Problem 1

For each of the following sequences:

(i) Determine the z-transform, $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$, if it exists.

(ii) Include with your answer the region of convergence of the z-transform in the z-plane.

(iii) Specify whether or not the DTFT of the sequence, $X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$, exists.

1. $x[n] = 3^n u[n]$

2. $x[n] = \cos(\frac{n}{2})u[n]$

3. $x[n] = \sin(\frac{n\pi}{4})u[n]$

4. $x[n] = \left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{8}\right)^n u[n]$

5. $x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1] + \left(\frac{4}{5}\right)^{n-1} u[n-2]$

6. $x[n] = 2^n \left(u[n] - u[n-10]\right)$

7. $x[n] = (1/8)^n u[n] + (1/2)^{n-1} u[n-1] + (1/8)^{n-2} u[n-2]$

8. $x[n] = n\cos(n/2)u[n]$

P1

1. $x[n] = 3^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{3}{z}\right)^n$$

$$= \frac{z}{z-3}, \quad |z| > 3$$

ROC : $|z| > 3$. DTFT does not exist

2. $x[n] = \cos\left(\frac{n}{2}\right) u[n]$

$$X(z) = \frac{1}{2}\sum_{n=0}^{\infty} e^{j\frac{n}{2}} z^{-n} + \frac{1}{2}\sum_{n=0}^{\infty} e^{-jn/2} z^{-n}$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{+j/2}} + \frac{z}{z - e^{-j/2}}\right], \quad |z| > 1$$

ROC : $|z| > 1$. DTFT doesnot exist

3. $x[n] = \sin\left(\frac{n\pi}{4}\right) u[n]$

$$X(z) = \frac{1}{2j} \sum_{n=0}^{\infty} e^{jn\pi/4} z^{-n} - \frac{1}{2j} \sum_{n=0}^{\infty} e^{-jn\pi/4} z^{-n}$$

$$= \frac{1}{2j} \frac{z}{z - e^{j\pi/4}} - \frac{1}{2j} \frac{z}{z - e^{-j\pi/4}}, \quad |z| > 1$$

ROC: $|z| > 1$. DTFT does not exist.

4. $x[n] = \left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{8}\right)^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{4z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{8z}\right)^n$$

$$= \frac{z}{z-1/4} + \frac{z}{z-1/8}, \quad |z| > 1/4$$

ROC : $|z| > 1/4$. DTFT exists

5. $x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1] + \left(\frac{4}{5}\right)^{n-1} u[n-2]$

$$X(z) = \frac{z}{z-1/2} z^{-1} + \frac{4}{5} \frac{z}{z-4/5} z^{-2}, \quad |z| > 4/5$$

ROC : $|z| > 4/5$. DTFT exists

6. $x[n] = 2^n (u[n] - u[n-10])$

$$X(z) = \sum_{n=0}^{9} 2^n z^{-n} = \sum_{n=0}^{9} \left(\frac{2}{z}\right)^n$$

$$= \frac{1 - \left(\frac{2}{z}\right)^{10}}{1 - 2/z} = \frac{z^{10} - 2^{10}}{z^9 (z-2)}$$

= f(z)/(z^9), since z = 2 is a factor of z^10 – 2^10

ROC: |z| > 0, DTFT exists

7. $x[n] = \left(\frac{1}{8}\right)^n u[n] + \left(\frac{1}{2}\right)^{n-1} u[n-1] + \left(\frac{1}{8}\right)^{n-2} u[n-2]$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^n u[n] + z^{-1} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + z^{-2} \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^n u[n]$$

$$= \frac{z}{z-1/8} + z^{-1} \frac{z}{z-1/2} + z^{-2} \frac{z}{z-1/8}$$

for $|z| > 1/8 \cap |z| > 1/2$

$\therefore$ ROC $|z| > 1/2$

DTFT exists since $|z|=1$ is inside ROC.

8. $x[n] = n \cos\left(\frac{n}{2}\right) u[n]$

Let $z[n] = \cos\left(\frac{n}{2}\right) u[n]$

$$Z(z) = \frac{z}{2(z-e^{j/2})} + \frac{z}{2(z-e^{-j/2})}$$

$$Z(z) = \sum_{n=0}^{\infty} \cos\frac{n}{2} \, z^{-n}$$

$$\frac{d\,Z(z)}{dz} = \sum_{n=1}^{\infty} -n\cos\left(\frac{n}{2}\right) z^{-n-1}$$

$$= -z^{-1} \sum_{n=1}^{\infty} +n \cos\left(\frac{n}{2}\right) z^{-n}$$

$$= -z^{-1} \sum_{n=0}^{\infty} n \cos\left(\frac{n}{2}\right) z^{-n}$$

$$\frac{d\,Z(z)}{dz} = -z^{-1} X(z)$$

$$X(z) = -z \frac{dZ(z)}{dz}$$

$$= -\frac{z}{2}\left[\frac{z-e^{j/2}-z}{(z-e^{j/2})^2} + \frac{z-e^{-j/2}-z}{(z-e^{-j/2})^2}\right]$$

$$X(z) = \cancel{+\frac{z}{2}} \quad \frac{z\, e^{j/2}}{2(z-e^{j/2})^2} + \frac{z\, e^{-j/2}}{2(z-e^{-j/2})^2}$$

ROC $|z| > 1$

DTFT does not exist.

Problem 2

For each of the following z-transforms:

- Sketch the pole-zero plot.
- Assume the sequences are causal (i.e. zero for $n < 0$) and sketch the region of convergence (ROC).
- Compute the inverse z-transform, corresponding to the ROC you determined.

1.
$$X_1(z) = \frac{z^{-1}}{1 + 2jz^{-1}}$$

2.
$$X_2(z) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

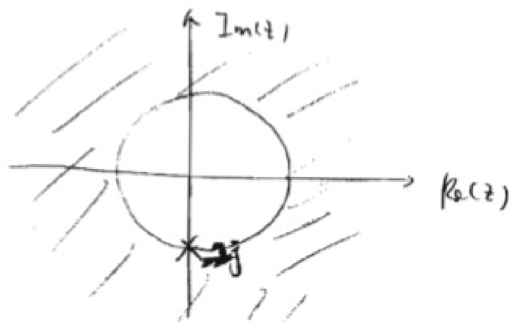
3.
$$X_3(z) = \frac{1 + z^{-2}}{1 - \frac{5}{2}z^{-1} + z^{-2}}$$

4.
$$X_4(z) = \frac{1 - 3z^{-1}}{1 + 0.5z^{-1} + 0.25z^{-2}}$$

#1. $X_1(z) = \frac{z^{-1}}{1+2jz^{-1}} = \frac{1}{z+2j}$ ⇒ Pole at $z=-2j$

(a) no zeros

Sketched as follows: (zeros as "o", poles as "x")



Sequence is Causal

⇒ ROC is outside of the outermost pole

i.e., $|z| > |2j| = |2|$

⇒ ROC: $|z| > 2$

The ROC is the hatched area (outside of the disk of radius 2).

(b) $X_1(z) = \dfrac{1}{z+2j} = z^{-1}\left(\dfrac{z}{z-(-2j)}\right)$

let $\mathcal{Z}$ and $\mathcal{Z}^{-1}$ denote z-transform and inverse z-transform respectively

① $\dfrac{z}{z+2j} \xleftrightarrow{\text{inverse Z-transform}} (-2j)^n u[n]$

② And according to the shifting property $x[n-k] \xleftrightarrow{Z} X(z)z^{-k}$

① and ② $\Rightarrow z^{-1}\left(\dfrac{z}{z+2j}\right) \xleftrightarrow{\text{inverse Z-trans.}} (-2j)^{n-1} u[n-1]$

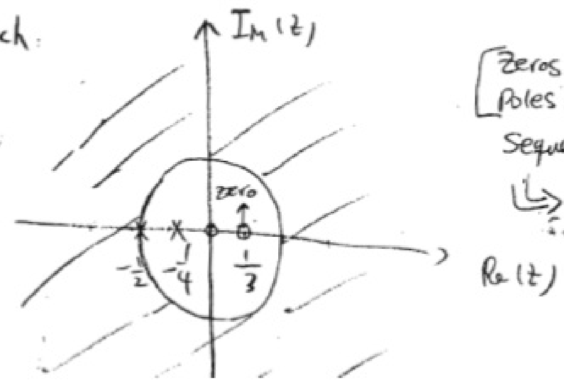
#2. $X_2(z) = \dfrac{1-\frac{1}{3}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$

$= \dfrac{z(z-\frac{1}{3})}{(z+\frac{1}{4})(z+\frac{1}{2})}$

Poles: $z = -\frac{1}{4}, -\frac{1}{2}$.

zeros: $z = 0, \frac{1}{3}$

Sketch:



Zeros: o
Poles: x

Sequence is causal

$\hookrightarrow$ ROC is outside of the outermost pole

$\Rightarrow$ ROC: $|z| > |-\frac{1}{2}|$

$\hookrightarrow$ ROC $|z| > |\frac{1}{2}|$

$$X_2(z) = \frac{z(z-\frac{1}{3})}{(z+\frac{1}{4})(z+\frac{1}{2})} \Rightarrow \frac{X_2(z)}{z} = \frac{z-\frac{1}{3}}{(z+\frac{1}{4})(z+\frac{1}{2})} = \frac{A}{z+\frac{1}{4}} + \frac{B}{z+\frac{1}{2}}$$

Partial Fraction Expansion [see page G.14 in the course notes]:

$$A = \left.\frac{z-\frac{1}{3}}{z+\frac{1}{2}}\right|_{z=-\frac{1}{4}} = -\frac{7}{3} \qquad B = \left.\frac{z-\frac{1}{3}}{z+\frac{1}{4}}\right|_{z=-\frac{1}{2}} = \frac{10}{3}$$

Thus

$$X_2(z) = -\frac{7}{3}\frac{z}{z+\frac{1}{4}} + \frac{10}{3}\frac{z}{z+\frac{1}{2}} \qquad (\text{ROC: } |z| > \tfrac{1}{2})$$

$$\xRightarrow{\text{inverse z-transform}} x_2[n] = -\frac{7}{3}\left(-\frac{1}{4}\right)^n u[n] + \frac{10}{3}\left(-\frac{1}{2}\right)^n u[n]$$

Used the linearity property and the following z-transform pair to obtain this:

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a} \qquad \text{ROC: } |z| > |a| \quad \text{[See Problem 3 solution for derivations]}$$

For $X_2(z)$, we have $a = -\frac{1}{4}$ for the first term and $a = -\frac{1}{2}$ in the second term

Problem 4

#3.

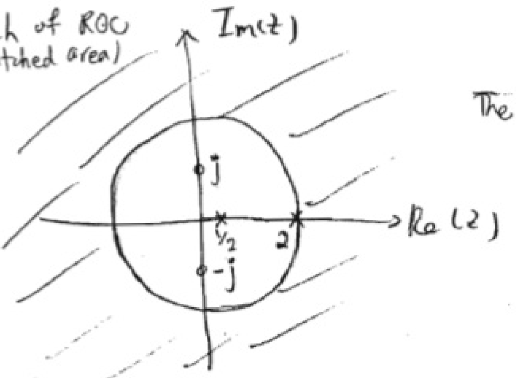
$$X_3(z) = \frac{1+z^{-2}}{1-\frac{5}{2}z^{-1}+z^{-2}} = \frac{z^2+1}{(z-\frac{1}{2})(z-2)} \Rightarrow \begin{cases} \text{poles: } z = \frac{1}{2}, 2 \\ \text{zeros: } z = \pm j \end{cases}$$

Since the numerator can be written as: $z^2+1 = (z-j)(z+j)$

Sketch of ROC (hatched area)

o: zeros

x: poles



The sequence is causal $\Rightarrow$ ROC is outside of the outermost pole

ROC: $|z| > 2$

$$X_3(z) = \frac{z^2+1}{(z-\frac{1}{2})(z-2)} \Rightarrow \frac{X_3(z)}{z} = \frac{z^2+1}{z(z-\frac{1}{2})(z-2)} = \frac{A}{z} + \frac{B}{z-\frac{1}{2}} + \frac{C}{z-2}$$

$$A = \left.\frac{z^2+1}{(z-\frac{1}{2})(z-2)}\right|_{z=0} = 1 ; \quad B = \left.\frac{z^2+1}{z(z-2)}\right|_{z=\frac{1}{2}} = -\frac{5}{3}$$

$$C = \left.\frac{z^2+1}{z(z-\frac{1}{2})}\right|_{z=2} = \frac{5}{3}$$

Thus
$$X_3(z) = 1 + (-\tfrac{5}{3})\frac{z}{z-\frac{1}{2}} + \frac{5}{3}\frac{z}{z-2} \quad (\text{ROC: } |z|>2)$$

$$\xrightarrow[\text{z-transform}]{\text{inverse}} x_3[n] = \delta[n] - \frac{5}{3}(\tfrac{1}{2})^n u[n] + \frac{5}{3} 2^n u[n]$$

Used the linearity property and following z-transform pairs to obtain the inverse z-transform:

① $\delta[n] \xleftrightarrow{z} 1$ ROC: all z

② $a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}$ ROC: $|z|>|a|$ → See Solution to Problem 3 for derivations.

4. $$X_4(z) = \frac{1-3z^{-1}}{1+0.5z^{-1}+0.25z^{-2}} = \frac{z^2-3z}{z^2+0.5z+0.25}$$

poles: $z^2+0.5z+0.25=0 \Rightarrow z_{1,2} = -\frac{1}{4} \pm j\frac{\sqrt{3}}{4} = \frac{1}{2}e^{\pm j\frac{2\pi}{3}}$

$$\therefore \frac{X_4(z)}{z} = \frac{z-3}{(z-\frac{1}{2}e^{j\frac{2\pi}{3}})(z+\frac{1}{2}e^{-j\frac{2\pi}{3}})}$$

$$= \frac{A}{z-\frac{1}{2}e^{j\frac{2\pi}{3}}} + \frac{B}{z-\frac{1}{2}e^{-j\frac{2\pi}{3}}}$$

$$\therefore \quad A = \frac{\frac{1}{2}e^{j\frac{2\pi}{3}} - 3}{\frac{1}{2}e^{j\frac{2\pi}{3}} - \frac{1}{2}e^{-j\frac{2\pi}{3}}} = \frac{-\sqrt{3} + \sqrt{3}j}{j2\sqrt{3}} = \frac{1}{2} + \frac{\sqrt{3}j}{2\sqrt{3}}$$

$$B = \frac{\frac{1}{2}e^{-j\frac{2\pi}{3}} - 3}{\frac{1}{2}e^{-j\frac{2\pi}{3}} - \frac{1}{2}e^{j\frac{2\pi}{3}}} = \frac{1}{2} - \frac{\sqrt{3}j}{2\sqrt{3}}$$

conjugate

$(B = A^*)$

$$\therefore H(z) = \left(\frac{1}{2} + \frac{\sqrt{3}j}{2\sqrt{3}}\right)\frac{z}{z - \frac{1}{2}e^{j\frac{2\pi}{3}}} + \left(\frac{1}{2} - \frac{\sqrt{3}j}{2\sqrt{3}}\right)\frac{z}{z - \frac{1}{2}e^{-j\frac{2\pi}{3}}}$$

$$\Rightarrow h[n] = \left(\frac{1}{2} + \frac{\sqrt{3}j}{2\sqrt{3}}\right)\left(\frac{1}{2}e^{j\frac{2\pi}{3}}\right)^n u[n] + \left(\frac{1}{2} - \frac{\sqrt{3}j}{2\sqrt{3}}\right)\left(\frac{1}{2}e^{-j\frac{2\pi}{3}}\right)^n u[n]$$

Alternatively

$\Rightarrow$ Can further simplify:

$$h[n] = \left(\frac{1}{2}\right)^n u[n]\left[\left(\frac{1}{2} + \frac{\sqrt{3}j}{2\sqrt{3}}\right)e^{j\frac{2\pi n}{3}} + \left(\frac{1}{2} - \frac{\sqrt{3}j}{2\sqrt{3}}\right)e^{-j\frac{2\pi n}{3}}\right]$$

$$= \left(\frac{1}{2}\right)^n u[n]\left(\frac{1}{2}2\cos\left(\frac{2\pi n}{3}\right) - \frac{\sqrt{3}}{2\sqrt{3}}2\sin\left(\frac{2\pi n}{3}\right)\right)$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n]\left(\cos\left(\frac{2\pi n}{3}\right) - \frac{\sqrt{3}}{\sqrt{3}}\sin\left(\frac{2\pi n}{3}\right)\right]$$

$$= \left(\frac{1}{2}\right)^n \cos\left(\frac{2\pi n}{3}\right)u[n] - \frac{\sqrt{3}}{\sqrt{3}}\left(\frac{1}{2}\right)^n \sin\left(\frac{2\pi n}{3}\right)u[n]$$

ROC: $|z| > |z_1| = \left|\frac{1}{2}e^{j\frac{2\pi}{3}}\right| = \frac{1}{2}$

Plot: