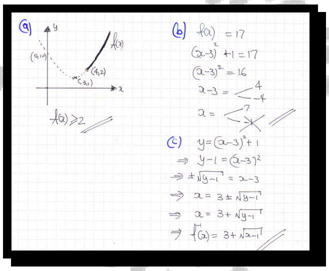
a) Sketch the graph of $f(x)$ and hence write down its range.

b) Solve the equation

$$f(x) = 17 .$$

c) Find an expression for $f^{-1}(x)$ in its simplest form.

$\boxed{f(x) \geq 2}$, $\boxed{x = 7, \; x \neq -1}$, $\boxed{f^{-1}(x) = 3 + \sqrt{x-1}}$,



Question 18 (***)

The functions f and g are given by

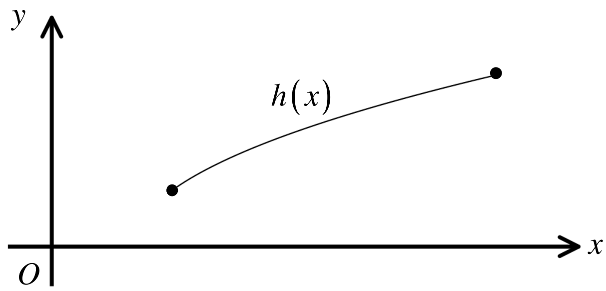
$$f(x) = \sqrt{x}, \ x \in \mathbb{R}, \ x \geq 0 .$$

$$g(x) = x - 2, \ x \in \mathbb{R} .$$

a) Find an expression for the function composition $fg(x)$.

The function h, whose graph is shown below, is defined by

$$h(x) = \sqrt{x-2}, \ x \in \mathbb{R}, \ 3 \leq x \leq 11 .$$

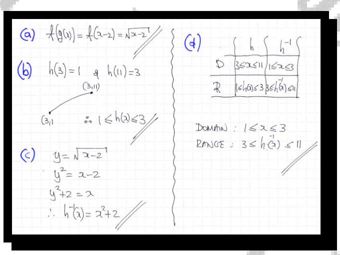


b) State the range of $h(x)$.

c) Determine an expression for the inverse function $h^{-1}(x)$.

d) State the domain and range of $h^{-1}(x)$.

$\boxed{}$, $\boxed{fg(x) = \sqrt{x-2}}$, $\boxed{1 \leq h(x) \leq 3}$, $\boxed{h^{-1}(x) = x^2 + 2}$, $\boxed{1 \leq x \leq 3 \ \ \& \ \ 3 \leq h^{-1}(x) \leq 11}$



Question 19 (***)

The functions f and g are defined by

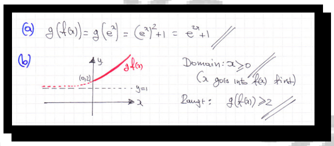
$$f(x) = \mathrm{e}^{x}, \quad x \in \mathbb{R}, \ x \geq 0.$$

$$g(x) = x^{2} + 1, \quad x \in \mathbb{R}.$$

a) Find an expression for $gf(x)$, in its simplest form.

b) Determine the domain and range of $gf(x)$.

$\boxed{gf(x) = \mathrm{e}^{2x} + 1}$, $\boxed{x \geq 0, \quad gf(x) \geq 2}$



Question 20 (***)

The functions f and g are defined by

$$f : x \mapsto x^2 - 2x - 3, \quad x \in \mathbb{R}, \ 0 \le x \le 5.$$

$$g : x \mapsto ax^2 + 2, \quad x \in \mathbb{R}, \ a \text{ is a real constant.}$$

a) Find the range of f.

b) Determine the value of a, if $gf(1) = 6$.

$\boxed{}$, $\boxed{-4 \le f(x) \le 12}$, $\boxed{a = \frac{1}{4}}$

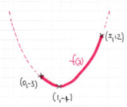
a) LOCATE THE MINIMUM OF THE QUADRATIC BY COMPLETING THE SQUARE

$f(x) = x^2 - x - 3$, $0 \le x \le 5$

$f(x) = (x-1)^2 - 1 - 3$

$f(x) = (x-1)^2 - 4$

SKETCH THE FUNCTION f



$\therefore$ RANGE IS:

$f(x) \in \mathbb{R}, \ -4 \le f(x) \le 12$

b) $f(x) = x^2 - 2x - 3$, $0 \le x \le 5$

$g(x) = ax^2 + 2$, $x \in \mathbb{R}$

$g(f(1)) = 6$

$g(1^2 - 2 \times 1 - 3) = 6$

$g(-4) = 6$

$a(-4)^2 + 2 = 6$

$16a + 2 = 6$

$16a = 4$

$a = \frac{1}{4}$

Question 21 (***)

The function f is defined as

$$f : x \mapsto \frac{2}{x-3} - \frac{4}{x^2-4x+3}, \ x \in \mathbb{R}, \ x > 1.$$

a) Show clearly that

$$f : x \mapsto \frac{2}{x-1}, \ x \in \mathbb{R}, \ x > 1.$$

b) Find an expression for f^{-1}, in its simplest form.

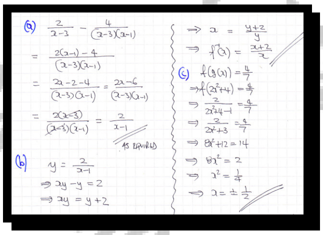
The function g is given by

$$g : x \mapsto 2x^2 + 4, \ x \in \mathbb{R}.$$

c) Solve the equation

$$fg(x) = \frac{4}{7}.$$

$\boxed{}$, $\boxed{f^{-1}(x) = \frac{x+2}{x} = 1 + \frac{2}{x}}$, $\boxed{x = \pm\frac{1}{2}}$



(a) $\frac{2}{x-3} - \frac{4}{(x-3)(x-1)}$

$= \frac{2(x-1)-4}{(x-3)(x-1)}$

$= \frac{2x-2-4}{(x-3)(x-1)} = \frac{2x-6}{(x-3)(x-1)}$

$= \frac{2\cancel{(x-3)}}{\cancel{(x-3)}(x-1)} = \frac{2}{x-1}$

AS REQUIRED

(b) $y = \frac{2}{x-1}$

$\Rightarrow xy - y = 2$

$\Rightarrow xy = y+2$

$\Rightarrow x = \frac{y+2}{y}$

$\Rightarrow f^{-1}(x) = \frac{x+2}{x}$

(c) $f(g(x)) = \frac{4}{7}$

$\Rightarrow f(2x^2+4) = \frac{4}{7}$

$\Rightarrow \frac{2}{2x^2+4-1} = \frac{4}{7}$

$\Rightarrow \frac{2}{2x^2+3} = \frac{4}{7}$

$\Rightarrow 8x^2+12 = 14$

$\Rightarrow 8x^2 = 2$

$\Rightarrow x^2 = \frac{1}{4}$

$\Rightarrow x = \pm\frac{1}{2}$

Question 22 (***)

The functions f and g are defined by

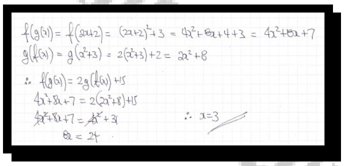
$$f : x \mapsto x^2 + 3, \quad x \in \mathbb{R}.$$

$$g : x \mapsto 2x + 2, \quad x \in \mathbb{R}.$$

Solve the equation

$$fg(x) = 2gf(x) + 15.$$

$\boxed{x = 3}$


$f(g(x)) = f(2x+2) = (2x+2)^2+3 = 4x^2+8x+4+3 = 4x^2+8x+7$

$g(f(x)) = g(x^2+3) = 2(x^2+3)+2 = 2x^2+8$

$\therefore \ f(g(x)) = 2g(f(x))+15$

$4x^2+8x+7 = 2(2x^2+8)+15$

$\cancel{4x^2}+8x+7 = \cancel{4x^2}+31$

$8x = 24$

$\therefore \ x=3$

Question 23 (***)

The functions f and g are defined as

$$f(x) = \frac{x+6}{x+2}, \quad x \in \mathbb{R}, \ x \neq -2$$

$$g(x) = 7 - 2x^2, \quad x \in \mathbb{R}.$$

a) State the range of $g(x)$.

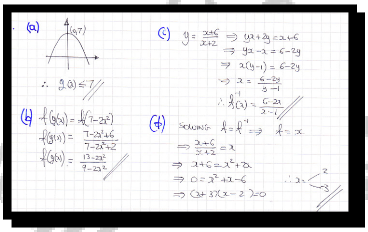
b) Find, as a simplified fraction, an expression for $fg(x)$.

c) Find, as a simplified fraction, an expression for $f^{-1}(x)$.

d) Solve the equation

$$f^{-1}(x) = f(x).$$

$\boxed{}$, $\boxed{g(x) \le 7}$, $\boxed{fg(x) = \frac{13-2x^2}{9-2x^2}}$, $\boxed{f^{-1}(x) = \frac{2x-6}{1-x}}$, $\boxed{x = -3, 2}$



Question 24 (***)

The function f is defined as

$$f : x \mapsto \frac{x}{x-1}, \quad x \in \mathbb{R},\ x \neq 1$$

a) Find in its simplest form the composition $ff(x)$.

b) Find an expression for $f^{-1}(x)$ in its simplest form.

$\boxed{ff(x) = x}$, $\boxed{f : x \mapsto \frac{x}{x-1}, \quad x \in \mathbb{R},\ x \neq 1}$

(a) $f(f(x)) = f\left(\frac{x}{x+1}\right) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \dots$ multiply top & bottom by $(x-1)$

$= \frac{x}{x-(x-1)} = \frac{x}{x-x+1} = \frac{x}{1} = x$

(b) $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ for all x $\therefore f^{-1}(x) = f(x) = \frac{x}{x+1}$

Question 25 (***)

The functions g and f are given by

$$g : x \mapsto 4 - 3x, \quad x \in \mathbb{R}$$

$$f : x \mapsto x^2 + ax + b, \quad x \in \mathbb{R},$$

where a and b are non zero constants.

Given that $fg(2) = -5$ and $gf(2) = -29$, find the value a and the value of b.

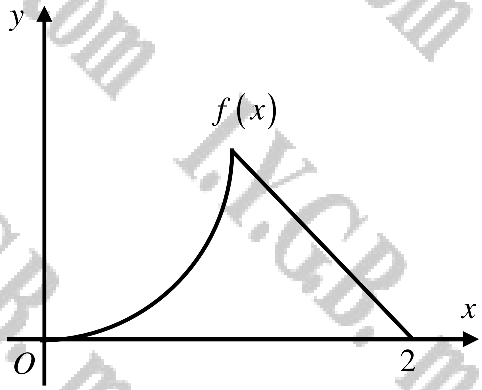
$\boxed{a = 4, b = -1}$

$f(g(2)) = -5$
$f(-2) = -5$
$4 - 2a + b = -5$
$-2a + b = -9$
$\boxed{2a - b = 9}$

$g(f(2)) = -29$
$g(4 + 2a + b) = -29$
$4 - 3(4 + 2a + b) = -29$
$4 - 12 - 6a - 3b = -29$
$-6a - 3b = -21$
$6a + 3b = 21$
$\boxed{2a + b = 7}$

Adding
$2a - b = 9$
$2a + b = 7$
$4a = 16$
$\therefore a = 4$
$b = -1$

Question 26 (***)

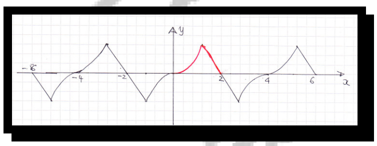


The figure above shows the **part** of the curve with equation

$$y = f(x), \text{ for } 0 \le x \le 2 .$$

Given that the curve is odd and periodic with period 4, sketch the curve for $-6 \le x \le 6$.

graph



Question 27 (***)

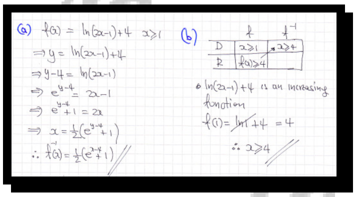
The function f is defined by

$$f(x) = \ln(2x-1) + 4, \;\; x \in \mathbb{R}, \; x \geq 1.$$

a) Find $f^{-1}(x)$ in its simplest form.

b) Determine the domain of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = \tfrac{1}{2}\left(1 + e^{x-4}\right)}$, $\boxed{x \in \mathbb{R}, \; x \geq 4}$



Question 28 (***)

The function f is given by

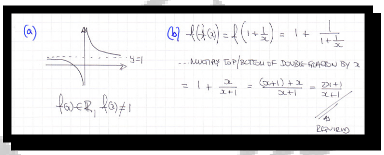
$$f : x \mapsto 1 + \frac{1}{x}, \;\; x \in \mathbb{R}, \; x \neq 0.$$

a) Find the range of f.

b) Show clearly that

$$ff : x \mapsto \frac{2x+1}{x+1}.$$

$\boxed{f(x) \in \mathbb{R}, \;\; f(x) \neq 1}$



Question 29 (***)

The functions f and g are defined as

$$f(x) = 2x - 1, \quad x \in \mathbb{R}$$

$$g(x) = \mathrm{e}^{\frac{x}{2}}, \quad x \in \mathbb{R}.$$

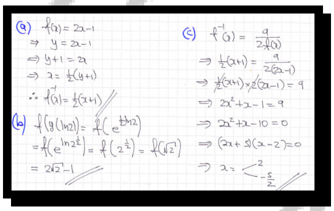
a) Find an expression for $f^{-1}(x)$.

b) Find, as an exact surd, the value of $fg(\ln 2)$.

c) Solve the equation

$$f^{-1}(x) = \frac{9}{2f(x)}.$$

$\boxed{f^{-1}(x) = \frac{x+1}{2}}$, $\boxed{fg(\ln 2) = 2\sqrt{2} - 1}$, $\boxed{x = -\frac{5}{2}, 2}$



Question 30 (***)

The functions f, g and h are defined as

$$f(x) = x^2 - 1, \quad x \in \mathbb{R}$$

$$g(x) = e^{\frac{3x}{2}}, \quad x \in \mathbb{R}$$

$$h(x) = fg(x), \quad x \in \mathbb{R}.$$

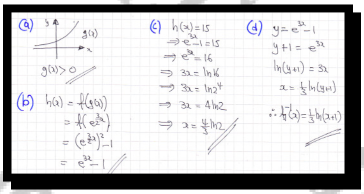
a) State the range of $g(x)$.

b) Find, in its simplest form, an expression for $h(x)$.

c) Solve the equation $h(x) = 15$, giving the answer in terms of $\ln 2$.

d) Find an expression for $h^{-1}(x)$, the inverse of $h(x)$.

$\boxed{g(x) > 0}$, $\boxed{fg(x) = e^{3x} - 1}$, $\boxed{x = \frac{4}{3}\ln 2}$, $\boxed{h^{-1}(x) = \frac{1}{3}\ln(x+1)}$



Question 31 (***)

The functions f and g are defined by

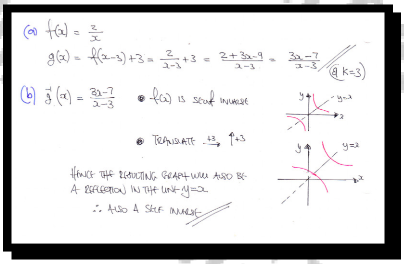
$$f(x) \equiv \frac{2}{x}, \ x \in \mathbb{R}, \ x \neq 0$$

$$g(x) \equiv f(x-3)+3, \ x \in \mathbb{R}, \ x \neq k.$$

a) Find an expression for $g(x)$, as a simplified fraction stating the value of the constant k.

b) Find an expression for $g^{-1}(x)$.

$\boxed{g(x) = \frac{3x-7}{x-3}}$, $\boxed{k=3}$, $\boxed{g^{-1}(x) = \frac{3x-7}{x-3}, \text{ self inverse}}$



Question 32 (***)

The functions f and g are defined by

$$f : x \mapsto 4 - x^2,\ x \in \mathbb{R}$$

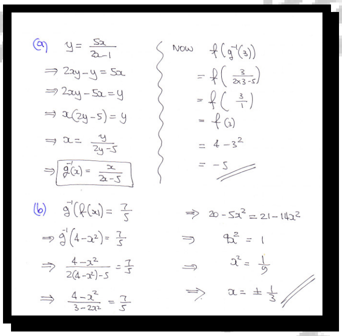
$$g : x \mapsto \frac{5x}{2x-1},\ x \in \mathbb{R},\ x \neq \frac{1}{2}.$$

a) Evaluate $fg^{-1}(3)$.

b) Solve the equation

$$g^{-1}f(x) = \frac{7}{5}.$$

$\boxed{}$, $\boxed{fg^{-1}(3) = -5}$, $\boxed{x = \pm\frac{1}{3}}$



(a) $y = \frac{5x}{2x-1}$

$\Rightarrow 2xy - y = 5x$

$\Rightarrow 2xy - 5x = y$

$\Rightarrow x(2y-5) = y$

$\Rightarrow x = \frac{y}{2y-5}$

$\Rightarrow \boxed{g^{-1}(x) = \frac{x}{2x-5}}$

Now $f(g^{-1}(3))$

$= f\left(\frac{3}{2\times3-5}\right)$

$= f\left(\frac{3}{1}\right)$

$= f(3)$

$= 4 - 3^2$

$= -5$

(b) $g^{-1}(f(x)) = \frac{7}{5}$

$\Rightarrow g^{-1}(4-x^2) = \frac{7}{5}$

$\Rightarrow \frac{4-x^2}{2(4-x^2)-5} = \frac{7}{5}$

$\Rightarrow \frac{4-x^2}{3-2x^2} = \frac{7}{5}$

$\Rightarrow 20 - 5x^2 = 21 - 14x^2$

$\Rightarrow 9x^2 = 1$

$\Rightarrow x^2 = \frac{1}{9}$

$\Rightarrow x = \pm\frac{1}{3}$

Question 33 (***)

The function f is defined as

$$f : x \mapsto \frac{1}{x+2} + \frac{2x+11}{2x^2+x-6} \quad x \in \mathbb{R}, \quad x > \frac{3}{2}.$$

a) Show clear that

$$f : x \mapsto \frac{4}{2x-3}, \quad x \in \mathbb{R}, \quad x > \frac{3}{2}.$$

b) Find an expression for f^{-1}, in its simplest form.

c) Find the domain of f^{-1}.

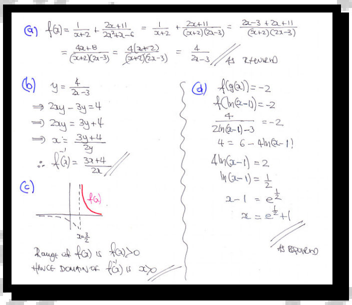
The function g is given by

$$g : x \mapsto \ln(x-1), \quad x \in \mathbb{R}, \quad x > 1.$$

d) Show that $x = 1 + \sqrt{e}$ is the solution of the equation

$$fg(x) = -2.$$

$\boxed{}$, $\boxed{f^{-1}(x) = \frac{3x+4}{2x}}$, $\boxed{x > 0}$



Question 34 (***)

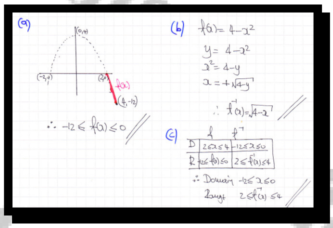
$$f(x) = 4 - x^2, \ x \in \mathbb{R}, \ 2 \le x \le 4.$$

a) Determine the range of $f(x)$.

b) Find an expression for the inverse function $f^{-1}(x)$.

c) State the domain and range of $f^{-1}(x)$.

$\boxed{-12 \le f(x) \le 0}$, $\boxed{f^{-1}(x) = \sqrt{4-x}}$, $\boxed{-12 \le x \le 0}$, $\boxed{2 \le f^{-1}(x) \le 4}$



Question 35 **(***+)**

A function is defined by

$$f(x) = \sqrt{e^x - 1}, \quad x \geq 0.$$

a) Find the values of …

i. … $f(\ln 5)$.

ii. … $f'(\ln 5)$.

The inverse function of $f(x)$ is $g(x)$.

b) Determine an expression for $g(x)$.

c) State the value of $g'(2)$.

$\boxed{f(\ln 5) = 2}$, $\boxed{f'(\ln 5) = \frac{5}{4}}$, $\boxed{g(x) = \ln(x^2 + 1)}$, $\boxed{g'(2) = \frac{4}{5}}$

a) I) JUST SUBSTITUTING

$f(\ln 5) = \sqrt{e^{\ln 5} - 1} = \sqrt{5-1} = 2$

II) DIFFERENTIATING FIRST

$f(x) = (e^x - 1)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(e^x - 1)^{-\frac{1}{2}} \times e^x$

$f'(x) = \dfrac{e^x}{2\sqrt{e^x - 1}}$

$f'(\ln 5) = \dfrac{e^{\ln 5}}{2\sqrt{e^{\ln 5} - 1}} = \dfrac{5}{2\sqrt{5-1}} = \dfrac{5}{4}$

b) BY THE "STANDARD" METHOD

$\Longrightarrow y = \sqrt{e^x - 1}$

$\Longrightarrow y^2 = e^x - 1$

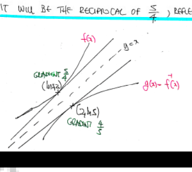
$\Longrightarrow y^2 + 1 = e^x$

$\Longrightarrow x = \ln(1 + y^2)$

$\therefore f^{-1}(x) = \ln(1 + x^2)$

$g(x)$

c) IT WILL BE THE RECIPROCAL OF $\frac{5}{4}$, REFLECTION IN THE LINE $y = x$



$\therefore g'(2) = \frac{4}{5}$

Question 36 (***+)

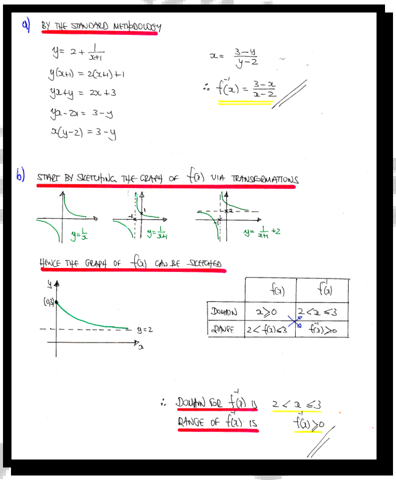
A function f is defined by

$$f(x) = 2 + \frac{1}{x+1}, \quad x \in \mathbb{R},\ x \geq 0.$$

a) Find an expression for $f^{-1}(x)$, as a simplified fraction.

b) Find the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = \frac{3-x}{x-2}}$, $\boxed{2 < x \leq 3}$, $\boxed{f^{-1}(x) \geq 0}$



a) BY THE STANDARD METHODOLOGY

$y = 2 + \frac{1}{x+1}$

$y(x+1) = 2(x+1)+1$

$yx + y = 2x + 3$

$yx - 2x = 3 - y$

$x(y-2) = 3 - y$

$x = \frac{3-y}{y-2}$

$\therefore f^{-1}(x) = \frac{3-x}{x-2}$

b) START BY SKETCHING THE GRAPH OF $f(x)$ VIA TRANSFORMATIONS

HENCE THE GRAPH OF $f(x)$ CAN BE SKETCHED

	$f(x)$	$f^{-1}(x)$
DOMAIN	$x \geq 0$	$2 < x \leq 3$
RANGE	$2 < f(x) \leq 3$	$f^{-1}(x) \geq 0$

$\therefore$ DOMAIN FOR $f^{-1}(x)$ IS $2 < x \leq 3$

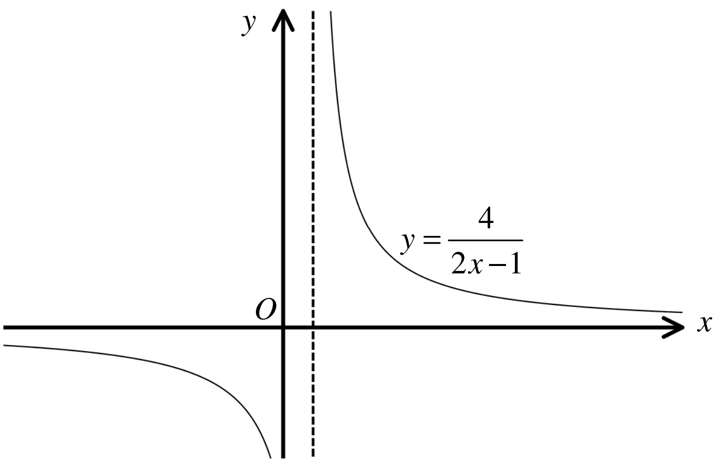
RANGE OF $f^{-1}(x)$ IS $f^{-1}(x) \geq 0$

Question 37 **(***+)**

$$y = \frac{2}{x-2} - \frac{6}{(x-2)(2x-1)}.$$

a) Show clearly that $y = \dfrac{4}{2x-1}$

The figure below shows the graph of $y = \dfrac{4}{2x-1}$, $x \neq a$.



b) State the equation of the vertical asymptote of the curve, shown dotted in the figure above.

The function f is defined

$$f(x) = \frac{4}{2x-1}, \quad x > 1$$

c) State the range of $f(x)$.

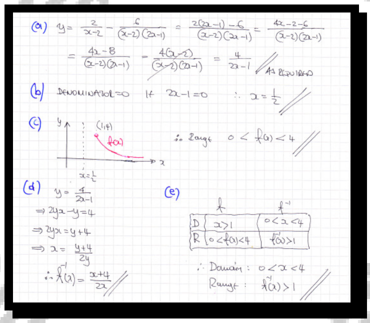
[continues overleaf]

[continued from overleaf]

d) Obtain an expression for the inverse of the function, $f^{-1}(x)$.

e) State the domain and range of $f^{-1}(x)$.

$\boxed{}$, $\boxed{x=\frac{1}{2}}$, $\boxed{0<f(x)<4}$, $\boxed{f^{-1}(x)=\frac{x+4}{2x}}$, $\boxed{0<x<4}$, $\boxed{f^{-1}(x)>1}$



(a) $y=\frac{2}{x-2}-\frac{6}{(x-2)(2x-1)}=\frac{2(2x-1)-6}{(x-2)(2x-1)}=\frac{4x-2-6}{(x-2)(2x-1)}$

$=\frac{4x-8}{(x-2)(2x-1)}=\frac{4\cancel{(x-2)}}{\cancel{(x-2)}(2x-1)}=\frac{4}{2x-1}$ // AS REQUIRED

(b) DENOMINATOR $=0$ IF $2x-1=0$ $\therefore x=\frac{1}{2}$ //

(c)

$\therefore$ Range $0<f(x)<4$ //

(d) $y=\frac{4}{2x-1}$

$\Rightarrow 2yx-y=4$

$\Rightarrow 2yx=y+4$

$\Rightarrow x=\frac{y+4}{2y}$

$\therefore f^{-1}(x)=\frac{x+4}{2x}$ //

(e)

	f	f^{-1}
D	$x>1$	$0<x<4$
R	$0<f(x)<4$	$f^{-1}(x)>1$

$\therefore$ Domain : $0<x<4$

Range : $f^{-1}(x)>1$ //

Question 38 (***+)

The function f is defined by

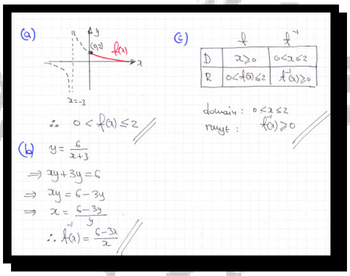
$$f(x) = \frac{6}{x+3}, \quad x \in \mathbb{R},\ x \geq 0.$$

a) Find the range of $f(x)$.

b) Determine an expression for $f^{-1}(x)$ in its simplest form.

c) Find the domain and range of $f^{-1}(x)$.

$$\boxed{0 < f(x) \leq 2}, \quad \boxed{f^{-1}(x) = \frac{6-3x}{x}}, \quad \boxed{0 < x \leq 2,\ f^{-1}(x) \geq 0}$$



Question 39 **(***+)**

The function $f(x)$ is defined by

$$f(x) = \sqrt{x+1}, \quad x \in \mathbb{R}, \; x \geq 0 .$$

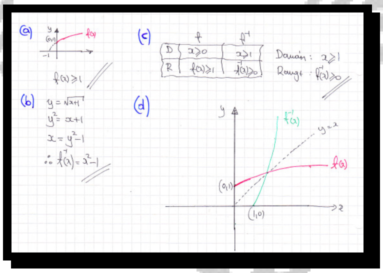
a) Find the range of $f(x)$.

b) Find an expression for $f^{-1}(x)$ in its simplest form.

c) State the domain and range of $f^{-1}(x)$.

d) Sketch in the same diagram $f(x)$ and $f^{-1}(x)$.

$\boxed{f(x) \geq 1}$, $\boxed{f^{-1}(x) = x^2 - 1}$, $\boxed{x \geq 1, \;\; f^{-1}(x) \geq 0}$



Question 40 (***+)

The function f is given by

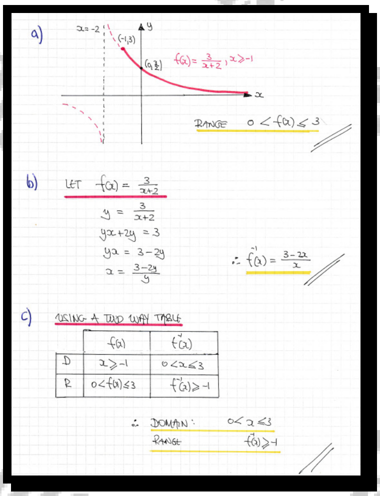
$$f : x \mapsto \frac{3}{x+2}, \; x \in \mathbb{R}, \; x \geq -1.$$

a) By sketching the graph of f, or otherwise, state its range.

b) Determine an expression for $f^{-1}(x)$, the inverse of f.

c) Find the domain and range of $f^{-1}(x)$.

$\boxed{}$, $\boxed{f(x) \in \mathbb{R}, \; 0 < f(x) \leq 3}$, $\boxed{f^{-1}(x) = \frac{3}{x} - 2 = \frac{3-2x}{x}}$, $\boxed{0 < x \leq 3, \; f^{-1}(x) \geq -1}$,



a)

RANGE $0 < f(x) \leq 3$

b) LET $f(x) = \frac{3}{x+2}$

$y = \frac{3}{x+2}$

$yx + 2y = 3$

$yx = 3 - 2y$

$x = \frac{3-2y}{y}$

$\therefore f^{-1}(x) = \frac{3-2x}{x}$

c) USING A TWO WAY TABLE

	$f(x)$	$f^{-1}(x)$
D	$x \geq -1$	$0 < x \leq 3$
R	$0 < f(x) \leq 3$	$f^{-1}(x) \geq -1$

$\therefore$ DOMAIN: $0 < x \leq 3$

RANGE $f^{-1}(x) \geq -1$

Question 41 **(***+)**

The function f is satisfies

$$f(x) = \sqrt{x} - 3, \; x \in \mathbb{R}, \; 0 \le x \le 9.$$

a) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

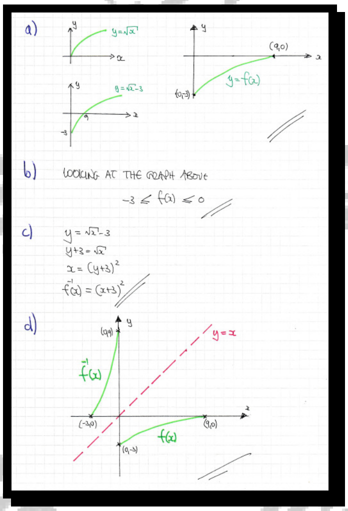
b) State the range of $f(x)$.

c) Find an expression for $f^{-1}(x)$.

d) Sketch in the same set of axes as that of part **(a)** the graph of $f^{-1}(x)$.

The sketch must include the coordinates of the points where the graph of $f^{-1}(x)$ meets the coordinate axes, and how $f^{-1}(x)$ is related graphically to $f(x)$.

$\boxed{}$, $\boxed{-3 \le f(x) \le 0}$, $\boxed{f^{-1}(x) = (x+3)^2}$



Question 42 (***+)

The function f satisfies

$$f : x \mapsto \frac{3x+1}{x+4}, \; x \in \mathbb{R}, \; x > -4 .$$

a) Find an expression for $f^{-1}(x)$ in its simplest form.

b) Determine the domain and the range of $f^{-1}(x)$.

The function g is given by

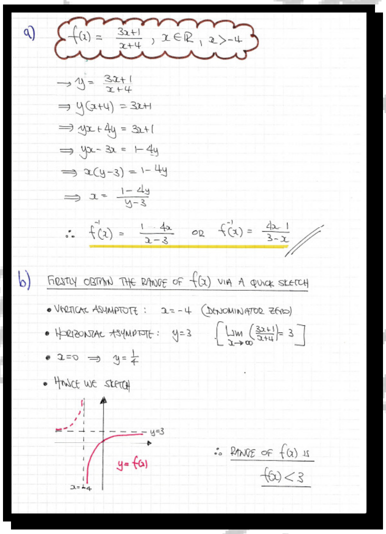
$$g : x \mapsto e^{x} - 3, \quad x \in \mathbb{R} .$$

c) Solve the equation

$$fg(x) = \tfrac{4}{5} ,$$

giving exact answers in terms of $\ln 2$.

$\boxed{}$, $\boxed{f^{-1} : x \mapsto \frac{1-4x}{x-3}}$, $\boxed{x \in \mathbb{R}, \quad x < 3}$, $\boxed{f^{-1}(x) \in \mathbb{R}, \quad f^{-1}(x) > -4}$, $\boxed{x = 2\ln 2}$



a) $f(x) = \frac{3x+1}{x+4}, \; x \in \mathbb{R}, \; x > -4$

$\rightarrow y = \frac{3x+1}{x+4}$

$\Rightarrow y(x+4) = 3x+1$

$\Rightarrow yx + 4y = 3x+1$

$\Rightarrow yx - 3x = 1 - 4y$

$\Rightarrow x(y-3) = 1 - 4y$

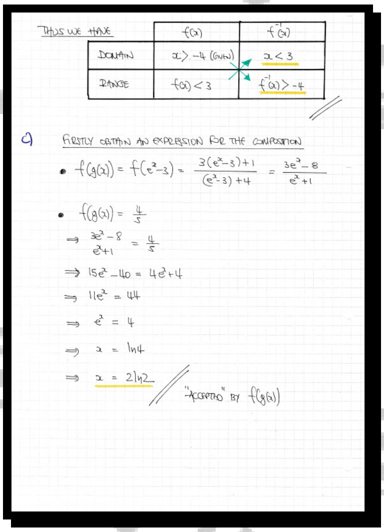
$\Rightarrow x = \frac{1-4y}{y-3}$

$\therefore f^{-1}(x) = \frac{1-4x}{x-3}$ OR $f^{-1}(x) = \frac{4x-1}{3-x}$

b) FIRSTLY OBTAIN THE RANGE OF $f(x)$ VIA A QUICK SKETCH

- VERTICAL ASYMPTOTE: $x = -4$ (DENOMINATOR ZERO)
- HORIZONTAL ASYMPTOTE: $y = 3$ $\left[\lim_{x \to \infty} \left(\frac{3x+1}{x+4}\right) = 3\right]$
- $x = 0 \Rightarrow y = \frac{1}{4}$
- HENCE WE SKETCH

$\therefore$ RANGE OF $f(x)$ IS $f(x) < 3$



THUS WE HAVE

	$f(x)$	$f^{-1}(x)$
DOMAIN	$x > -4$ (GIVEN)	$x < 3$
RANGE	$f(x) < 3$	$f^{-1}(x) > -4$

c) FIRSTLY OBTAIN AN EXPRESSION FOR THE COMPOSITION

- $f(g(x)) = f(e^x - 3) = \frac{3(e^x-3)+1}{(e^x-3)+4} = \frac{3e^x - 8}{e^x + 1}$

- $f(g(x)) = \frac{4}{5}$

$\Rightarrow \frac{3e^x - 8}{e^x + 1} = \frac{4}{5}$

$\Rightarrow 15e^x - 40 = 4e^x + 4$

$\Rightarrow 11e^x = 44$

$\Rightarrow e^x = 4$

$\Rightarrow x = \ln 4$

$\Rightarrow x = 2\ln 2$

"ACCEPTED" BY $f(g(x))$

Question 43 (***+)

The function f is defined as

$$f : x \mapsto \frac{2x-1}{x^2-x-2} - \frac{1}{x-2}, \ x \in \mathbb{R}, \ x > 4.$$

a) Show clearly that

$$f : x \mapsto \frac{1}{x+1}, \ x \in \mathbb{R}, \ x > 4.$$

b) Find the range of f.

c) Determine an expression for the inverse function, $f^{-1}(x)$.

d) State the domain and range of $f^{-1}(x)$.

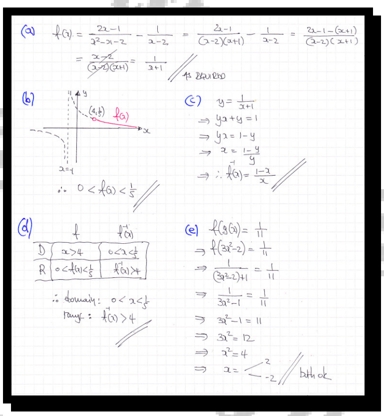
The function g is given by

$$g : x \mapsto 3x^2 - 2, \ x \in \mathbb{R}.$$

e) Solve the equation

$$fg(x) = \frac{1}{11}.$$

$\boxed{\quad}$, $\boxed{0 < f(x) < \frac{1}{5}}$, $\boxed{f^{-1}(x) = \frac{1-x}{x}}$, $\boxed{0 < x < \frac{1}{5}}$, $\boxed{f^{-1}(x) > 4}$, $\boxed{x = \pm 2}$



Question 44 (***+)

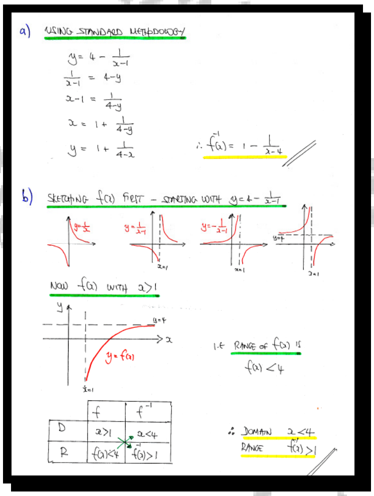
A function f is defined by

$$f(x) = 4 - \frac{1}{x-1}, \quad x \in \mathbb{R},\ x > 1.$$

a) Determine an expression for the inverse, $f^{-1}(x)$.

b) Find the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = 1 - \frac{1}{x-4} = \frac{x-5}{x-4}}$, $\boxed{x < 4}$, $\boxed{f^{-1}(x) > 1}$


a) USING STANDARD METHODOLOGY

$y = 4 - \frac{1}{x-1}$

$\frac{1}{x-1} = 4 - y$

$x - 1 = \frac{1}{4-y}$

$x = 1 + \frac{1}{4-y}$

$y = 1 + \frac{1}{4-x}$

$\therefore f^{-1}(x) = 1 - \frac{1}{x-4}$

b) SKETCHING $f(x)$ FIRST – STARTING WITH $y = 4 - \frac{1}{x-1}$

$y = \frac{1}{x}$ $y = \frac{1}{x-1}$ $y = -\frac{1}{x-1}$ $y = 4$ $x = 1$

NOW $f(x)$ WITH $x > 1$

$y = f(x)$ $y = 4$ $x = 1$

I.E RANGE OF $f(x)$ IS

$f(x) < 4$

	f	f^{-1}
D	$x > 1$	$x < 4$
R	$f(x) < 4$	$f^{-1}(x) > 1$

$\therefore$ DOMAIN $x < 4$

RANGE $f^{-1}(x) > 1$

Question 45 (***+)

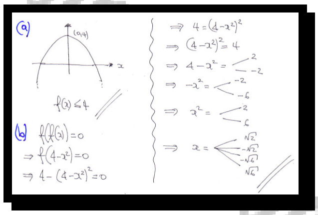
$$f(x) = 4 - x^2,\ \ x \in \mathbb{R}.$$

a) State the range of $f(x)$.

b) Solve the equation

$$ff(x) = 0.$$

$\boxed{f(x) \leq 4}$, $\boxed{x = \pm\sqrt{2}, \pm\sqrt{6}}$



Question 46 (***+)

$$f(x) = 4(x+1)^2, \ x \in \mathbb{R}, \ x \le -2.$$

a) State the range of $f(x)$.

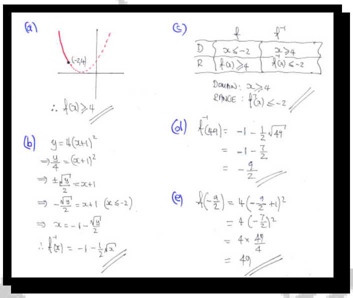
b) Find an expression for the inverse function $f^{-1}(x)$.

c) State the domain and range of $f^{-1}(x)$.

d) Evaluate $f^{-1}(49)$.

e) Verify that the answer to part (d) is correct by carrying an appropriate calculation involving $f(x)$.

$\boxed{f(x) \ge 4}$, $\boxed{f^{-1}(x) = -1 - \frac{1}{2}\sqrt{x}}$, $\boxed{x \ge 4, \ f^{-1}(x) \le -2}$, $\boxed{f^{-1}(49) = -\frac{9}{2}}$, $\boxed{f\left(-\frac{9}{2}\right) = 49}$



Question 47 (***+)

The function f is given by

$$f : x \mapsto 3 + \frac{2}{x-2}, \quad x \in \mathbb{R}, \ x > 2 .$$

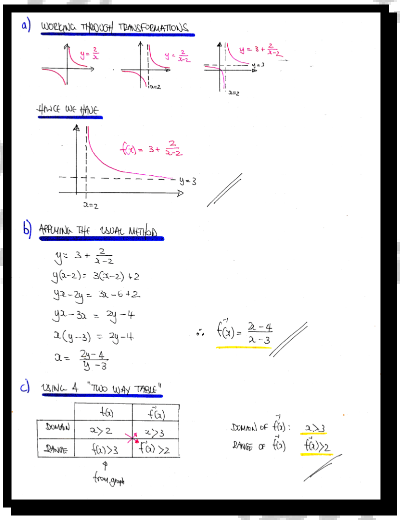
a) Sketch the graph of f.

b) Find an expression for $f^{-1}(x)$ as a single fraction, in its simplest form.

c) Find the domain and range of $f^{-1}(x)$.

d) Find the value of x that satisfy the equation $f(x) = f^{-1}(x)$

$\boxed{}$, $\boxed{f^{-1}(x) = \frac{2x-4}{x-3}}$, $\boxed{x \in \mathbb{R}, \ x > 3}$, $\boxed{f(x) \in \mathbb{R}, \ f^{-1}(x) > 2}$, $\boxed{x = 4, x \neq 1}$



a) WORKING THROUGH TRANSFORMATIONS

HENCE WE HAVE

b) APPLYING THE USUAL METHOD

$y = 3 + \frac{2}{x-2}$

$y(x-2) = 3(x-2) + 2$

$yx - 2y = 3x - 6 + 2$

$yx - 3x = 2y - 4$

$x(y-3) = 2y - 4$

$x = \frac{2y-4}{y-3}$

$\therefore \ f^{-1}(x) = \frac{2x-4}{x-3}$

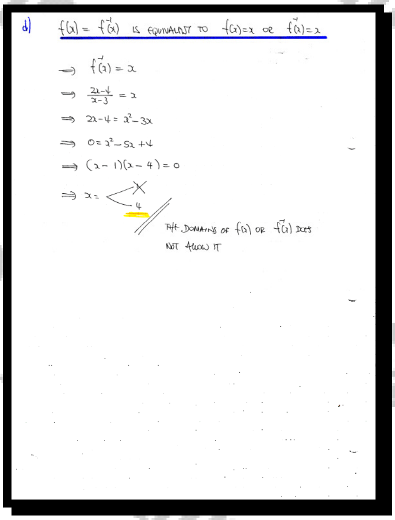
c) USING A "TWO WAY TABLE"

	$f(x)$	$f^{-1}(x)$
DOMAIN	$x > 2$	$x > 3$
RANGE	$f(x) > 3$	$f^{-1}(x) > 2$

from graph

DOMAIN OF $f^{-1}(x)$: $x > 3$

RANGE OF $f^{-1}(x)$: $f^{-1}(x) > 2$



d) $f(x) = f^{-1}(x)$ IS EQUIVALENT TO $f(x) = x$ OR $f^{-1}(x) = x$

$\Rightarrow f^{-1}(x) = x$

$\Rightarrow \frac{2x-4}{x-3} = x$

$\Rightarrow 2x - 4 = x^2 - 3x$

$\Rightarrow 0 = x^2 - 5x + 4$

$\Rightarrow (x-1)(x-4) = 0$

$\Rightarrow x = \cancel{1}, \ 4$

THE DOMAINS OF $f(x)$ OR $f^{-1}(x)$ DOES NOT ALLOW IT

Question 48 **(***+)**

The functions f and g are defined below

$$f(x) = x^2 + 2, \ x \in \mathbb{R}, \ x > 0$$

$$g(x) = 3x - 1, \ x \in \mathbb{R}, \ x > 4 .$$

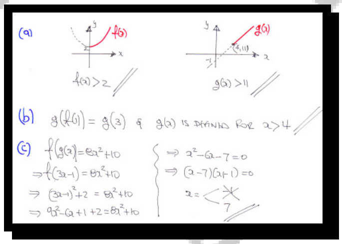
a) Write down the range of $f(x)$ and the range of $g(x)$.

b) Explain why $gf(1)$ cannot be evaluated.

c) Solve the equation

$$fg(x) = 8x^2 + 10 .$$

$\boxed{f(x) \in \mathbb{R}, \ f(x) > 2}$, $\boxed{g(x) \in \mathbb{R}, \ g(x) > 11}$ $\boxed{x = 7, \ x \neq 1}$



Question 49 (***+)

The functions f and g are defined below

$$f(x) = x^2 - 2, \ x \in \mathbb{R}$$

$$g(x) = 2x + 3, \ x \in \mathbb{R}, \ x > 0.$$

a) Write down the range of $f(x)$.

b) Find, in its simplest form, an expression for $fg(x)$.

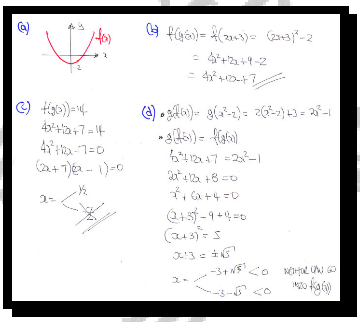
c) Solve the equation

$$fg(x) = 14.$$

d) Show that there is no solution for the equation

$$fg(x) = gf(x).$$

$\boxed{f(x) \in \mathbb{R}, \ f(x) \geq -2}$, $\boxed{fg(x) = 4x^2 + 12x + 7}$, $\boxed{x = \frac{1}{2}, x \neq -\frac{7}{2}}$



Question 50 (***+)

The functions f and g are defined as

$$f(x) = 4 + \ln x, \; x \in \mathbb{R}, \; x > 0.$$

$$g(x) = \mathrm{e}x^2, \; x \in \mathbb{R}.$$

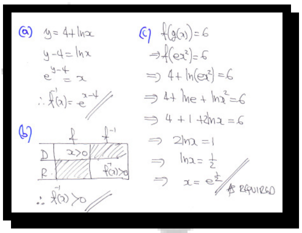
a) Find an expression for $f^{-1}(x)$.

b) State the range of $f^{-1}(x)$.

c) Show that $x = \sqrt{\mathrm{e}}$ is a solution of the equation

$$fg(x) = 6.$$

$\boxed{f^{-1}(x) = \mathrm{e}^{x-4}}$, $\boxed{f^{-1}(x) > 0}$



Question 51 **(***+)**

The function f is given by

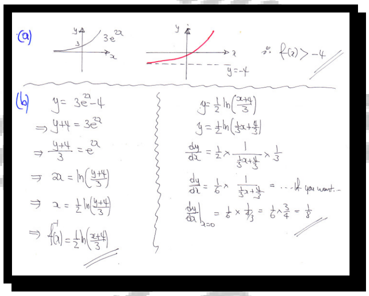
$$f(x) = 3e^{2x} - 4, \; x \in \mathbb{R}.$$

a) State the range of $f(x)$.

b) Find an expression for $f^{-1}(x)$.

c) Find the value of the gradient on $f^{-1}(x)$ at the point where $x = 0$.

$\boxed{f(x) > -4}$, $\boxed{f^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+4}{3}\right)}$, $\boxed{\frac{1}{8}}$



Question 52 (***+)

The functions f and g are defined

$$f(x) = x^2 - 10x, \ x \in \mathbb{R}$$

$$g(x) = e^x + 5, \ x \in \mathbb{R}.$$

a) Find, showing all steps in the calculation, the value of $g(3\ln 2)$.

b) Find, in its simplest form, an expression for $fg(x)$.

c) Show clearly that

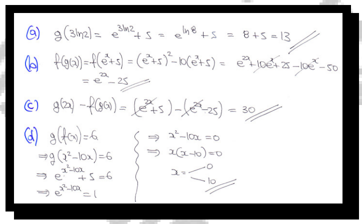
$$g(2x) - fg(x) = k,$$

stating the value of the constant k.

d) Solve the equation

$$gf(x) = 6.$$

$\boxed{g(3\ln 2) = 13}$, $\boxed{fg(x) = e^{2x} - 25}$, $\boxed{k = 30}$, $\boxed{x = 0, \ x = 10}$



(a) $g(3\ln 2) = e^{3\ln 2} + 5 = e^{\ln 8} + 5 = 8 + 5 = 13$

(b) $f(g(x)) = f(e^x + 5) = (e^x + 5)^2 - 10(e^x + 5) = e^{2x} + \cancel{10e^x} + 25 - \cancel{10e^x} - 50$
$= e^{2x} - 25$

(c) $g(2x) - f(g(x)) = (\cancel{e^{2x}} + 5) - (\cancel{e^{2x}} - 25) = 30$

(d) $g(f(x)) = 6$
$\Rightarrow g(x^2 - 10x) = 6$
$\Rightarrow e^{x^2 - 10x} + 5 = 6$
$\Rightarrow e^{x^2 - 10x} = 1$
$\Rightarrow x^2 - 10x = 0$
$\Rightarrow x(x - 10) = 0$
$x = 0, \ 10$

Question 53 (***+)

$$f(x) = \mathrm{e}^{x}, \; x \in \mathbb{R}, \; x > 0.$$

$$g(x) = 2x^{3} + 11, \; x \in \mathbb{R}.$$

a) Find and simplify an expression for the composite function $gf(x)$.

b) State the domain and range of $gf(x)$.

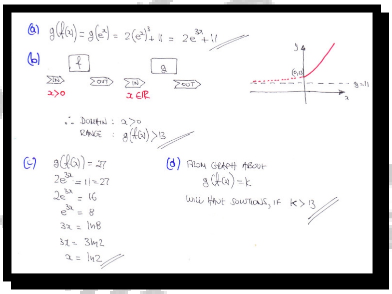
c) Solve the equation

$$gf(x) = 27.$$

The equation $gf(x) = k$, where k is a constant, has solutions.

d) State the range of the possible values of k.

$\boxed{}$, $\boxed{gf(x) = 2\mathrm{e}^{3x} + 11}$, $\boxed{x > 0, \; gf(x) > 13}$, $\boxed{x = \ln 2}$, $\boxed{k > 13}$



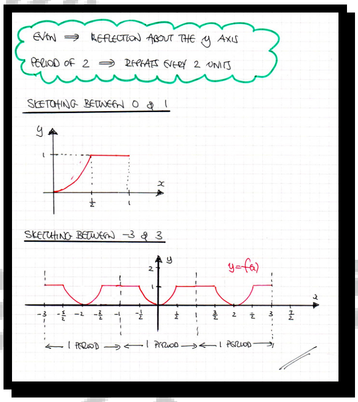
Question 54 (***+)

An **even** function f, of period 2 is defined by

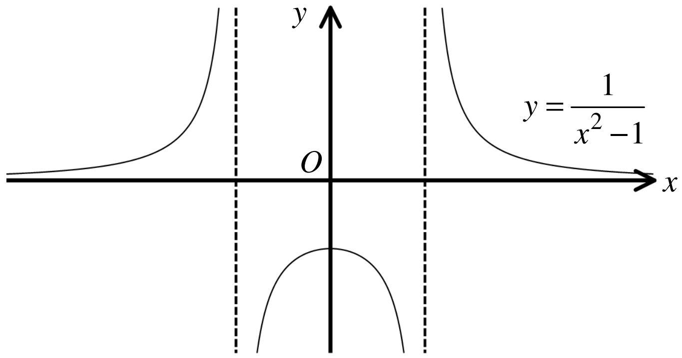
$$f(x) \equiv \begin{cases} 4x^2 & 0 \le x \le \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \end{cases}$$

Sketch the graph of $f(x)$ for $-3 \le x \le 3$.

$\boxed{}$, $\boxed{\text{graph}}$



Question 55 (***+)



The figure above shows the graph of the curve C with equation

$$y = \frac{1}{x^2 - 1}, \quad x \neq \pm 1.$$

a) State the equations of the vertical asymptotes of the curve, marked with dotted lines in the diagram.

The function f is defined as

$$f(x) = \frac{1}{x^2 - 1}, \quad x \in \mathbb{R}, \; x > 1.$$

b) Write down the range of $f(x)$.

c) Find an expression for $f^{-1}(x)$.

[continues overleaf]

[continued from overleaf]

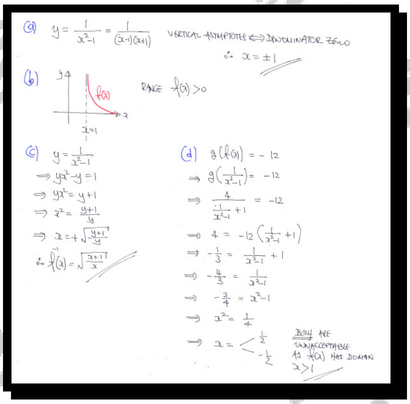
The function g is defined as

$$g(x) = \frac{4}{x+1}, \quad x \in \mathbb{R}, \quad x \neq -1.$$

d) Show no value of x satisfies the equation

$$gf(x) = -12.$$

$$\boxed{x = \pm 1}, \quad \boxed{f(x) \in \mathbb{R},\ f(x) > 0}, \quad \boxed{f^{-1}(x) = \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}}}, \quad \boxed{x = \pm\frac{1}{2}}$$



Question 56 (***+)

The functions f and g are defined by

$$f(x) = x - \frac{1}{x}, \ x \in \mathbb{R}, \ x \geq 1$$

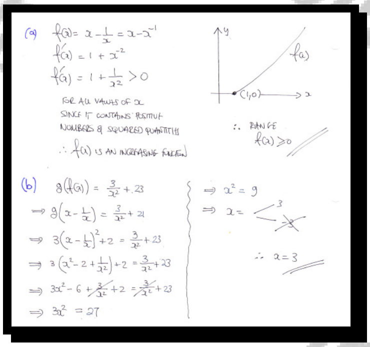
$$g(x) = 3x^2 + 2, \ x \in \mathbb{R}, \ x \geq 0.$$

a) By showing that $f(x)$ is an increasing function, find its range.

b) Solve the equation

$$gf(x) = \frac{3}{x^2} + 23.$$

$\boxed{f(x) \geq 0}$, $\boxed{x = 3}$


(a) $f(x) = x - \frac{1}{x} = x - x^{-1}$

$f'(x) = 1 + x^{-2}$

$f'(x) = 1 + \frac{1}{x^2} > 0$

FOR ALL VALUES OF x
SINCE IT CONTAINS POSITIVE
NUMBERS & SQUARED QUANTITIES

$\therefore f(x)$ IS AN INCREASING FUNCTION

$\therefore$ RANGE
$f(x) \geq 0$

(b) $g(f(x)) = \frac{3}{x^2} + 23$

$\Rightarrow g\left(x - \frac{1}{x}\right) = \frac{3}{x^2} + 21$

$\Rightarrow 3\left(x - \frac{1}{x}\right)^2 + 2 = \frac{3}{x^2} + 23$

$\Rightarrow 3\left(x^2 - 2 + \frac{1}{x^2}\right) + 2 = \frac{3}{x^2} + 23$

$\Rightarrow 3x^2 - 6 + \cancel{\frac{3}{x^2}} + 2 = \cancel{\frac{3}{x^2}} + 23$

$\Rightarrow 3x^2 = 27$

$\Rightarrow x^2 = 9$

$\Rightarrow x = 3, \cancel{-3}$

$\therefore x = 3$

Question 57 **(***+)**

The functions f and g are given by

$$f(x) = 3x + \ln 2, \ x \in \mathbb{R}$$

$$g(x) = e^{2x}, \ x \in \mathbb{R}.$$

a) Show clearly that

$$gf(x) = 4e^{6x}.$$

b) Show further that $x = \ln(2e)$ is the solution of the equation

$$\frac{d}{dx}\left[\frac{1}{2}gf(x-1)\right] = 768.$$

$\boxed{\text{proof}}$

(a) $g(f(x)) = g(3x + \ln 2) = e^{2(3x+\ln 2)} = e^{6x + 2\ln 2} = e^{6x} \times e^{2\ln 2}$
$= e^{6x} \times e^{\ln 4} = 4e^{6x}$

(b) $\frac{d}{dx}\left[\frac{1}{2}gf(x-1)\right] = 768$
$\Rightarrow \frac{d}{dx}\left[\frac{1}{2} \times 4e^{6(x-1)}\right] = 768$
$\Rightarrow \frac{d}{dx}\left[2e^{6x-6}\right] = 768$
$\Rightarrow 2e^{6x-6} \times 6 = 768$
$\Rightarrow 12e^{6x-6} = 768$
$\Rightarrow e^{6x-6} = 64$
$\Rightarrow 6x - 6 = \ln 64$
$\Rightarrow 6x = 6 + \ln 64$
$\Rightarrow x = 1 + \frac{1}{6}\ln 64$
$\Rightarrow x = 1 + \frac{1}{6}\ln 2^6$
$\Rightarrow x = 1 + \ln 2$
$\Rightarrow x = \ln e + \ln 2$
$\Rightarrow x = \ln(2e)$

Question 58 (***+)

The piecewise continuous function f is **odd** with domain all real numbers.

It is defined by

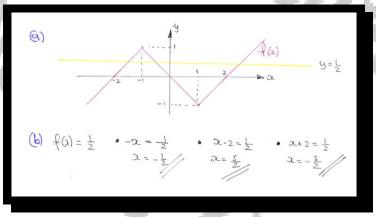
$$f(x) \equiv \begin{cases} -x & 0 \le x \le 1 \\ x-2 & x > 1 \end{cases}$$

a) Sketch the graph of f for all values of x.

b) Solve the equation

$$f(x) = \tfrac{1}{2}.$$

$\boxed{}$, $\boxed{x = -\frac{3}{2}, -\frac{1}{2}, \frac{5}{2}}$



Question 59 (***+)

The functions f and g are given by

$$f(x) = x^2 + 2kx + 4, \quad x \in \mathbb{R}$$

$$g(x) = 3 - kx, \quad x \in \mathbb{R}.$$

where k is a non zero constant.

a) Find, in terms of k, the range of f.

b) Given further that $fg(2) = 4$, determine the value of k.

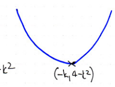
$\boxed{}$, $\boxed{f(x) \geq 4 - k^2}$, $\boxed{k = \frac{3}{2}}$

a) COMPLETING THE SQUARE

$f(x) = x^2 + 2kx + 4, \quad x \in \mathbb{R}$

$f(x) = (x+k)^2 - k^2 + 4$

$f(x)$ HAS A MINIMUM VALUE OF $4 - k^2$



$f(x) \geq 4 - k^2$

b) $f(g(2)) = 4$

$\Rightarrow f(3 - k \times 2) = 4$

$\Rightarrow f(3 - 2k) = 4$

$\Rightarrow (3-2k)^2 + 2k(3-2k) + \cancel{4} = \cancel{4}$

$\Rightarrow 9 - 12k + \cancel{4k^2} + 6k - \cancel{4k^2} = 0$

$\Rightarrow 9 = 6k$

$\Rightarrow k = \frac{3}{2}$

Question 60 (****)

The function f is defined by

$$f(x) = 2 + \sqrt{x}, \quad x \in \mathbb{R}, \quad x \geq 0.$$

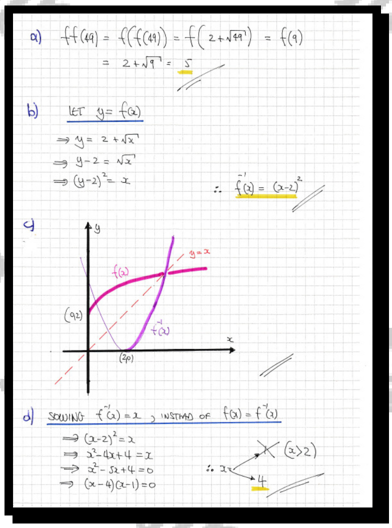
a) Evaluate $ff(49)$.

b) Find an expression for the inverse function, $f^{-1}(x)$.

c) Sketch in the same set of axes the graph of $f(x)$ and the graph of $f^{-1}(x)$, clearly marking the line of reflection between the two graphs.

d) Show that $x = 4$ is the only solution of the equation $f(x) = f^{-1}(x)$.

$\boxed{ff(49) = 5}$, $\boxed{f^{-1}(x) = (x-2)^2}$



Question 61 (****)

The functions f and g are defined by

$$f(x) = x^2, \ x \in \mathbb{R}, \ x \geq 1$$

$$g(x) = x - 6, \ x \in \mathbb{R}, \ x \leq 10.$$

a) Find the domain and range of $fg(x)$.

b) Show the following equation has no solutions

$$fg(x) = g^{-1}(x).$$

$\boxed{}$, $\boxed{7 \leq x \leq 10}$, $\boxed{1 \leq fg(x) \leq 16}$

a) WE START WITH THE DOMAIN OF $f(g(x))$

$\xrightarrow{\text{IN}} \boxed{g(x)} \xrightarrow{\text{OUT}}$ $\quad \xrightarrow{\text{IN}} \boxed{f(x)} \xrightarrow{\text{OUT}}$

$x \leqslant 10 \qquad x-6 \qquad x \geqslant 1$

THE DOMAIN MUST SATISFY

$x \leqslant 10$ AND $x - 6 \geqslant 1$

$x \geqslant 7$

COMBINING WE OBTAIN

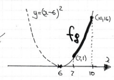
$7 \leqslant x \leqslant 10$ //

TO FIND THE RANGE

$f(g(x)) = f(x-6) = (x-6)^2$

SKETCHING NOTING THE DOMAIN

$\therefore \quad 1 \leqslant f(g(x)) \leqslant 16$ //



b) SOLVING THE EQUATION

$\Rightarrow f(g(x)) = g^{-1}(x)$

$\Rightarrow (x-6)^2 = x+6$

$\Rightarrow x^2 - 12x + 36 = x + 6$

$g(x) = x - 6$
$y = x - 6$
$y + 6 = x$
$g^{-1}(x) = x + 6$

$\Rightarrow x^2 - 13x + 30 = 0$

$\Rightarrow (x-10)(x-3) = 0$

$\Rightarrow x = \begin{cases} 3 \\ 10 \end{cases}$

LOOKING AT THE DOMAIN OF $f(g(x))$

ONLY SOLUTION IS $x = 10$ AS $7 \leqslant x \leqslant 10$.

NOW LOOKING AT $g(x)$ & ITS INVERSE

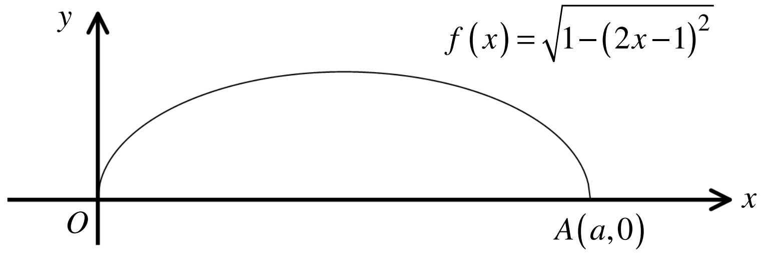
	$g(x)$	$g^{-1}(x)$
DOMAIN	$x \leqslant 10$	$x \leqslant 4$
RANGE	$g(x) \leqslant 4$	$g^{-1}(x) \leqslant 10$

$\therefore$ DOMAIN OF $g^{-1}(x) \leqslant 4$

$\therefore \quad x \neq 10$

$\therefore$ NO SOLUTIONS

Question 62 (****)



The figure above shows the graph of the function

$$f(x) \equiv \sqrt{1-(2x-1)^2}\,,\ x \in \mathbb{R}\,,\ 0 \le x \le a\,.$$

a) Find the value of the constant a.

b) State the range of $f(x)$.

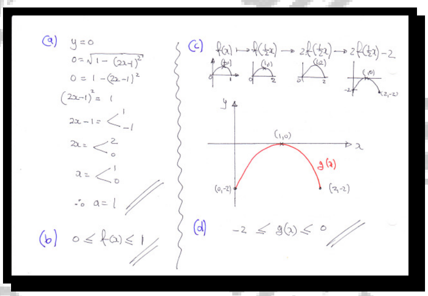
The function g is suitably defined by

$$g(x) = 2f\left(\tfrac{1}{2}x\right) - 2\,.$$

c) Sketch the graph of $g(x)$.

d) State the domain and range of $g(x)$.

$\boxed{}$, $\boxed{a=1}$, $\boxed{0 \le f(x) \le 1}$, $\boxed{0 \le x \le 2,\quad -2 \le g(x) \le 0}$



Question 63 (****)

The function f is defined by

$$f(x) = 1 + \sqrt{x}, \; x \in \mathbb{R}, \; x \geq 0 .$$

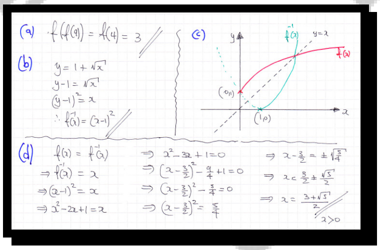
a) Evaluate $ff(9)$.

b) Find an expression for the inverse function, $f^{-1}(x)$.

c) Sketch in the same diagram the graph of $f(x)$ and the graph of $f^{-1}(x)$, clearly marking the line of reflection between the two graphs.

d) Show that $x = \dfrac{3+\sqrt{5}}{2}$ is the only solution of the equation $f(x) = f^{-1}(x)$.

$\boxed{}$, $\boxed{ff(9) = 3}$, $\boxed{f^{-1}(x) = (x-1)^2}$



Question 64 (****)

The functions f and g are defined by

$$f(x) = x^2 - 4, \ x \in \mathbb{R}, \ x > 8$$

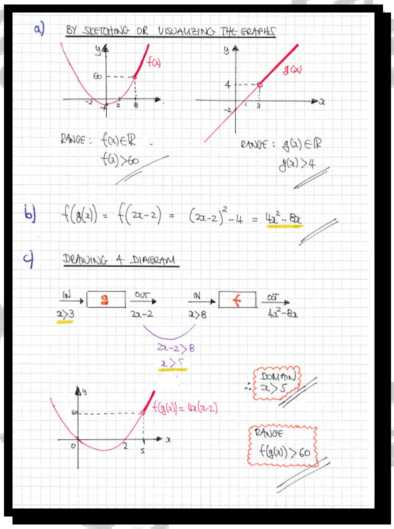
$$g(x) = 2x - 2, \ x \in \mathbb{R}, \ x > 3.$$

a) State the range of $f(x)$ and the range of $g(x)$.

b) Find a simplified expression for $fg(x)$.

c) Determine the domain and range of $fg(x)$.

$\boxed{}$, $\boxed{f(x) > 60}$, $\boxed{g(x) > 4}$, $\boxed{fg(x) = 4x^2 - 8x}$, $\boxed{x > 5}$, $\boxed{fg(x) > 60}$



Question 65 (****)

The functions f and g are defined by

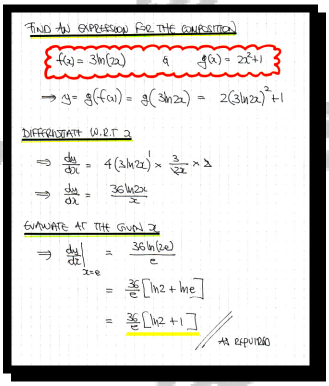
$$f(x) = 3\ln 2x, \quad x \in \mathbb{R}, \quad x > 0$$

$$g(x) = 2x^2 + 1, \quad x \in \mathbb{R}.$$

Show that the value of the gradient on the curve $y = gf(x)$ at the point where $x = \mathrm{e}$ is

$$\frac{36}{\mathrm{e}}(1 + \ln 2).$$

proof



FIND AN EXPRESSION FOR THE COMPOSITION

$f(x) = 3\ln(2x)$ & $g(x) = 2x^2+1$

$\Rightarrow y = g(f(x)) = g(3\ln 2x) = 2(3\ln 2x)^2 + 1$

DIFFERENTIATE W.R.T x

$\Rightarrow \frac{dy}{dx} = 4(3\ln 2x)^1 \times \frac{3}{\cancel{2}x} \times \cancel{2}$

$\Rightarrow \frac{dy}{dx} = \frac{36\ln 2x}{x}$

EVALUATE AT THE GIVEN x

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=e} = \frac{36\ln(2e)}{e}$

$= \frac{36}{e}[\ln 2 + \ln e]$

$= \frac{36}{e}[\ln 2 + 1]$ // AS REQUIRED

Question 66 (****)

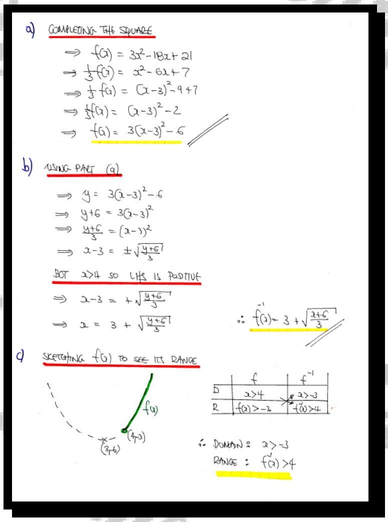
$$f(x) = 3x^2 - 18x + 21, \; x \in \mathbb{R}, \; x > 4 .$$

a) Express $f(x)$ in the form $A(x+B)^2 + C$, where A, B and C are integers ans hence find the range of $f(x)$.

b) Find a simplified expression for $f^{-1}(x)$, the inverse of $f(x)$.

c) Determine the domain and range of $f^{-1}(x)$.

$\boxed{}$, $\boxed{A=3, \;\; B=-3, \;\; C=-6}$, $\boxed{f(x) > -3}$, $\boxed{f^{-1}(x) = 3 + \sqrt{\dfrac{x+6}{3}}}$, $\boxed{x > -3, \; f^{-1}(x) > 4}$



Question 67 (****)

The piecewise continuous function f is **even** with domain $x \in \mathbb{R}$.

It is defined by

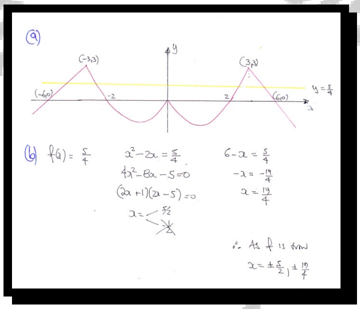
$$f(x) \equiv \begin{cases} x^2 - 2x & 0 \le x \le 3 \\ 6 - x & x > 3 \end{cases}$$

a) Sketch the graph of f for all values of x.

b) Solve the equation

$$f(x) = \frac{5}{4}$$

$\boxed{}$, $\boxed{x = \pm\frac{5}{2}, \pm\frac{19}{4}}$



Question 68 (****)

$$f(x) = x^2 - 4x - 5,\ x \in \mathbb{R},\ x \geq 2.$$

a) Find the range of $f(x)$.

b) State the domain and range of $f^{-1}(x)$.

c) Sketch the graph of $f^{-1}(x)$, marking clearly the coordinates of any points where the graph meets the coordinate axes.

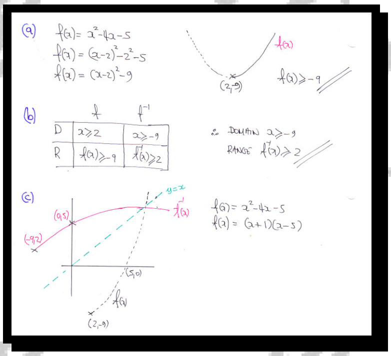
The function g is given

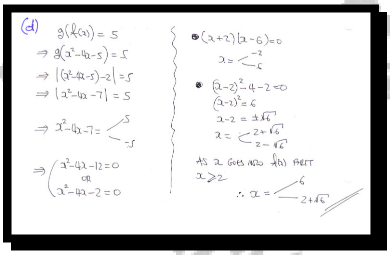
$$g(x) = |x-2|,\ x \in \mathbb{R}.$$

d) Find, in exact form where appropriate, the solutions of the equation

$$gf(x) = 5.$$

$\boxed{f(x) \geq -9}$, $\boxed{x \geq -9,\quad f^{-1}(x) \geq 2}$, $\boxed{x = 2+\sqrt{6},\ x = 6}$





Question 68 (****)

The function f is given by

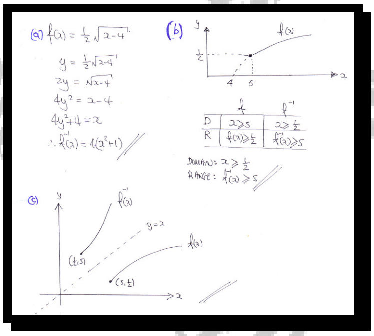
$$f(x)=\frac{1}{2}\sqrt{x-4}\,,\ x\in\mathbb{R},\ x\geq 5.$$

a) Determine an expression for $f^{-1}(x)$, in its simplest form.

b) Find the domain and range of $f^{-1}(x)$.

c) Sketch in the same diagram the graph of $f(x)$ and the graph of $f^{-1}(x)$.

$\boxed{f^{-1}(x)=4\left(x^2+1\right)}$, $\boxed{x\in\mathbb{R},\ x\geq\frac{1}{2}}$, $\boxed{f^{-1}(x)\in\mathbb{R},\ f^{-1}(x)\geq 5}$



Question 70 (****)

The functions f and g are defined by

$$f(x) = \sqrt{x+4}, \ x \in \mathbb{R}, \ x \geq -3$$

$$g(x) = 2x^2 - 3, \ x \in \mathbb{R}, \ x \leq 47 .$$

a) Find a simplified expression for $gf(x)$.

b) Determine the domain and range of $gf(x)$.

c) Solve the equation

$$fg(x) = 17 .$$

$\boxed{}$, $\boxed{gf(x) = 2x+5}$, $\boxed{-3 \leq x \leq 5}$, $\boxed{-1 \leq gf(x) \leq 15}$, $\boxed{x = -12}$

a) STANDARD METHODOLOGY

$g(f(x)) = g(\sqrt{x+4}) = 2\left[\sqrt{x+4}\right]^2 - 3 = 2(x+4) - 3 = 2x+5$

b) LOOKING AT A DIAGRAM

$\rightarrow \boxed{f} \rightarrow \quad \rightarrow \boxed{g} \rightarrow$

$x \geq -3 \qquad x \leq 47$

$\Downarrow$

$f(x) \geq 1$ BEFORE IT GOES INTO g — COMBINING

$g(x) \leq 47$

$2x^2 - 3 \leq 47$

$2x^2 \leq 50$

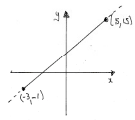
$x^2 \leq 25$

$-5 \leq x < 5$

COMBINING THE INEQUALITIES $x \geq -3$ TO GO INTO $f(x)$ AND THE OUTPUT OF f TO GO INTO $g(x)$ WE NEED $-5 \leq x \leq 5$

$\therefore$ DOMAIN $-3 \leq x \leq 5$

FOR THE RANGE OF $g(f(x)) = 2x+5$



RANGE $-1 \leq g(f(x)) \leq 15$

c) FORMING $f(g(x))$ AS PART OF THE REQUIRED EQUATION

$\Rightarrow f(g(x)) = 17$

$\Rightarrow f(2x^2 - 3) = 17$

$\Rightarrow \sqrt{(2x^2-3)+4} = 17$

$\Rightarrow \sqrt{2x^2+1} = 17$

$\Rightarrow 2x^2 + 1 = 17^2$

$\Rightarrow 2x^2 = 288$

$\Rightarrow x^2 = 144$

$\Rightarrow x = \begin{cases} \cancel{12} & \text{CANNOT GO INTO } g(x) \\ -12 \end{cases}$

Question 71 (****)

The function $f(x)$ is given by

$$f(x) = 2x^2 + 3, \ \ x \in \mathbb{R}, \ x \le 0.$$

a) Sketch the graph of $f(x)$.

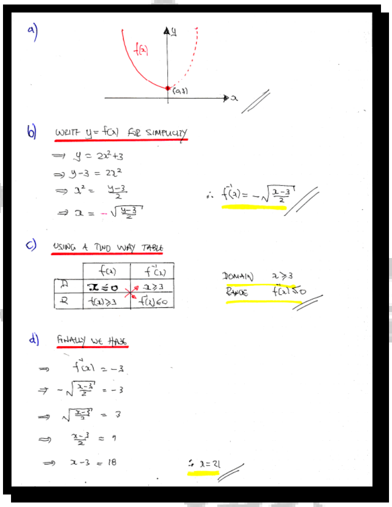
b) Find $f^{-1}(x)$ in its simplest form.

c) Find the domain and range of $f^{-1}(x)$.

d) Solve the equation

$$f^{-1}(x) = -3.$$

$\boxed{}$, $\boxed{f^{-1}(x) = -\sqrt{\frac{x-3}{2}}}$, $\boxed{x \in \mathbb{R}, \ x \ge 3}$, $\boxed{f(x) \in \mathbb{R}, \ f(x) \le 0}$, $\boxed{x = 21}$



Question 72 (****)

The function f is defined by

$$f(x) = \begin{cases} 4 - x, & x \in \mathbb{R},\ x \leq 2 \\ 2(x-1)^2, & x \in \mathbb{R},\ x \geq 2 \end{cases}$$

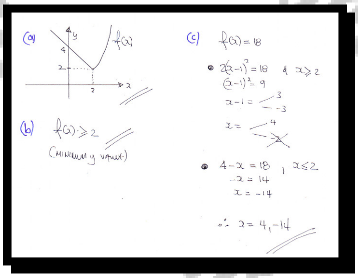
a) Sketch the graph of $f(x)$.

b) State the range of $f(x)$.

c) Solve the equation

$$f(x) = 18.$$

$\boxed{}$, $\boxed{f(x) \geq 2}$, $\boxed{x = -14, 4}$



Question 73 (****)

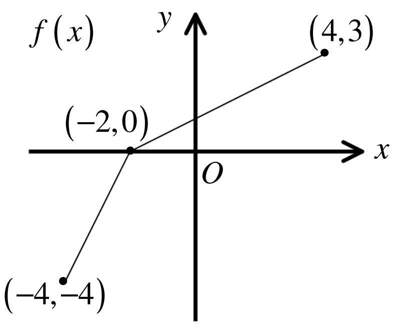


figure 1

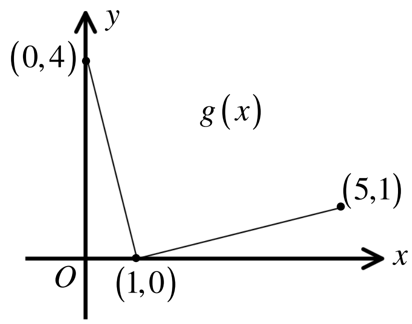


figure 2

Figure 1 and figure 2 above, show the graphs of two piecewise continuous functions $f(x)$ and $g(x)$, respectively.

Each graph consists of two straight line segments joining the points with the coordinates shown in each figure.

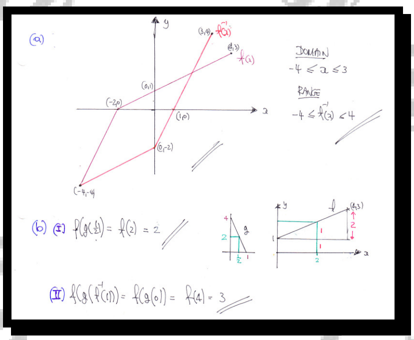
a) Sketch on the same set of axes the graphs of $f(x)$ and its inverse $f^{-1}(x)$, stating the domain and range of $f^{-1}(x)$.

b) Evaluate …

i. … $fg\left(\frac{1}{2}\right)$.

ii. … $fgf^{-1}(1)$.

$\boxed{\text{C3M}}$, $\boxed{-4 \le x \le 3}$, $\boxed{-4 \le f^{-1}(x) \le 4}$, $\boxed{fg\left(\frac{1}{2}\right) = 2}$, $\boxed{fgf^{-1}(1) = 3}$



Question 74 (****)

$$f(x) = 2 - \frac{1}{x}, \quad x \in \mathbb{R},\ x \neq 0 .$$

a) Sketch the graph of $f(x)$.

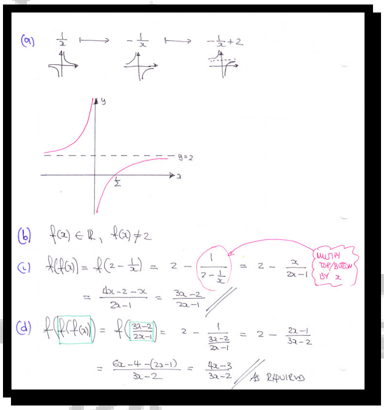
b) State the range of $f(x)$.

c) Find as a simplified fraction, an expression for $ff(x)$.

d) Hence, show that

$$fff(x) = \frac{4x-3}{3x-2} .$$

$\boxed{f(x) \in \mathbb{R},\ f(x) \neq 2}$, $\boxed{ff(x) = \frac{3x-2}{2x-1}}$



Question 75 (****)

The functions f and g satisfy

$$f(x) = 2e^{\frac{1}{2}x}, \ x \in \mathbb{R}$$

$$g(x) = \ln 4x \ \ x \in \mathbb{R}, \ x > \frac{1}{4}.$$

a) Find $fg(x)$ in its simplest form.

b) Find the domain and range of $fg(x)$.

c) Solve the equation

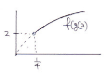
$$fg(x) = 3x + 1.$$

$\boxed{}$, $\boxed{fg(x) = 4\sqrt{x}}$, $\boxed{x \in \mathbb{R}, \ x > \frac{1}{4}}$, $\boxed{fg(x) \in \mathbb{R}, \ f(x) > 2}$, $\boxed{x = 1, \ x \neq \frac{1}{9}}$

(a) $f(g(x)) = f(\ln 4x) = 2e^{\frac{1}{2}\ln(4x)} = 2e^{\ln(4x)^{\frac{1}{2}}} = 2 \times \sqrt{4x}$
$= 4\sqrt{x}$

(b) DOMAIN $x > \frac{1}{4}$

$\text{IN} \xrightarrow[x > \frac{1}{4}]{g} \frac{\text{OUT}}{\text{IN}} \xrightarrow[x \in \mathbb{R}]{f} \text{OUT}$



$\therefore$ RANGE $f(g(x)) > 2$

(c) $4\sqrt{x} = 3x + 1$
$\Rightarrow (4\sqrt{x})^2 = (3x+1)^2$
$\Rightarrow 16x = 9x^2 + 6x + 1$
$\Rightarrow 0 = 9x^2 - 10x + 1$
$\Rightarrow 0 = (9x-1)(x-1)$
$\Rightarrow x = \begin{cases} 1 \\ \cancel{\frac{1}{9}} \end{cases} \quad x > \frac{1}{4}$

OR $4\sqrt{x} = 3x + 1$
$\Rightarrow 0 = 3x - 4\sqrt{x} + 1$
$\Rightarrow 0 = 3(\sqrt{x})^2 - 4\sqrt{x} + 1$
$\Rightarrow 0 = (3\sqrt{x} - 1)(\sqrt{x} - 1)$
$\Rightarrow \sqrt{x} = \begin{cases} 1 \\ \frac{1}{3} \end{cases}$
$\Rightarrow x = \begin{cases} 1 \\ \cancel{\frac{1}{9}} \end{cases} \quad x > \frac{1}{4}$

Question 76 (****)

The function $f(x)$ is given by

$$f(x) = \frac{4}{x-2}, \quad x \in \mathbb{R},\ x \neq 2.$$

a) Find an expression for $f^{-1}(x)$, in its simplest form.

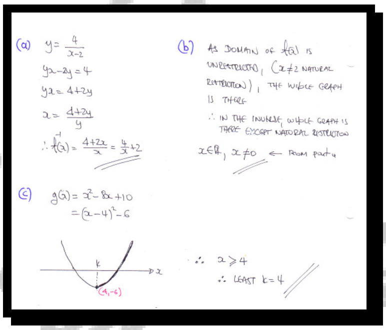
b) State the domain of $f^{-1}(x)$.

The function g is defined as

$$g(x) = x^2 - 8x + 10, \quad x \in \mathbb{R},\ x \geq k.$$

c) Given that $g^{-1}(x)$ exists, find the least value of k.

$\boxed{f^{-1}(x) = \frac{2x+4}{x}}$, $\boxed{x \in \mathbb{R},\ x \neq 0}$, $\boxed{k = 4}$



(a) $y = \frac{4}{x-2}$

$yx - 2y = 4$

$yx = 4 + 2y$

$x = \frac{4+2y}{y}$

$\therefore f^{-1}(x) = \frac{4+2x}{x} = \frac{4}{x} + 2$

(b) AS DOMAIN OF $f(x)$ IS UNRESTRICTED, ($x \neq 2$ NATURAL RESTRICTION), THE WHOLE GRAPH IS THERE

$\therefore$ IN THE INVERSE, WHOLE GRAPH IS THERE EXCEPT NATURAL RESTRICTION

$x \in \mathbb{R}, x \neq 0$ ← FROM part (a)

(c) $g(x) = x^2 - 8x + 10$

$= (x-4)^2 - 6$

$\therefore x \geq 4$

$\therefore$ LEAST $k = 4$

Question 77 **(****)**

The functions f and g are given below

$$f(x) = \frac{1}{2-2x}, \quad x \in \mathbb{R}, \quad x \neq 0, x \neq \tfrac{1}{2}, x \neq 1$$

$$g(x) = ff(x).$$

a) Find a simplified expression for $g(x)$.

b) Hence show clearly that

$$ffff(x) = x.$$

c) Find an expression for the inverse function $g^{-1}(x)$.

$\boxed{g(x) = \frac{1-x}{1-2x}}$, $\boxed{g^{-1}(x) = \frac{1-x}{1-2x}}$

(a) $g(x) = f(f(x)) = f\left(\frac{1}{2-2x}\right) = \frac{1}{2-2\left(\frac{1}{2-2x}\right)} = \frac{1}{2-\frac{1}{1-x}}$ MULTIPLY TOP BOTTOM BY $1-x$

$= \frac{1-x}{2(1-x)-1} = \frac{1-x}{2-2x-1} = \frac{1-x}{1-2x}$

(b) $ffff(x) = g(g(x)) = g\left(\frac{1-x}{1-2x}\right) = \frac{1-\frac{1-x}{1-2x}}{1-2\left(\frac{1-x}{1-2x}\right)}$ MULTIPLY TOP & BOTTOM BY $1-2x$

$= \frac{(1-2x)-(1-x)}{1-2x-2(1-x)} = \frac{-x}{-1} = x$

(c) SINCE $g(g(x)) = x \Rightarrow g(x) = g^{-1}(x) \quad \therefore g^{-1}(x) = \frac{1-x}{1-2x}$

Question 78 (****)

$$f(x) = x^2 - 6x, \ x \in \mathbb{R}, \ x \le 3.$$

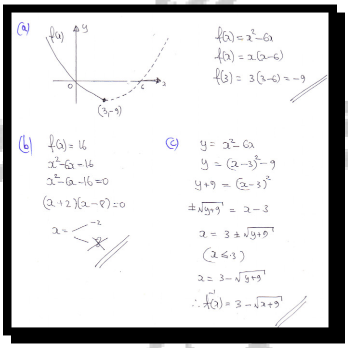
a) Find the range of $f(x)$.

b) Solve the equation

$$f(x) = 16.$$

c) Find an expression for the inverse function $f^{-1}(x)$.

$\boxed{f(x) \ge -9}$, $\boxed{x = -2}$, $\boxed{f^{-1}(x) = 3 - \sqrt{x+9}}$



Question 79 (****)

The following functions are defined as follows

$$f(x) = 3 - x^2, \; x \in \mathbb{R}$$

$$g(x) = \frac{2}{x+1}, \; x \in \mathbb{R}, \; x \neq -1.$$

a) Find the range of $f(x)$.

b) Find $g^{-1}(x)$ in its simplest form, further stating its range.

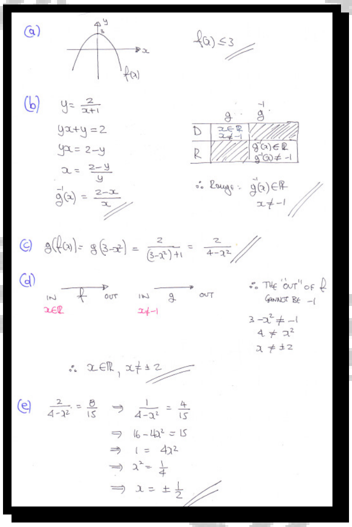
c) Determine the composite function $gf(x)$.

d) Find the domain of $gf(x)$.

e) Solve the equation

$$gf(x) = \frac{8}{15}.$$

$\boxed{f(x) \leq 3}$, $\boxed{g^{-1}(x) = \frac{2}{x} - 1 = \frac{2-x}{x}}$, $\boxed{g^{-1}(x) \neq -1}$, $\boxed{gf(x) = \frac{2}{4-x^2}}$, $\boxed{x \in \mathbb{R}, x \neq \pm 2}$, $\boxed{x = \pm\frac{1}{2}}$



Question 80 (****)

The function f is given by

$$f : x \mapsto \frac{3-x}{1+x}, \ x \in \mathbb{R}, \ x \le -2.$$

a) Show that for some constants a and b

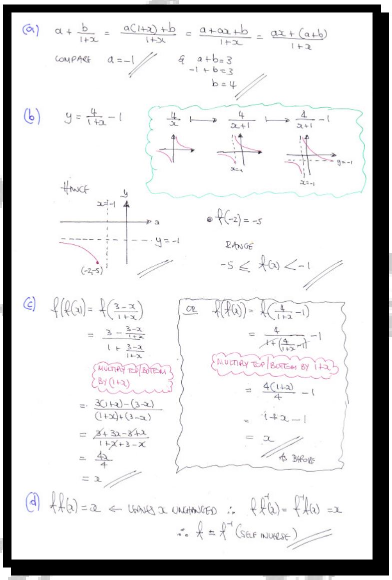
$$\frac{3-x}{1+x} \equiv a + \frac{b}{1+x}.$$

b) Sketch the graph of f and hence state its range.

c) Show that $ff(x) = x$, for all $x \le -2$.

d) Without finding $f^{-1}(x)$ explain how part (c) can be used to deduce $f^{-1}(x)$.

$\boxed{a=-1}$, $\boxed{b=4}$, $\boxed{-5 \le f(x) < -1}$



Question 81 (****)

The function $f(x)$ is defined by

$$f(x) = (x-2)^2 - 1,\ x \in \mathbb{R},\ x \le 2.$$

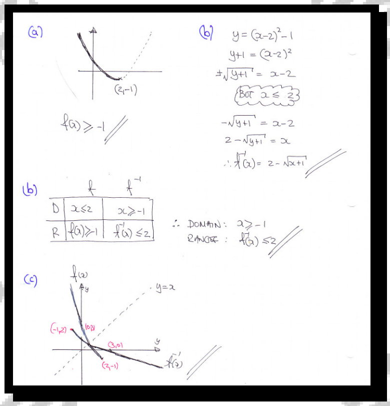
a) Find the range of $f(x)$.

b) Find $f^{-1}(x)$ in its simplest form.

c) Determine the domain and range of $f^{-1}(x)$.

d) Sketch in the same diagram $f(x)$ and $f^{-1}(x)$.

$\boxed{f(x) \ge -1}$, $\boxed{f^{-1}(x) = 2 - \sqrt{x+1}}$, $\boxed{x \ge -1,\ \ f^{-1}(x) \le 2}$



Question 82 (****)

The functions f and g are defined by

$$f(x) = e^{x}, \; x \in \mathbb{R}$$

$$g(x) = \ln\left(x^{2} - 4\right), \; x \in \mathbb{R}, \; x > 2 \; .$$

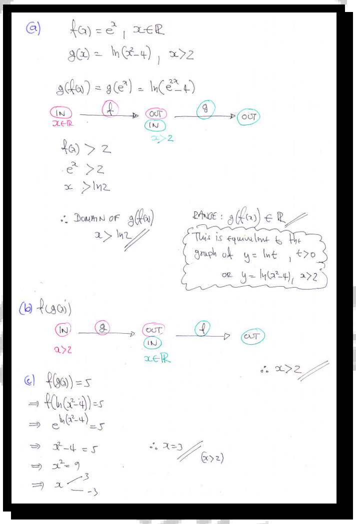
a) Find the domain and range of $gf(x)$.

b) Find the domain of $fg(x)$.

c) Solve the equation

$$fg(x) = 5 \, .$$

$\boxed{x > \ln 2, \; gf(x) \in \mathbb{R}}$, $\boxed{x > 2}$, $\boxed{x = 3}$



Question 83 (****)

The functions f and g are defined by

$$f(x) = e^{x} - 3, \quad x \in \mathbb{R}$$

$$g(x) = x + 1, \quad x \in \mathbb{R}.$$

a) Find an expression for $f^{-1}(x)$, the inverse of $f(x)$.

b) State the domain and range of $f^{-1}(x)$.

c) Solve the equation

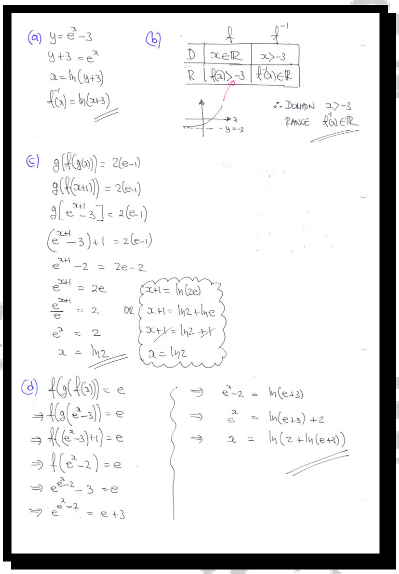
$$gfg(x) = 2(e-1),$$

giving the final answer in terms of logarithms in its simplest form.

d) Find an exact solution of the equation

$$fgf(x) = e.$$

$\boxed{f^{-1}(x) = \ln(x+3)}$, $\boxed{x \in \mathbb{R},\ x > -3}$, $\boxed{f^{-1}(x) \in \mathbb{R}}$, $\boxed{x = \ln 2}$, $\boxed{x = \ln\left[2 + \ln(3+e)\right]}$


(a) $y = e^{x} - 3$

$y + 3 = e^{x}$

$x = \ln(y+3)$

$f^{-1}(x) = \ln(x+3)$

(b)

	f	f^{-1}
D	$x \in \mathbb{R}$	$x > -3$
R	$f(x) > -3$	$f^{-1}(x) \in \mathbb{R}$

$\therefore$ DOMAIN $x > -3$

RANGE $f^{-1}(x) \in \mathbb{R}$

(c) $g(f(g(x))) = 2(e-1)$

$g(f(x+1)) = 2(e-1)$

$g\left[e^{x+1} - 3\right] = 2(e-1)$

$\left(e^{x+1} - 3\right) + 1 = 2(e-1)$

$e^{x+1} - 2 = 2e - 2$

$e^{x+1} = 2e$

$\frac{e^{x+1}}{e} = 2$

$e^{x} = 2$

$x = \ln 2$

OR

$x + 1 = \ln(2e)$

$x + 1 = \ln 2 + \ln e$

$x + \cancel{1} = \ln 2 + \cancel{1}$

$x = \ln 2$

(d) $f(g(f(x))) = e$

$\Rightarrow f(g(e^{x} - 3)) = e$

$\Rightarrow f((e^{x} - 3) + 1) = e$

$\Rightarrow f(e^{x} - 2) = e$

$\Rightarrow e^{e^{x} - 2} - 3 = e$

$\Rightarrow e^{e^{x} - 2} = e + 3$

$\Rightarrow e^{x} - 2 = \ln(e+3)$

$\Rightarrow e^{x} = \ln(e+3) + 2$

$\Rightarrow x = \ln(2 + \ln(e+3))$

Question 84 (****)

The functions f and g are defined by

$$f(x) = x^2 + 3x - 7, \ x \in \mathbb{R}$$

$$g(x) = ax + b, \ x \in \mathbb{R},$$

where a and b are positive constants.

When the composition $fg(x)$ is divided by $(x+2)$ the remainder is 21, while $(x-1)$ is a factor of the composition $gf(x)$.

Determine the value of a and the value of b.

$\boxed{}$, $\boxed{a=4}$, $\boxed{b=12}$

- $f(g(x)) = f(ax+b)$
  $= (ax+b)^2 + 3(ax+b) - 7$
  $= a^2x^2 + 2abx + b^2 + 3ax + 3b - 7$
  $= a^2x^2 + (2ab+3a)x + (b^2+3b-7)$

- $g(f(x)) = g(x^2+3x-7)$
  $= a(x^2+3x-7) + b$
  $= ax^2 + 3ax - 7a + b$

- $f(g(-2)) = 21$
  $4a^2 - 2(2ab+3a) + b^2 + 3b - 7 = 21$
  $4a^2 - 4ab - 6a + b^2 + 3b - 28 = 0$

- $g(f(1)) = 0$
  $a + 3a - 7a + b = 0$
  $\boxed{b = 3a}$

$4a^2 - 4a(3a) - 6a + (3a)^2 + 3(3a) - 28 = 0$
$4a^2 - 12a^2 - 6a + 9a^2 + 9a - 28 = 0$
$a^2 + 3a - 28 = 0$
$(a-4)(a+7) = 0$

$a = 4, \cancel{-7}$ $\quad b = 12, \cancel{-21}$ $\quad a,b>0$

Question 85 (****)

The function f is defined as

$$f(x) = \frac{1}{1+\tan x}, \quad 0 \le x < \frac{\pi}{2}.$$

a) Use differentiation to show that f is a one to one function.

b) Find a simplified expression for the inverse of f.

c) Determine the range of f.

$\boxed{}$, $\boxed{f^{-1}(x) = \arctan\left(\frac{1-x}{x}\right)}$, $\boxed{0 < f(x) \le 1}$

(a)

$f(x) = \frac{1}{1+\tan x} = (1+\tan x)^{-1}$

$f'(x) = -(1+\tan x)^{-2} \times \sec^2 x$

$f'(x) = -\frac{\sec^2 x}{(1+\tan x)^2}$

SINCE $f'(x) < 0$ FOR THE ENTIRE DOMAIN, THE FUNCTION IS DECREASING, SO THE FUNCTION IS ONE TO ONE

(b)

$y = \frac{1}{1+\tan x}$

$1+\tan x = \frac{1}{y}$

$\tan x = \frac{1}{y} - 1$

$\tan x = \frac{1-y}{y}$

$x = \arctan\left(\frac{1-y}{y}\right)$

$\therefore f^{-1}(x) = \arctan\left(\frac{1-x}{x}\right)$

(c) FOR $0 \le x < \frac{\pi}{2}$

$\tan x \ge 0$

$1+\tan x \ge 1$

$0 < \frac{1}{1+\tan x} \le 1$

$\therefore$ RANGE $0 < f(x) \le 1$

Question 86 (****)

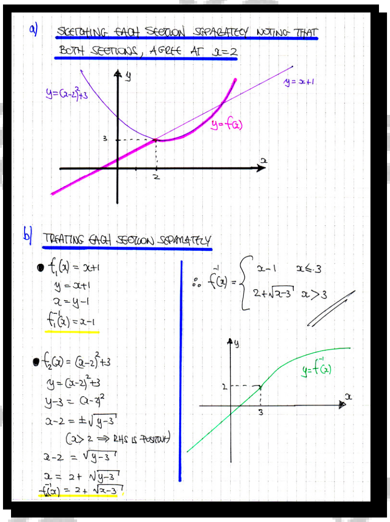
The function f is defined by

$$f(x)=\begin{cases} x+1, & x\in\mathbb{R},\ x\le 2 \\ (x-2)^2+3, & x\in\mathbb{R},\ x>2 \end{cases}$$

a) Sketch the graph of $f(x)$.

b) Find an expression for $f^{-1}(x)$, fully specifying its domain.

$\boxed{}$, $\boxed{f(x)=\begin{cases} x-1, & x\in\mathbb{R},\ x\le 3 \\ 2+\sqrt{x-3}, & x\in\mathbb{R},\ x\ge 3 \end{cases}}$



Question 87 (****)

The function $f(x)$ is defined by

$$f(x) = \ln(3x-2) + 3, \ x \in \mathbb{R}, \ x \geq 1.$$

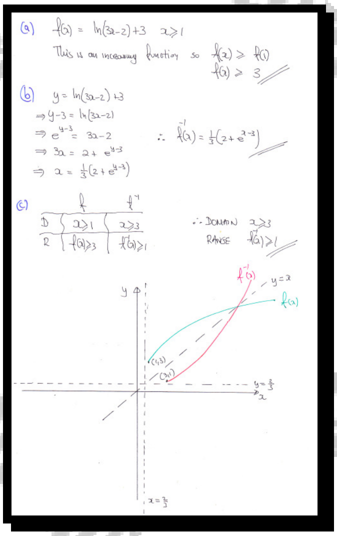
a) Find the range of $f(x)$.

b) Find $f^{-1}(x)$ in its simplest form.

c) Find the domain and range of $f^{-1}(x)$.

d) Sketch in the same diagram $f(x)$ and $f^{-1}(x)$.

$\boxed{f(x) \geq 3}$, $\boxed{f^{-1}(x) = \frac{1}{3}\left(2 + e^{x-3}\right)}$, $\boxed{x \geq 3, \ f^{-1}(x) \geq 1}$



Question 88 **(****)**

The functions $f(x)$ and $g(x)$ are defined by

$$f(x) = \ln x, \ \ x \in \mathbb{R}, \ x > 0$$

$$g(x) = e^{3x}, \ \ x \in \mathbb{R}, \ x > 1 .$$

a) Find, in its simplest form, the function compositions

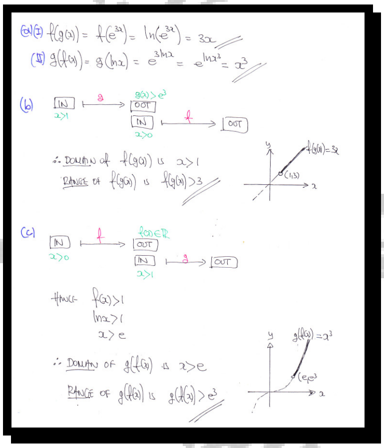
i. $fg(x)$.

ii. $gf(x)$.

b) Find the domain and range of $fg(x)$.

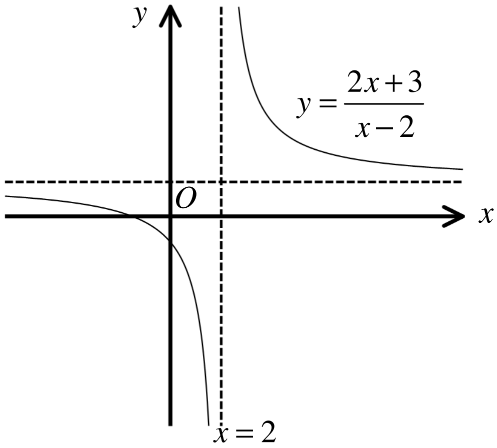
c) Find the domain and range of $gf(x)$.

$\boxed{fg(x) = 3x}$, $\boxed{gf(x) = x^3}$, $\boxed{x > 1, \ fg(x) > 3}$, $\boxed{x > e, \ gf(x) > e^3}$



Question 89 (****)

The figure below shows the graph of the curve with equation $y = \dfrac{2x+3}{x-2}$.



a) Write down the equation of the horizontal asymptote to the curve.

The function f is defined as

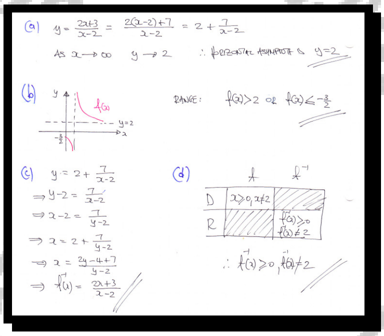
$$f(x) = \frac{2x+3}{x-2}, \quad x \in \mathbb{R}, \ x \geq 0, x \neq 2 .$$

b) Find the range of $f(x)$.

c) Find $f^{-1}(x)$ in its simplest form.

d) State the range of $f^{-1}(x)$.

$\boxed{y=2}$, $\boxed{f(x) \leq -\frac{3}{2} \ \text{ or } \ f(x) > 2}$, $\boxed{f^{-1}(x) = \frac{2x+3}{x-2}}$, $\boxed{f^{-1}(x) \geq 0, \ f^{-1}(x) \neq 2}$



Question 90 (****)

The function $f(x)$ is defined by

$$f(x) = \frac{1}{\sqrt{x-2}}, \quad x \in \mathbb{R}, \; x > 2.$$

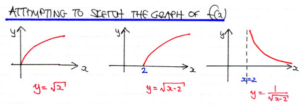
a) Find the range of $f(x)$.

b) Determine a simplified expression for $f^{-1}(x)$, further stating the domain and range of $f^{-1}(x)$.

c) Show that the equation $f^{-1}(x) = -\frac{3}{x}$ has no real solutions.

$\boxed{}$, $\boxed{f(x) > 0}$, $\boxed{f^{-1}(x) = \frac{1}{x^2} + 2}$, $\boxed{x > 0, \; f^{-1}(x) > 2}$

a) ATTEMPTING TO SKETCH THE GRAPH OF $f(x)$



THE RANGE OF $f(x)$ IS

$f(x) \in \mathbb{R}, \; f(x) > 0$

b) LET $y = f(x)$ FOR SIMPLICITY

$\Rightarrow y = \frac{1}{\sqrt{x-2}}$

$\Rightarrow y^2 = \frac{1}{x-2}$

$\Rightarrow x - 2 = \frac{1}{y^2}$

$\Rightarrow x = \frac{1}{y^2} + 2$

$\Rightarrow f^{-1}(x) = \frac{1}{x^2} + 2$

	$f(x)$	$f^{-1}(x)$
D	$x > 2$	$x > 0$
R	$f(x) > 0$	$f^{-1}(x) > 2$

DOMAIN OF $f^{-1}(x)$: $x > 0$

RANGE OF $f^{-1}(x)$: $f^{-1}(x) > 2$

c) SOLVING THE GIVEN EQUATION, IN ORDER TO DETERMINE "WHAT IS THE PROBLEM" WITH THE ROOTS

$\Rightarrow \frac{1}{x^2} + 2 = -\frac{3}{x}$

$\Rightarrow 1 + 2x^2 = -3x$

$\Rightarrow 2x^2 + 3x + 1 = 0$

$\Rightarrow (2x+1)(x+1) = 0$

$\Rightarrow x = -\frac{1}{2}, \; -1$

NEITHER SOLUTION IS POSSIBLE AS THE DOMAIN OF $f^{-1}(x)$ IS $x > 0$

Question 91 (****)

The functions $f(x)$ and $g(x)$ are defined by

$$f(x) = 2e^{x}, \;\; x \in \mathbb{R}$$

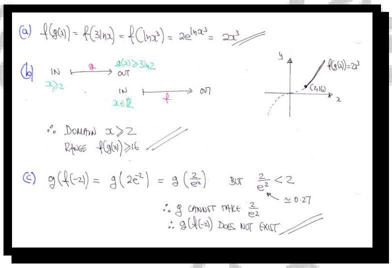
$$g(x) = 3\ln x, \;\; x \in \mathbb{R}, \; x \geq 2 .$$

a) Find, in its simplest form, the function composition $fg(x)$.

b) Find the domain and range of $fg(x)$.

c) Show that $gf(-2)$ does not exist.

$\boxed{fg(x) = 2x^3}$, $\boxed{x \geq 2, \; fg(x) \geq 16}$



Question 92 (****)

$$f(x) = 2\cos 2x, \quad x \in \mathbb{R}, \ 0 \le x \le \frac{\pi}{2}$$

$$g(x) = |x|, \quad x \in \mathbb{R}.$$

a) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the curve crosses the coordinate axes.

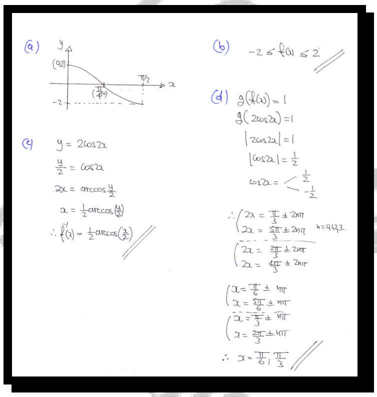
b) State the range of $f(x)$.

c) Find an expression for $f^{-1}(x)$.

d) Solve the equation

$$gf(x) = 1.$$

$\boxed{\left(\frac{\pi}{4},0\right),(0,2)}$, $\boxed{-2 \le f(x) \le 2}$, $\boxed{f^{-1}(x) = \frac{1}{2}\arccos\left(\frac{x}{2}\right)}$, $\boxed{x = \frac{\pi}{6}, \frac{\pi}{3}}$



Question 93 (****)

The functions f and g are defined as

$$f(x) = 3\left(2^{-x}\right) - 1, \quad x \in \mathbb{R}, \quad x \geq 0$$

$$g(x) = \log_2 x, \quad x \in \mathbb{R}, \quad x \geq 1.$$

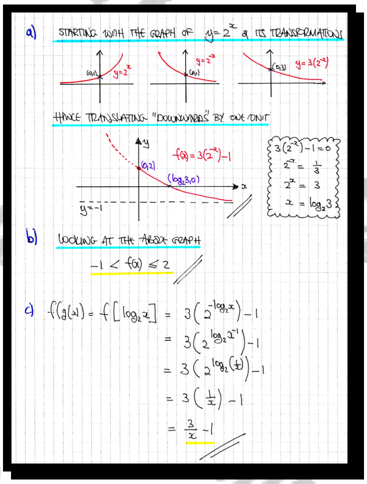
a) Sketch the graph of f.

- Mark clearly the exact coordinates of any points where the curve meets the coordinate axes. Give the answers, where appropriate, in exact form in terms of logarithms base 2.
- Mark and label the equation of the asymptote to the curve.

b) State the range of f.

c) Find $f(g(x))$ in its simplest form.

$\boxed{}$, $\boxed{(0,2)}$, $\boxed{(\log_2 3, 0)}$, $\boxed{y=-1}$, $\boxed{-1 < f(x) \leq 2}$, $\boxed{f(g(x)) = \frac{3}{x} - 1}$



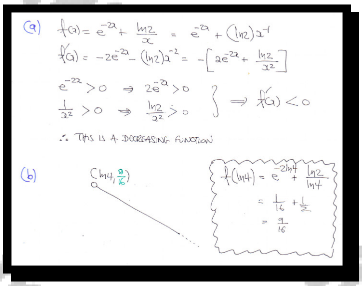
Question 94 (****)

$$f(x) = e^{-2x} + \frac{\ln 2}{x}, \ x \in \mathbb{R}, \ x > \ln 4 .$$

a) Show that $f(x)$ in a decreasing function.

b) Find the range of $f(x)$ in its simplest form.

$\boxed{\text{C3}}$, $\boxed{f(x) \in \mathbb{R}, \ f(x) < \frac{9}{16}}$



(a) $f(x) = e^{-2x} + \frac{\ln 2}{x} = e^{-2x} + (\ln 2)x^{-1}$

$f'(x) = -2e^{-2x} - (\ln 2)x^{-2} = -\left[2e^{-2x} + \frac{\ln 2}{x^2}\right]$

$e^{-2x} > 0 \Rightarrow 2e^{-2x} > 0$

$\frac{1}{x^2} > 0 \Rightarrow \frac{\ln 2}{x^2} > 0$

$\Rightarrow f'(x) < 0$

∴ THIS IS A DECREASING FUNCTION

(b) $(\ln 4, \frac{9}{16})$

$f(\ln 4) = e^{-2\ln 4} + \frac{\ln 2}{\ln 4} = \frac{1}{16} + \frac{1}{2} = \frac{9}{16}$

Question 95 (****)

The function $f(x)$ satisfies

$$f(x)=\frac{2x+1}{x-1}, \quad x\in\mathbb{R}, \quad x\geq 2.$$

a) Show that

$$f(x)=A+\frac{B}{x-1},$$

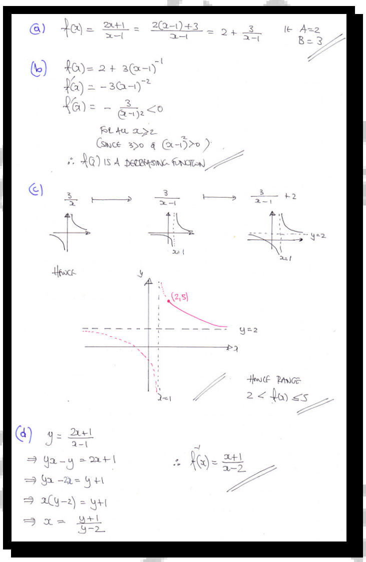
where A and B are positive constants to be found.

b) Show that $f(x)$ is s decreasing function.

c) Sketch the graph of $f(x)$ and hence find its range.

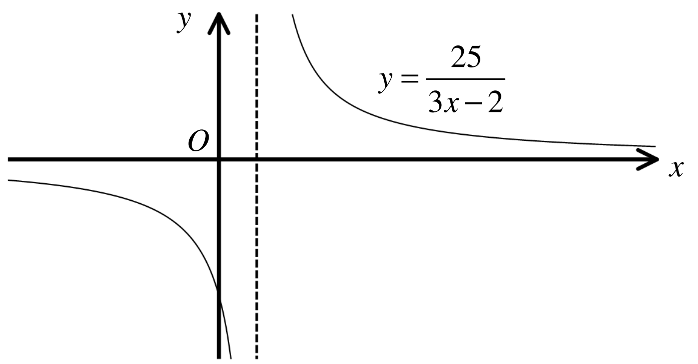
d) Find $f^{-1}(x)$, in its simplest form.

$\boxed{A=2, B=3}$, $\boxed{f(x)\in\mathbb{R},\ 2<f(x)\leq 5}$, $\boxed{f^{-1}(x)=\frac{x+1}{x-2}}$



Question 96 (****)

The figure below shows the graph of the curve C with equation $y = \dfrac{25}{3x-2}$.



a) State the equation of the vertical asymptote of the curve, marked with a dotted line in the diagram.

The function f is defined as

$$f(x) = \frac{25}{3x-2}, \quad x \in \mathbb{R}, \quad 1 < x \le 9.$$

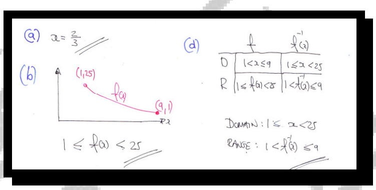
b) Write down the range of $f(x)$.

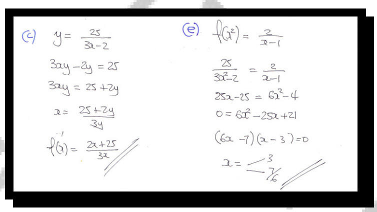
c) Find an expression for $f^{-1}(x)$.

d) State the domain and range of $f^{-1}(x)$.

e) Solve the equation $f(x^2) = \dfrac{2}{x-1}$.

$\boxed{x = \frac{2}{3}}$, $\boxed{1 \le f(x) < 25}$, $\boxed{f^{-1}(x) = \frac{2x+25}{3x}}$, $\boxed{1 \le x < 25}$, $\boxed{1 < f^{-1}(x) \le 9}$, $\boxed{x = \frac{7}{6}, 3}$





Question 97 **(****)**

The functions $f(x)$ and $g(x)$ are defined as

$$f(x)=\frac{2x^2-50}{x+5}, \; x\in\mathbb{R}, \; x\neq -6.$$

$$g(x)=x^2+1, \; x\in\mathbb{R}.$$

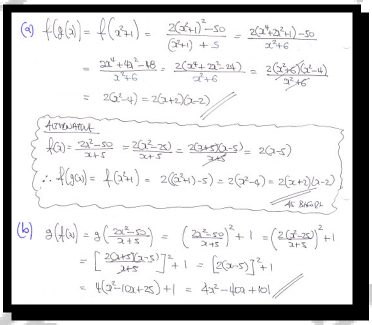
Show that …

a) … $fg(x)=k(x+k)(x-k)$,

stating the value of the constant k.

b) … $gf(x)=4x^2-40x+101$.

$\boxed{k=2}$



(a) $f(g(x))=f(x^2+1)=\frac{2(x^2+1)^2-50}{(x^2+1)+5}=\frac{2(x^4+2x^2+1)-50}{x^2+6}$

$=\frac{2x^4+4x^2-48}{x^2+6}=\frac{2(x^4+2x^2-24)}{x^2+6}=\frac{2\cancel{(x^2+6)}(x^2-4)}{\cancel{x^2+6}}$

$=2(x^2-4)=2(x+2)(x-2)$

ALTERNATIVE

$f(x)=\frac{2x^2-50}{x+5}=\frac{2(x^2-25)}{x+5}=\frac{2\cancel{(x+5)}(x-5)}{\cancel{x+5}}=2(x-5)$

$\therefore f(g(x))=f(x^2+1)=2((x^2+1)-5)=2(x^2-4)=2(x+2)(x-2)$

AS BEFORE

(b) $g(f(x))=g\left(\frac{2x^2-50}{x+5}\right)=\left(\frac{2x^2-50}{x+5}\right)^2+1=\left(\frac{2(x^2-25)}{x+5}\right)^2+1$

$=\left[\frac{2\cancel{(x+5)}(x-5)}{\cancel{x+5}}\right]^2+1=\left[2(x-5)\right]^2+1$

$=4(x^2-10x+25)+1=4x^2-40x+101$

Question 98 (****)

The functions f and g are defined as

$$f(x) = x^2 - 16,\ x \in \mathbb{R},\ x < 0$$

$$g(x) = 12 - \frac{1}{2}x,\ x \in \mathbb{R},\ x > 8.$$

a) Find, in any order, ...

i. ... the range of $f(x)$ and the range of $g(x)$.

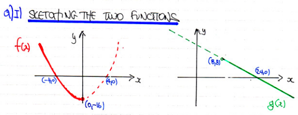
ii. ... the domain and range of $fg(x)$.

b) Solve the equation

$$fg(x) = g(2x - 22).$$

$\boxed{}$, $\boxed{f(x) > -16}$, $\boxed{g(x) < 8}$, $\boxed{x > 24,\ fg(x) > -16}$, $\boxed{x = 30}$

a)I) SKETCHING THE TWO FUNCTIONS



RANGE OF $f(x)$ — $f(x) > -16$

RANGE OF $g(x)$ — $g(x) < 8$

a)II)

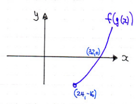
$\xrightarrow{x>8} \boxed{g} \xrightarrow{g(x)} \quad \xrightarrow{x<0} \boxed{f} \longrightarrow$

- $f(g(x)) = f(12 - \frac{1}{2}x) = (12 - \frac{1}{2}x)^2 - 16$
  $= \frac{1}{4}(x-24)^2 - 16$
- THE DOMAIN MUST SATISFY $x > 8$ AND $g(x) < 0$

$12 - \frac{1}{2}x < 0$

$-\frac{1}{2}x < -12$

$x > 24$



- THE RANGE CAN BE FOUND BY INSPECTION OR BY LOOKING AT THE GRAPH OPPOSITE

$f(g(x)) > -16$

c) SOLVING THE EQUATION

$\Rightarrow f(g(x)) = f(2x-22)$

$\Rightarrow \frac{1}{4}(x-24)^2 - 16 = 12 - \frac{1}{2}(2x-22)$ $\quad \times 4$

(FOUND IN PART b II)

$\Rightarrow (x-24)^2 - 64 = 48 - 2(2x-22)$

$\Rightarrow x^2 - 48x + 576 - 64 = 48 - 4x + 44$

$\Rightarrow x^2 - 44x + 420 = 0$

BY THE QUADRATIC FORMULA OR FACTORIZATION

$\Rightarrow (x-30)(x-14) = 0$

$\Rightarrow x = \cancel{14},\ 30$ $\quad x > 24$

Question 99 (****)

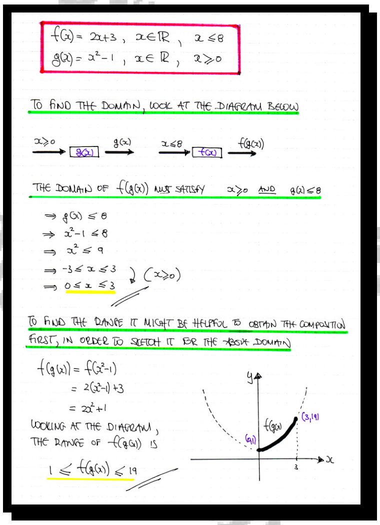
The functions f and g are defined by

$$f(x) = 2x + 3, \; x \in \mathbb{R}, \; x \le 8$$

$$g(x) = x^2 - 1, \; x \in \mathbb{R}, \; x \ge 0 .$$

Find the domain and range of $fg(x)$.

$\boxed{}$, $\boxed{0 \le x \le 3}$, $\boxed{1 \le fg(x) \le 19}$



$f(x) = 2x+3, \; x \in \mathbb{R}, \; x \le 8$
$g(x) = x^2-1, \; x \in \mathbb{R}, \; x \ge 0$

TO FIND THE DOMAIN, LOOK AT THE DIAGRAM BELOW

$\xrightarrow{x \ge 0} \boxed{g(x)} \xrightarrow{g(x)} \quad \xrightarrow{x \le 8} \boxed{f(x)} \xrightarrow{f(g(x))}$

THE DOMAIN OF $f(g(x))$ MUST SATISFY $x \ge 0$ AND $g(x) \le 8$

$\Rightarrow g(x) \le 8$
$\Rightarrow x^2 - 1 \le 8$
$\Rightarrow x^2 \le 9$
$\Rightarrow -3 \le x \le 3$ $\quad (x \ge 0)$
$\Rightarrow 0 \le x \le 3$

TO FIND THE RANGE IT MIGHT BE HELPFUL TO OBTAIN THE COMPOSITION FIRST, IN ORDER TO SKETCH IT FOR THE ABOVE DOMAIN

$f(g(x)) = f(x^2-1)$
$= 2(x^2-1)+3$
$= 2x^2+1$

LOOKING AT THE DIAGRAM, THE RANGE OF $f(g(x))$ IS

$1 \le f(g(x)) \le 19$

Question 100 (****)

The functions f and g are given by

$$f : x \mapsto x^2, \ x \in \mathbb{R}.$$

$$g : x \mapsto 2x+1, \ x \in \mathbb{R}.$$

a) Solve the equation

$$fg(x) = gf(x).$$

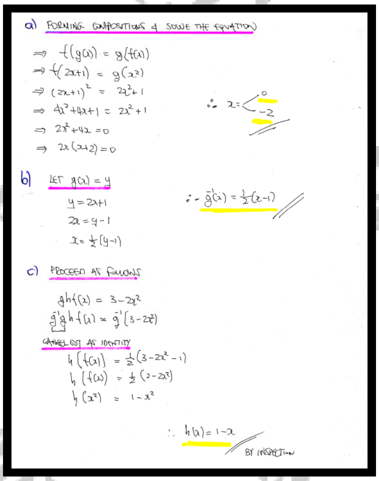
b) Find the inverse function of g.

The function h is defined on a suitable domain such so that

$$ghf(x) = 3 - 2x^2, \ x \in \mathbb{R}.$$

c) Determine an equation of h.

$\boxed{}$, $\boxed{x = -2 \ \cup \ x = 0}$, $\boxed{g^{-1}(x) = \frac{x-1}{2}}$, $\boxed{h(x) = 1 - x}$


a) FORMING COMPOSITIONS & SOLVE THE EQUATION

$\Rightarrow f(g(x)) = g(f(x))$
$\Rightarrow f(2x+1) = g(x^2)$
$\Rightarrow (2x+1)^2 = 2x^2+1$
$\Rightarrow 4x^2+4x+1 = 2x^2+1$
$\Rightarrow 2x^2+4x = 0$
$\Rightarrow 2x(x+2) = 0$

$\therefore x = \begin{cases} 0 \\ -2 \end{cases}$

b) LET $g(x) = y$

$y = 2x+1$
$2x = y-1$
$x = \frac{1}{2}(y-1)$

$\therefore g^{-1}(x) = \frac{1}{2}(x-1)$

c) PROCEED AS FOLLOWS

$ghf(x) = 3-2x^2$
$\underbrace{g^{-1}g}\,hf(x) = g^{-1}(3-2x^2)$
CANCEL OUT AS IDENTITY
$h(f(x)) = \frac{1}{2}(3-2x^2-1)$
$h(f(x)) = \frac{1}{2}(2-2x^2)$
$h(x^2) = 1-x^2$

$\therefore h(x) = 1-x$

BY INSPECTION

Question 101 (****)

The piecewise continuous function f is **even** with domain $x \in \mathbb{R}$, $-6 \le x \le 6$.

It is defined by

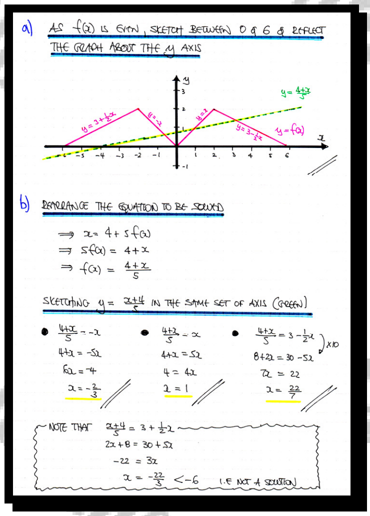
$$f(x) \equiv \begin{cases} x & 0 \le x \le 2 \\ 3 - \frac{1}{2}x & 2 \le x \le 6 \end{cases}$$

a) Sketch the graph of f for $-6 \le x \le 6$.

b) Hence, solve the equation

$$x = 4 + 5f(x).$$

$\boxed{}$, $\boxed{x = -\frac{2}{3} \cup x = 1 \cup x = \frac{22}{7}}$


a) AS $f(x)$ IS EVEN, SKETCH BETWEEN 0 & 6 & REFLECT THE GRAPH ABOUT THE y AXIS

b) REARRANGE THE EQUATION TO BE SOLVED

$\Rightarrow x = 4 + 5f(x)$
$\Rightarrow 5f(x) = 4 + x$
$\Rightarrow f(x) = \frac{4+x}{5}$

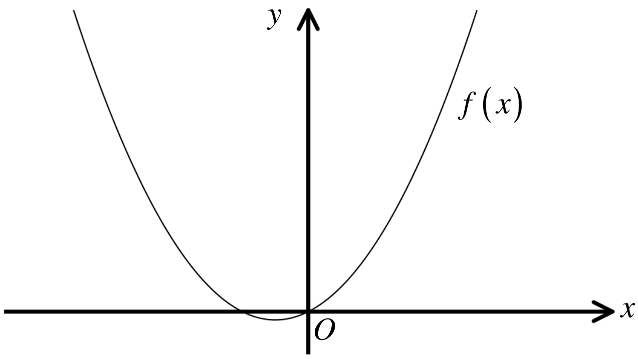
SKETCHING $y = \frac{x+4}{5}$ IN THE SAME SET OF AXIS (GREEN)

- $\frac{4+x}{5} = -x$; $4+x = -5x$; $6x = -4$; $x = -\frac{2}{3}$
- $\frac{4+x}{5} = x$; $4+x = 5x$; $4 = 4x$; $x = 1$
- $\frac{4+x}{5} = 3 - \frac{1}{2}x$) $\times 10$; $8+2x = 30 - 5x$; $7x = 22$; $x = \frac{22}{7}$

NOTE THAT $\frac{x+4}{5} = 3 + \frac{1}{2}x$
$2x + 8 = 30 + 5x$
$-22 = 3x$
$x = -\frac{22}{3} < -6$ I.E NOT A SOLUTION

Question 102 **(****)**

The graph below shows the graph of a function $f(x)$.



The function f is defined by

$$f(x) = \begin{cases} ax^2 + x, & x \in \mathbb{R},\ x \le 1 \\ bx^3 + 2, & x \in \mathbb{R},\ x > 1 \end{cases}$$

The function is **continuous** and **smooth.**

Find the value of a and the value of b.

$\boxed{}$, $\boxed{a = 4,\ b = 3}$

$$f(x) = \begin{cases} ax^2 + x & x \le 1 \\ bx^3 + 2 & x > 1 \end{cases} \Rightarrow f'(x) = \begin{cases} 2ax + 1 & x \le 1 \\ 3bx^2 & x > 1 \end{cases}$$

- CONTINUOUS

$ax^2 + x = bx^3 + 2$ at $x = 1$

$a + 1 = b + 2$

$\boxed{a - b = 1}$

- SMOOTH

$2ax + 1 = 3bx^2$ at $x = 1$

$2a + 1 = 3b$

$\boxed{2a - 3b = -1}$

$a = b + 1$

$2(b+1) - 3b = -1$

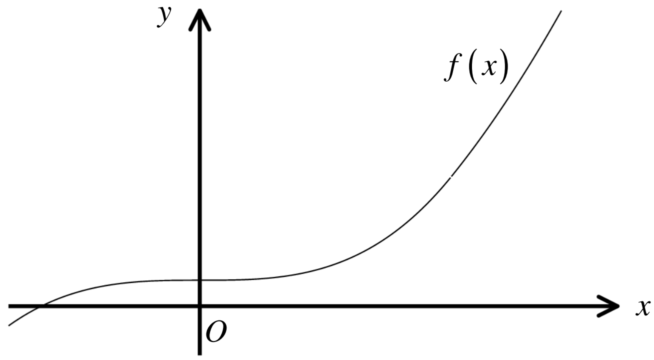
$2b + 2 - 3b = -1$

$3 = b$

$\therefore b = 3$

$a = 4$

Question 103 (****)



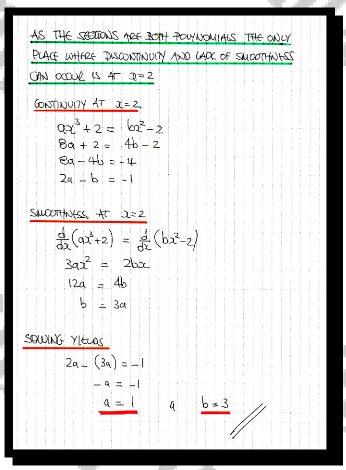
The figure above shows the graph of a function $f(x)$, defined by

$$f(x) = \begin{cases} ax^3 + 2, & x \in \mathbb{R}, \ x \le 2 \\ bx^2 - 2, & x \in \mathbb{R}, \ x > 2 \end{cases}$$

The function is **continuous** and **smooth.**

Find the value of a and the value of b.

$\boxed{}$, $\boxed{a = 1}$, $\boxed{b = 3}$



AS THE SECTIONS ARE BOTH POLYNOMIALS THE ONLY PLACE WHERE DISCONTINUITY AND LACK OF SMOOTHNESS CAN OCCUR IS AT $x=2$

CONTINUITY AT $x=2$

$ax^3 + 2 = bx^2 - 2$

$8a + 2 = 4b - 2$

$8a - 4b = -4$

$2a - b = -1$

SMOOTHNESS AT $x=2$

$\frac{d}{dx}(ax^3+2) = \frac{d}{dx}(bx^2-2)$

$3ax^2 = 2bx$

$12a = 4b$

$b = 3a$

SOLVING YIELDS

$2a - (3a) = -1$

$-a = -1$

$a = 1$ & $b = 3$ //

Question 104 (****)

The function $f(x)$ is defined by

$$f(x) \equiv 3 - 2x^2, \ x \in \mathbb{R}, \ x \le 0.$$

a) State the range of $f(x)$.

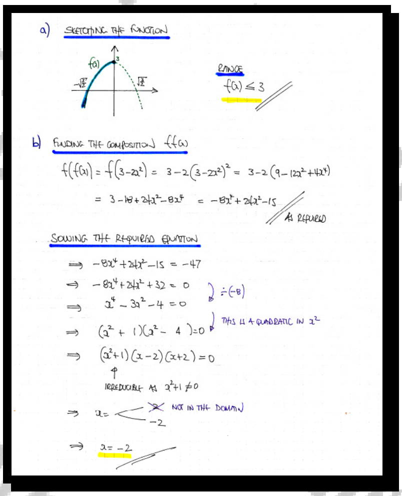
b) Show that $ff(x) = -8x^4 + 24x^2 - 15$ and hence solve the equation $ff(x) = -47$.

c) Find an expression for the inverse function, $f^{-1}(x)$.

d) Solve the equation

$$f(x) = f^{-1}(x).$$

$\boxed{\text{MP2-U}}$, $\boxed{f(x) \le 3}$, $\boxed{x = -2}$, $\boxed{f^{-1}(x) = -\sqrt{\dfrac{3-x}{2}}}$, $\boxed{x = -\dfrac{3}{2}}$


a) SKETCHING THE FUNCTION

RANGE

$f(x) \le 3$

b) FINDING THE COMPOSITION $ff(x)$

$f(f(x)) = f(3-2x^2) = 3-2(3-2x^2)^2 = 3-2(9-12x^2+4x^4)$

$= 3-18+24x^2-8x^4 = -8x^4+24x^2-15$

AS REQUIRED

SOLVING THE REQUIRED EQUATION

$\Rightarrow -8x^4+24x^2-15 = -47$

$\Rightarrow -8x^4+24x^2+32 = 0$ $\quad \div(-8)$

$\Rightarrow x^4-3x^2-4=0$

THIS IS A QUADRATIC IN x^2

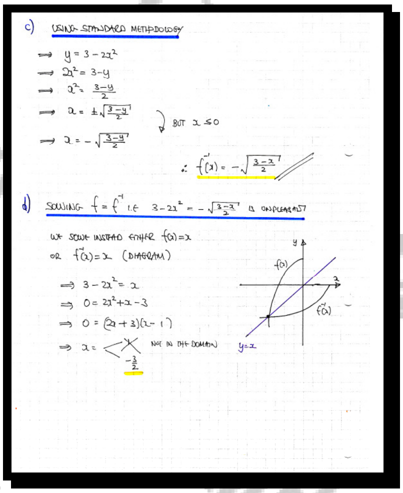
$\Rightarrow (x^2+1)(x^2-4)=0$

$\Rightarrow (x^2+1)(x-2)(x+2)=0$

IRREDUCIBLE AS $x^2+1 \ne 0$

$\Rightarrow x = \cancel{2}$ NOT IN THE DOMAIN, -2

$\Rightarrow x = -2$


c) USING STANDARD METHODOLOGY

$\Rightarrow y = 3-2x^2$

$\Rightarrow 2x^2 = 3-y$

$\Rightarrow x^2 = \frac{3-y}{2}$

$\Rightarrow x = \pm\sqrt{\frac{3-y}{2}}$

BUT $x \le 0$

$\Rightarrow x = -\sqrt{\frac{3-y}{2}}$

$\therefore f^{-1}(x) = -\sqrt{\frac{3-x}{2}}$

d) SOLVING $f = f^{-1}$ I.E. $3-2x^2 = -\sqrt{\frac{3-x}{2}}$ IS UNPLEASANT

WE SOLVE INSTEAD EITHER $f(x)=x$

OR $f^{-1}(x)=x$ (DIAGRAM)

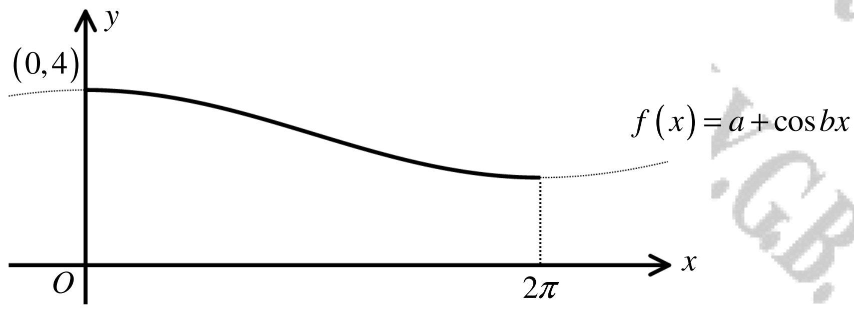
$\Rightarrow 3-2x^2 = x$

$\Rightarrow 0 = 2x^2+x-3$

$\Rightarrow 0 = (2x+3)(x-1)$

$\Rightarrow x = \cancel{1}$ NOT IN THE DOMAIN, $-\frac{3}{2}$

Question 105 (****)



The figure above shows the graph of the function

$$f(x) = a + \cos bx, \ 0 \le x \le 2\pi,$$

where a and b are non zero constants.

The stationary points $(0,4)$ and $(2\pi, 2)$ are the endpoints of the graph.

a) State the range of $f(x)$ and hence find the value of a and the value of b.

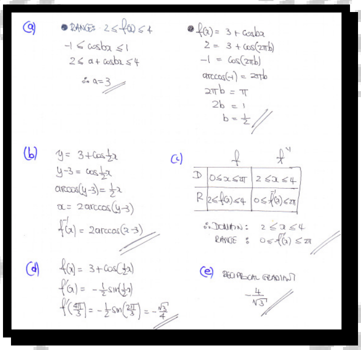
b) Find an expression for $f^{-1}(x)$, the inverse function of $f(x)$.

c) State the domain and range of $f^{-1}(x)$.

d) Find the gradient at the point on $f(x)$ with coordinates $\left(\frac{4\pi}{3}, \frac{5}{2}\right)$.

e) State the gradient at the point on $f^{-1}(x)$ with coordinates $\left(\frac{5}{2}, \frac{4\pi}{3}\right)$.

$\boxed{}$, $\boxed{2 \le f(x) \le 4}$, $\boxed{a = 3, \ b = \frac{1}{2}}$, $\boxed{f^{-1}(x) = 2\arccos(x-3)}$, $\boxed{2 \le x \le 4}$, $\boxed{0 \le f^{-1}(x) \le 2\pi}$, $\boxed{-\frac{\sqrt{3}}{4}}$, $\boxed{-\frac{4}{\sqrt{3}}}$



Question 106 (****)

The function f is defined in a suitable domain of real numbers and satisfies

$$f(x)=\ln\left(\frac{\mathrm{e}-x}{\mathrm{e}+x}\right).$$

a) Show that f is odd.

b) Determine the largest possible domain of f.

c) Solve the equation

$$f(x)+f(x+1)=0.$$

$\boxed{}$, $\boxed{x\in\mathbb{R},\quad -\mathrm{e}<x<\mathrm{e}}$, $\boxed{x=-\frac{1}{2}}$

a) $\boxed{f(x)=\ln\left(\frac{e-x}{e+x}\right)}$

$f(-x)=\ln\left(\frac{e-(-x)}{e+(-x)}\right)=\ln\left(\frac{e+x}{e-x}\right)=\ln\left(\frac{e-x}{e+x}\right)^{-1}$

$=-\ln\left(\frac{e-x}{e+x}\right)=-f(x)$

$\therefore$ $f(x)$ IS INDEED ODD

b) TO FIND THE LARGEST POSSIBLE DOMAIN

- $e+x\neq 0$
  $x\neq -e$
- THE LOG'S ARGUMENT MUST BE POSITIVE

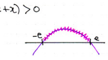
$\frac{e-x}{e+x}>0$

$\frac{(e-x)(e+x)}{(e+x)(e+x)}>0$

$\frac{(e-x)(e+x)}{(e+x)^2}>0$

AS THE DENOMINATOR IS ALWAYS POSITIVE

$(e-x)(e+x)>0$



$-e<x<e$

$\therefore$ LARGEST REAL DOMAIN

$x\in\mathbb{R},\ -e<x<e$

FINALLY SOLVING THE EQUATION

$\Rightarrow f(x)+f(x+1)=0$

$\Rightarrow \ln\left(\frac{e-x}{e+x}\right)+\ln\left[\frac{e-(x+1)}{e+(x+1)}\right]=0$

$\Rightarrow \ln\left(\frac{e-x}{e+x}\right)+\ln\left(\frac{e-x-1}{e+x+1}\right)=0$

$\Rightarrow \ln\left[\frac{e-x}{e+x}\times\frac{e-x-1}{e+x+1}\right]=0$

$\Rightarrow \frac{(e-x)(e-x-1)}{(e+x)(e+x+1)}=1$

$\Rightarrow (e-x)\left[(e-x)-1\right]=(e+x)\left[(e+x)+1\right]$

$\Rightarrow (e-x)^2-(e-x)=(e+x)^2+(e+x)$

$\Rightarrow \cancel{e^2}-2ex+\cancel{x^2}-e+\cancel{x}=\cancel{e^2}+2ex+\cancel{x^2}+e+\cancel{x}$

$\Rightarrow -2\cancel{e}=4\cancel{e}x$

$\Rightarrow x=-\frac{1}{2}$

Question 107 (****)

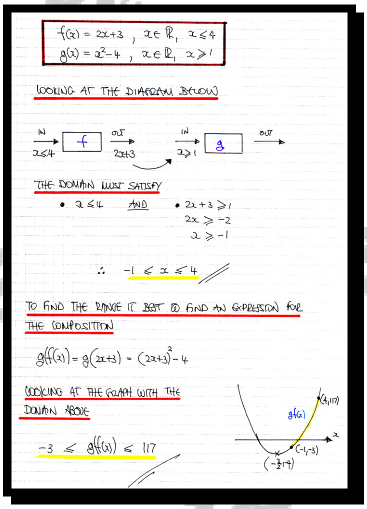
The functions f and g are defined by

$$f(x) = 2x + 3, \; x \in \mathbb{R}, \; x \leq 4$$

$$g(x) = x^2 - 4, \; x \in \mathbb{R}, \; x \geq 1.$$

Find the domain and range of $gf(x)$.

$\boxed{}$, $\boxed{-1 \leq x \leq 4}$, $\boxed{-3 \leq gf(x) \leq 117}$



$f(x) = 2x+3$, $x \in \mathbb{R}$, $x \leq 4$
$g(x) = x^2-4$, $x \in \mathbb{R}$, $x \geq 1$

LOOKING AT THE DIAGRAM BELOW

$\xrightarrow[x \leq 4]{\text{IN}} \boxed{f} \xrightarrow[2x+3]{\text{OUT}} \quad \xrightarrow[x \geq 1]{\text{IN}} \boxed{g} \xrightarrow{\text{OUT}}$

THE DOMAIN MUST SATISFY

- $x \leq 4$ AND
- $2x+3 \geq 1$
  $2x \geq -2$
  $x \geq -1$

$\therefore \; -1 \leq x \leq 4$

TO FIND THE RANGE IT BEST TO FIND AN EXPRESSION FOR THE COMPOSITION

$gf(x) = g(2x+3) = (2x+3)^2 - 4$

LOOKING AT THE GRAPH WITH THE DOMAIN ABOVE

$-3 \leq gf(x) \leq 117$

Question 108 (****)

The function f satisfies

$$f(x) \equiv x^2 - 4x + 1, \quad x \in \mathbb{R}, \quad x > 4.$$

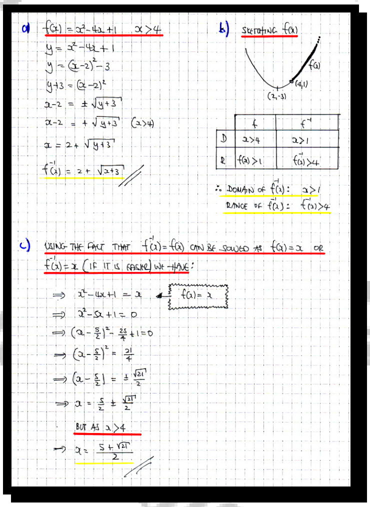
a) Find an expression for $f^{-1}(x)$.

b) Determine the domain and range of $f^{-1}(x)$.

c) Solve the equation

$$f(x) = f^{-1}(x).$$

$\boxed{f^{-1}(x) = 2 + \sqrt{x+3}}$, $\boxed{x \in \mathbb{R},\ x > 1}$, $\boxed{f^{-1}(x) \in \mathbb{R},\ f^{-1}(x) > 4}$, $\boxed{x = \dfrac{5+\sqrt{21}}{2}}$



a) $f(x) = x^2 - 4x + 1 \quad x > 4$

$y = x^2 - 4x + 1$

$y = (x-2)^2 - 3$

$y + 3 = (x-2)^2$

$x - 2 = \pm\sqrt{y+3}$

$x - 2 = +\sqrt{y+3} \quad (x > 4)$

$x = 2 + \sqrt{y+3}$

$f^{-1}(x) = 2 + \sqrt{x+3}$

b) SKETCHING $f(x)$

$f(x)$ $(4,1)$ $(2,-3)$

	f	f^{-1}
D	$x>4$	$x>1$
R	$f(x)>1$	$f^{-1}(x)>4$

$\therefore$ DOMAIN OF $f^{-1}(x)$: $x > 1$

RANGE OF $f^{-1}(x)$: $f^{-1}(x) > 4$

c) USING THE FACT THAT $f^{-1}(x) = f(x)$ CAN BE SOLVED AS $f(x) = x$ OR $f^{-1}(x) = x$ (IF IT IS EASIER) WE HAVE:

$\Rightarrow x^2 - 4x + 1 = x \quad \leftarrow \boxed{f(x) = x}$

$\Rightarrow x^2 - 5x + 1 = 0$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 1 = 0$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 = \frac{21}{4}$

$\Rightarrow \left(x - \frac{5}{2}\right) = \pm\frac{\sqrt{21}}{2}$

$\Rightarrow x = \frac{5}{2} \pm \frac{\sqrt{21}}{2}$

BUT AS $x > 4$

$\Rightarrow x = \frac{5+\sqrt{21}}{2}$

Question 109 (****)

The functions $f(x)$ and $g(x)$ are given by

$$f(x) = 3x - k, \ x \in \mathbb{R}, \ x \geq 1, \ k \in \mathbb{R}$$

$$g(x) = 2x^2 + 4, \ x \in \mathbb{R}, \ x \geq 0$$

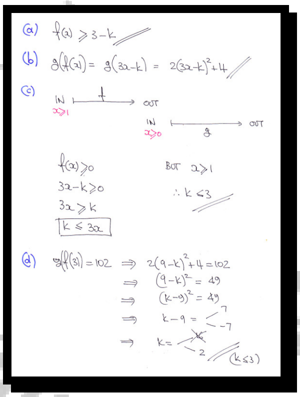
a) State the range of $f(x)$.

b) Find an expression for $gf(x)$ in terms of k.

c) Find the range of values of k which allows $gf(x)$ to be formed.

d) Find the value of k, given that $gf(3) = 102$.

$\boxed{f(x) \in \mathbb{R}, \ f(x) \geq 3 - k}$, $\boxed{f(x) \in \mathbb{R}, \ 2(3x-k)^2 + 4}$, $\boxed{k \leq 3}$, $\boxed{k = 2, \ k \neq 16}$



(a) $f(x) \geq 3 - k$

(b) $g(f(x)) = g(3x-k) = 2(3x-k)^2 + 4$

(c) IN $\xrightarrow{f}$ OUT, $x \geq 1$

IN $\xrightarrow{g}$ OUT, $x \geq 0$

$f(x) \geq 0$

$3x - k \geq 0$

$3x \geq k$

$\boxed{k \leq 3x}$

BUT $x \geq 1$

$\therefore k \leq 3$

(d) $g(f(3)) = 102 \Rightarrow 2(9-k)^2 + 4 = 102$

$\Rightarrow (9-k)^2 = 49$

$\Rightarrow (k-9)^2 = 49$

$\Rightarrow k - 9 = 7, -7$

$\Rightarrow k = \cancel{16}, 2$ $(k \leq 3)$

Question 110 (****)

The function $f(x)$ has domain $x \in \mathbb{R}$, $-1 \le x \le 5$.

It is further given that $f'(x) > 0$ and $f''(x) < 0$

Find a possible equation of $f(x)$, which ***does not*** contain exponentials.

$\boxed{}$, $\boxed{f(x) = \frac{1}{x+2}}$

NEGATIVE POWER OF x HAVE THE PROPERTY OF OF REVERSING SIGN ON DIFFERENTIATION (SO DOES THE NEGATIVE EXPONENTIAL)

E.G. $x^{-1}, -x^{-2}, +2x^{-3}, -6x^{-4}, +24x^{-5}$ ETC

AS WE NEED THE FUNCTION TO HAVE POSITIVE GRADIENT FUNCTION WE MAY START WITH

$-\frac{1}{x} \qquad +\frac{1}{x^2} \qquad -\frac{2}{x^3}$

BUT THIS IN NOT DEFINED AT $x=0$, SO WE MAY TRANSLATE BY 2 UNITS TO THE LEFT TO INCLUDE $x=-1$

HENCE $f(x) = -\frac{1}{x+2}$

$f'(x) = \frac{1}{(x+2)^2}$ WHICH IS POSITIVE $-1 \le x \le 5$

$f''(x) = \frac{-2}{(x+2)^3}$ WHICH IS NEGATIVE $-1 \le x \le 5$

Question 111 **(****+)**

The function f is defined by

$$f(x) = \begin{cases} -x^2 + 8x - 5, & x \in \mathbb{R},\ x \le 2 \\ x^2 - 2x + 8, & x \in \mathbb{R},\ x > 2 \end{cases}$$

a) Show that f …

i. … is not continuous.

ii. … is an increasing function.

Let the set A be defined

$$A = \{x \in \mathbb{R} : 1 \le x \le 3\}.$$

b) Determine the range of $f(A)$.

c) Find an expression for $f^{-1}(x)$, indicating clearly its domain.

$\boxed{}$, $\boxed{f(A) \in [2,7] \cup (8,11]}$, $\boxed{f^{-1}(x) = \begin{cases} 4 - \sqrt{11-x} & x \le 7 \\ 1 + \sqrt{1+x} & x > 8 \end{cases}}$

$$\boxed{f(x) = \begin{cases} -x^2 + 8x - 5 & , x \in \mathbb{R},\ x \le 2 \\ x^2 - 2x + 8 & , x \in \mathbb{R},\ x > 2 \end{cases}}$$

a) I) AS POLYNOMIALS ARE CONTINUOUS THE ONLY PLACE WHERE DISCONTINUITY MIGHT OCCUR IS AT $x=2$

$f(2) = -2^2 + 8 \times 2 - 5 = -4 + 16 - 5 = 7$

$\lim_{x \to 2^-} [f(x)] = 2^2 - 2 \times 2 + 8 = 4 - 4 + 8 = 8$

OR SIMPLY SUBSTITUTE $x=2$ INTO THE "SECOND" SECTION

$\therefore$ NOT CONTINUOUS AS THERE IS A "JUMP" FROM 7 TO 8 AT $x=2$

II) CONSIDERING TWO SEPARATE SECTIONS

- $f_1(x) = -x^2 + 8x - 5,\ x \le 2$

  $f_1'(x) = -2x + 8$

- NOW $x \le 2$

  $-2x \ge -4$

  $-2x + 8 \ge 4$

  $f_1'(x) > 4$

- $f_2(x) = x^2 - 2x + 8,\ x > 2$

  $f_2'(x) = 2x - 2$

- NOW $x > 2$

  $2x > 4$

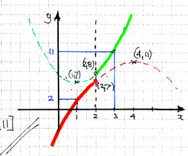
  $2x - 2 > 2$

  $f_2'(x) > 2$

$f'(x) > 0$ FOR ALL x, SO f IS INCREASING

b) LOOKING AT THE GRAPH OF f

$f(A)$ CAN TAKE VALUES BETWEEN 2 & 11, EXCLUDING THE GAP



$\therefore f(A) \in [2,7] \cup (8,11]$

c) TREATING EACH SECTION SEPARATELY

- $y = -x^2 + 8x - 5,\ x \le 2$

  $\Rightarrow -y = x^2 - 8x + 5$

  $\Rightarrow -y = (x-4)^2 - 11$

  $\Rightarrow 11 - y = (x-4)^2$

  $\Rightarrow x - 4 = -\sqrt{11-y}$

  $\Rightarrow x = 4 - \sqrt{11-y}$

- $y = x^2 - 2x - 5,\ x > 2$

  $\Rightarrow y = (x-1)^2 - 6$

  $\Rightarrow y + 6 = (x-1)^2$

  $\Rightarrow x - 1 = +\sqrt{y+1}$

  $\Rightarrow x = 1 + \sqrt{y+1}$

$$\therefore f^{-1}(x) = \begin{cases} 4 - \sqrt{11-x} & x \le 7 \\ 1 + \sqrt{1+x} & x > 8 \end{cases}$$

Question 112 (****+)

The function $f(x)$ is defined

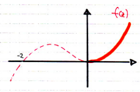
$$f(x) = x^2(x+2), \; x \in \mathbb{R}, \; x > 0.$$

a) Show that $f(x)$ is invertible.

b) Solve the equation

$$f(x) = f^{-1}(x).$$

$\boxed{\text{MP2-E}}$, $\boxed{x = -1+\sqrt{2}}$



a) EASIEST IS TO SHOW THAT f IS AN INCREASING FUNCTION IN ITS DOMAIN

$\Rightarrow f(x) = x^2(x+2) \quad x>0$

$\Rightarrow f(x) = x^3 + 2x^2$

$\Rightarrow f'(x) = 3x^2 + 2x$

IF $x>0$, $f'(x)>0$

$\therefore$ f IS AN INCREASING FUNCTION, & HENCE INVERTIBLE

b) THE SOLUTION SET OF $f(x) = f^{-1}(x)$ IS IDENTICAL TO THAT OF $f(x) = x$ OR INDEED $f^{-1}(x) = x$

$\Rightarrow f(x) = x$

$\Rightarrow x^2(x+2) = x$

$\Rightarrow x(x+2) = 1 \quad (x \neq 0)$

$\Rightarrow x^2 + 2x = 1$

$\Rightarrow (x+1)^2 - 1 = 1$

$\Rightarrow (x+1)^2 = 2$

$\Rightarrow x+1 = \pm\sqrt{2}$

$\Rightarrow x = -1+\sqrt{2}$ or $\cancel{-1-\sqrt{2}}$ $\quad x>0$

Question 113 (****+)

The function f is defined as

$$f : x \mapsto 6 - \ln(x+3), \; x \in \mathbb{R}, \; x \geq -2 .$$

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\ln x \xrightarrow{T_1} \ln(x+3) \xrightarrow{T_2} -\ln(x+3) \xrightarrow{T_3} -\ln(x+3)+6 .$$

a) Describe geometrically T_1, T_2 and T_3, and hence sketch the graph of $f(x)$. Indicate clearly any intersections with the axes and the graph's starting point.

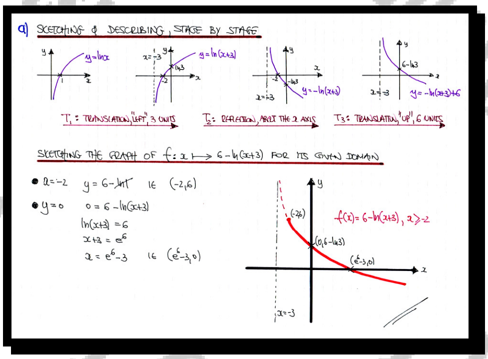
b) Find, in its simplest form, an expression for $f^{-1}(x)$, stating further the domain and range of $f^{-1}(x)$.

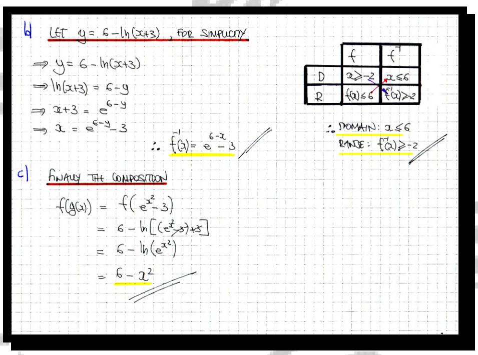
The function g satisfies

$$g : x \rightarrow e^{x^2} - 3, \quad x \in \mathbb{R} .$$

c) Find, in its simplest form, an expression for the composition $fg(x)$.

$\boxed{}$, $\boxed{T_1 = \text{translation, "left", 3 units}}$, $\boxed{T_2 = \text{reflection about } x\text{-axis}}$, $\boxed{T_3 = \text{translation, "up", 6 units}}$, $\boxed{(0, 6 - \ln 3)}$, $\boxed{(-3 + e^6, 0)}$, $\boxed{(-2, 6)}$, $\boxed{f^{-1} : x \mapsto -3 + e^{6-x}}$, $\boxed{x \leq 6, \quad f^{-1}(x) \geq -2}$, $\boxed{fg : x \mapsto 6 - x^2}$





Question 114 (****+)

The functions $f(x)$ and $g(x)$ are defined by

$$f(x) = \frac{4}{x+3}, \quad x \in \mathbb{R}, \ x > 0$$

$$g(x) = 9 - 2x^2, \quad x \in \mathbb{R}, \ x \geq 2.$$

a) Find, in its simplest form, the function $fg(x)$.

b) Find the domain of $fg(x)$.

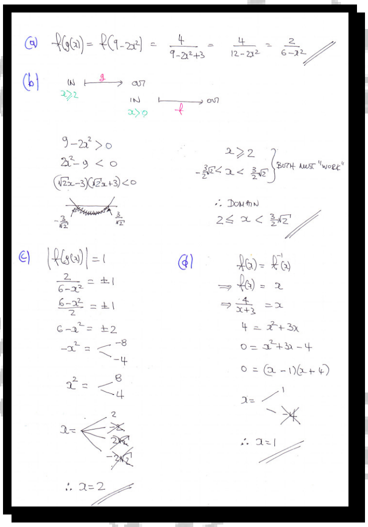
c) Find in exact form where appropriate the solutions of the equation

$$|fg(x)| = 1.$$

d) Solve the equation

$$f(x) = f^{-1}(x).$$

$\boxed{fg(x) = \frac{2}{6-x^2}}$, $\boxed{2 \leq x < \frac{3\sqrt{2}}{2}}$, $\boxed{x = 2, \ x \neq \pm 2\sqrt{2}, \ x \neq -2}$, $\boxed{x = 1, \ x \neq -4}$



Question 115 (****+)

The functions f and g are defined by

$$f(x) = 2x+1, \quad x \in \mathbb{R}, \quad x \le 5$$

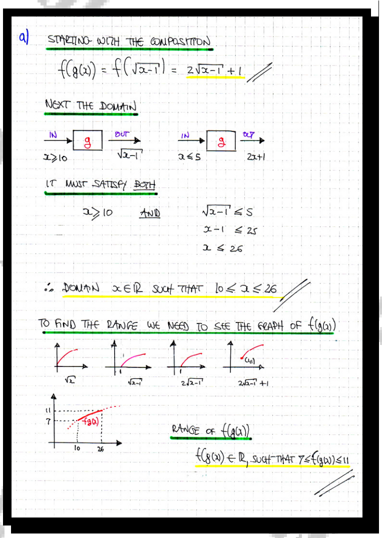
$$g(x) = \sqrt{x-1}, \quad x \in \mathbb{R}, \quad x \ge 10 .$$

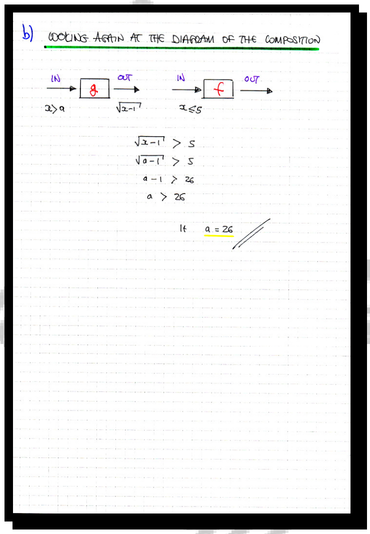
a) Find an expression for the composite function $fg(x)$, further stating its domain and range.

The domain of $g(x)$ is next changed to $x > a$.

b) Given that now $gf(x)$ **cannot** be formed, determine the smallest possible value of the constant a.

$\boxed{}$, $\boxed{fg(x) = 1 + 2\sqrt{x-1}}$, $\boxed{x \in \mathbb{R}, \ 10 \le x \le 26}$, $\boxed{fg(x) \in \mathbb{R}, \quad 7 \le fg(x) \le 11}$, $\boxed{a = 26}$





Question 116 (****+)

The function f satisfies

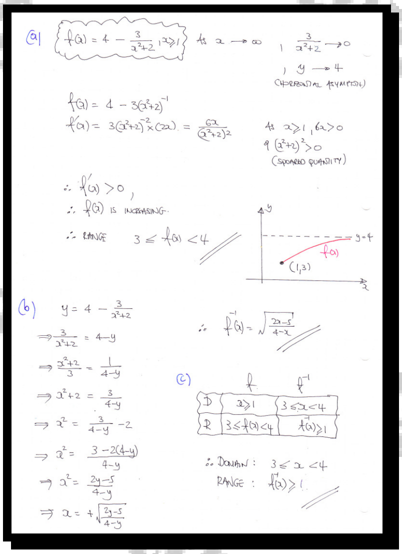
$$f(x) = 4 - \frac{3}{x^2+2}, \quad x \in \mathbb{R}, \ x \geq 1.$$

a) By considering the horizontal asymptote of $f(x)$ and showing further it is an increasing function, find its range.

b) Find $f^{-1}(x)$, in its simplest form.

c) Find the domain and range of $f^{-1}(x)$.

$\boxed{f(x) \in \mathbb{R},\ 3 \leq f(x) < 4}$, $\boxed{f^{-1}(x) = \sqrt{\frac{2x-5}{4-x}}}$, $\boxed{x \in \mathbb{R},\ 3 \leq x < 4}$, $\boxed{f(x) \in \mathbb{R},\ f(x) \geq 1}$



(a) $f(x) = 4 - \frac{3}{x^2+2}, x \geq 1$ As $x \to \infty$, $\frac{3}{x^2+2} \to 0$, $y \to 4$ (HORIZONTAL ASYMPTOTE)

$f(x) = 4 - 3(x^2+2)^{-1}$

$f'(x) = 3(x^2+2)^{-2} \times (2x) = \frac{6x}{(x^2+2)^2}$

As $x \geq 1$, $6x > 0$ & $(x^2+2)^2 > 0$ (SQUARED QUANTITY)

$\therefore f'(x) > 0$,

$\therefore f(x)$ IS INCREASING.

$\therefore$ RANGE $3 \leq f(x) < 4$

(b) $y = 4 - \frac{3}{x^2+2}$

$\Rightarrow \frac{3}{x^2+2} = 4 - y$

$\Rightarrow \frac{x^2+2}{3} = \frac{1}{4-y}$

$\Rightarrow x^2 + 2 = \frac{3}{4-y}$

$\Rightarrow x^2 = \frac{3}{4-y} - 2$

$\Rightarrow x^2 = \frac{3-2(4-y)}{4-y}$

$\Rightarrow x^2 = \frac{2y-5}{4-y}$

$\Rightarrow x = +\sqrt{\frac{2y-5}{4-y}}$

$\therefore f^{-1}(x) = \sqrt{\frac{2x-5}{4-x}}$

(c)

	f	f^{-1}
D	$x \geq 1$	$3 \leq x < 4$
R	$3 \leq f(x) < 4$	$f^{-1}(x) \geq 1$

$\therefore$ DOMAIN : $3 \leq x < 4$

RANGE : $f^{-1}(x) \geq 1$

Question 117 (****+)

The function f is given by

$$f(x) = 1 + \sqrt{x+1}, \quad x \in \mathbb{R}, \; x \geq 0.$$

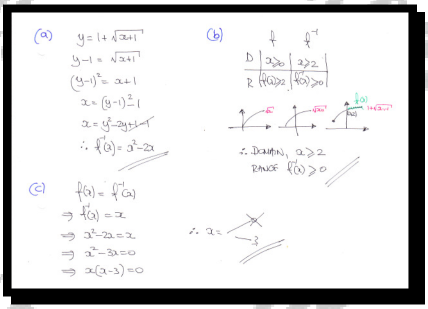
a) Find an expression for the inverse function $f^{-1}(x)$.

b) Determine the domain and range of $f^{-1}(x)$.

c) Solve the equation

$$f(x) = f^{-1}(x).$$

$\boxed{f^{-1}(x) = x^2 - 2x}$, $\boxed{x \in \mathbb{R}, \; x \geq 0}$, $\boxed{f^{-1}(x) \in \mathbb{R}, \; f^{-1}(x) \geq 0}$, $\boxed{x = 3}$


(a) $y = 1 + \sqrt{x+1}$

$y - 1 = \sqrt{x+1}$

$(y-1)^2 = x + 1$

$x = (y-1)^2 - 1$

$x = y^2 - 2y + \cancel{1} - \cancel{1}$

$\therefore f^{-1}(x) = x^2 - 2x$

(b)

	f	f^{-1}
D	$x \geq 0$	$x \geq 2$
R	$f(x) \geq 2$	$f^{-1}(x) \geq 0$

$\therefore$ DOMAIN, $x \geq 2$

RANGE $f^{-1}(x) \geq 0$

(c) $f(x) = f^{-1}(x)$

$\Rightarrow f^{-1}(x) = x$

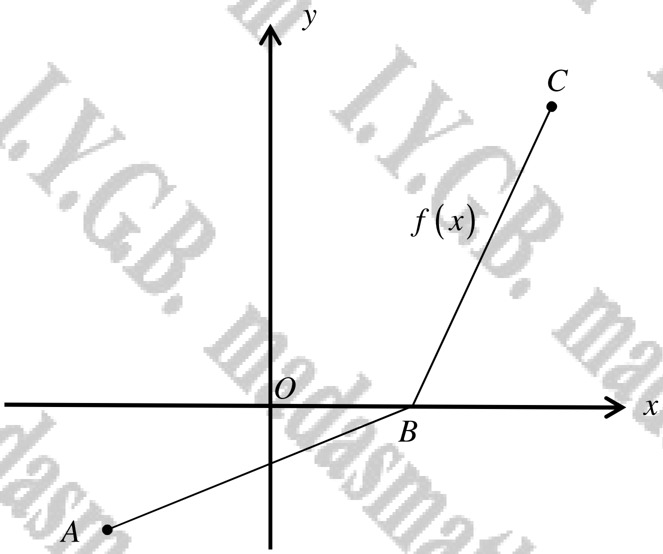
$\Rightarrow x^2 - 2x = x$

$\Rightarrow x^2 - 3x = 0$

$\Rightarrow x(x-3) = 0$

$\therefore x = \cancel{0}$ or 3

Question 118 (****+)



The above figure shows the graph of the function $f(x)$, consisting of two straight line segments starting at $A(-4,-4)$ and $B(6,8)$ meeting at the point $C(4,0)$.

a) State the range of $f(x)$.

b) Evaluate $ff(4)$.

c) Hence find $fff(5)$.

[continues overleaf]

[continued from overleaf]

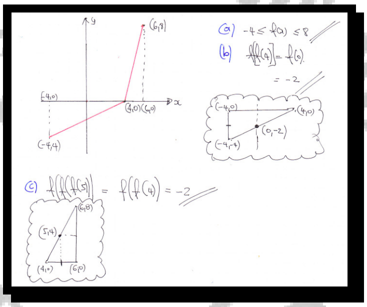
The inverse of $f(x)$ is $f^{-1}(x)$.

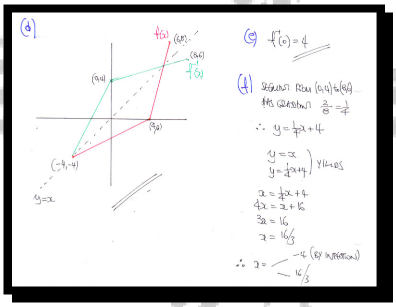
d) Sketch the graph of $f^{-1}(x)$.

e) State the value of $f^{-1}(x)$.

f) Solve the equation $f(x) = f^{-1}(x)$.

$\boxed{-4 \le f(x) \le 6}$, $\boxed{ff(4) = -2}$, $\boxed{fff(5) = -2}$, $\boxed{f^{-1}(0) = 4}$, $\boxed{x = -4, \frac{16}{3}}$





Question 119 (****+)

The functions f and g are given by

$$f(x) = 5\mathrm{e}^{-x} + 1, \ x \in \mathbb{R}, \ x \geq 0$$

$$g(x) = 2x + 1, \ x \in \mathbb{R}.$$

a) Find …

i. … an expression for $gf(x)$.

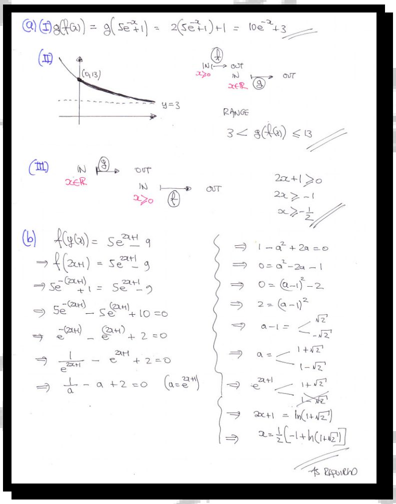
ii. … the range of $gf(x)$.

iii. … the domain of $fg(x)$.

b) Show that the only solution of the equation $fg(x) = 5\mathrm{e}^{2x+1} - 9$ can be written as

$$x = \tfrac{1}{2}\left[-1 + \ln\left(1 + \sqrt{2}\right)\right].$$

$\boxed{gf(x) = 10\mathrm{e}^{-x} + 3}$, $\boxed{3 < gf(x) \leq 13}$, $\boxed{x \geq -\tfrac{1}{2}}$



(a)(I) $g(f(x)) = g(5e^{-x}+1) = 2(5e^{-x}+1)+1 = 10e^{-x}+3$

(II) $(0,13)$ $y=3$

RANGE $3 < g(f(x)) \leq 13$

(III) $2x+1 \geq 0$, $2x \geq -1$, $x \geq -\frac{1}{2}$

(b) $f(g(x)) = 5e^{2x+1} - 9$

$\Rightarrow 5e^{-(2x+1)} - 5e^{2x+1} + 10 = 0$

$\Rightarrow e^{-(2x+1)} - e^{2x+1} + 2 = 0$

$\Rightarrow \frac{1}{a} - a + 2 = 0 \quad (a = e^{2x+1})$

$\Rightarrow 1 - a^2 + 2a = 0$

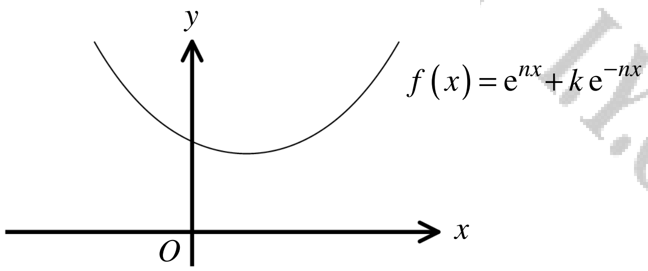
$\Rightarrow 0 = (a-1)^2 - 2$

$\Rightarrow a = 1 \pm \sqrt{2}$

$\Rightarrow 2x+1 = \ln(1+\sqrt{2})$

$\Rightarrow x = \frac{1}{2}\left(-1 + \ln(1+\sqrt{2})\right)$

Question 120 (****+)



The figure above shows the graph of the function with equation

$$f(x) = e^{nx} + k e^{-nx}, \quad x \in \mathbb{R}, \quad k > 1, \quad n > 0.$$

Find the range of $f(x)$ in exact form.

$\boxed{}$, $\boxed{f(x) \geq 2\sqrt{k}}$

LOCATE THE CO-ORDINATES OF THE MINIMUM BY DIFFERENTIATION

$f(x) = e^{nx} + ke^{-nx}$
$f'(x) = ne^{nx} - nke^{-nx}$

SOLVE $f'(x) = 0$

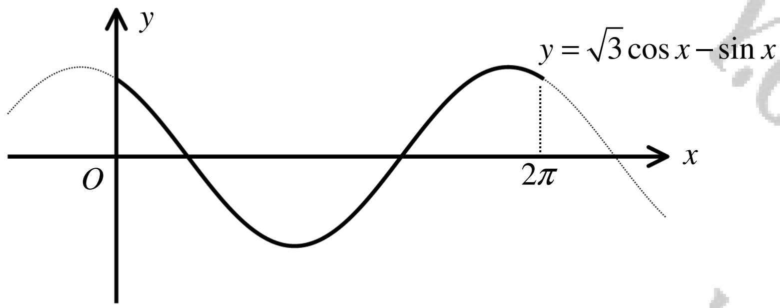
$\Rightarrow ne^{nx} - nke^{-nx} = 0$
$\Rightarrow e^{nx} - ke^{-nx} = 0 \qquad n \neq 0$
$\Rightarrow e^{nx} = ke^{-nx}$
$\Rightarrow e^{nx} = \frac{k}{e^{nx}}$
$\Rightarrow (e^{nx})^2 = k$
$\Rightarrow e^{nx} = +\sqrt{k} \qquad e^{nx} > 0$

NEXT WE CAN FIND THE y CO-ORDINATE – WE DON'T REQUIRE x

$\Rightarrow y = e^{nx} + ke^{-nx}$
$\Rightarrow y = e^{nx} + \frac{k}{e^{nx}}$
$\Rightarrow y = \sqrt{k} + \frac{k}{\sqrt{k}}$
$\Rightarrow y = \sqrt{k} + \sqrt{k}$
$\Rightarrow y = 2\sqrt{k}$

$\therefore$ THE RANGE IS $f(x) \geq 2\sqrt{k}$

Question 121 (****+)



The graph of $y = \sqrt{3}\cos x - \sin x$ for $0 \le x \le 2\pi$ is shown in the figure above.

a) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

The function f is defined as

$$f(x) = \sqrt{3}\cos x - \sin x, \; x \in \mathbb{R}, \; 0 \le x \le 2\pi .$$

b) State the range of $f(x)$.

c) Explain why $f(x)$ does not have an inverse.

[continues overleaf]

[continued from overleaf]

The function $g(x)$ is defined as

$$g(x) = \sqrt{3}\cos x - \sin x, \ x \in \mathbb{R}, \ 0 < x_1 \le x \le x_2 < 2\pi .$$

The ranges of $f(x)$ and $g(x)$ are the same and the inverse function $g^{-1}(x)$ exists.

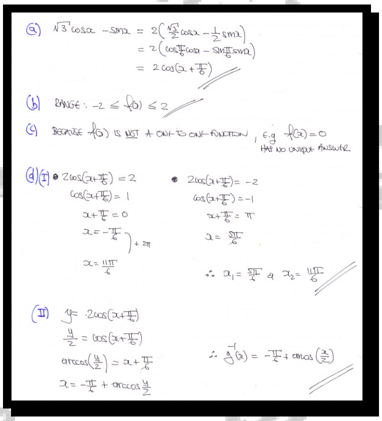
d) Find …

i. …the value of x_1 and the value of x_2.

ii. … an expression for $g^{-1}(x)$.

$\boxed{}$, $\boxed{\sqrt{3}\cos x - \sin x = 2\cos\left(x+\frac{\pi}{6}\right)}$, $\boxed{2 \le f(x) \le 2}$, $\boxed{x_1 = \frac{5\pi}{6}}$, $\boxed{x_2 = \frac{11\pi}{6}}$,

$\boxed{g^{-1}(x) = -\frac{\pi}{6} + \arccos\left(\frac{1}{2}\right)}$



(a) $\sqrt{3}\cos x - \sin x = 2\left(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right)$
$= 2\left(\cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\sin x\right)$
$= 2\cos\left(x+\frac{\pi}{6}\right)$

(b) RANGE: $-2 \le f(x) \le 2$

(c) BECAUSE $f(x)$ IS NOT A ONE-TO-ONE FUNCTION, e.g. $f(x)=0$ HAS NO UNIQUE ANSWER

(d)(I) • $2\cos\left(x+\frac{\pi}{6}\right) = 2$
$\cos\left(x+\frac{\pi}{6}\right) = 1$
$x+\frac{\pi}{6} = 0$
$x = -\frac{\pi}{6}$ $+2\pi$
$x = \frac{11\pi}{6}$

• $2\cos\left(x+\frac{\pi}{6}\right) = -2$
$\cos\left(x+\frac{\pi}{6}\right) = -1$
$x+\frac{\pi}{6} = \pi$
$x = \frac{5\pi}{6}$

$\therefore x_1 = \frac{5\pi}{6}$ & $x_2 = \frac{11\pi}{6}$

(II) $y = 2\cos\left(x+\frac{\pi}{6}\right)$
$\frac{y}{2} = \cos\left(x+\frac{\pi}{6}\right)$
$\arccos\left(\frac{y}{2}\right) = x+\frac{\pi}{6}$
$x = -\frac{\pi}{6} + \arccos\frac{y}{2}$

$\therefore g^{-1}(x) = -\frac{\pi}{6} + \arccos\left(\frac{x}{2}\right)$

Question 122 (****+)

The function f is defined as

$$f(x) = \ln\left(\frac{1+x}{1-x}\right), \ |x| < 1.$$

a) Show that $f(x)$ is an odd function.

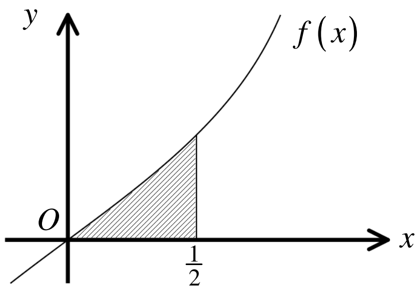
b) Find an expression for $f'(x)$ as a single simplified fraction, showing further that $f'(x)$ is an even function.

c) Determine an expression for $f^{-1}(x)$.

d) Use the substitution $u = e^x + 1$ to find the exact value of

$$\int_0^{\ln 3} f^{-1}(x) \ dx.$$

The figure below shows part of the graph of $f(x)$.



e) Find an exact value for the area of the shaded region, bounded by $f(x)$, the coordinate axes and the straight line with equation $x = \frac{1}{2}$.

$\boxed{}$, $\boxed{f'(x) = \frac{2}{1-x^2}}$, $\boxed{f'(x) = \frac{e^x - 1}{e^x + 1}}$, $\boxed{\ln\left(\frac{4}{3}\right)}$, $\boxed{\text{area} = \frac{3}{2}\ln 3 - 2\ln 2 \approx 0.262}$

[solution overleaf]

a) USING THE STANDARD METHOD

$$f(-x) = \ln\left[\frac{1+(-x)}{1-(-x)}\right] = \ln\left(\frac{1-x}{1+x}\right) = \ln\left(\frac{1+x}{1-x}\right)^{-1}$$

$$= -\ln\left(\frac{1+x}{1-x}\right) = -f(x)$$

AS $f(-x) = -f(x)$ THE FUNCTION IS ODD

b) DIFFERENTIATING AFTER MANIPULATING

$$f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$f'(x) = \frac{1}{1+x} + \frac{1}{1-x}$$

$$f'(x) = \frac{(1-\cancel{x}) + (1+\cancel{x})}{(1+x)(1-x)}$$

$$f'(x) = \frac{2}{1-x^2}$$

CHECKING $f'(-x)$

$$f'(-x) = \frac{2}{1-(-x)^2} = \frac{2}{1-x^2} = f'(x)$$

AS $f'(-x) = f'(x)$, $f'(x)$ IS EVEN

c) WRITE $f(x)$ AS y & REARRANGE

$$y = \ln\left(\frac{1+x}{1-x}\right) \implies e^y = \frac{1+x}{1-x}$$

$$\implies e^y(1-x) = 1+x$$

$$\implies e^y - xe^y = 1+x$$

$$\implies e^y - 1 = x + xe^y$$

$$\implies x(1+e^y) = e^y - 1$$

$$\implies x = \frac{e^y-1}{e^y+1}$$

$$\therefore f^{-1}(x) = \frac{e^x-1}{e^x+1}$$

d)

$$\int_0^{\ln 3} f^{-1}(x)\,dx = \int_0^{\ln 3} \frac{e^x-1}{e^x+1}\,dx$$

$$= \int_2^4 \frac{e^x-1}{u}\left(\frac{du}{e^x}\right) = \int_2^4 \frac{u-2}{u(u-1)}\,du$$

$u = e^x+1$

$\frac{du}{dx} = e^x$

$dx = \frac{du}{e^x}$

$e^x = u-1$

$x=0,\ u=2$

$x=\ln 3,\ u=4$

... PARTIAL FRACTIONS ...

$$\frac{u-2}{u(u-1)} \equiv \frac{A}{u} + \frac{B}{u-1}$$

$$\boxed{u-2 \equiv A(u-1) + Bu}$$

- IF $u=0$: $-2 = -A$, $A = 2$
- IF $u=1$: $-1 = B$, $B = -1$

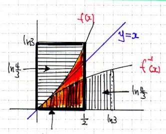
$$= \int_2^4 \frac{2}{u} - \frac{1}{u-1}\,du = \Big[2\ln|u| - \ln|u-1|\Big]_2^4$$

$$= (2\ln 4 - \ln 3) - (2\ln 2 - \cancel{\ln 1}) = 2\ln 4 - \ln 3 - 2\ln 2$$

$$= \ln 16 - \ln 3 - \ln 4 = \ln\left(\frac{16}{3\times 4}\right) = \ln\left(\frac{16}{12}\right)$$

$$= \ln\left(\frac{4}{3}\right)$$

e) LOOKING AT THE DIAGRAM BELOW



RECTANGLE $= \frac{1}{2}\ln 3$

$$\therefore \text{REQUIRED AREA} = \frac{1}{2}\ln 3 - \ln\frac{4}{3}$$

$$= \frac{1}{2}\ln 3 - (\ln 4 - \ln 3)$$

$$= \frac{1}{2}\ln 3 - 2\ln 2 + \ln 3$$

$$= \frac{3}{2}\ln 3 - \ln 4$$

Question 123 (****+)

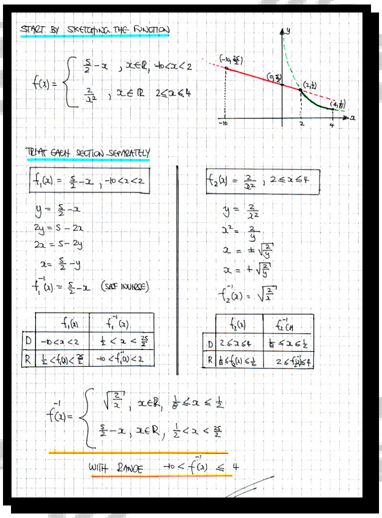
The piecewise continuous function f is defined by

$$f(x) \equiv \begin{cases} \frac{5}{2} - x, & x \in \mathbb{R}, \ -10 < x < 2 \\ \frac{2}{x^2}, & x \in \mathbb{R}, \quad 2 \le x \le 4 \end{cases}$$

Determine an expression, similar to the one above, for the inverse of f.

You must also give the range of the inverse of f.

$\boxed{}$, $\boxed{f^{-1}(x) \equiv \begin{cases} \sqrt{\frac{2}{x}}, & x \in \mathbb{R}, \ \frac{1}{8} \le x \le \frac{1}{2} \\ \frac{5}{2} - x, & x \in \mathbb{R}, \ \frac{1}{2} < x < \frac{25}{2} \end{cases}}$, $\boxed{-10 < f^{-1}(x) \le 4}$



START BY SKETCHING THE FUNCTION

$$f(x) = \begin{cases} \frac{5}{2} - x, & x \in \mathbb{R}, \ -10 < x < 2 \\ \frac{2}{x^2}, & x \in \mathbb{R} \ \ 2 \le x \le 4 \end{cases}$$

TREAT EACH SECTION SEPARATELY

$\boxed{f_1(x) = \frac{5}{2} - x, \ -10 < x < 2}$

$y = \frac{5}{2} - x$

$2y = 5 - 2x$

$2x = 5 - 2y$

$x = \frac{5}{2} - y$

$f_1^{-1}(x) = \frac{5}{2} - x$ (SELF INVERSE)

	$f_1(x)$	$f_1^{-1}(x)$
D	$-10 < x < 2$	$\frac{1}{2} < x < \frac{25}{2}$
R	$\frac{1}{2} < f_1(x) < \frac{25}{2}$	$-10 < f_1^{-1}(x) < 2$

$\boxed{f_2(x) = \frac{2}{x^2}, \ 2 \le x \le 4}$

$y = \frac{2}{x^2}$

$x^2 = \frac{2}{y}$

$x = \pm\sqrt{\frac{2}{y}}$

$x = +\sqrt{\frac{2}{y}}$

$f_2^{-1}(x) = \sqrt{\frac{2}{x}}$

	$f_2(x)$	$f_2^{-1}(x)$
D	$2 \le x \le 4$	$\frac{1}{8} \le x \le \frac{1}{2}$
R	$\frac{1}{8} \le f_2(x) \le \frac{1}{2}$	$2 \le f_2^{-1}(x) \le 4$

$$f^{-1}(x) = \begin{cases} \sqrt{\frac{2}{x}}, & x \in \mathbb{R}, \ \frac{1}{8} \le x \le \frac{1}{2} \\ \frac{5}{2} - x, & x \in \mathbb{R}, \ \frac{1}{2} < x < \frac{25}{2} \end{cases}$$

WITH RANGE $-10 < f^{-1}(x) \le 4$

Question 124 (****+)

$$f(x) = \ln(4x-8), \ x \in \mathbb{R}, \ x > 2.$$

a) Find an expression for the inverse function, $f^{-1}(x)$.

b) Find the domain and range of $f^{-1}(x)$.

The function g is defined as

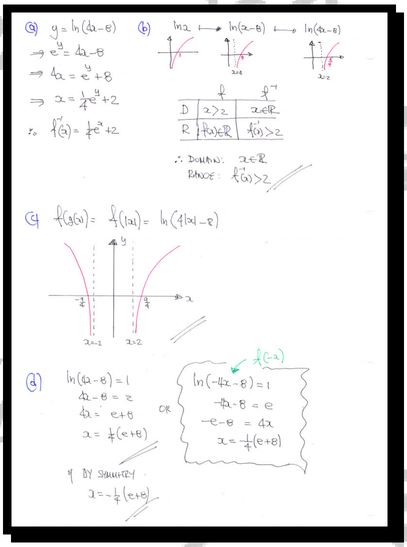
$$g(x) = |x|, \ x \in \mathbb{R}.$$

c) Sketch the graph of $fg(x)$, indicating clearly the equations of any asymptotes and the coordinates of the points where the graph meets the coordinate axes.

d) Hence solve the equation

$$fg(x) = 1.$$

$\boxed{f^{-1}(x) = 2 + \tfrac{1}{4}e^x}$, $\boxed{x \in \mathbb{R}, \ f^{-1}(x) > 2}$, $\boxed{\left(\tfrac{9}{4}, 0\right), \left(-\tfrac{9}{4}, 0\right), \ x = \pm 2}$, $\boxed{x = \pm\tfrac{1}{2}(e+8)}$



Question 125 (****+)

Information about the functions f, g and h are given by

$$f(x) \equiv 1 - \frac{1}{x},$$

$$g(x) \equiv ff(x),$$

$$fh(x) = \frac{x-3}{x-4}.$$

All the above functions are defined for all real numbers except for values of x for which the functions are undefined.

Find simplified expressions for …

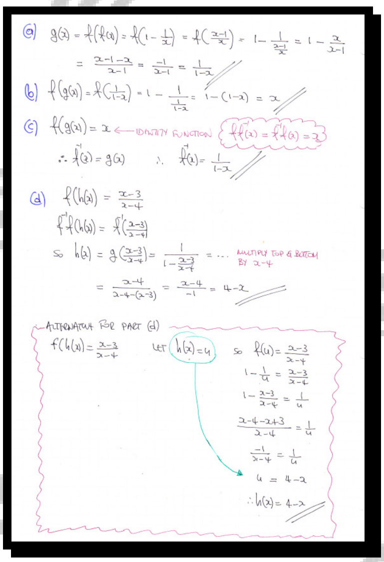
a) … $g(x)$.

b) … $fg(x)$.

c) … $f^{-1}(x)$.

d) … $h(x)$.

$$\boxed{g(x) = \frac{1}{1-x}}, \quad \boxed{fg(x) = x}, \quad \boxed{f^{-1}(x) = \frac{1}{1-x}}, \quad \boxed{h(x) = 4-x}$$



(a) $g(x) = f(f(x)) = f\left(1 - \frac{1}{x}\right) = f\left(\frac{x-1}{x}\right) = 1 - \frac{1}{\frac{x-1}{x}} = 1 - \frac{x}{x-1}$

$= \frac{x-1-x}{x-1} = \frac{-1}{x-1} = \frac{1}{1-x}$

(b) $f(g(x)) = f\left(\frac{1}{1-x}\right) = 1 - \frac{1}{\frac{1}{1-x}} = 1-(1-x) = x$

(c) $f(g(x)) = x \leftarrow$ IDENTITY FUNCTION $ff^{-1}(x) = f^{-1}f(x) = x$

$\therefore f^{-1}(x) = g(x)$ $\therefore f^{-1}(x) = \frac{1}{1-x}$

(d) $f(h(x)) = \frac{x-3}{x-4}$

$f^{-1}f(h(x)) = f^{-1}\left(\frac{x-3}{x-4}\right)$

so $h(x) = g\left(\frac{x-3}{x-4}\right) = \frac{1}{1-\frac{x-3}{x-4}} = \dots$ MULTIPLY TOP & BOTTOM BY $x-4$

$= \frac{x-4}{x-4-(x-3)} = \frac{x-4}{-1} = 4-x$

ALTERNATIVE FOR PART (d)

$f(h(x)) = \frac{x-3}{x-4}$ LET $h(x) = u$ so $f(u) = \frac{x-3}{x-4}$

$1 - \frac{1}{u} = \frac{x-3}{x-4}$

$1 - \frac{x-3}{x-4} = \frac{1}{u}$

$\frac{x-4-x+3}{x-4} = \frac{1}{u}$

$\frac{-1}{x-4} = \frac{1}{u}$

$u = 4-x$

$\therefore h(x) = 4-x$

Question 126 (****+)

The functions f and g are defined by

$$f(x) \equiv \sin x, \; x \in \mathbb{R}, \; -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

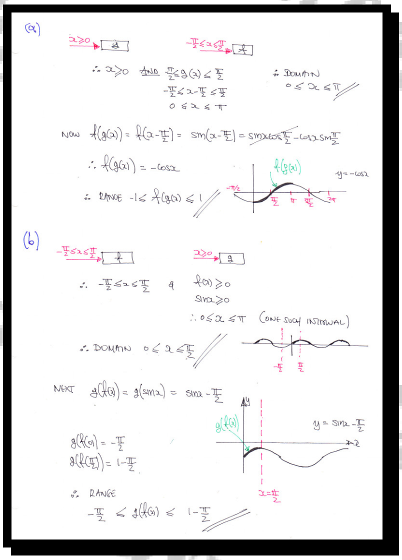
$$g(x) \equiv x - \frac{\pi}{2}, \; x \in \mathbb{R}, \; x \ge 0.$$

Determine, showing a clear method, the domain and range of the compositions

a) $fg(x)$.

b) $gf(x)$.

$\boxed{0 \le x \le \pi}$, $\boxed{-1 \le fg(x) \le 1}$, $\boxed{0 \le x \le \frac{\pi}{2}}$, $\boxed{-\frac{\pi}{2} \le gf(x) \le 1 - \frac{\pi}{2}}$



Question 127 **(****+)**

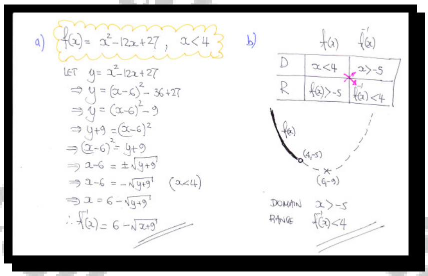
A function f is defined by

$$f(x) = x^2 - 12x + 27, \ x \in \mathbb{R}, \ x < 4.$$

a) Find an expression for $f^{-1}(x)$.

b) State the domain and range of $f^{-1}(x)$.

$\boxed{f^{-1}(x) = 6 - \sqrt{x+9}}$, $\boxed{x \geq -5}$, $\boxed{f^{-1}(x) < 4}$



Question 128 (****+)

The functions f and g are defined by

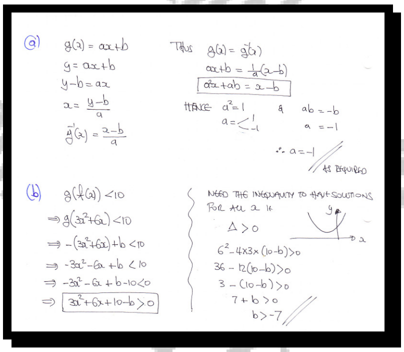
$$f(x) \equiv 3x^2 + 6x, \; x \in \mathbb{R},$$

$$g(x) \equiv ax + b, \; x \in \mathbb{R}.$$

a) Given that $g(x)$ is a self inverse function show that $a = -1$.

b) Given that $gf(x) < 10$ for all values of x, determine the range of values of b.

$\boxed{b > -7}$



(a) $g(x) = ax+b$

$y = ax+b$

$y-b = ax$

$x = \frac{y-b}{a}$

$g^{-1}(x) = \frac{x-b}{a}$

Thus $g(x) = g^{-1}(x)$

$ax+b = \frac{1}{a}(x-b)$

$\boxed{a^2x+ab = x-b}$

HENCE $a^2=1$ & $ab = -b$

$a = \pm 1$ $\quad a = -1$

$\therefore a = -1$ // AS REQUIRED

(b) $g(f(x)) < 10$

$\Rightarrow g(3x^2+6x) < 10$

$\Rightarrow -(3x^2+6x)+b < 10$

$\Rightarrow -3x^2-6x+b < 10$

$\Rightarrow -3x^2-6x+b-10 < 0$

$\Rightarrow \boxed{3x^2+6x+10-b > 0}$

NEED THE INEQUALITY TO HAVE SOLUTIONS FOR ALL x IF

$\Delta > 0$

$6^2 - 4\times3\times(10-b) > 0$

$36 - 12(10-b) > 0$

$3 - (10-b) > 0$

$7 + b > 0$

$b > -7$ //

Question 129 (****+)

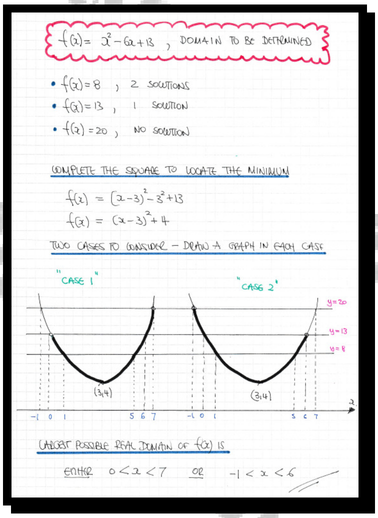
A function f is defined in a restricted real domain and has equation

$$f(x) \equiv x^2 - 6x + 13.$$

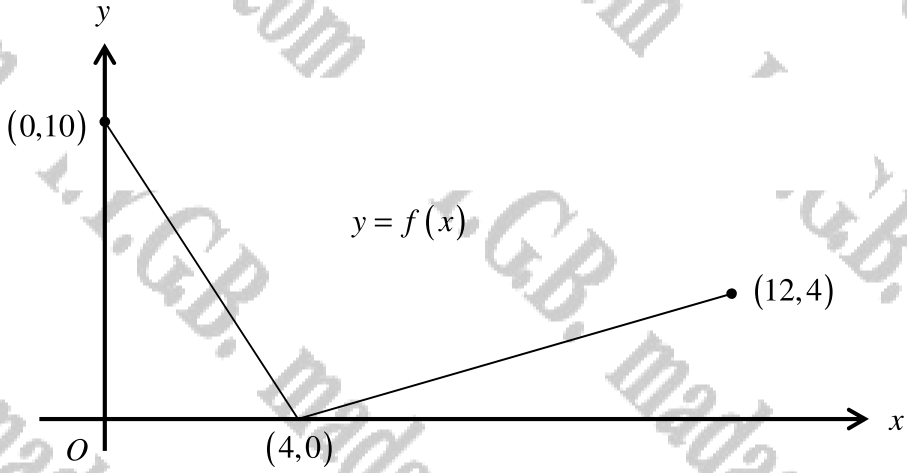
It is further given that the equations $f(x) = 8$, $f(x) = 13$ and $f(x) = 20$ have 2 distinct solutions, 1 solution and no solutions, respectively.

Determine the possible domain of f.

$\boxed{}$, $\boxed{0 < x < 7 \quad \text{or} \quad -1 < x < 6}$



Question 130 (****+)



The graph of the function $f(x)$ consists of two straight line segments joining the point $(0,10)$ to $(4,0)$ and the point $(12,4)$ to $(4,0)$, as shown in the figure above.

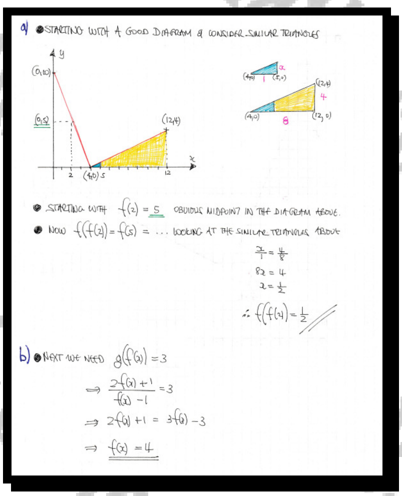
a) Find the value of $ff(2)$.

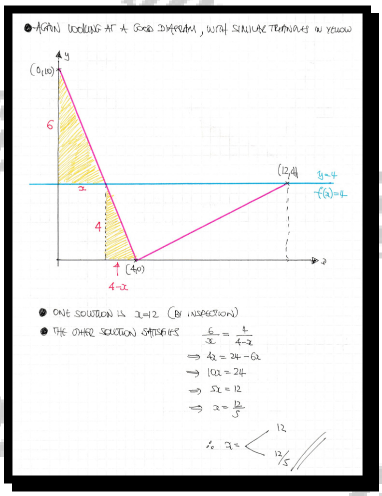
The function g is defined as

$$g(x) \equiv \frac{2x+1}{x-1}, \quad x \in \mathbb{R}, \; x \neq 1.$$

b) Determine the solutions of the equation $gf(x) = 3$.

$\boxed{}$, $\boxed{ff(2) = \frac{1}{2}}$, $\boxed{x = \frac{12}{5}, 12}$





Question 131 (****+)

The function f is defined as

$$f(x) = 3 - \ln 4x, \quad x \in \mathbb{R}, \; x > 0$$

a) Determine, in exact form, the coordinates of the point where the graph of f crosses the x axis.

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\ln x \xrightarrow{T_1} \ln 4x \xrightarrow{T_2} -\ln 4x \xrightarrow{T_3} 3 - \ln 4x$$

b) Describe geometrically each of the transformations T_1, T_2 and T_3, and hence sketch the graph of $f(x)$.

Indicate clearly any intersections with the coordinate axes.

The function g is defined by

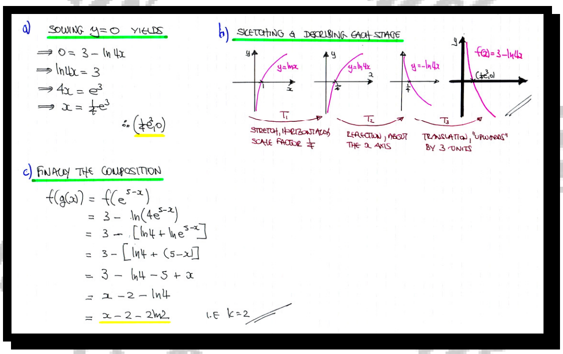
$$g(x) = e^{5-x}, \quad x \in \mathbb{R}.$$

c) Show that

$$fg(x) = x - k - k\ln k,$$

where k is a positive integer.

$\boxed{}$, $\boxed{\left(\frac{1}{4}e^3, 0\right)}$, $\boxed{T_1 = \text{stretch in } x \text{, scale factor } \frac{1}{4}}$, $\boxed{T_2 = \text{reflection in the } x\text{-axis}}$, $\boxed{T_3 = \text{translation, "up", 4 units}}$, $\boxed{k = 2}$



Question 132 (****+)

The function f is defined as

$$f(x) = \ln(4-2x), \; x \in \mathbb{R}, \; x < 2 .$$

a) Find in exact form the coordinates of the points where the graph of $f(x)$ crosses the coordinate axes.

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\ln x \xrightarrow{T_1} \ln(x+4) \xrightarrow{T_2} \ln(2x+4) \xrightarrow{T_3} \ln(-2x+4)$$

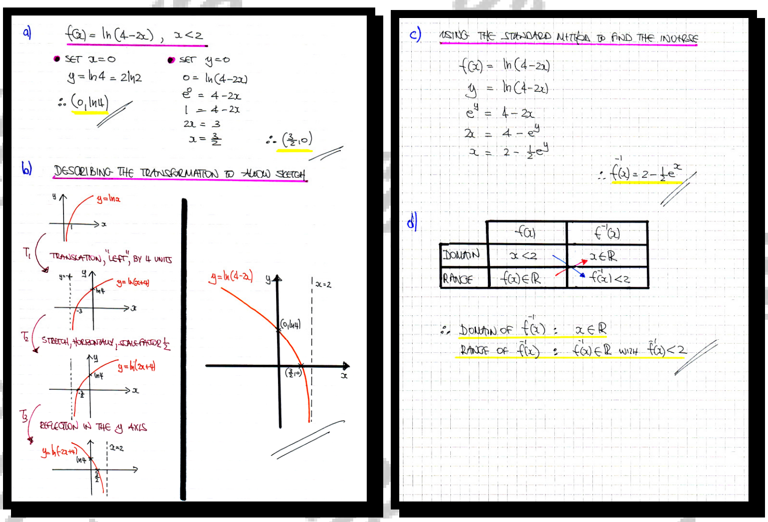
b) Describe geometrically the transformations T_1, T_2 and T_3, and hence sketch the graph of $f(x)$.

Indicate clearly any asymptotes and coordinates of intersections with the axes.

c) Find, an expression for $f^{-1}(x)$, the inverse function of $f(x)$.

d) State the domain and range of $f^{-1}(x)$.

$\boxed{\left(\frac{3}{2},0\right), \; (0,\ln 4)}$, $\boxed{T_1 = \text{translation, "left", 4 units}}$, $\boxed{T_2 = \text{stretch in } x \text{, scale factor } \frac{1}{2}}$, $\boxed{T_3 = \text{reflection in the } y\text{-axis}}$, $\boxed{\text{asymptote } x = 2}$, $\boxed{f^{-1}(x) = 2 - \frac{1}{2}e^x}$, $\boxed{x \in \mathbb{R}}$, $\boxed{f^{-1}(x) < 2}$



Question 133 (****+)

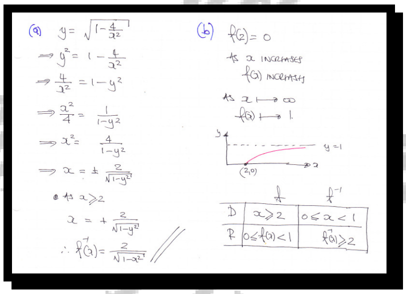
The function f is defined by

$$f(x)=\sqrt{1-\frac{4}{x^2}}\,,\ x\in\mathbb{R},\ x\geq 2\,.$$

a) Find an expression for $f^{-1}(x)$, in its simplest form.

b) Determine the domain and range of $f^{-1}(x)$.

$$\boxed{f^{-1}(x)=\frac{2}{\sqrt{1-x^2}}},\ \boxed{x\in\mathbb{R},\ 0\leq x<1},\ \boxed{f^{-1}(x)\in\mathbb{R},\ f^{-1}(x)\geq 2}$$



(a) $y=\sqrt{1-\frac{4}{x^2}}$

$\Rightarrow y^2=1-\frac{4}{x^2}$

$\Rightarrow \frac{4}{x^2}=1-y^2$

$\Rightarrow \frac{x^2}{4}=\frac{1}{1-y^2}$

$\Rightarrow x^2=\frac{4}{1-y^2}$

$\Rightarrow x=\pm\frac{2}{\sqrt{1-y^2}}$

• As $x\geq 2$

$x=+\frac{2}{\sqrt{1-y^2}}$

$\therefore f^{-1}(x)=\frac{2}{\sqrt{1-x^2}}$ //

(b) $f(2)=0$

As x INCREASES $f(x)$ INCREASES

As $x\mapsto\infty$ $f(x)\mapsto 1$.

	f	f^{-1}
D	$x\geq 2$	$0\leq x<1$
R	$0\leq f(x)<1$	$f^{-1}(x)\geq 2$

Question 134 (****+)

The function f is defined on a suitable domain, so that the functions g and h satisfy the following relationships.

$$g(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x)$$

$$h(x) = \frac{1}{2}f(x) - \frac{1}{2}f(-x).$$

a) Show clearly that g is an even function and h is an odd function.

It is now given that

$$f(x) = \frac{x+1}{x-1}, \quad x \in \mathbb{R}, \quad x \neq \pm 1.$$

b) Express $f(x)$ as the sum of an even and an odd function.

$$\boxed{f(x) = \frac{x^2+1}{x^2-1} + \frac{2x}{x^2-1}}$$

(a) $g(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x)$

$g(-x) = \frac{1}{2}f(-x) + \frac{1}{2}f(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x) = g(x) \Rightarrow g(-x) = g(x)$

SO EVEN

$h(x) = \frac{1}{2}f(x) - \frac{1}{2}f(-x)$

$h(-x) = \frac{1}{2}f(-x) - \frac{1}{2}f(x) = -\left[\frac{1}{2}f(x) - \frac{1}{2}f(-x)\right] = -h(x) \Rightarrow h(-x) = h(x)$

SO ODD

(b) NOW $g(x) + h(x) = \frac{1}{2}f(x) + \cancel{\frac{1}{2}f(-x)} + \frac{1}{2}f(x) - \cancel{\frac{1}{2}f(-x)} = f(x)$

$f(x) = g(x) + h(x)$

$$\frac{x+1}{x-1} = \underbrace{\frac{1}{2}\left[\frac{x+1}{x-1} + \frac{-x+1}{-x-1}\right]}_{\frac{1}{2}(f(x)+f(-x))} + \underbrace{\frac{1}{2}\left[\frac{x+1}{x-1} - \frac{-x+1}{-x-1}\right]}_{\frac{1}{2}(f(x)-f(-x))}$$

$$\frac{x+1}{x-1} = \frac{1}{2}\left[\frac{x+1}{x-1} + \frac{x-1}{x+1}\right] + \frac{1}{2}\left[\frac{x+1}{x-1} - \frac{x-1}{x+1}\right]$$

$$\frac{x+1}{x-1} = \frac{1}{2}\left[\frac{x^2+\cancel{2x}+1+x^2-\cancel{2x}+1}{(x-1)(x+1)}\right] + \frac{1}{2}\left[\frac{\cancel{x^2}+2x+\cancel{1}-\cancel{x^2}+2x-\cancel{1}}{(x-1)(x+1)}\right]$$

$$\frac{x+1}{x-1} = \frac{1}{2}\left[\frac{2x^2+2}{x^2-1}\right] + \frac{1}{2}\left[\frac{4x}{x^2-1}\right]$$

$$\frac{x+1}{x-1} = \underbrace{\frac{x^2+1}{x^2-1}}_{\text{EVEN}} + \underbrace{\frac{2x}{x^2-1}}_{\text{ODD}}$$

Question 135 (****+)

The functions f and g are defined by

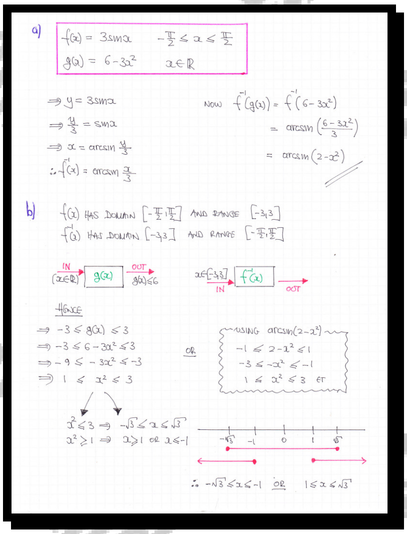
$$f(x) \equiv 3\sin x, \quad x \in \mathbb{R}, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$g(x) \equiv 6 - 3x^2, \quad x \in \mathbb{R}.$$

a) Find an expression for $f^{-1}g(x)$.

b) Determine the domain of $f^{-1}g(x)$.

$\boxed{}$, $\boxed{f^{-1}g(x) = \arcsin\left(2 - x^2\right)}$, $\boxed{-\sqrt{3} \le x \le -1 \ \text{ or } \ 1 \le x \le \sqrt{3}}$



a) $f(x) = 3\sin x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$g(x) = 6 - 3x^2 \quad x \in \mathbb{R}$

$\Rightarrow y = 3\sin x$

$\Rightarrow \frac{y}{3} = \sin x$

$\Rightarrow x = \arcsin \frac{y}{3}$

$\therefore f^{-1}(x) = \arcsin \frac{x}{3}$

NOW $f^{-1}(g(x)) = f^{-1}(6 - 3x^2)$

$= \arcsin\left(\frac{6 - 3x^2}{3}\right)$

$= \arcsin(2 - x^2)$

b) $f(x)$ HAS DOMAIN $[-\frac{\pi}{2}, \frac{\pi}{2}]$ AND RANGE $[-3, 3]$

$f^{-1}(x)$ HAS DOMAIN $[-3, 3]$ AND RANGE $[-\frac{\pi}{2}, \frac{\pi}{2}]$

IN $(x \in \mathbb{R})$ $\boxed{g(x)}$ OUT $g(x) \le 6$

$x \in [-3, 3]$ IN $\boxed{f^{-1}(x)}$ OUT

HENCE

$\Rightarrow -3 \le g(x) \le 3$

$\Rightarrow -3 \le 6 - 3x^2 \le 3$

$\Rightarrow -9 \le -3x^2 \le -3$

$\Rightarrow 1 \le x^2 \le 3$

OR

USING $\arcsin(2 - x^2)$

$-1 \le 2 - x^2 \le 1$

$-3 \le -x^2 \le -1$

$1 \le x^2 \le 3$ ETC

$x^2 \le 3 \Rightarrow -\sqrt{3} \le x \le \sqrt{3}$

$x^2 \ge 1 \Rightarrow x \ge 1$ OR $x \le -1$

$-\sqrt{3} \quad -1 \quad 0 \quad 1 \quad \sqrt{3}$

$\therefore -\sqrt{3} \le x \le -1$ OR $1 \le x \le \sqrt{3}$

Question 136 (****+)

A function f is defined as

$$y = 3x^4 - 8x^3 - 6x^2 + 24x - 8, \; x \in \mathbb{R}, \; -2 \le x \le 3.$$

Sketch the graph of f, and hence state its range.

The sketch must include the coordinates of any stationary points and any intersections with the coordinate axes.

$\boxed{-27 \le f(x) \le 37}$

START WITH THE STATIONARY POINTS

$$f(x) = 3x^4 - 8x^3 - 6x^2 + 24x - 8$$
$$f'(x) = 12x^3 - 24x^2 - 12x + 24$$

SOLVING FOR ZERO

$$\Rightarrow 12x^3 - 24x^2 - 12x + 24 = 0$$
$$\Rightarrow x^3 - 2x^2 - x + 2 = 0$$
$$\Rightarrow x^2(x-2) - (x-2) = 0$$
$$\Rightarrow (x-2)(x^2-1) = 0$$
$$\Rightarrow (x-2)(x-1)(x+1) = 0$$

$x = 2, 1, -1$ $\quad$ $y = 0, 5, -27$ $\quad$ $\therefore \boxed{(2,0) \; (1,5) \; (-1,-27)}$

NOW AS THE FUNCTION IS STATIONARY ON THE x AXIS IT MUST HAVE A REPEATED ROOT AT $x=2$ (NO INFLEXION AS THERE ARE TWO MORE STATIONARY VALUES).

HENCE DIVIDE BY $(x-2)^2$

$$\begin{array}{r} 3x^2 + 4x - 2 \\ x^2 - 4x + 4 \,\big)\, \overline{3x^4 - 8x^3 - 6x^2 + 24x - 8} \\ \underline{-3x^4 + 12x^3 - 12x^2 } \\ 4x^3 - 18x^2 + 24x - 8 \\ \underline{-4x^3 + 16x^2 - 16x } \\ -2x^2 + 8x - 8 \\ \underline{2x^2 - 8x + 8} \end{array}$$

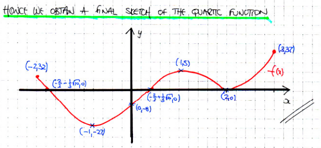
$$\therefore f(x) = (x-2)^2(3x^2+4x-2)$$

$$b^2 - 4ac = 16 - 4 \times 3 \times (-2) = 40$$

GETTING THE "ZEROS" OF THE QUADRATIC

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{40}}{2 \times 3} = \frac{-4 \pm 2\sqrt{10}}{6} = -\frac{2}{3} \pm \frac{1}{3}\sqrt{10}$$

HENCE WE OBTAIN A FINAL SKETCH OF THE QUARTIC FUNCTION



AND FINALLY THE RANGE LOOKING AT THE ABOVE GRAPH IS

$$-27 \le f(x) \le 37$$

Question 137 **(****+)**

The function f is defined as

$$f(x) \equiv \frac{x+1}{2x-1}, \quad x \in \mathbb{R}, \quad x \neq \frac{1}{2}.$$

The function g is suitably defined so that

$$f\left(g(x)\right) \equiv \frac{3x+2}{3x-5}, \quad x \in \mathbb{R}, \quad x \neq \frac{5}{3}.$$

a) Determine an expression for $g(x)$.

The function h is suitably defined so that

$$h\left(f(x)\right) \equiv \frac{2x-7}{x-2}, \quad x \in \mathbb{R}, \quad x \neq 2.$$

b) Determine an expression for $h(x)$.

$\boxed{}$, $\boxed{g(x) = \frac{2x-1}{x+3}}$, $\boxed{h(x) = \frac{4x-3}{x-1}}$

$f(x) = \frac{x+1}{2x-1}$, $x \in \mathbb{R}$, $x \neq \frac{1}{2}$ $\quad$ $f(g(x)) = \frac{3x+2}{3x-5}$, $x \in \mathbb{R}$, $x \neq \frac{5}{3}$ $\quad$ $h(f(x)) = \frac{2x-7}{x-2}$, $x \in \mathbb{R}$, $x \neq 2$

a) $f(g(x)) = \frac{3x+2}{3x-5}$

$$\Rightarrow \frac{g(x)+1}{2g(x)-1} = \frac{3x+2}{3x-5}$$
$$\Rightarrow (3x-5)g + 3x - 5 = 2(3x+2)g - 3x - 2$$
$$\Rightarrow 6x - 3 = (6x+4-3x+5)g$$
$$\Rightarrow 6x - 3 = (3x+9)g$$
$$\Rightarrow g = \frac{6x-3}{3x+9}$$
$$\therefore g(x) = \frac{2x-1}{x+3}$$

ALTERNATIVE USING INVERSES

find the inverse of f first

$$\Rightarrow y = \frac{x+1}{2x-1}$$
$$\Rightarrow 2xy - y = x + 1$$
$$\Rightarrow 2xy - x = y + 1$$
$$\Rightarrow x(2y-1) = y + 1$$
$$\Rightarrow x = \frac{y+1}{2y-1}$$
$$\therefore f^{-1}(x) = \frac{x+1}{2x-1} \text{ (SELF INVERSE)}$$

Next we proceed as follows

$$f(g(x)) = \frac{3x+2}{3x-5}$$
$$\Rightarrow f^{-1}\left(f\left(g(x)\right)\right) = f^{-1}\left(\frac{3x+2}{3x-5}\right)$$
$$\Rightarrow g(x) = \frac{\frac{3x+2}{3x-5} + 1}{2\left(\frac{3x+2}{3x-5}\right) - 1}$$
$$\Rightarrow g(x) = \frac{3x+2+3x-5}{2(3x+2)-3x+5}$$
$$\Rightarrow g(x) = \frac{6x-3}{3x+9}$$
$$\Rightarrow g(x) = \frac{2x-1}{x+3} \text{ AS BEFORE}$$

b) $h(f(x)) = \frac{2x-7}{x-2}$

- LET $f(x) = u$

$$\Rightarrow u = \frac{x+1}{2x-1}$$
$$\Rightarrow 2xu - u = x + 1$$
$$\Rightarrow 2xu - x = u + 1$$
$$\Rightarrow x(2u-1) = u + 1$$
$$\Rightarrow x = \frac{u+1}{2u-1}$$

- THUS WE HAVE

$$\Rightarrow h(u) = \frac{2\left(\frac{u+1}{2u-1}\right) - 7}{\frac{u+1}{2u-1} - 2}$$
$$\Rightarrow h(u) = \frac{2(u+1) - 7(2u-1)}{u+1-2(2u-1)}$$
$$\Rightarrow h(u) = \frac{-12u+9}{-3u+3}$$
$$\Rightarrow h(u) = \frac{4u-3}{u-1} \text{ (NOTHING SPECIAL ABOUT } u\text{)}$$
$$\therefore h(x) = \frac{4x-3}{x-1}$$

Question 138 (*****)

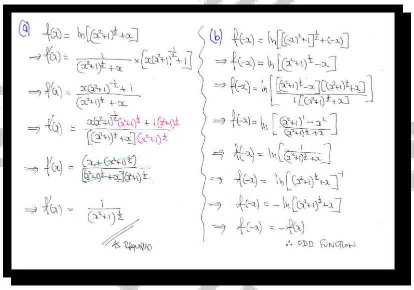
$$f(x)=\ln\left[\left(x^2+1\right)^{\frac{1}{2}}+x\right],\ x\in\mathbb{R}.$$

Show clearly that …

a) … $f'(x)=\dfrac{1}{\sqrt{x^2+1}}$.

b) … $f(x)$ is an odd function.

proof



(a) $f(x)=\ln\left[(x^2+1)^{\frac{1}{2}}+x\right]$

$\Rightarrow f'(x)=\dfrac{1}{(x^2+1)^{\frac{1}{2}}+x}\times\left[x(x^2+1)^{-\frac{1}{2}}+1\right]$

$\Rightarrow f'(x)=\dfrac{x(x^2+1)^{-\frac{1}{2}}+1}{(x^2+1)^{\frac{1}{2}}+x}$

$\Rightarrow f'(x)=\dfrac{x(x^2+1)^{-\frac{1}{2}}(x^2+1)^{\frac{1}{2}}+1(x^2+1)^{\frac{1}{2}}}{\left[(x^2+1)^{\frac{1}{2}}+x\right](x^2+1)^{\frac{1}{2}}}$

$\Rightarrow f'(x)=\dfrac{\cancel{\left(x+(x^2+1)^{\frac{1}{2}}\right)}}{\cancel{\left[(x^2+1)^{\frac{1}{2}}+x\right]}(x^2+1)^{\frac{1}{2}}}$

$\Rightarrow f'(x)=\dfrac{1}{(x^2+1)^{\frac{1}{2}}}$

// AS REQUIRED

(b) $f(-x)=\ln\left[\left[(-x)^2+1\right]^{\frac{1}{2}}+(-x)\right]$

$\Rightarrow f(-x)=\ln\left[(x^2+1)^{\frac{1}{2}}-x\right]$

$\Rightarrow f(-x)=\ln\left[\dfrac{\left[(x^2+1)^{\frac{1}{2}}-x\right]\left[(x^2+1)^{\frac{1}{2}}+x\right]}{1\left[(x^2+1)^{\frac{1}{2}}+x\right]}\right]$

$\Rightarrow f(-x)=\ln\left[\dfrac{(x^2+1)^1-x^2}{(x^2+1)^{\frac{1}{2}}+x}\right]$

$\Rightarrow f(-x)=\ln\left[\dfrac{1}{(x^2+1)^{\frac{1}{2}}+x}\right]$

$\Rightarrow f(-x)=\ln\left[(x^2+1)^{\frac{1}{2}}+x\right]^{-1}$

$\Rightarrow f(-x)=-\ln\left[(x^2+1)^{\frac{1}{2}}+x\right]$

$\Rightarrow f(-x)=-f(x)$

$\therefore$ ODD FUNCTION

Question 139 (*****)

$$f(x) = \frac{4x^2 - 10x + 7}{x^2 - 3x + 2}, \quad x \in \mathbb{R}, \; x \neq 1, \; x \neq 2.$$

Determine the range of $f(x)$.

$\boxed{\text{SP-C}}$, $\boxed{f(x) \le -2\sqrt{3} \cup f(x) \ge 2\sqrt{3}}$

$$\boxed{f(x) = \frac{4x^2 - 10x + 7}{x^2 - 3x + 2} \quad x \in \mathbb{R}, \; x \neq 1, x \neq 2}$$

- LET $y = f(x)$ & USE A DISCRIMINANT METHOD

$$\Rightarrow y = \frac{4x^2 - 10x + 7}{x^2 - 3x + 2}$$

$$\Rightarrow yx^2 - 3xy + 2y = 4x^2 - 10x + 7$$

$$\Rightarrow (y-4)x^2 + (10-3y)x + (2y-7) = 0$$

- NOW FOR REAL ROOTS $b^2 - 4ac \geqslant 0$

$$(10-3y)^2 - 4(y-4)(2y-7) \geqslant 0$$

$$9y^2 - 60y + 100 - 4(2y^2 - 7y - 8y + 28) \geqslant 0$$

$$9y^2 - \cancel{60y} + 100 - 8y^2 + \cancel{60y} - 112 \geqslant 0$$

$$y^2 - 12 \geqslant 0$$

$$y^2 \geqslant 12$$

$$y \leqslant -\sqrt{12} \quad \text{OR} \quad y \geqslant \sqrt{12}$$

- HENCE THE RANGE IS

$$f(x) \leqslant -2\sqrt{3} \quad \text{OR} \quad f(x) \geqslant 2\sqrt{3}$$

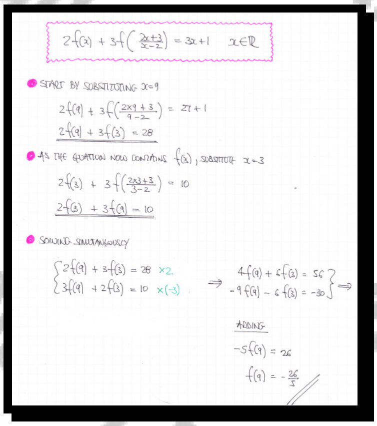
Question 140 (*****)

The function f satisfies

$$2f(x)+3f\left(\frac{2x+3}{x-2}\right)=3x+1,\ x\in\mathbb{R}.$$

Find the value of $f(9)$.

$\boxed{}$, $\boxed{f(9)=-\frac{26}{5}}$



$\boxed{2f(x)+3f\left(\frac{2x+3}{x-2}\right)=3x+1 \quad x\in\mathbb{R}}$

- START BY SUBSTITUTING $x=9$

$$2f(9)+3f\left(\frac{2\times9+3}{9-2}\right)=27+1$$
$$2f(9)+3f(3)=28$$

- AS THE EQUATION NOW CONTAINS $f(3)$, SUBSTITUTE $x=3$

$$2f(3)+3f\left(\frac{2\times3+3}{3-2}\right)=10$$
$$2f(3)+3f(9)=10$$

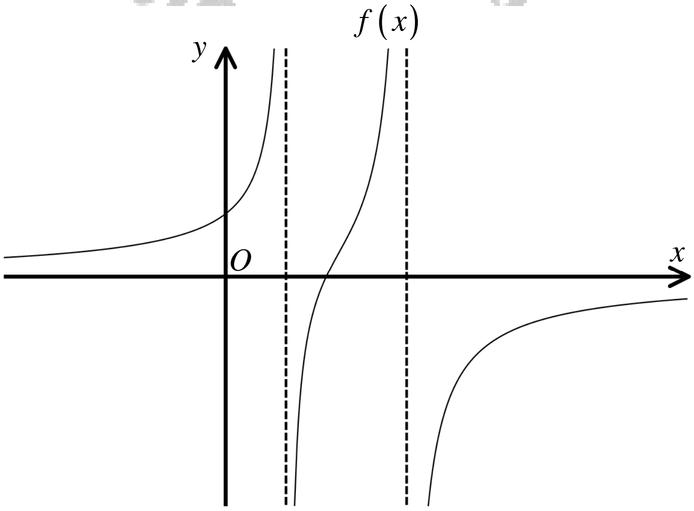
- SOLVING SIMULTANEOUSLY

$$\left\{\begin{array}{l}2f(9)+3f(3)=28 \quad \times 2\\ 3f(9)+2f(3)=10 \quad \times(-3)\end{array}\right. \Rightarrow \left.\begin{array}{l}4f(9)+6f(3)=56\\ -9f(9)-6f(3)=-30\end{array}\right\}\Rightarrow$$

ADDING

$$-5f(9)=26$$
$$f(9)=-\frac{26}{5}$$

Question 141 (*****)



The figure above shows the graph of

$$f(x)=\frac{5-3x}{(x-1)(x-3)}, \quad x \in \mathbb{R}, \; x \neq 1,3.$$

a) State the equations of the vertical asymptotes of $f(x)$, which are shown as dotted lines in the figure.

[continues overleaf]

[continued from overleaf]

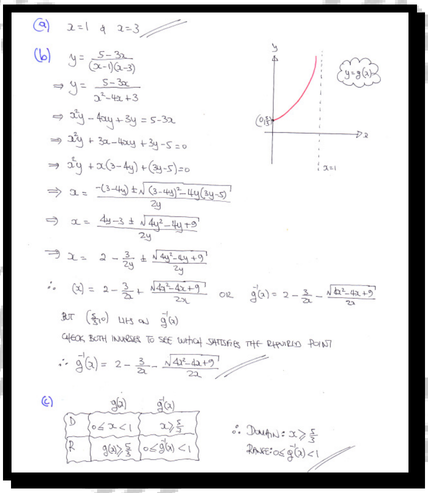
The function g is defined by as

$$g(x) = \frac{5-3x}{(x-1)(x-3)}, \quad x \in \mathbb{R}, \ 0 \le x < 1.$$

b) Find an expression for $g^{-1}(x)$.

c) State the domain and range of $g^{-1}(x)$.

$$\boxed{g^{-1}(x) = \frac{4x-3-\sqrt{4x^2-4x+9}}{2x}}, \ \boxed{x \in \mathbb{R}, \ \ x \ge \frac{5}{3}}, \ \boxed{g^{-1}(x) \in \mathbb{R}, \ \ 0 \le g^{-1}(x) \le 1}$$



(a) $x = 1$ & $x = 3$

(b) $y = \frac{5-3x}{(x-1)(x-3)}$

$\Rightarrow y = \frac{5-3x}{x^2-4x+3}$

$\Rightarrow x^2y - 4xy + 3y = 5-3x$

$\Rightarrow x^2y + 3x - 4xy + 3y - 5 = 0$

$\Rightarrow x^2y + x(3-4y) + (3y-5) = 0$

$\Rightarrow x = \frac{-(3-4y) \pm \sqrt{(3-4y)^2 - 4y(3y-5)}}{2y}$

$\Rightarrow x = \frac{4y-3 \pm \sqrt{4y^2-4y+9}}{2y}$

$\Rightarrow x = 2 - \frac{3}{2y} \pm \frac{\sqrt{4y^2-4y+9}}{2y}$

$\therefore (x) = 2 - \frac{3}{2x} + \frac{\sqrt{4x^2-4x+9}}{2x}$ OR $g^{-1}(x) = 2 - \frac{3}{2x} - \frac{\sqrt{4x^2-4x+9}}{2x}$

BUT $(\frac{5}{3}, 0)$ LIES ON $g^{-1}(x)$

CHECK BOTH INVERSES TO SEE WHICH SATISFIES THE REQUIRED POINT

$\therefore g^{-1}(x) = 2 - \frac{3}{2x} - \frac{\sqrt{4x^2-4x+9}}{2x}$

(c)

	$g(x)$	$g^{-1}(x)$
D	$0 \le x < 1$	$x \ge \frac{5}{3}$
R	$g(x) \ge \frac{5}{3}$	$0 \le g^{-1}(x) < 1$

$\therefore$ DOMAIN: $x \ge \frac{5}{3}$

RANGE: $0 \le g^{-1}(x) < 1$

Question 142 (*****)

The function f is defined below.

$$f(x) \equiv \ln\left[\sin x + \sqrt{2-\cos^2 x}\right], \quad x \in \mathbb{R}.$$

Prove that f is odd.

$\boxed{}$, $\boxed{\text{proof}}$

$$f(x) = \ln\left[\sin x + \sqrt{2-\cos^2 x}\right]$$

LET US NOTE THAT $\sin(-x) \equiv -\sin x$

$\cos(-x) \equiv \cos x$

THUS WE NOW HAVE

$$\begin{aligned} f(-x) &= \ln\left[\sin(-x) + \sqrt{2-\cos^2(-x)}\right] \\ &= \ln\left[-\sin x + \sqrt{2-\cos^2 x}\right] \\ &= \ln\left[\sqrt{2-\cos^2 x} - \sin x\right] \\ &= \ln\left[\frac{\left(\sqrt{2-\cos^2 x} - \sin x\right)\left(\sqrt{2-\cos^2 x} + \sin x\right)}{\sqrt{2-\cos^2 x} + \sin x}\right] \\ &= \ln\left[\frac{(2-\cos^2 x) - \sin^2 x}{\sqrt{2-\cos^2 x} + \sin x}\right] \\ &= \ln\left[\frac{1}{\sqrt{2-\cos^2 x} + \sin x}\right] \\ &= \ln\left[\sqrt{2-\cos^2 x} + \sin x\right]^{-1} \\ &= -\ln\left[\sqrt{2-\cos^2 x} + \sin x\right] \\ &= -f(x) \end{aligned}$$

AS $f(-x) = -f(x)$, f IS ODD //

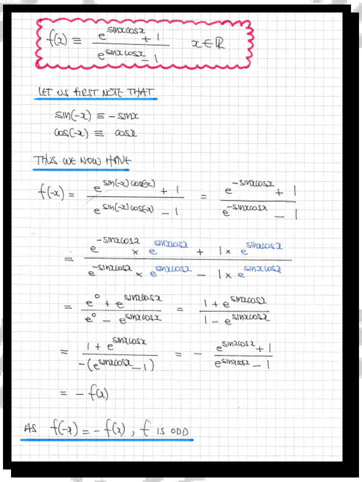
Question 143 (*****)

The function f is defined below.

$$f(x) \equiv \frac{e^{\sin x \cos x}+1}{e^{\sin x \cos x}-1}, \ x \in \mathbb{R}.$$

Prove that f is odd.

$\boxed{}$, $\boxed{\text{proof}}$



$$f(x) \equiv \frac{e^{\sin x \cos x}+1}{e^{\sin x \cos x}-1} \quad x \in \mathbb{R}$$

LET US FIRST NOTE THAT

$\sin(-x) \equiv -\sin x$

$\cos(-x) \equiv \cos x$

THUS WE NOW HAVE

$$f(-x) = \frac{e^{\sin(-x)\cos(-x)}+1}{e^{\sin(-x)\cos(-x)}-1} = \frac{e^{-\sin x \cos x}+1}{e^{-\sin x \cos x}-1}$$

$$= \frac{e^{-\sin x \cos x} \times e^{\sin x \cos x} + 1 \times e^{\sin x \cos x}}{e^{-\sin x \cos x} \times e^{\sin x \cos x} - 1 \times e^{\sin x \cos x}}$$

$$= \frac{e^{0} + e^{\sin x \cos x}}{e^{0} - e^{\sin x \cos x}} = \frac{1 + e^{\sin x \cos x}}{1 - e^{\sin x \cos x}}$$

$$= \frac{1 + e^{\sin x \cos x}}{-(e^{\sin x \cos x}-1)} = -\frac{e^{\sin x \cos x}+1}{e^{\sin x \cos x}-1}$$

$$= -f(x)$$

AS $f(-x) = -f(x)$, f IS ODD

Question 144 (*****)

$$f(x) = \frac{e^x - 1}{e^x + 1}, \quad x \in \mathbb{R}.$$

a) Show clearly that …

i. … $f(-x) = -f(x)$.

ii. … $f'(x) = \dfrac{2e^x}{\left(e^x + 1\right)^2}$.

b) Explain how the results of part **(a)** show that $f^{-1}(x)$ exists.

c) Find an expression for $f^{-1}(x)$.

The function $g(x)$ is defined in a suitable domain, so that

$$fg(x) = \frac{x^2 + 6x + 8}{x^2 + 6x + 10}.$$

d) Determine the equation of $g(x)$, in its simplest form.

$\boxed{}$, $\boxed{f^{-1}(x) = \ln\left(\dfrac{1+x}{1-x}\right)}$, $\boxed{g(x) = 2\ln(x+3)}$

(a)(I) $f(-x) = \dfrac{e^{-x}-1}{e^{-x}+1}$ = MULTIPLY TOP BOTTOM BY e^x … $= \dfrac{1-e^x}{1+e^x}$

$= \dfrac{-(e^x-1)}{e^x+1} = -f(x)$ AS REQUIRED

(II) $f'(x) = \dfrac{(e^x+1)e^x - e^x(e^x-1)}{(e^x+1)^2} = \dfrac{e^{2x}+e^x-e^{2x}+e^x}{(e^x+1)^2} = \dfrac{2e^x}{(e^x+1)^2}$ AS REQUIRED

(b) THE FUNCTION IS ODD AND HAS NO TURNING POINTS

$\therefore$ IT IS A ONE TO ONE FUNCTION, SO INVERTIBLE

(c) $y = \dfrac{e^x-1}{e^x+1}$

$\Rightarrow ye^x + y = e^x - 1$

$\Rightarrow ye^x - e^x = -y-1$

$\Rightarrow e^x(y-1) = -y-1$

$\Rightarrow e^x = \dfrac{-y-1}{y-1}$

$\Rightarrow e^x = \dfrac{1+y}{1-y}$

$\Rightarrow x = \ln\left(\dfrac{1+y}{1-y}\right)$

$\therefore f^{-1}(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

(d) $f(g(x)) = \dfrac{x^2+6x+8}{x^2+6x+10} \Rightarrow f^{-1}(f(g(x))) = f^{-1}\left(\dfrac{x^2+6x+8}{x^2+6x+10}\right)$

$\Rightarrow g(x) = \ln\left[\dfrac{1+\frac{x^2+6x+8}{x^2+6x+10}}{1-\frac{x^2+6x+8}{x^2+6x+10}}\right]$

$\Rightarrow g(x) = \ln\left[\dfrac{x^2+6x+10+x^2+6x+8}{x^2+6x+10-x^2-6x-8}\right]$

$\Rightarrow g(x) = \ln\left(\dfrac{2x^2+12x+18}{2}\right)$

$\Rightarrow g(x) = \ln(x^2+6x+9)$

$\Rightarrow g(x) = \ln(x+3)^2$

$\Rightarrow g(x) = 2\ln(x+3)$

Question 145 (*****)

The function f satisfies the following three relationships

i. $f(3n-2) \equiv f(3n)-2,\ n \in \mathbb{N}$.

ii. $f(3n) \equiv f(n),\ n \in \mathbb{N}$.

iii. $f(1) = 25$.

Determine the value of $f(25)$.

$\boxed{f(25) = 23}$

$f(3n-2) \equiv f(3n) - 2$ — I
$f(3n) \equiv f(n)$ — II
$f(1) = 25$ — III

- $n = 9 \Rightarrow f(25) = f(27) - 2$ (BY I)

$= f(9) - 2$ (BY II)

$= f(3) - 2$ (BY II)

$= f(1) - 2$ (BY II)

$= 25 - 2$ (BY III)

$= 23$

Question 146 **(*****)**

The function f is defined as

$$f(x) = -4 + \sqrt{mx+12}, \ x \in \mathbb{R}, \ x \geq -\frac{m}{12},$$

where m is a positive constant.

It is given that the graph of $f(x)$ and the graph of $f^{-1}(x)$ touch each other.

Solve the equation

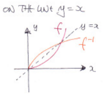
$$f(x) = f^{-1}(x).$$

$\boxed{\text{SP O}}$, $\boxed{x = \pm 2}$

$f(x) = -4 + \sqrt{mx+12}$, $x \in \mathbb{R}$, $x \geq -\frac{m}{12}$

- IF $f(x)$ & $f^{-1}(x)$ MEET, THEY MUST MEET ON THE LINE $y = x$

THUS WE MAY SOLVE



$f(x) = x = f^{-1}(x)$

$\Rightarrow -4 + \sqrt{mx+12} = x$

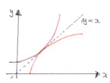
$\Rightarrow \sqrt{mx+12} = x+4$

$\Rightarrow mx + 12 = (x+4)^2$

$\Rightarrow mx + 12 = x^2 + 8x + 16$

$\Rightarrow \boxed{0 = x^2 + (8-m)x + 4}$

- IF THE TWO GRAPHS TOUCH EACH OTHER, THEY MUST ALSO THE LINE $y = x$



$\Rightarrow b^2 - 4ac = 0$

$\Rightarrow (8-m)^2 - 4 \times 1 \times 4 = 0$

$\Rightarrow (8-m)^2 - 16 = 0$

$\Rightarrow (8-m)^2 = 16$

$\Rightarrow 8 - m = \begin{cases} 4 \\ -4 \end{cases}$

$\Rightarrow -m = \begin{cases} -4 \\ -12 \end{cases}$

$\Rightarrow m = \begin{cases} 4 \\ 12 \end{cases}$

- IF $m = 4$

$x^2 + (8-m)x + 4 = 0$

$x^2 + 4x + 4 = 0$

$(x+2)^2 = 0$

$x = -2$

$x \geq -\frac{m}{12}$

$x \geq -\frac{1}{3}$

$\therefore x \neq -2$

- IF $m = 12$

$x^2 + (8-m)x + 4 = 0$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x = 2$

$x \geq -\frac{m}{12}$

$x \geq -1$

$\therefore$ ONLY SOLUTION $x = 2$

Question 147 (*****)

The functions f and g are defined by

$$f(x) \equiv \cos x, \; x \in \mathbb{R}, \; 0 \le x \le \pi$$

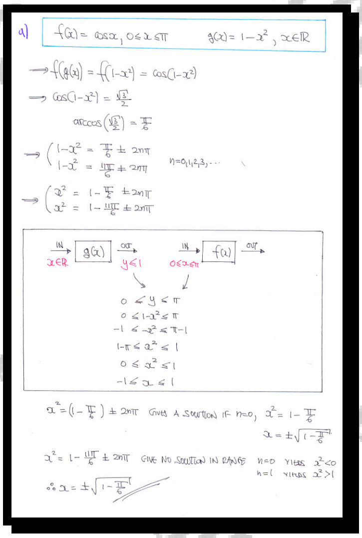
$$g(x) \equiv 1 - x^2, \; x \in \mathbb{R}.$$

a) Solve the equation

$$fg(x) = \frac{1}{2}.$$

b) Determine the values of x for which $f^{-1}g(x)$ is **not** defined.

$$\boxed{x = \pm\sqrt{1 - \frac{\pi}{6}}}, \quad \boxed{x < -\sqrt{2} \;\text{ or }\; x > \sqrt{2}}$$



a) $f(x) = \cos x, \; 0 \le x \le \pi \qquad g(x) = 1 - x^2, \; x \in \mathbb{R}$

$\Rightarrow f(g(x)) = f(1-x^2) = \cos(1-x^2)$

$\Rightarrow \cos(1-x^2) = \frac{\sqrt{3}}{2}$

$\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

$\Rightarrow \begin{cases} 1-x^2 = \frac{\pi}{6} \pm 2n\pi \\ 1-x^2 = \frac{11\pi}{6} \pm 2n\pi \end{cases} \quad n = 0,1,2,3,\dots$

$\Rightarrow \begin{cases} x^2 = 1 - \frac{\pi}{6} \pm 2n\pi \\ x^2 = 1 - \frac{11\pi}{6} \pm 2n\pi \end{cases}$

$0 \le y \le \pi$

$0 \le 1 - x^2 \le \pi$

$-1 \le -x^2 \le \pi - 1$

$1 - \pi \le x^2 \le 1$

$0 \le x^2 \le 1$

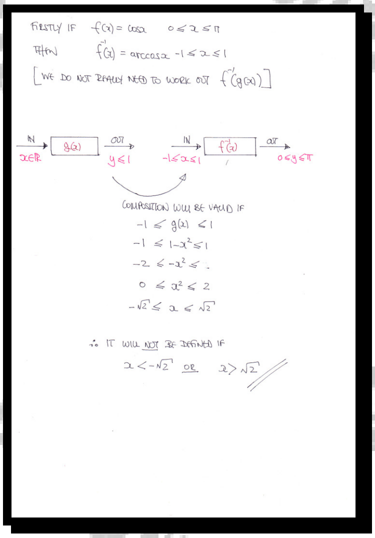
$-1 \le x \le 1$

$x^2 = \left(1 - \frac{\pi}{6}\right) \pm 2n\pi$ GIVES A SOLUTION IF $n=0$, $x^2 = 1 - \frac{\pi}{6}$

$x = \pm\sqrt{1 - \frac{\pi}{6}}$

$x^2 = 1 - \frac{11\pi}{6} \pm 2n\pi$ GIVE NO SOLUTION IN RANGE $n=0$ YIELDS $x^2 < 0$, $n=1$ YIELDS $x^2 > 1$

$\therefore x = \pm\sqrt{1 - \frac{\pi}{6}}$



FIRSTLY IF $f(x) = \cos x \quad 0 \le x \le \pi$

THEN $f^{-1}(x) = \arccos x \quad -1 \le x \le 1$

[WE DO NOT REALLY NEED TO WORK OUT $f^{-1}(g(x))$]

COMPOSITION WILL BE VALID IF

$-1 \le g(x) \le 1$

$-1 \le 1 - x^2 \le 1$

$-2 \le -x^2 \le 0$

$0 \le x^2 \le 2$

$-\sqrt{2} \le x \le \sqrt{2}$

$\therefore$ IT WILL NOT BE DEFINED IF

$x < -\sqrt{2}$ OR $x > \sqrt{2}$

Question 148 (*****)

The function f is defined

$$f(x) = \sqrt{4-x}\,,\ x \in \mathbb{R},\ x \le 4\,.$$

It is further given that

$$fg(x) = \sqrt{4+2x}\,,\ x \in \mathbb{R},\ x \ge -2\,,$$

$$hf(x) = x-4\,,\ x \in \mathbb{R},\ x \le 4\,.$$

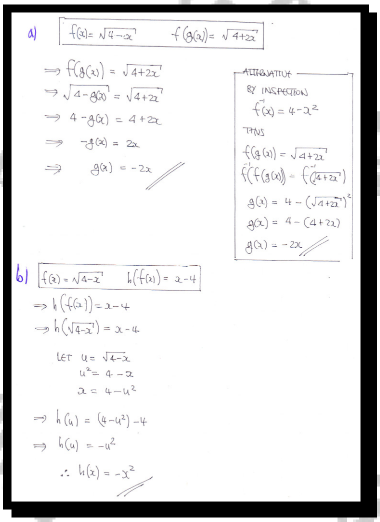
for some functions $g(x)$, $x \in \mathbb{R}$ and $h(x)$, $x \in \mathbb{R}$.

Find simplified expressions for …

a) … $g(x)$.

b) … $h(x)$.

$\boxed{}$, $\boxed{g(x) = -2x}$, $\boxed{h(x) = -x^2}$



a) $\boxed{f(x) = \sqrt{4-x} \qquad f(g(x)) = \sqrt{4+2x}}$

$\Rightarrow f(g(x)) = \sqrt{4+2x}$

$\Rightarrow \sqrt{4-g(x)} = \sqrt{4+2x}$

$\Rightarrow 4-g(x) = 4+2x$

$\Rightarrow -g(x) = 2x$

$\Rightarrow g(x) = -2x$

ALTERNATIVE

BY INSPECTION

$f^{-1}(x) = 4-x^2$

THUS

$f(g(x)) = \sqrt{4+2x}$

$f^{-1}(f(g(x))) = f^{-1}(\sqrt{4+2x})$

$g(x) = 4-(\sqrt{4+2x})^2$

$g(x) = 4-(4+2x)$

$g(x) = -2x$

b) $\boxed{f(x) = \sqrt{4-x} \qquad h(f(x)) = x-4}$

$\Rightarrow h(f(x)) = x-4$

$\Rightarrow h(\sqrt{4-x}) = x-4$

LET $u = \sqrt{4-x}$

$u^2 = 4-x$

$x = 4-u^2$

$\Rightarrow h(u) = (4-u^2)-4$

$\Rightarrow h(u) = -u^2$

$\therefore h(x) = -x^2$

Question 149 (*****)

The functions f and g are defined by

$$f(x) = \begin{cases} x^2, & x \in \mathbb{R},\ 0 < x \le 1 \\ 2 - x, & x \in \mathbb{R},\ 1 < x \le 2 \end{cases}$$

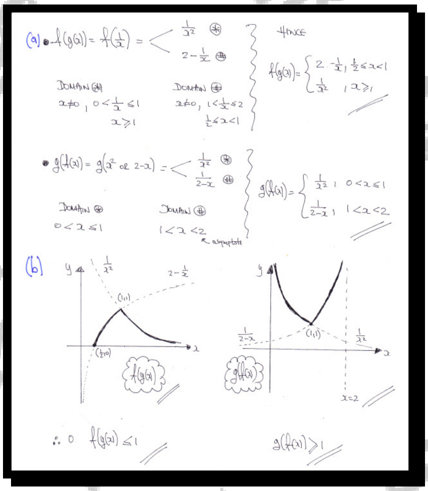
$$g(x) = \frac{1}{x},\ x \in \mathbb{R},\ x \neq 0.$$

a) Find expressions for the function compositions $fg(x)$ and $gf(x)$, giving full descriptions of their domains.

b) Sketch the graphs of the function compositions $fg(x)$ and $gf(x)$, and hence state the ranges of $fg(x)$ and $gf(x)$.

$$\boxed{},\ \boxed{fg(x) = \begin{cases} \dfrac{1}{x^2}, & x \in \mathbb{R},\ \dfrac{1}{2} \le x < 1 \\ \dfrac{1}{x^2}, & x \in \mathbb{R},\ x \ge 1 \end{cases}},\ \boxed{gf(x) = \begin{cases} \dfrac{1}{x^2}, & x \in \mathbb{R},\ 0 < x \le 1 \\ \dfrac{1}{2-x}, & x \in \mathbb{R},\ 1 < x < 2 \end{cases}}$$

$$\boxed{\begin{array}{l} fg(x) \in \mathbb{R},\ 0 \le fg(x) \le 1 \\ gf(x) \in \mathbb{R},\ \quad gf(x) \ge 1 \end{array}}$$



Question 150 (*****)

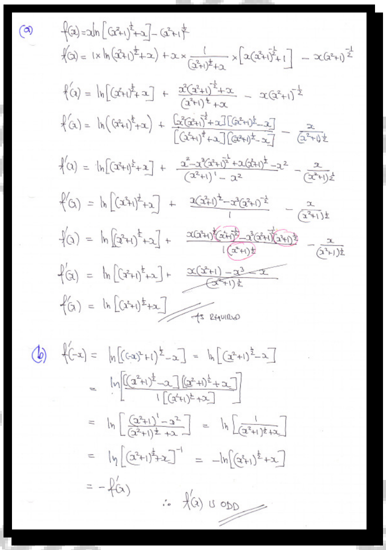
$$f(x) = x\ln\left[\left(x^2+1\right)^{\frac{1}{2}} + x\right] - \left(x^2+1\right)^{\frac{1}{2}},\ x \in \mathbb{R}.$$

Show clearly that …

a) … $f'(x) = \ln\left[\left(x^2+1\right)^{\frac{1}{2}} + x\right]$.

b) … $f'(x)$ is an odd function.

$\boxed{}$, $\boxed{\text{proof}}$



(a) $f(x) = x\ln\left[(x^2+1)^{\frac{1}{2}}+x\right] - (x^2+1)^{\frac{1}{2}}$

$f'(x) = 1 \times \ln\left((x^2+1)^{\frac{1}{2}}+x\right) + x \times \frac{1}{(x^2+1)^{\frac{1}{2}}+x} \times \left[x(x^2+1)^{-\frac{1}{2}}+1\right] - x(x^2+1)^{-\frac{1}{2}}$

$f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right] + \frac{x^2(x^2+1)^{-\frac{1}{2}}+x}{(x^2+1)^{\frac{1}{2}}+x} - x(x^2+1)^{-\frac{1}{2}}$

$f'(x) = \ln\left((x^2+1)^{\frac{1}{2}}+x\right) + \frac{\left[x^2(x^2+1)^{-\frac{1}{2}}+x\right]\left[(x^2+1)^{\frac{1}{2}}-x\right]}{\left[(x^2+1)^{\frac{1}{2}}+x\right]\left[(x^2+1)^{\frac{1}{2}}-x\right]} - \frac{x}{(x^2+1)^{\frac{1}{2}}}$

$f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right] + \frac{x^2 - x^3(x^2+1)^{-\frac{1}{2}} + x(x^2+1)^{\frac{1}{2}} - x^2}{(x^2+1)^1 - x^2} - \frac{x}{(x^2+1)^{\frac{1}{2}}}$

$f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right] + \frac{x(x^2+1)^{\frac{1}{2}} - x^3(x^2+1)^{-\frac{1}{2}}}{1} - \frac{x}{(x^2+1)^{\frac{1}{2}}}$

$f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right] + \frac{x(x^2+1)^{\frac{1}{2}}(x^2+1)^{\frac{1}{2}} - x^3(x^2+1)^{-\frac{1}{2}}(x^2+1)^{\frac{1}{2}}}{1(x^2+1)^{\frac{1}{2}}} - \frac{x}{(x^2+1)^{\frac{1}{2}}}$

$f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right] + \cancel{\frac{x(x^2+1) - x^3 - x}{(x^2+1)^{\frac{1}{2}}}}$

$f'(x) = \ln\left[(x^2+1)^{\frac{1}{2}}+x\right]$ as required

(b) $f'(-x) = \ln\left[((-x)^2+1)^{\frac{1}{2}}-x\right] = \ln\left[(x^2+1)^{\frac{1}{2}}-x\right]$

$= \ln\left[\frac{\left[(x^2+1)^{\frac{1}{2}}-x\right]\left[(x^2+1)^{\frac{1}{2}}+x\right]}{1\left[(x^2+1)^{\frac{1}{2}}+x\right]}\right]$

$= \ln\left[\frac{(x^2+1)^1-x^2}{(x^2+1)^{\frac{1}{2}}+x}\right] = \ln\left[\frac{1}{(x^2+1)^{\frac{1}{2}}+x}\right]$

$= \ln\left[(x^2+1)^{\frac{1}{2}}+x\right]^{-1} = -\ln\left[(x^2+1)^{\frac{1}{2}}+x\right]$

$= -f'(x)$

$\therefore f'(x)$ is odd

Question 151 (*****)

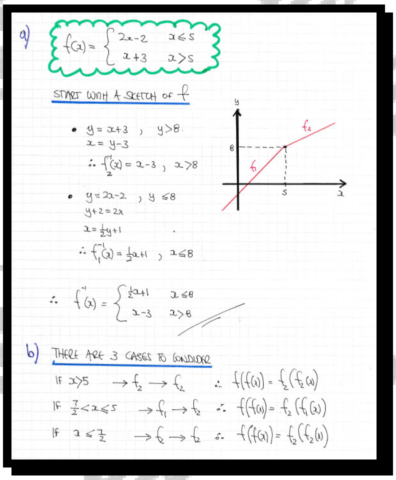
The piecewise continuous function f is given below.

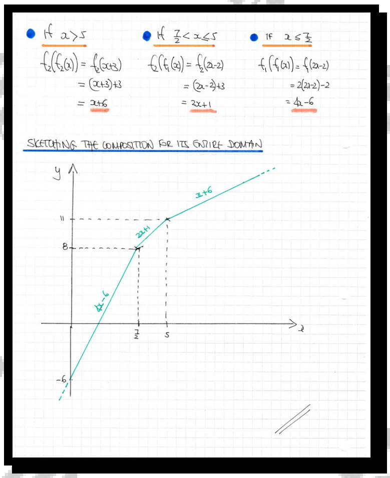
$$f(x) \equiv \begin{cases} 2x-2 & x \le 5 \\ x+3 & x > 5 \end{cases}$$

a) Determine an expression, in similar form to that of $f(x)$ above, for the inverse function, $f^{-1}(x)$.

b) Sketch a detailed graph for the composition $ff(x)$.

$\boxed{}$, $\boxed{f^{-1}(x) \equiv \begin{cases} \frac{1}{2}x+1 & x \le 8 \\ x-3 & x > 8 \end{cases}}$





Question 152 (*****)

The function f is defined as

$$f(x) \equiv 4x^3 - 12x^2 + 8x, \quad x \in \mathbb{R}, \ 0 \le x \le 3.$$

Find the range of f, and hence sketch its graph, showing clearly the coordinates of any relevant points.

$\boxed{}$, $\boxed{-\frac{8}{9}\sqrt{3} \le f(x) \le 24}$

$$\boxed{f(x) = 4x^3 - 12x^2 + 8x, \ x \in \mathbb{R}; \ 0 \le x \le 3}$$

- FIRSTLY LOOK FOR LOCAL MINIMA/MAXIMA

$\Rightarrow f'(x) = 12x^2 - 24x + 8$

$\Rightarrow 0 = 12x^2 - 24x + 8$

$\Rightarrow 0 = 3x^2 - 6x + 2$

$\therefore x = \frac{6 \pm \sqrt{(-6)^2 - 4\times3\times2}}{2\times3} = \frac{6 \pm \sqrt{36-24}}{6} = \frac{6 \pm \sqrt{12}}{6}$

$= \frac{6 \pm 2\sqrt{3}}{6} = 1 \pm \frac{\sqrt{3}}{3}$ (BOTH ARE IN THE DOMAIN)

- CHECK THESE POINTS FIRST

$f(x) = 4x^3 - 12x^2 + 8x$

$f(x) = 4x(x^2 - 3x + 2)$

$f(x) = 4x(x-2)(x-1)$

$f\left(1+\frac{\sqrt{3}}{3}\right) = 4\left(1+\frac{\sqrt{3}}{3}\right)\left(1+\frac{\sqrt{3}}{3}-2\right)\left(1+\frac{\sqrt{3}}{3}-1\right)$

$= 4\left(1+\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{3}}{3}-1\right)\times\frac{\sqrt{3}}{3} = 4\times\frac{\sqrt{3}}{3}\times\left(\frac{\sqrt{3}}{3}+1\right)\left(\frac{\sqrt{3}}{3}-1\right)$

$= 4\times\frac{\sqrt{3}}{3}\times\left(\frac{1}{3}-1\right) = 4\times\frac{\sqrt{3}}{3}\times\left(-\frac{2}{3}\right) = -\frac{8}{9}\sqrt{3}$

$f\left(1-\frac{\sqrt{3}}{3}\right) = 4\left(1-\frac{\sqrt{3}}{3}\right)\left(1-\frac{\sqrt{3}}{3}-2\right)\left(1-\frac{\sqrt{3}}{3}-1\right)$

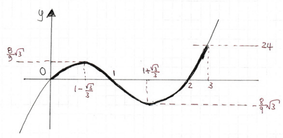
$= 4\left(1-\frac{\sqrt{3}}{3}\right)\left(-1-\frac{\sqrt{3}}{3}\right)\left(-\frac{\sqrt{3}}{3}\right) = 4\left(1-\frac{\sqrt{3}}{3}\right)\left(1+\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{3}}{3}\right)$

$= 4\times\left(1-\frac{1}{3}\right)\times\frac{\sqrt{3}}{3} = 4\times\frac{2}{3}\times\frac{\sqrt{3}}{3} = \frac{8}{9}\sqrt{3}$

- FINALLY CHECK THE "END-POINTS"

$f(0) = 0$

$f(3) = 4\times3\times(3-2)\times(3-1) = 4\times3\times1\times2 = 24$



$\therefore$ RANGE $-\frac{8}{9}\sqrt{3} \le f(x) \le 24$

Question 153 (*****)

$$f(x) \equiv \frac{x-k}{x^2-4x-k}, \quad x \in \mathbb{R}, \quad x^2-4x-k \neq 0,$$

where k is a constant.

Given that the range of the function is all the real numbers determine the range of possible values of k.

$\boxed{}$, $\boxed{0 \le k \le 5}$

$\boxed{f(x) = \dfrac{x-k}{x^2-4x-k}, \ x \in \mathbb{R}, \ x^2-4x-k \neq 0}$

- WRITE IN y NOTATION FOR SIMPLICITY

$$y = \frac{x-k}{x^2-4x-k}$$
$$yx^2 - 4xy - ky = x - k$$
$$yx^2 - 4yx - x + k - ky = 0$$
$$yx^2 - x(4y+1) + k(1-y) = 0$$

- FORMING THE DISCRIMINANT OF THIS QUADRATIC IN x, YIELDS

$$b^2 - 4ac = [-(4y+1)]^2 - 4yk(1-y)$$
$$= 16y^2 + 8y + 1 - 4ky + 4ky^2$$
$$= (16+4k)y^2 + (8-4k)y + 1$$

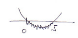
- THIS DISCRIMINANT MUST PRODUCE SOLUTIONS FOR THE QUADRATIC IN x FOR ALL VALUES OF OF $\underline{y \, \& \, k}$

THUS $\boxed{(16+4k)y^2 + (8-4k)y + 1 \geq 0, \ \text{NOW FOR } \underline{\text{ALL}} \ y}$

- SO THE GRAPH OF THE ABOVE, WRITTEN AS AN EQUATION, IS EITHER TOUCHING THE x AXIS OR LIES ENTIRELY ABOVE
- THIS IMPLIES $\quad b^2 - 4ac \leq 0$

$$\Rightarrow (8-4k)^2 - 4(16+4k) \times 1 \leq 0$$
$$\Rightarrow 16(2-k)^2 - 16(4+k) \leq 0$$
$$\Rightarrow (2-k)^2 - (4+k) \leq 0$$
$$\Rightarrow 4 - 4k + k^2 - 4 - k \leq 0$$
$$\Rightarrow k^2 - 5k \leq 0$$
$$\Rightarrow k(k-5) \leq 0$$

C.V $<$ 0, 5



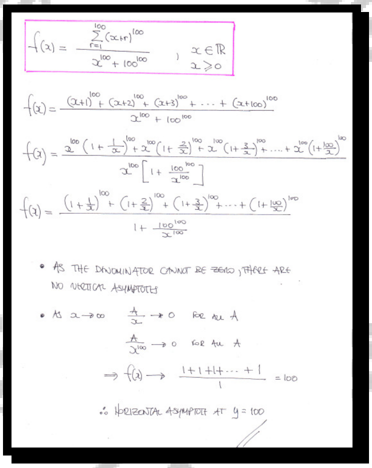
$0 \leq k \leq 5$

Question 154 (*****)

$$f(x) \equiv \frac{1}{x^{100}+100^{100}} \sum_{r=1}^{100} (x+r)^{100}, \quad x \in \mathbb{R}, \; x \geq 0 .$$

Use a formal method to find the equations of any asymptotes of $f(x)$.

$\boxed{}$, $\boxed{y=100}$



$$f(x) = \frac{\sum_{r=1}^{100}(x+r)^{100}}{x^{100}+100^{100}}, \quad x \in \mathbb{R}, \; x \geq 0$$

$$f(x) = \frac{(x+1)^{100}+(x+2)^{100}+(x+3)^{100}+\dots+(x+100)^{100}}{x^{100}+100^{100}}$$

$$f(x) = \frac{x^{100}\left(1+\frac{1}{x}\right)^{100}+x^{100}\left(1+\frac{2}{x}\right)^{100}+x^{100}\left(1+\frac{3}{x}\right)^{100}+\dots+x^{100}\left(1+\frac{100}{x}\right)^{100}}{x^{100}\left[1+\frac{100^{100}}{x^{100}}\right]}$$

$$f(x) = \frac{\left(1+\frac{1}{x}\right)^{100}+\left(1+\frac{2}{x}\right)^{100}+\left(1+\frac{3}{x}\right)^{100}+\dots+\left(1+\frac{100}{x}\right)^{100}}{1+\frac{100^{100}}{x^{100}}}$$

- AS THE DENOMINATOR CANNOT BE ZERO, THERE ARE NO VERTICAL ASYMPTOTES
- AS $x \to \infty$ $\quad \frac{A}{x} \to 0$ FOR ALL A

$\frac{A}{x^{100}} \to 0$ FOR ALL A

$$\Rightarrow f(x) \to \frac{1+1+1+\dots+1}{1} = 100$$

$\therefore$ HORIZONTAL ASYMPTOTE AT $y=100$

Question 155 (*****)

The functions f and g are defined in the largest possible real domain and their equations are given in terms of a constant k by

$$f(x)=\frac{\left(3k^2+1\right)x-k+1}{x-k+3} \quad \text{and} \quad g(x)=\frac{7x+4k}{4x+10}.$$

Given that f and g are identical, determine the possible value or values of k.

$\boxed{}$, $\boxed{k=\frac{1}{2}}$

$\boxed{f(x)=\frac{(3k^2+1)x-k+1}{x-k+3}}$ $\boxed{g(x)=\frac{7x+4k}{4x+10}}$

METHOD A

- LOOKING AT THE ASYMPTOTE (VERTICALLY) OF THE $g(x)$ WE OBTAIN $x=-\frac{5}{2}$

$\Rightarrow x-k+3=0$

$\Rightarrow -\frac{5}{2}-k+3=0$

$\Rightarrow \frac{1}{2}=k$

- VERIFY THIS VALUE FOR EACH OF THE TWO FUNCTIONS

$$f(x)=\frac{\left(3\left(\frac{1}{2}\right)^2+1\right)x-\frac{1}{2}+1}{x-\frac{1}{2}+3}=\frac{\frac{7}{4}x+\frac{1}{2}}{x+\frac{5}{2}}=\frac{7x+2}{4x+10}$$

$$g(x)=\frac{7x+4\left(\frac{1}{2}\right)}{4x+10}=\frac{7x+2}{4x+10}$$

- THE FUNCTIONS ARE IDENTICAL IF $k=\frac{1}{2}$, WITH DOMAIN $x\in\mathbb{R}$, $x\neq-\frac{5}{2}$

METHOD B

- SET THE TWO FUNCTIONS EQUAL TO ONE ANOTHER & COMPARE THE COEFFICIENTS OF x

$$\Rightarrow \frac{(3k^2+1)x+(1-k)}{x+(3-k)}=\frac{7x+4k}{4x+10}$$

$$\Rightarrow \left[(3k^2+1)x+(1-k)\right]\left[4x+10\right]=\left[x+(3-k)\right]\left[7x+4k\right]$$

$$\Rightarrow 4(3k^2+1)x^2+\left[10(3k^2+1)+4(1-k)\right]x+10(1-k)=7x^2+\left[4k+7(3-k)\right]x+4k(3-k)$$

- EQUATING $[x^2]$

$\Rightarrow 4(3k^2+1)=7$

$\Rightarrow 3k^2+1=\frac{7}{4}$

$\Rightarrow 3k^2=\frac{3}{4}$

$\Rightarrow k^2=\frac{1}{4}$

$\Rightarrow k=\frac{1}{2}, -\frac{1}{2}$

- EQUATING $[x^1]$

$\Rightarrow 10(3k^2+1)+4(1-k)=4k+7(3-k)$

$\Rightarrow 30k^2+10+4-4k=4k+21-7k$

$\Rightarrow 30k^2-k-7=0$

$\Rightarrow (15k+7)(2k-1)=0$

$\Rightarrow k=-7/15, \frac{1}{2}$

- EQUATING $[x^0]$

$\Rightarrow 10(1-k)=4k(3-k)$

$\Rightarrow 10-10k=12k-4k^2$

$\Rightarrow 4k^2-22k+10=0$

$\Rightarrow 2k^2-11k+5=0$

$\Rightarrow (2k-1)(k-5)=0$

$\Rightarrow k=\frac{1}{2}, 5$

$\therefore$ function are identical if $k=\frac{1}{2}$, with domain $x\in\mathbb{R}$, $x\neq-\frac{5}{2}$

Question 156 (*****)

The function f is defined by

$$f(x) = 2 - \frac{1}{x}, \quad x \in \mathbb{R},\ x \neq 0.$$

a) Prove that

$$f^{n}(x) = \frac{(n+1)x - n}{nx - (n-1)}, \quad n \geq 1,$$

where $f^{n}(x)$ denotes the n^{th} composition of $f(x)$ by itself.

b) State an expression for the domain of $f^{n}(x)$.

$\boxed{}$, $\boxed{x \in \mathbb{R},\ x \neq \frac{n-1}{n}}$

(a) • $f^{1}(x) = \frac{(1+1)x - 1}{1x - (1-1)} = \frac{2x-1}{x} = 2 - \frac{1}{x} = f(x)$

• $f^{2}(x) = f(f(x)) = f\left(2 - \frac{1}{x}\right) = 2 - \frac{1}{2 - \frac{1}{x}} = 2 - \frac{x}{2x-1} = \frac{4x - 2 - x}{2x-1}$

$= \frac{3x-2}{2x-1}$

ALSO

$f^{2}(x) = \frac{(2+1)x - 2}{2x - (2-1)} = \frac{3x-2}{2x-1}$ $\quad \therefore$ RESULT HOLDS FOR $n = 1, 2$

SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$

• $f^{k}(x) = \frac{(k+1)x - k}{kx - (k-1)}$

• $f^{k+1}(x) = f\left[\frac{(k+1)x - k}{kx - (k-1)}\right] = 2 - \frac{1}{\frac{(k+1)x - k}{kx - (k-1)}} = 2 - \frac{kx - (k-1)}{(k+1)x - k}$

$= \frac{2(k+1)x - 2k - kx + (k-1)}{(k+1)x - k} = \frac{(k+2)x - k - 1}{(k+1)x - k} = \frac{(k+1+1)x - (k+1)}{(k+1)x - (k+1-1)}$

THUS IF THE RESULT HOLDS FOR $n = k \in \mathbb{N} \Rightarrow$ THE RESULT ALSO HOLDS FOR $n = k+1$

SINCE THE RESULT HOLDS FOR $n = 1, 2 \Rightarrow$ THE RESULT MUST HOLD $\forall n \in \mathbb{N}$

(b) RESTRICTION IN DOMAIN OF $f(x)$ IS 'NATURAL'

$\therefore \ nx - (n-1) \neq 0$

$x \neq \frac{n}{n-1}$ $\quad \therefore \ x \in \mathbb{R},\ x \neq \frac{n}{n-1}$

Question 157 (*****)

The real functions f and g have a common domain $0 \le x \le 4$, and defined as

$$f(x) \equiv (x-1)(x-2)(x-3) \quad \text{and} \quad g(x) \equiv \int_0^x f(t)\, dt .$$

Use a detailed algebraic method to determine the range of g.

$\boxed{}$, $\boxed{-\frac{9}{4} \le g(x) \le 0}$

$$f(x) = (x-1)(x-2)(x-3) \qquad 0 \le x \le 4$$

$$g(x) = \int_0^x f(t)\, dt \qquad 0 \le x \le 4$$

- FIRSTLY TIDY $f(x)$

$$f(x) = (x-1)(x^2-5x+6) = \begin{array}{r} x^3 - 5x^2 + 6x \\ -x^2 + 5x - 6 \\ \hline x^3 - 6x^2 + 11x - 6 \end{array}$$

- NEXT FIND THE VALUE OF THE FUNCTION AT ITS ENDPOINTS

$$g(0) = \int_0^0 f(t)\, dt = 0$$

$$g(4) = \int_0^4 f(t)\, dt = \int_0^4 t^3 - 6t^2 + 11t - 6 \, dt$$

$$= \left[\tfrac{1}{4}t^4 - 2t^3 + \tfrac{11}{2}t^2 - 6t\right]_0^4$$

$$= \left(\tfrac{1}{4} \times 4^4 - 2 \times 4^3 + \tfrac{11}{2} \times 16 - 6 \times 4\right) - 0$$

$$= 4^3 - 2 \times 4^3 + 88 - 24$$

$$= -4^3 + 88 - 24$$

$$= -64 + 88 - 24$$

$$= 0$$

$$\therefore \underline{g(0) = g(4) = 0}$$

- NEXT LOOK FOR STATIONARY POINTS

$g'(x) = f(x)$ $\qquad \therefore$ STATIONARY AT $x = 1, 2, 3$

$$\therefore g(1) = \left[\tfrac{1}{4}t^4 - 2t^3 + \tfrac{11}{2}t^2 - 6t\right]_0^1$$

$$= \left(\tfrac{1}{4} - 2 + \tfrac{11}{2} - 6\right) - 0 = \frac{1 - 8 + 22 - 24}{4} = -\frac{9}{4}$$

$$g(2) = \left[\tfrac{1}{4}t^4 - 2t^3 + \tfrac{11}{2}t^2 - 6t\right]_0^2$$

$$= (4 - 16 + 22 - 12) - 0 = -2$$

$$g(3) = \left[\tfrac{1}{4}t^4 - 2t^3 + \tfrac{11}{2}t^2 - 6t\right]_0^3$$

$$= \left(\tfrac{81}{4} - 54 + \tfrac{99}{2} - 18\right) - 0 = \frac{81 - 216 + 198 - 72}{4} = -\frac{9}{4}$$

- AS $g(x)$ IS CONTINUOUS THE VALUES OF g ARE SUFFICIENT TO DETERMINE THE RANGE

$$\therefore -\frac{9}{4} \le g(x) \le 0$$

<u>ALTERNATIVELY</u> DETERMINE THE NATURE VIA CALCULUS

$$g''(x) = f'(x) = (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)$$

$$g''(1) = (-1)(-2) = 2 > 0 \implies$$

$$g''(2) = 1 \times (-1) = -1 < 0 \implies$$

$$g''(3) = 2 \times 1 = 1 > 0 \implies$$

Question 158 (*****)

The functions f and g are each defined in the largest possible real number domain and given by

$$f(x)=\sqrt{x-\sqrt{x^2-x-2}} \quad \text{and} \quad g(x)=\sqrt{x-\sqrt{x+6}}\ .$$

By considering the domains of f and g, show that $fg(x)$ cannot be formed.

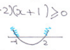
$\boxed{}$, $\boxed{\text{proof}}$

- FIRSTLY INVESTIGATE THE LARGEST POSSIBLE REAL DOMAIN OF EACH OF THE TWO FUNCTIONS SEPARATELY

$$\boxed{y=f(x)=\sqrt{x-\sqrt{x^2-x-2}}}$$

- FIRSTLY $x^2-x-2 \geqslant 0$

$$(x-2)(x+1) \geqslant 0$$



$$x \leqslant -1 \ \underline{\text{OR}} \ x \geqslant 2$$

- ALSO $x-\sqrt{x^2-x-2} \geqslant 0$

SOLVE THE "CORRESPONDING" EQUATION INSTEAD

$$x=\sqrt{x^2-x-2}$$
$$x^2=x^2-x-2$$
$$\cancel{x=-2}$$ DOES NOT SATISFY ORIGINAL SO NO EXTRA CRITICAL VALUES.

- CHECKING THE INDIVIDUAL REGIONS, $x \leqslant -1$ OR $x \geqslant 2$, WITH $x-\sqrt{x^2-x-2}$ WE DEDUCE THAT THE SOLUTION SET IS

$$\underline{\underline{x \geqslant 2}}$$

- NEXT CHECK THE LARGEST POSSIBLE DOMAIN OF $g(x)$

$$\boxed{y=g(x)=\sqrt{x-\sqrt{x+6}}}$$

- FIRSTLY $x+6 \geqslant 0$

$$x \geqslant -6.$$

- ALSO $x-\sqrt{x+6} \geqslant 0$

SOLVE THE CORRESPONDING EQUATION INSTEAD

$$\Rightarrow x=\sqrt{x+6}$$
$$\Rightarrow x^2=x+6$$
$$\Rightarrow x^2-x-6=0$$
$$\Rightarrow (x+2)(x-3)=0$$
$$x=\begin{cases}\cancel{-2}\\3\end{cases} \leftarrow \text{ONLY SOLUTION OF THE ORIGINAL EQUATION}$$

- CHECKING AGAIN THE INEQUALITY THE ABOVE CRITICAL VALUE WE OBTAIN $x \geqslant 3$.
- IN ORDER TO SATISFY THE PREVIOUS CONDITION $x \geqslant -6$ THE NEW CONDITION, THE DOMAIN IS

$$\underline{\underline{x \geqslant 3}}$$

- NEXT TRY TO FORM $f(g(x))$

$x \geqslant 2$ ⟩ f: $\sqrt{x-\sqrt{x^2-x-2}}$ $\qquad$ $x \geqslant 3$ ⟩ g: $\sqrt{x-\sqrt{x+6}}$

- WE REQUIRE $\sqrt{x-\sqrt{x^2-x-2}} \geqslant 3$ $\underline{\text{AND}}$ $x \geqslant 2$
- SOLVE THE CORRESPONDING EQUATION

$$\sqrt{x-\sqrt{x^2-x-2}}=3$$
$$x-\sqrt{x^2-x-2}=9$$
$$x-9=\sqrt{x^2-x-2}$$
$$\cancel{x^2}-18x+81=\cancel{x^2}-x-2$$
$$-17x=-83$$
$$x=\frac{83}{17} \leftarrow \text{DOES } \underline{\text{NOT}} \text{ SATISFY THE ORIGINAL}$$

- LOOKING FURTHER TO CHECK WHETHER IT WORKS FOR THE INTERVAL $x^2-x-2 \geqslant 0$ WHICH YIELDED $x \leqslant -1$ OR $x \geqslant 2$
  - CHECKING THE $x=-1$ $\sqrt{-1-\sqrt{0}}$ NOT EVEN POSSIBLE
  - CHECKING THE $x=2$ $\sqrt{2-\sqrt{4-2-2}} \ngeqslant 3$

$\therefore \sqrt{x-\sqrt{x^2-x-2}} \geqslant 3$ $\underline{\text{CANNOT}}$ BE SATISFIED

$\therefore f(g(x))$ CANNOT BE FORMED

Question 159 (*****)

The function f, defined for all real numbers, satisfies the following relationship

$$f(x)+4f(-x)\equiv 1+x^2\int_{-1}^{1} f(u)\ du\,.$$

Determine as an exact fraction the value of

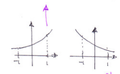
$$\int_{-1}^{1} f(x)\ dx\,.$$

$\boxed{}$, $\boxed{\frac{6}{13}}$

$\boxed{f(x)+4f(-x)=1+x^2\int_{-1}^{1}f(u)\,du}$

INTEGRATE THE EQUATION WITH RESPECT TO x, BETWEEN -1 & 1

$$\Rightarrow \int_{-1}^{1}f(x)\,dx+4\int_{-1}^{1}f(-x)\,dx=\int_{-1}^{1}1\,dx+\int_{x=-1}^{x=1}x^2\underbrace{\left[\int_{u=-1}^{u=1}f(u)\,du\right]}_{\text{THIS IS A CONSTANT ONCE EVALUATED}}dx$$



SAME AS VALUE AS $\int_{-1}^{1}f(x)\,dx$

$$\Rightarrow \int_{-1}^{1}f(x)\,dx+4\int_{-1}^{1}f(x)\,dx=\left[x\right]_{-1}^{1}+\left[\int_{-1}^{1}f(u)\,du\right]\int_{-1}^{1}x^2\,dx$$

$$\Rightarrow 5\int_{-1}^{1}f(x)\,dx=1-(-1)+\left[\int_{-1}^{1}f(u)\,du\right]\left[\tfrac{1}{3}x^3\right]_{-1}^{1}$$

$$\Rightarrow 5\int_{-1}^{1}f(x)\,dx=2+\left[\int_{-1}^{1}f(u)\,du\right]\left(\tfrac{1}{3}-\left(-\tfrac{1}{3}\right)\right)$$

$$\Rightarrow 5\int_{-1}^{1}f(x)\,dx=2+\tfrac{2}{3}\int_{-1}^{1}f(u)\,du$$

BUT $\int_{-1}^{1}f(x)\,dx=\int_{-1}^{1}f(u)\,du=\int_{-1}^{1}f(t)\,dt=\int_{-1}^{1}f(y)\,dy=\dots$

$$\Rightarrow \tfrac{13}{3}\int_{-1}^{1}f(x)\,dx=2$$

$$\Rightarrow \int_{-1}^{1}f(x)\,dx=\frac{6}{13}$$

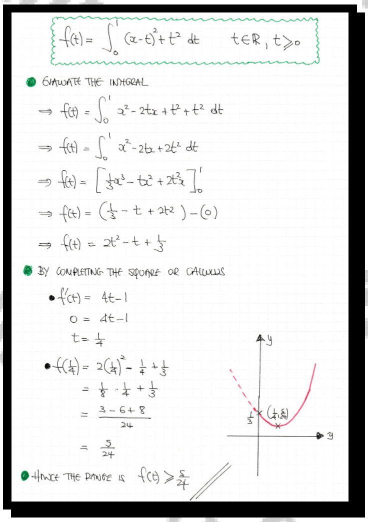
Question 160 (*****)

The function $y = f(t)$ is defined by the integral

$$f(t) \equiv \int_0^1 (x-t)^2 + t^2 \ dx\,, \quad t \in \mathbb{R}\,, \quad t \geq 0\,.$$

Determine the range of y.

$\boxed{}$, $\boxed{f(t) \geq \frac{5}{24}}$



$f(t) = \int_0^1 (x-t)^2 + t^2 \, dt \qquad t \in \mathbb{R},\ t \geqslant 0$

• EVALUATE THE INTEGRAL

$\Rightarrow f(t) = \int_0^1 x^2 - 2tx + t^2 + t^2 \, dt$

$\Rightarrow f(t) = \int_0^1 x^2 - 2tx + 2t^2 \, dt$

$\Rightarrow f(t) = \left[\frac{1}{3}x^3 - tx^2 + 2t^2 x\right]_0^1$

$\Rightarrow f(t) = \left(\frac{1}{3} - t + 2t^2\right) - (0)$

$\Rightarrow f(t) = 2t^2 - t + \frac{1}{3}$

• BY COMPLETING THE SQUARE OR CALCULUS

• $f'(t) = 4t - 1$

$0 = 4t - 1$

$t = \frac{1}{4}$

• $f\left(\frac{1}{4}\right) = 2\left(\frac{1}{4}\right)^2 - \frac{1}{4} + \frac{1}{3}$

$= \frac{1}{8} - \frac{1}{4} + \frac{1}{3}$

$= \frac{3 - 6 + 8}{24}$

$= \frac{5}{24}$

• HENCE THE RANGE IS $f(t) \geqslant \frac{5}{24}$

Question 161 (*****)

The function $y = f(x)$, $x \in \mathbb{R}$ satisfies

$$f(x) + 2f(2-x) = x^2, \quad t \in \mathbb{R}, \quad t \geq 0.$$

Determine a simplified expression for $y = f(x)$.

$\boxed{}$, $\boxed{f(x) = \frac{1}{3}x^2 - \frac{8}{3}x + \frac{8}{3}}$

$\boxed{f(x) + 2f(2-x) = x^2}$

- LET $x = 2-y \implies y = 2-x$

$f(2-y) + 2f(y) = (2-y)^2$

- REWRITE BOTH, IN A COMMON VARIABLE SAY u

$$\left.\begin{aligned} f(u) + 2f(2-u) &= u^2 \\ f(2-u) + 2f(u) &= (2-u)^2 \end{aligned}\right\} \implies f(2-u) = (2-u)^2 - 2f(u)$$

- BY SUBSTITUTION

$\implies f(u) + 2\left[(2-u)^2 - 2f(u)\right] = u^2$

$\implies f(u) + 2(2-u)^2 - 4f(u) = u^2$

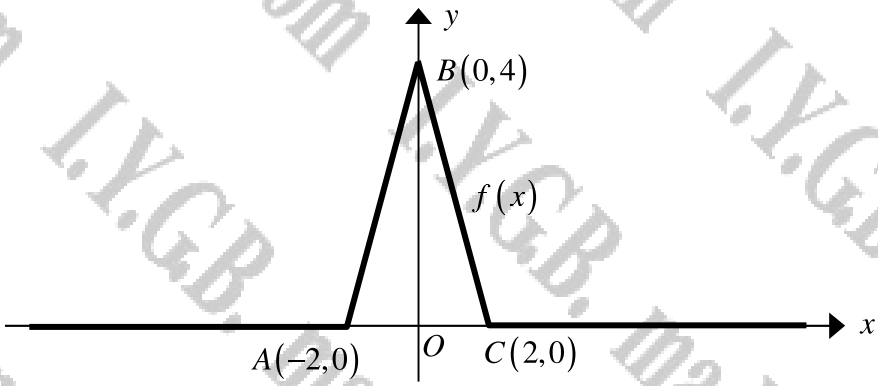
$\implies 2(u^2 - 4u + 4) - u^2 = 3f(u)$

$\implies u^2 - 8u + 8 = 3f(u)$

$\implies f(u) = \frac{1}{3}u^2 - \frac{8}{3}u + \frac{8}{3}$

IE $f(x) = \frac{1}{3}x^2 - \frac{8}{3}x + \frac{8}{3}$

Question 162 (*****)



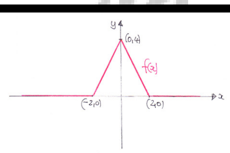
The figure above shows the graph of the function $f(x)$, consisting entirely of straight line sections. The coordinates of the joints of these straight line sections which make up the graph of $f(x)$ are also marked in the figure.

Given further that

$$\int_{-2}^{2} k + f\left(x^2 - 4\right)\, dx = 0,$$

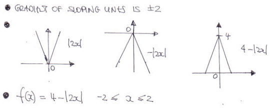
determine as an exact fraction the value of the constant k.

$\boxed{}$, $\boxed{k = \frac{4}{3}}$



FORMULATE AN EQUATION FOR $f(x)$

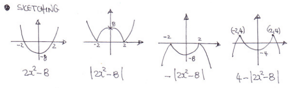
- GRADIENT OF SLOPING LINES IS ± 2
-



- $f(x) = 4 - |2x| \quad -2 \leqslant x \leqslant 2$

HENCE $f(x^2 - 4) = 4 - |2(x^2 - 4)|$
$= 4 - |2x^2 - 8|$

- SKETCHING



- SO FOR $-2 \leqslant x \leqslant 2$

$4 - |2x^2 - 8| \equiv 2x^2 - 4$
$f(x^2 - 4) \equiv 2x^2 - 4$

NOW CONSIDERING THE INTEGRAL

$\Rightarrow \int_{-2}^{2} k + f(x^2 - 4)\, dx = 0$

$\Rightarrow \int_{-2}^{2} k + 2x^2 - 4\, dx = 0$

AS THE INTEGRAND IS EVEN

$\Rightarrow 2\int_{0}^{2} (k-4) + 2x^2\, dx = 0$

$\Rightarrow \left[(k-4)x + \frac{2}{3}x^3\right]_0^2 = 0$

$\Rightarrow 2(k-4) + \frac{16}{3} = 0$

$\Rightarrow k - 4 + \frac{8}{3} = 0$

$\Rightarrow k = 4 - \frac{8}{3}$

$\Rightarrow k = \frac{4}{3}$

Question 163 (*****)

$$f(x) \equiv \frac{1}{k}\left(x^2-1\right)\left(x^2-9\right), \ x \in \mathbb{R}, \ k \in \mathbb{N}.$$

Determine the solution interval (n,k), $n \in \mathbb{N}$, so that the equation

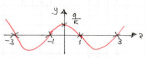
$$|f(x)| = n,$$

has exactly n distinct real roots.

$\boxed{}$, $\boxed{(n,k) = (8,1) = (6,2) = (4,4) = (5,2) = (6,2) = (7,2)}$

$$f(x) = \frac{1}{k}(x^2-1)(x^2-9) \quad x \in \mathbb{R}$$

- START WITH A QUICK SKETCH OF THE CURVE, WHICH EVIDENTLY IT IS EVEN

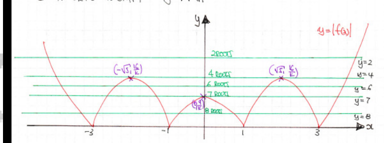


- NEXT WORK THE LOCAL MINIMA BY DIFFERENTIATION, IGNORING $x=0$

$$f'(x) = \frac{1}{k}(2x)(x^2-9) + \frac{1}{k}(x^2-1)(2x)$$
$$f'(x) = \frac{2x}{k}\left[(x^2-9)+(x^2-1)\right]$$
$$f'(x) = \frac{2x}{k}\left[2x^2-10\right]$$
$$f'(x) = \frac{4x}{k}(x^2-5)$$

- HENCE WE HAVE MINIMA AT $x = \pm\sqrt{5}$, WITH $f(\pm\sqrt{5}) = \frac{1}{k}(5-1)(5-9)$
  $f(\pm\sqrt{5}) = -\frac{16}{k}$

- NEXT DRAW THE GRAPH OF $y = |f(x)|$



- NOW LOOKING AT THE GRAPH OF $|f(x)| = n \leftarrow$ ROOTS 8, 7, 6, 4, 2

IF $n=2$ $\quad \frac{16}{k} > 2 \Rightarrow \frac{k}{16} < \frac{1}{2}$
$\Rightarrow k < 8$

IF $n=4$ $\quad \frac{16}{k} = 4 \Rightarrow 4k = 16$
$k = 4$

IF $n=6$ $\quad \frac{9}{k} < 6 < \frac{16}{k}$
$\frac{9}{k} < 6$ OR $\frac{16}{k} > 6$
$k > \frac{3}{2}$ OR $k < \frac{8}{3}$

IF $n=7$ $\quad \frac{9}{k} = 7 \Rightarrow \cancel{k = \frac{9}{7}}$ (NOT AN INTEGER)

IF $n=8$ $\quad \frac{9}{k} > 8 \Rightarrow k < \frac{9}{8}$

- HENCE WE HAVE

K	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
n	8	6		4	2	2	2									

I.E

$(n,k) = (8,1), (6,2), (4,4), (5,2), (6,2), (7,2)$