Linear Systems

Lecture Summary # 1

What is a system?

Several definitions exist, two of them are provided below.

- Definition 1: A set of components connected together to perform a desired task.
- Definition 2: Process or entity with well-defined inputs and outputs.

It is possible to combine the two definitions as follows

- Definition 3: A set of components with well-defined inputs and outputs, connected together to perform a desired task.

Systems can be very simple or extremely complex. Some examples of systems are below

- The world's economy
- Solar system
- RC filter
- Human body

When we talk about a system we may refer to the actual physical system or to its mathematical representation. Systems are typically grouped into linear and nonlinear systems. Real world systems are typically nonlinear but many of these nonlinear systems can be approximated by linear systems. This process is called linearization. Block diagrams are widely used to represent systems such as the one shown in figure 1. The mathematical description of a system is simply a function relating the inputs and outputs such as

$$y(t) = F[u(t)] \tag{1}$$

It is possible to have a system of a system as shown in figure 1-bottom. In this case

$$y(t) = G[F[u(t)]] \tag{2}$$

In general, F is not a simple algebraic function. For example, a dynamic system is represented by differential equations or difference equations, and $y(t)$ is not a function of $u(t)$ only, but a function of the history as well.

Time invariant systems

This is an important class of systems where the output satisfies

$$y(t-\tau) = F[u(t-\tau)] \tag{3}$$

If the input signal is delayed by amount τ, the output is the same without delay and delayed by the same amount as the input. Example: Apply an impulse (hand clap)

- At time t, we measure the response
- At time $t-\tau$, we obtain the same response.

This is illustrated in figure 2.

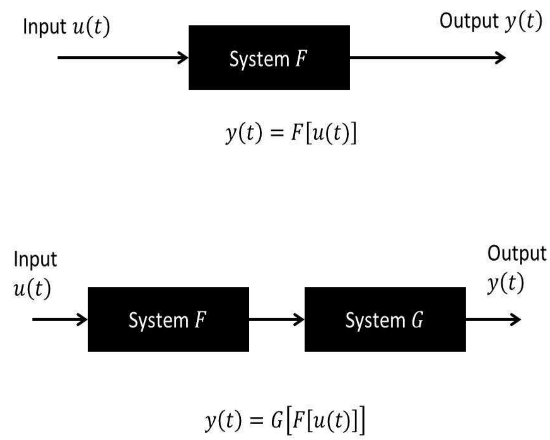

Fig. 1. Block diagram representation if a system

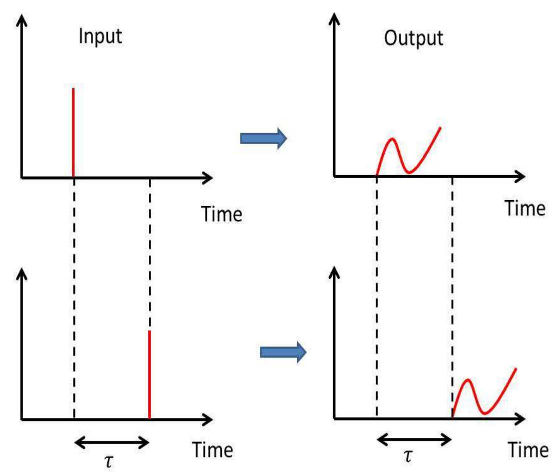

Fig. 2. Illustration of time invariance

Linearity

Linear systems obey the superposition principle, which consists of two properties: Homogeneity and additivity

- Homogeneity: If we increase the strength of the input, the output increases by the same factor, for example if we double the input, we expect the output to double.
- Additivity: If input u_1 produces y_1, and u_2 produces y_2, then the application of $u_1 + u_2$ will produce $y_1 + y_2$.

These two conditions define the necessary and sufficient conditions for linearity. Combining these two properties together,

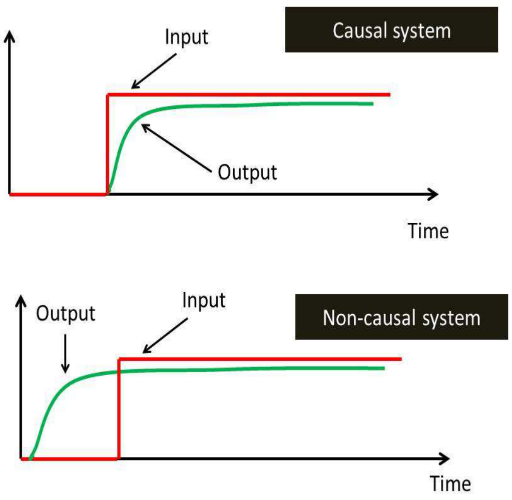

Fig. 3. Illustration of causal system response

it is possible to write

$$F(au_1 + bu_2) = aF(u_1) + bF(u_2) \tag{4}$$

Causality

The output before time t does not depend on the input after t. In other words, the output of the system depends on the present and past values of the input but not the future inputs. This is illustrated in figure 3

Basic examples of systems

Some examples of systems are provided below.

- Squarer:
$$y(t) = (u(t))^2 \tag{5}$$
- Time delay:
$$y(t) = u(t-\tau) \tag{6}$$
- Differentiator
$$y(t) = \frac{du(t)}{dt} \tag{7}$$
- Integrator:
$$y(t) = \int_0^t u(\tau)d\tau \tag{8}$$

Impulse response

Linear time invariant systems are characterized by their impulse response. Consider a LTI system with input u and output y. The impulse response is the output $h(t) = y(t)$ when the input is an impulse: $u(t) = \delta(t)$. By definition

$$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \tag{9}$$

The output of the system can be found using the impulse response of the system as follows

$$\begin{aligned} y(t) &= u(t) * h(t) = h(t) * u(t) && (10)\\ &= \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau && (11)\\ &= \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau && (12) \end{aligned}$$

This operation is called convolution and represents one of the most important properties of LTI systems.

Classification of state models

State models can be grouped under several categories

Nonlinear time varying

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) && (13)\\ y(t) &= h(x(t), u(t), t) && (14) \end{aligned}$$

Nonlinear time invariant

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) && (15)\\ y(t) &= h(x(t), u(t)) && (16) \end{aligned}$$

Linear time varying

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) && (17)\\ y(t) &= C(t)x(t) + D(t)u(t) && (18) \end{aligned}$$

Linear time invariant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) && (19)\\ y(t) &= Cx(t) + Du(t) && (20) \end{aligned}$$

Linear systems in state space

A continuous time linear system can be written as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) && (21)\\ y(t) &= Cx(t) + Du(t) && (22) \end{aligned}$$

The first equation is called the state equation and the second one is called the output or measurement equation, where $x \in \mathbb{R}^n, u \in \mathbb{R}^k, y \in \mathbb{R}^m$. The system represents an input output relationship. For a given input the output can be obtained by solving for $x(t)$ and plug in the solution in the output equation. Some particular cases are as follows

- When u is a scalar ($k = 1$), the system is called single input
- When y is a scalar ($m = 1$), the system is called single output
- When both u and y are scalar, the system is called single input-single output (SISO)

- When both u and y are not scalar, the system is called multiple input-multiple output (MIMO)
- When $n = 0$, there is no state and the system simply becomes $y(t) = Du(t)$. This system is called memoryless. A system is called memoryless if the output at any time depends on the value of the input at the same time.

Discrete time systems

A LTI discrete state space system is given by

$$x(k+1) = Ax(k) + Bu(k) \quad (23)$$
$$y(k) = Cx(k) + Du(k) \quad (24)$$

Discrete time systems have the exact same properties as continuous time systems.

Solution of LTI system

Consider a LTI system of the form

$$\dot{x}(t) = Ax(t) \quad (25)$$
$$x(t_0) = x_0 \quad (26)$$

The solution of the system is

$$x(t) = \phi(t, t_0)x_0 = x(t, t_0, x_0) = e^{A(t-t_0)}x_0 \quad (27)$$

The solution is unique and $\phi(t, t_0)$ is called the fundamental matrix. For the general LTI case

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (28)$$
$$y(t) = Cx(t) + Du(t) \quad (29)$$
$$x(t_0) = x_0 \quad (30)$$

The solution is

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \quad (31)$$

The second term in the equation represents the convolution. Recall that e^{At} is a matrix exponential. One way to calculate this matrix is by using the Taylor series.

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + ... \quad (32)$$

Matlab has a built in function "expm" that can be used to calculate the exponential of a matrix.

Linearization

Most real world systems are nonlinear. Linear systems are the exception, not the rule. A time invariant nonlinear system can be written as

$$\dot{x}(t) = f(x(t), u(t)) \quad (33)$$
$$y(t) = h(x(t), u(t))$$

where f, h are nonlinear functions. We can study certain classes of linear systems by simply linearizing them. Recall that f and h are vector functions. For example, it is possible to write for f:

$$f = [f_1, f_2, ..., f_n]^T \quad (34)$$

Since x is also a vector, the derivative of f with respect to x is a matrix of partial derivatives given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (35)$$

This matrix is called the Jacobian matrix. The following notation is used

$$\frac{\partial f}{\partial x} = \left[\frac{\partial f_i}{\partial x_j}\right]_{ij} \quad (36)$$

Linearization uses the Jacobian matrix.

Definition

A pair $(x_{eq}, u_{eq}) \in \mathbb{R}^n \times \mathbb{R}^k$ is called equilibrium point of (33) if

$$f(x_{eq}, u_{eq}) = 0 \quad (37)$$

In this case

$$y_{eq} = h(x_{eq}, u_{eq}) \quad (38)$$

The equilibrium points are special solutions for which the state of the system does not change $\dot{x} = 0$. Suppose now we apply an input given by

$$u = u_{eq} + \delta u \quad (39)$$

where δu is small, it is possible to write

$$x = x_{eq} + \delta x \quad (40)$$

We can write

$$\dot{x} = \dot{x}_{eq} + \dot{\delta x} \quad (41)$$

Knowing that

$$\dot{x} = f(x_{eq} + \delta x, u_{eq} + \delta u) \quad (42)$$

and

$$\dot{x}_{eq} = f(x_{eq}, u_{eq}) \quad (43)$$

we can write

$$\dot{\delta x} = \dot{x} - \dot{x}_{eq} = f(x_{eq} + \delta x, u_{eq} + \delta u) - f(x_{eq}, u_{eq}) \quad (44)$$

$$\dot{\delta x} = \frac{\partial f(x_{eq}, u_{eq})}{\partial x}\delta x + \frac{\partial f(x_{eq}, u_{eq})}{\partial u}\delta u + o(||\delta x||^2) + o(||\delta u||^2) \quad (45)$$

where

$$\frac{\partial f}{\partial x} = \left[\frac{\partial f_i}{\partial x_j}\right]_{ij} \quad (46)$$

$$\frac{\partial f}{\partial u} = \left[\frac{\partial f_i}{\partial u_j}\right]_{ij} \quad (47)$$

It is possible to ignore the higher order terms in (45). In this case, we can write:

$$\dot{\delta x} = A\delta x + B\delta u \quad (48)$$

where

$$A = \left[\frac{\partial f_i}{\partial x_j}\right]_{ij} \quad (49)$$

$$B = \left[\frac{\partial f_i}{\partial u_j}\right]_{ij} \tag{50}$$

It is possible to apply the same approach to the output equation as follows

$$y = y_{eq} + \delta y \tag{51}$$

Knowing that

$$y = h(x_{eq} + \delta x, u_{eq} + \delta u) \tag{52}$$

and

$$y_{eq} = h(x_{eq}, u_{eq}) \tag{53}$$

Thus

$$\delta y = y - y_{eq} = h(x_{eq} + \delta x, u_{eq} + \delta u) - h(y_{eq}, u_{eq}) \tag{54}$$

$$\delta y = \frac{\partial h(x_{eq}, u_{eq})}{\partial x}\delta x + \frac{\partial h(x_{eq}, u_{eq})}{\partial u}\delta u + o(||\delta x||^2) + o(||\delta u||^2) \tag{55}$$

where

$$\frac{\partial h}{\partial x} = \left[\frac{\partial h_i}{\partial x_j}\right]_{ij} \tag{56}$$

$$\frac{\partial h}{\partial u} = \left[\frac{\partial h_i}{\partial u_j}\right]_{ij} \tag{57}$$

It is possible to ignore the higher order terms in (55). In this case, we can write:

$$\delta y = C\delta x + D\delta u \tag{58}$$

where

$$C = \left[\frac{\partial h_i}{\partial x_j}\right]_{ij} \tag{59}$$

$$D = \left[\frac{\partial h_i}{\partial u_j}\right]_{ij} \tag{60}$$

LINEARIZATION EXAMPLES

Example 1

The simple pendulum can be described by the following equation

$$\ddot{x} + k \sin x = 0; k > 0 \tag{61}$$

We want to linearize the system near its operating points. The system can be written as

$$\dot{x}_1 = x_2 \tag{62}$$

$$\dot{x}_2 = -ksinx_1 \tag{63}$$

where $x_1 = x$ represents the angle as shown in figure 4. First, we find the operating points. By putting $\dot{x}_1 = 0, \dot{x}_2 = 0$ and solving, we get two solutions:

- Operating point 1: $(x_1, x_2) = (0, 0)$
- Operating point 2: $(x_1, x_2) = (\pi, 0)$

Now, we know that

$$f_1(x_1, x_2) = x_2 \tag{64}$$

$$f_2(x_1, x_2) = -k \sin x_1 \tag{65}$$

The Jacobian matrix is given by

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -k\cos x & 0 \end{bmatrix} \tag{66}$$

which gives

- Operating point 1:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \tag{67}$$

The linearized system is then

$$\dot{x}_1 = x_2 \tag{68}$$

$$\dot{x}_2 = -kx_1 \tag{69}$$

- Operating point 2:

$$A_2 = \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix} \tag{70}$$

The linearized system is then

$$\dot{x}_1 = x_2 \tag{71}$$

$$\dot{x}_2 = kx_1 \tag{72}$$

Example 2

Consider the water tank shown in figure 4. We assume that the nominal inflow rate is constant and equal to u_{eq}. It can be proven that the output is related to the water level by the following equation

$$y(t) = C_v\sqrt{x(t)} \tag{73}$$

where C_v is a positive constant. The water level changes according to the following equation

$$\dot{x}(t) = -\frac{C_v}{a}\sqrt{x(t)} + \frac{1}{a}u(t) \tag{74}$$

$$y(t) = C_v\sqrt{x(t)} \tag{75}$$

$$a\dot{x} = u(t) - y(t) \tag{76}$$

where a is the cross sectional area of the tank. Since u_{eq} is given, we can find the operating point by putting $\dot{x} = 0$ and solving, which gives

$$x_{eq} = \frac{u_{eq}^2}{C_v^2} \tag{77}$$

The equilibrium for the output is

$$y_{eq} = u_{eq} \tag{78}$$

Linearizing about the operating points gives us

$$\frac{\partial f}{\partial x} = -\frac{C_v}{a}\frac{1}{2\sqrt{x}}|_{x_{eq}} = -\frac{C_v^2}{2au_{eq}} \tag{79}$$

$$\frac{\partial h}{\partial x} = \frac{C_v}{2\sqrt{x}}|_{x_{eq}} = \frac{C_v^2}{2u_{eq}} \tag{80}$$

Therefore, the linearized system is given by

$$\dot{\delta x} = -\frac{C_v^2}{2au_{eq}}\delta x + \frac{1}{a}\delta u \tag{81}$$

$$\delta y = \frac{C_v^2}{2u_{eq}}\delta x \tag{82}$$

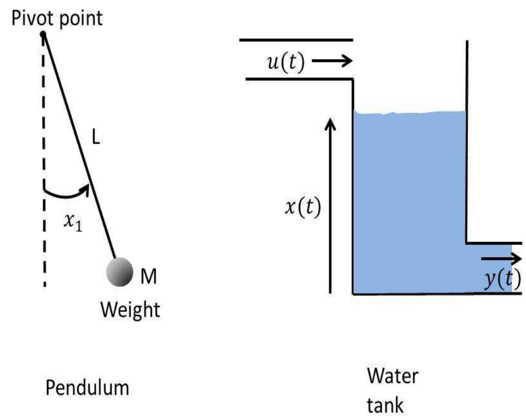

Fig. 4. Examples for linearization

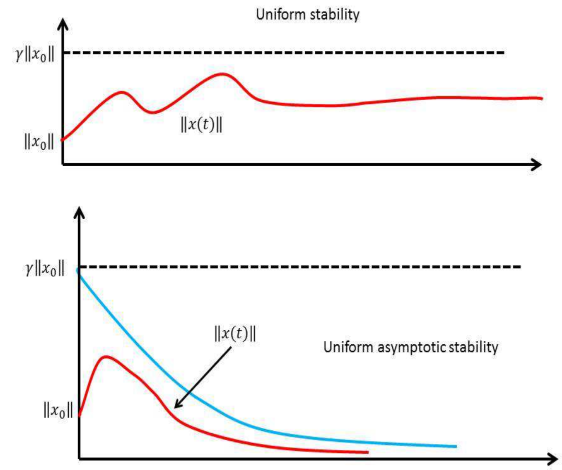

Fig. 5. Illustration of uniform stability and uniform asymptotic stability

I. Stability

Stability is among the most important concepts and properties in dynamic systems and controls. There are two types of stability:

- Internal stability or Lyapunov stability: Stability of the unforced system (no input).
- Input/output stability: Concerned with the effect of the inputs on the outputs.

A. Internal stability

Internal stability deals with the boundedness and the asymptotic behavior if the solution of

$$\begin{aligned} \dot{x} &= A(t)x(t) \\ x(t_0) &= x_0 \end{aligned} \tag{83}$$

We want the solution to be bounded regardless of the initial conditions x_0, t_0. Some definitions are provided below:

- Definition 1: Uniform stability:
  The linear system of equation (83) is called uniformly stable if there exist a finite positive constant γ such that for any x_0, t_0, the corresponding solution satisfies
  $$||x(t)|| \leq \gamma ||x_0|| \tag{84}$$
  for $t \geq t_0$
  Constant γ does not depend on the choice of the initial state.
- Definition 2: Uniform exponential stability:
  The linear system of equation (83) is called uniformly exponentially stable if there exist finite positive constants β, γ such that for any x_0, t_0, the corresponding solution satisfies
  $$||x(t)|| \leq \gamma e^{-\beta(t-t_0)} ||x_0|| \tag{85}$$
  for $t \geq t_0$. Both uniform stability and uniform exponential stability are internal stability concepts.
- Definition 3: Lyapunov stability:
  The equilibrium point is said to be Lyapunov stable if for every $\varepsilon > 0$, there exist $\delta > 0$, such that if
  $$||x(0) - x_{eq}|| < \delta \tag{86}$$
  then
  $$||x(t) - x_{eq}|| < \varepsilon \tag{87}$$
  for $t \geq 0$.
- Definition 4: Asymptotic stability:
  The equilibrium point is said to be asymptotically stable if it is Lyapunov stable and here exist $\delta > 0$, such that if
  $$||x(0) - x_{eq}|| < \delta \tag{88}$$
  then
  $$\lim_{t \to \infty} ||x(t) - x_{eq}|| = 0 \tag{89}$$
  Lyapunov stability implies that a solution that start close enough to x_{eq} will stay close enough for ever. Asymptotic stability implies that the solution will go to its equilibrium point with time.

Lyapunov first method

This method is valid for linear systems and can be extended to nonlinear systems by using linearization. The method uses the eigenvalues of the state transition matrix A.

B. Theorem

Consider a linear system of the form

$$\dot{x} = Ax \tag{90}$$

Let $\lambda_i, i = 1, 2, \ldots, n$ be the eigenvalues of matrix A.

- The linear is asymptotically stable if $Re\{\lambda_i\} < 0$, $i = 1, 2, \ldots, n$.
- The system is unstable if there exist at least one λ_i with $Re\{\lambda_i\} > 0$.
- The system is stable if $Re\{\lambda_i\} \leq 0$, $i = 1, 2, \ldots, n$, and there are no repeated eigenvalues on the imaginary axis.

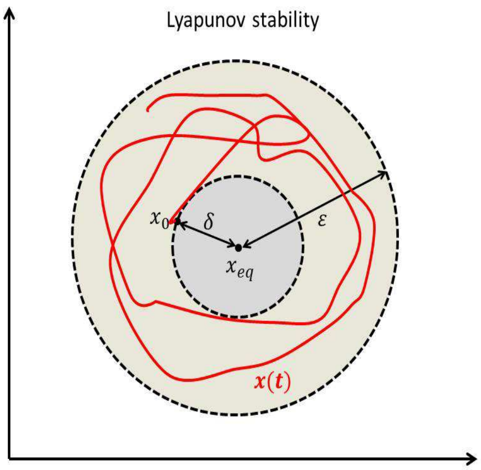

Fig. 6. Illustration of Lyapunov stability

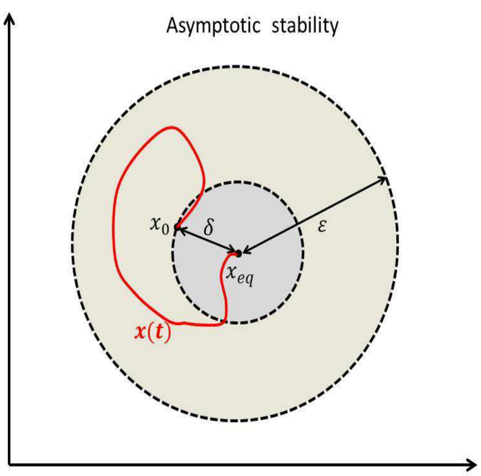

Fig. 7. Illustration of asymptotic stability

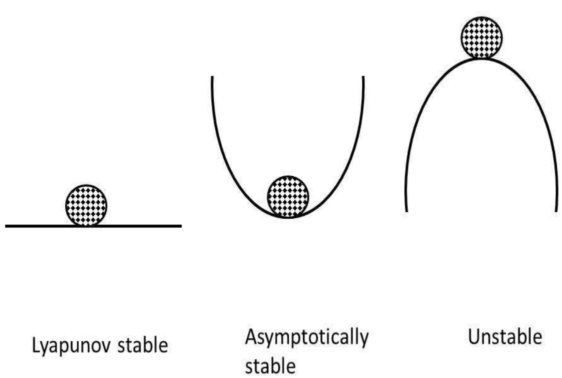

Fig. 8. Illustration of the concept of stability

Lyapunov second method

Lyapounov second method is also called Lyapunov direct method. It uses a concept similar to the energy of the system. We consider a nonlinear system as follows

$$\dot{x}(t) = f(x(t)) \tag{91}$$

- Definition 1: Positive definite function:
  Function $V : \mathbb{R}^n \to \mathbb{R}$ is positive definite if
  - $V(x) > 0$ for all $x \neq 0$
  - $V(x) = 0$ if and only if $x = 0$
- Definition 2: Negative definite function :
  Function $V : \mathbb{R}^n \to \mathbb{R}$ is negative definite if
  - $V(x) < 0$ for all $x \neq 0$
  - $V(x) = 0$ if and only if $x = 0$
- Definition 3: Positive semi definite function:
  Function $V : \mathbb{R}^n \to \mathbb{R}$ is positive semi definite if
  - $V(x) \geq 0$ for all $x \neq 0$
  - $V(x) = 0$ if and only if $x = 0$
- Definition 4: Negative semi definite function:
  Function $V : \mathbb{R}^n \to \mathbb{R}$ is negative semi definite if
  - $V(x) \leq 0$ for all $x \neq 0$
  - $V(x) = 0$ if and only if $x = 0$

Note that $V(x)$ is a scalar function of n variables, that is $V(x_1, x_2, ..., x_n)$. Its time derivative is

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + ... && (92) \\ &= \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 + ... && (93) \\ &= \nabla V(x) f(x) && (94) \end{aligned}$$

where ∇ is the gradient symbol.

Stability criterion

Let x_{eq} be an equilibrium point of system (91). Let $V : \mathbb{R}^n \to\in \mathbb{R}$ be a positive definite and continuously differentiable function

- If $\dot{V}$ is negative semi definite, then x_{eq} is stable
- If $\dot{V}$ is negative definite, then x_{eq} is asymptotically stable
- If $\dot{V}$ is positive definite, then x_{eq} is unstable

Example

Lets consider the following linear scalar system

$$\dot{x}(t) = -ax(t) \tag{95}$$

where $a > 0$. Lets define $V(x)$ as follows

$$V(x) = x^T 2x = 2x^2 \tag{96}$$

Clearly, $V(x)$ is positive definite function, its time derivative is

$$\dot{V}(x) = 4x(-ax) = -4ax^2 \tag{97}$$

Clearly, $\dot{V}(x)$ is negative definite, therefore, the system is asymptotically stable.

Linear time invariant case

We start with a simple definition

Definition

Matrix P is positive definite if

- $x^T Px > 0$ for all nonzero values of x
- All eigenvalues of P are positive

Example 1

Let

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{98}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{99}$$

and

$$x^T Px = x_1^2 + x_2^2 \tag{100}$$

which is positive when $x \neq 0$. Therefore, P is positive definite. Clearly, the eigenvalues of P are $\{1, 1\}$ and thus both positive.

The following result is concerned with the stability of LTI system

Theorem

A LTI system is asymptotically stable if and only if for any positive definite matrix Q there exist a positive definite symmetric matrix solution to the Lyapunov equation

$$A^T P + PA = -Q \tag{101}$$

Proof

We use a quadratic Lyapunov function

$$\begin{aligned} V(x) &= x^T Px \quad (102) \\ P &> 0 \quad (103) \end{aligned}$$

We have

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T Px + x^T P\dot{x} \quad (104) \\ &= (xA)^T Px + x^T PAx \quad (105) \\ &= x^T A^T PX + X^T PAx \quad (106) \\ &= x^T \left[A^T P + PA \right] x \quad (107) \\ &= -x^T Qx \quad (108) \end{aligned}$$

with

$$A^T P + PA = -Q \tag{109}$$

Therefore, for $\dot{V}(x)$ to be negative definite, matrix Q has to be positive definite.

Bounded input-bounded output stability

A system is said to be bounded input bounded output stable if every bounded input produces a bounded output. Bounded input means that there exist a finite constant K such that $u(t) \leq K$.

- Theorem 1: A SISO system is BIBO stable if and only if its impulse response $h(t)$ is absolutely integrable in the interval $[0, \infty)$, that is if

$$\int_0^\infty |h(\tau)| d\tau \leq M \tag{110}$$

  where M is finite constant.
- Theorem 2: BIBO stability and steady state response If a system with transfer function $G(s)$ is BIBO stable, then as $t \to \infty$, the output excited by a step $u(t) = a$ for $t \geq 0$ approaches $G(0)$.
  This result specifies the response of a BIBO stable system to a step input.
- Theorem 3: A LTI system transfer matrix is BIBO stable if and only if every pole of every entry of G_{ij} has negative real part.
  It is clear that there is a relationship between the poles and the eigenvalues of the state transition matrix since

$$G(s) = C(sI - A)^{-1}B + D \tag{111}$$

  Thus, if there is no pole–zero cancellation, the eigenvalues of A are the poles of the transfer function. However, not all eigenvalues of A will appear in $G(s)$ if there is zero–pole cancellation. The following example illustrates this case.

Example

Consider the following example

$$\dot{x} = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u \tag{112}$$

$$y = \begin{bmatrix} -1 & 3 \end{bmatrix} x - 2u \tag{113}$$

The eigenvalues of A are $\{-1, 1\}$, thus the system is not stable in the sense of Lyapunov. The system's transfer function is given by

$$G(s) = \frac{2(1-s)}{s+1} \tag{114}$$

The system is BIBO because it has a negative pole at -1. In conclusion

- Asymptotic stability $\Rightarrow$ BIBO stability
- BIBO stability $\not\Rightarrow$ Asymptotic stability

Figure 9 shows a comparison between internal and BIBO stability.

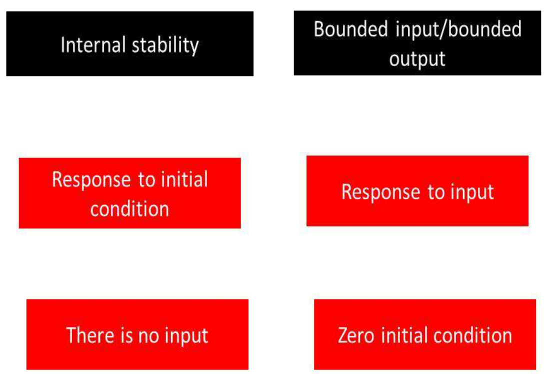

Fig. 9. Comparison between internal and BIBO stability